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SIZE EFFECT IN TENSILE AND COMPRESSION FRACTURE OF CONCRETE STRUCTURES: COMPUTATIONAL MODELING AND DESIGN

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Abstract

The paper has a two-fold purpose: (1) to review the main recent results on the problems of size effect in tensile and compression fracture of concrete structures, and (2) to present some new developments. The review focuses on simple approximate size effect formulae suitable for design, as well as the computational modeling of fracture required for the assessment of size effect. One new form of the formula for size effect in failures occurring at crack initiation is presented in this context. The new results include analysis of scaling of quasibrittle fracture through an extension of the J-integral and its use for the formulation of asymptotic size effect formula.

Key Words: Concrete, fracture, tension, compression, size effect.

1 Introduction

The energetic size effect, along with the energetic effects of shape such as the slenderness effect, is from the design viewpoint the most important feature of fracture mechanics of quasibrittle structures, i.e., structures failing only after a large stable crack growth of development of large cracking zones. Its understanding is essential for

efficient and safe design of concrete structures of large dimensions and structures made of high-strength concrete which are notorious for their brittleness.

The problem of size effect is an old one, in fact older than the mechanics of materials itself. The effect of the length of ropes under strength was discussed by Leonardo da Vinci (1500s) and Galileo (1638). The idea of a statistical size effect due to randomness of material strength was enunciated by Mariott (1686). However, the mathematical formulation of the statistical size effect had to await the formulation of appropriate probability concepts, which appeared during the 1920's with the formulation of the weakest link model and extreme value statistics (Peirce, 1926, Tippett, 1925, Fréchet, 1927, Fisher and Tippett, 1928, von Mises, 1936) and the discovery of the proper probability distribution bearing the name of Weibull (1939), with which the basic concept of the statistical size effect became complete. Applications of Weibull theory and the weakest link model to various types of structures have been developed subsequently and until about 1980 it was generally assumed that if a size effect was observed in experiments it had to be of statistical type. Today we know that this is not the case. In quasibrittle structures, the size effect is caused by energy release associated with localization of strain softening damage into a sizable fracture pressure zone or a long crack, and appears not only for tension but also for compression. This energetic size effect overwhelms the statistical size effect (Bažant and Planas, 1998).

A suggestion that the size effect observed in concrete structures might be non-statistical appeared already in the work of Leicester (1969, 1973); see also Karihaloo (1995). Taking analogy with the size effect of linear elastic fracture mechanics, well known from the inception of this theory, Leicester suggested that the nominal strength of concrete structure $\sigma_N = \text{const.}/D^s$ where D is a characteristic dimension of the structure and s is a constant between 0 and $\frac{1}{2}$. Such s -values were inferred by assuming failure to be caused by notches of a finite angle, due to the fact that their stress singularity exponent is larger than $-\frac{1}{2}$. But such notches could not be an acceptable justification of Leicester's formula, for two reasons: (1) notches of a finite angle cannot propagate (rather, a crack must emanate from the notch tip), (2) the singular stress field of such notches gives a zero flux of energy into the notch tip. Leicester's power law is in fact equivalent to Weibull theory, which also leads to a power law with an exponent larger than $-\frac{1}{2}$, and does not represent the quasibrittle energetic size effect. A power law can be true only for materials without a characteristic length and structures without a characteristic dimension

(see Bažant and Chen, 1997, Eqs. 1-3), and thus fails to capture this salient feature of quasibrittle energetic size effect, causing the transition from no size effect (as in plastic limit analysis) to the LEM size effect. Of course, this transition was not known to exist in 1969, being first detected in the experimental studies of Walsh (1972, 1976).

Finite element fracture models for concrete structures based either on the cohesive (fictitious) crack model (Hillerborg et al. 1976, Petersson, 1981) or the smeared crack band model (Bažant, 1976, 1982, Bažant and Cedolin, 1979, and Bažant and Oh, 1983) demonstrated the characteristic transition of the size effect from that of plasticity of small structures to that of LEM for large structures. These numerical results as well as the experimental evaluations of Walsh then led to the formulation of the energetic theory of quasibrittle size effect and a simple formula for the size effect law (Bažant, 1983, 1984).

For a long time engineers assumed that the use of fracture mechanics can be avoided by adopting the 'no-tension' design. However, the no-tension design is a plasticity approach, and so it exhibits no size effect, whereas fracture mechanics does. Therefore, in general, for a sufficiently large structure, fracture mechanics must predict a lower load capacity than the no-tension design. Detailed explanation of such behavior and numerical examples relevant to the design of dams have recently been given (Bažant 1996). The 'no-tension' design nevertheless remains a valuable simple design tool, but does not guarantee the required safety margin. Evaluation of the safety of dams requires fracture analysis (which is already a requirement of the U.S. Corps of Engineers).

Aside from the energetic quasibrittle size effect, suggestions have been made that the size effects observed in unnotched concrete structures might have a fractal origin, either in the invasive fractality of crack surfaces or the lacunar fractality of microcrack distributions (Carpinteri, 1994, 1995, Carpinteri et al. 1994, 1995). However, during the six years since the emergence of this idea, the mathematical justification of this idea has not progressed beyond a strictly geometrical argument, lacking mechanics. On the contrary, mechanical analysis showed inconsistencies indicating that fractality is not an acceptable explanation, whether invasive or lacunar (Bažant, 1996).

The purpose of the present paper is to survey the main results on the size effect and fracture of concrete. However, since an extensive review has been published very recently (Bažant and Chen 1997), the present review will be very brief. Some new results on generalized size effect analysis by means of the J-integral, applicable to both tensile and compression fractures, will also be presented.

2 Selective Survey of Basic Results

The design of a structure is an inverse boundary value problem, in which the material characteristics must be determined from the desired solution, such as the load carrying capacity, the ductility or the structure proportions. However, the inverse problem is far more difficult than the direct problem. The sophisticated computational approaches that exist today, such as the nonlocal finite element analysis, are difficult enough for solving the direct problem and are extremely cumbersome and ineffective for the inverse problem of design. One can of course play with the values of structure dimensions and material parameters in a trial-and-error fashion until a desired solution is luckily obtained, however, the task is immensely complicated by the fact that the computational models do not provide a clear understanding of the influence of structure shape, dimensions and material parameters. Such understanding can be provided only by simple formulae. They are of vital help to the designer, even if they are only approximate and sometimes very crude.

There are two kinds of simple formulae in concrete design: those purely empirical, and those supported by some theory. The latter are preferable by far, because only they offer the possibility of extrapolating beyond the range of the existing (statistically significant) test results and field experience. This is especially important for innovative designs.

2.1 Size and Shape Effects via Equivalent LEFM

Since the size effect in concrete and other quasibrittle structures represents a transition between plastic limit analysis for small sizes and linear elastic fracture mechanics (LEFM) for large sizes, the best strategy to obtain simple formulae is to use some sort of intelligent interpolation between these two extreme cases for which simple solutions exist. Such interpolation can be obtained by deducing asymptotic expansions of the size effect and then truncating them and matching them so as to obtain an approximation for the immediate range, which is of practical interest.

The fracture analysis in this approach relies on the concept of equivalent LEFM, in which a structure with a crack and a large fracture pressure zone is approximated by a structure with a sharp crack, and the effect of the sizable fracture pressure zone is approximated by certain finite extension of the actual traction-free crack. Application of the equivalent LEFM yields for the nominal strength of the

structure the expression

$$\sigma_N = \sqrt{\frac{EG_f}{Dg(\alpha_0 + \frac{c_f}{D})}}, \quad \alpha_0 = \frac{a_0}{D} \quad (1)$$

(Bažant, 1996, Bažant and Planas, 1998) in which E = Young's modulus, G_f = fracture energy of the material (asymptotic value of the R -curve) D = structure size (characteristic dimension), g = non-dimensionalized energy release function characterizing the structure geometry (shape), a_0 = notch length or traction-free crack length, and c_f = effective length of fracture pressure zone (roughly $\frac{1}{2}$ of the actual length). Expanding function g into a Taylor series, representing an asymptotic expansion with respect to D , provides

$$\begin{aligned} \sigma_N &= \sqrt{\frac{EG_f}{[g(\alpha_0) + g'(\alpha_0)(c_f/D) + \frac{1}{2!}g''(\alpha_0)(c_f/D)^2 + \dots]}} \\ &\approx \frac{Bf'_t}{\sqrt{1 + (D/D_0)}} \end{aligned} \quad (2)$$

in which,

$$D_0 = c_f \frac{g'(\alpha_0)}{g(\alpha_0)}, \quad Bf'_t = \sqrt{\frac{EG_f}{c_f g'(\alpha_0)}} \quad (3)$$

Here f'_t = tensile strength of material, introduced for convenience, B = dimensionless constant, and D_0 = transitional size. The ratio $\beta = D/D_0$ is called the brittleness number, characterizing the structure brittleness in a way that is independent of the structure geometry. It is important to realize that Eq. (2) describes not only the size effect but also the shape effect, which is embedded in the LEFM function $g(\alpha)$, which is available for many situations in handbooks and can be relatively easily obtained by linear elastic finite element analysis.

The theoretical support of Eq. (2) is now quite broad. This law has been derived by five different theoretical approaches: (1) asymptotic expansions and asymptotic matching, as mentioned above; (2) by simple energy release analysis based on the concept of stress relief zones (Bažant 1983, 1984); (3) as a second-order asymptotic approximation of the Jenq-Shah model for concrete fracture (which is similar to the Wells-Cottrell model for metal fracture) (e.g. Bažant and Planas 1998); (4) as a deterministic limit of a nonlocal generalization of Weibull theory of statistical size effect; and (5) by J-integral analysis, as shown in the sequel. There also exists extensive computational validation of this formula by means of (1) nonlocal finite

element analysis of notched or unnotched structures of various geometries; (2) calculations by the cohesive (fictitious) crack model of Hillerborg type; and (3) various lattice models, random particle models or discrete element models of heterogeneous microstructure.

It must be emphasized that Eq. (2) is valid only for notched specimens or for structures that obtain the maximum load after a large stable crack growth, which is typical of most reinforced concrete structures. For failures at crack initiation, (an undesirable type of failure which is however inevitable in many unreinforced concrete structures), a different law for the size effect ensues from Eq. (1). In this case, $\alpha_0 = 0$, and because the energy release rate for a zero crack length is zero, $g(0) = 0$, the first term of the series expansion in (2) vanishes and one must truncate the series no earlier than after the third, quadratic term. This yields the expansion

$$\begin{aligned} \sigma_N &= \lim_{\alpha_0 \rightarrow 0} \sqrt{\frac{EG_f}{Dg(\alpha_0 + \frac{c_f}{D})}} \\ &= \sqrt{\frac{EG_f}{g'(0)c_f + \frac{1}{2!}g''(0)c_f^2D^{-1} + \frac{1}{3!}g'''(0)c_f^3D^{-2} + \dots}} \end{aligned} \quad (4)$$

$$= \frac{f_{r\infty}}{\sqrt{1 - (q_1/D) + (q_2/D)^2 - (q_3/D)^3 + \dots}} \quad (5)$$

in which $f_{r\infty} = \sqrt{EG_f/c_f g'(0)}$, $q_1 = -c_f g''(0)/2!g'(0)$, and $q_2^2 = c_f^2 g'''(0)/3!g'(0), \dots$. Denoting $x = (q_1/D) - (q_2/D)^2 + (q_3/D)^3 - \dots$, and using the binomial series expansions $1/\sqrt{1-x} = 1 + (1/2)x + (3/8)x^2 - (5/16)x^3 + \dots$ and $1/(1-x) = 1 + x + x^2 + x^3 + \dots$, the last expression can be converted to two different kinds of expansions (both of which are exact and equal to each other if the number of terms approaches infinity):

$$\begin{aligned} \sigma_N &= f_{r\infty} \left[1 + \frac{q_1}{2} \frac{1}{D} + \left(\frac{3q_1^2}{8} - \frac{q_2^2}{2} \right) \frac{1}{D^2} + (\dots) \frac{1}{D^3} + \dots \right] \\ &\approx f_{r\infty} \left(1 + \frac{D_b}{D} \right) \end{aligned} \quad (6)$$

$$\begin{aligned} \text{or } \sigma_N &= \sqrt{f_{r\infty}^2 \left[1 + q_1 \frac{1}{D} + (q_1^2 - q_2^2) \frac{1}{D^2} + (\dots) \frac{1}{D^3} + \dots \right]} \\ &\approx \sqrt{A_1 + \frac{A_2}{D}} \end{aligned} \quad (7)$$

Here $D_b = -c_f g''(0)/4g'(0)$ ($D_b > 0$ has the meaning of the thickness

of the boundary layer of cracking, and

$$A_1 = f_{r\infty}^2 = \frac{EG_f}{[c_f g'(0)]^2}, \quad A_2 = f_{r\infty}^2 q_1 = 2f_{r\infty} D_b = -\frac{EG_f g''(0)}{2c_f [g'(0)]^3} \quad (8)$$

The approximate size effect formulae in (6) and (7) are asymptotically equivalent up to terms of the order of $1/D$, i.e. the derivatives $d\sigma_N/d(1/D)$ at $1/D \rightarrow 0$ are equal. These formulae have been obtained by truncating the two different infinite series expansions after the second term. Eq. (6) has been validated by numerous test results for the size effect on the modulus of rupture (bending strength of beams) and appears to agree with the data better than (7). Eq. (7) was recently proposed and used to describe some size effect data by Carpinteri (1994, 1995) and Carpinteri et al. (1995) under the name 'multifractal' scaling law (this name, however, is questionable because, according to Bažant (1996), the mechanical analysis of the effect of lacunar fractal microcracks leads to a formula of Weibull type, which is different from Eq. (8)).

The foregoing simple formulae for the size effect should, in principle, be applicable to various types of brittle failures of concrete structures, however, the precise manner of application still needs to be clarified for many cases. The difficulty is to determine the equivalent LEFM model that describes the extrapolation to sizes larger than those feasible in practice, in particular, the location of the major crack causing failure, its shape and length, the effective length of the fracture process zone, the value of the transitional size, and in some cases possible branching of the main crack (as recently discovered by Jirásek for the case of a curved dipping crack in a concrete dam). For this reason, the parameters of Eq. (2), B and D_0 , need to be determined for the time being in many situations from experiments.

2.2 Compression Fracture

The term 'compression fracture' still occasionally induces bewilderment since compression is normally not imagined to cause an open crack. This is of course literally true, however, the failure of quasibrittle materials in compression exhibits all the basic features captured by fracture mechanics, except when the confining hydrostatic pressure is extremely high. The fact that the material exhibits softening damage means that there is a localization of damage strain into the compression fracture band, accompanied by stress reduction on the flanks of the band, which in turn causes energy release. The rate of energy release must be equal to the rate of energy consumption and

dissipation at the front of a propagating compression fracture band. Because of the localization, a strength criterion would be unobjective in describing the propagation of the band. It must be either an energy criterion, or a relation between the stress across the band and the relative displacement between the flanks. Thus, all the basic attributes of fracture mechanics need to be used to describe compression fracture.

Compression fracture can have a greater variety of forms and is a richer phenomenon than tensile fracture. There are many microscopic fracture mechanisms that can produce a zone of axial splitting microcracks in the direction of the principle compressive stress of the greatest magnitude (Ashby and Hallam 1986, Kemeny and Cook 1987, 1991, Ingraffea 1977, Nesetová and Lajtai 1973, Kachanov 1982, Lehner and Kachanov 1996, Batto and Schulson 1993). They include cracks emanating from voids, cracks near inclusions, wedging actions in groups of inclusions and the so-called wing tip cracks. For global fracture analysis, these axial splitting microcracks must be treated in a smeared manner, as a continuum with damage.

Although the global fracture can take many forms, there are basically two: (1) the axial propagation of a band of splitting microcracks, and (2) the lateral propagation (either octagonal or inclined) of a band of splitting microcracks. As shown previously, the former mechanism cannot engender any size effect because it does not induce energy release from the entire structure, only from the damaged band. The latter mechanism, on the other hand, does engender a size effect, similar to that of tensile fracture, because the energy release zones grow faster than linearly with the length of the fracture band. Because axial propagation is locally easier, the splitting fracture should dominate for small enough structures, while the lateral propagation of a damaged band should dominate for large enough structures. The structure shape and boundary conditions, of course, have a large influence as well.

The principal difference from tensile fracture is that there is a residual compressive stress across the fracture band, σ_Y . In the simplest analysis (sometimes probably too crude), σ_Y can be assumed to be constant, i.e., uniformly distributed along the fracture band. An asymptotic analysis of the energy release that leads to the following formulae (Bažant and Chen, 1997, Bažant and Planas, 1998): for the case when a long fracture band develops before the maximum load

$$\sigma_N = \frac{\sqrt{EG_f + \sigma_Y \sqrt{\gamma'(\alpha_0)c_f + \gamma(\alpha_0)D}}}{\sqrt{g'(\alpha_0)c_f + g(\alpha_0)D}} \quad (9)$$

and for the case when the maximum load occurs at the initiation of

compression fracture (as soon as the fracture pressure zone, of non-negligible size, has formed):

$$\sigma_N = \frac{\sqrt{EG_f + \sigma_Y \sqrt{\gamma'(0)c_f + \frac{1}{2}\gamma''(0)\frac{c_f^2}{D}}}}{\sqrt{g'(0)c_f + \frac{1}{2}g''(0)\frac{c_f^2}{D}}} \quad (10)$$

Here g is the same non-dimensionalized energy release function for the given applied load, as already introduced for tensile fracture (Eq. 2), and γ is an analogous non-dimensionalized energy release function for a uniform traction applied at the flanks of the band, treated as a crack. This function can be easily determined by linear elastic finite element analysis. Another difference from tensile fracture is the value of the fracture energy of the band, G_b which may be expressed as $G_b = G_f h/s$ where h is the width of the band, s is a typical spacing of the axial splitting cracks of the band, and G_f is the fracture energy of the axial splitting cracks. G_b is much larger than the tensile fracture energy, typically by one or two orders of magnitude.

Although the experimental evidence for compression fracture is still very limited, the existence of size effect in certain important cases is already clear. Size effect was demonstrated in the reduced scale tests of tied square reinforced concrete columns of various sizes and slendernesses, carried out at Northwestern University (Bažant and Kwon, 1993), and in similar but larger scale test of columns made from normal concrete by Barr and Şener at University of Wales, Cardiff. These results have been modeled analytically in Bažant and Xiang (1997).

An analogous fracture analysis applied to the breakout of boreholes in rock under high compression indicates also size effect, which is verified by tests of Carter et al. (1992), Haimson and Herrick (1989) and others. A special type of compression fracture probably governs the maximum load in the diagram of shear failure of reinforced concrete beams with or without stirrups, for which the existence of size effect has already been clearly established. Much more research, however, needs to be done to master the problem of size effect in compression fracture.

2.3 Design Formulae and Energetic Fracturing Truss Model

The formulae for designing against failure after large crack growth in various types of structures should in general have the form:

$$\sigma_N = \sigma_{N0} + \sigma_{N1} \sqrt{\frac{1 + \beta_0}{1 + \beta}} \quad (11)$$

in which $\beta = D/D_0 =$ brittleness number (geometry independent), $\beta_0 =$ reference value of β , and $\sigma_{N0}, \sigma_{N1} =$ positive constants. The value of σ_{N0} can be obtained by plastic limit analysis based on the yield strength of steel or the residual stress in compression fracture band. However, practical prediction of the values of these parameters is still not clear for various types of failure. Of course, the values of these parameters can be calibrated by experimental results.

One type of failure for which extensive studies of the size effect and fracture mechanism have already been made is the diagonal shear failure of reinforced concrete beams without or with stirrup. Recently, a new type of analysis of this type of failure has been proposed (Bažant 1998) as a modification of the classical truss model (strut-and-tie model) (Ritter 1899, Mörsch 1902, Collins 1978, Vecchio and Collins 1986, Schlaich et al. 1987, Thürlimann 1976). In this model it is recognized that the failure of the idealized truss cannot be simultaneous but must be progressive. It is logical to assume that, at maximum load, it consists of propagation of a compression fracture band across the compression strut of concrete, the band being localized within a small portion of the length of the truss and having a thickness essentially determined by the heterogeneity of the material. In contrast to the classical plastic limit analysis, the analysis is conducted on the basis of the energy release, which captures the progressive nature of failure. The result of the analysis is the formula which is basically of the form of Eq. (11) (Bažant, 1998). In a separate paper in these proceedings (Bažant and Becq-Giraudon), the formula resulting from the energetic fracturing stress model is compared to extensive experimental results and its parameters are calibrated.

2.4 Computational Approaches to Validating Design

In the case of sensitive or innovative structural designs, it is necessary to verify the design computationally, for example, by finite element analysis of damage evolution and fracture. This is a subject that has been intensely studied for almost two decades and extensive knowledge has been acquired. The methods that are available, with various degrees of sophistication and physical justification, can be classified in six basic types:

1. The cohesive (fictitious) crack model pioneered for concrete by Hillerborg et al. (1976).
2. The smeared crack band model (Bažant, 1976, 1982, Bažant and Cedolin, 1979, Bažant and Oh, 1983).

3. Nonlocal integral type models, which average the cracking damage over a zone characterized by a certain material length.
4. Gradient models, which can be regarded as approximations of the nonlocal models obtained by Taylor series expansion of the integration kernel.
5. Nonlocal model based on crack interactions (Bažant, 1994, Bažant and Ožbolt, 1996).
6. Generalization (by Jirásek) of the element-free Galerkin (EFG) method (pioneered by Nayrolles, and Belytchko and Tabbara) to cohesive fracture or smeared band cracking, which utilizes moving least-square estimates to extract from nodal values not only the stress intensity factor but also further nonlinear fracture propagation characteristics.

The simplest form of the smeared crack model involves damage spread over the entire finite element. A more precise description of fracture with not too small elements is possible by introducing elements with discontinuities, pioneered by Ortiz et al. (1987) and Belytschko et al. (1988). Many models with discontinuities of strain or displacement have been developed during the last several years. An important study of these models has recently been contributed by Jirásek (1998), who presented a systematic approach to various types of kinematic enhancement and the formulation of the stress continuity condition. He distinguished statically optimal symmetric formulations, kinematically optimal symmetric formulations and mixed formulations, the last leading to a non-symmetric stiffness matrix of the element. He extended the formulation of the elements with discontinuity by introducing a combination with the smeared crack model which, together with the concept of rotating crack, can correct initially erroneous fracture directions in the finite element in the initial stage of fracturing. The discontinuity is introduced only later, after a certain degree of smeared cracking. The smeared part should be properly formulated in a nonlocal form. Jirásek showed that his new approach is essentially insensitive to the layout of the mesh and avoids the problems with stress locking.

Among the gradient models, those employing the first gradients (which correspond to the continuum of Cosserat or micropolar type) are ineffective for tensile fracture of concrete. The current studies focus generally on the second gradient models. Such models can be best looked at as approximations of the nonlocal formulation obtained by Taylor series expansion of the kernel, which is how the second gradient model was originally obtained by Bažant (1984b). A questionable

aspect of the gradient models is their neglect of long range interactions, due to the truncation of the Taylor series expansion of the kernel. According to the nonlocal model based on crack interactions (Bažant, 1994), such long range interactions are not negligible because the crack interactions decay rather weakly with the distance between cracks.

A large amount of studies have dealt with the constitutive relations for strain softening cracking materials, which are more properly characterized as the constitutive relations for the fracture pressure zone. The classical approaches consist of various adaptations of phenomenological plasticity or continuum damage mechanics models expressed in terms of tensorial invariants. Recently, due to large increase of computational power, the microplane model is gaining ground and has been successfully used in a hydrocode for the modeling of impact involving several hundred thousand finite elements, and an explicit integration procedure. Improved versions of the microplane model for concrete are emerging, capable of a general description of fracture phenomena. It is a particular advantage of the microplane model that it can be realistically applied not only to tensile fracture, but also to compression fracture (failure with localization of compressive strain softening into a band); Bažant et al. 1996.

The microplane models are limited by the assumption of kinematic constraint between the inelastic phenomena on the microscale and the macroscopic continuum. To relax this assumption, it seems advantageous if the methods of formulation of equivalent elastic properties of composites could be transplanted to the analysis of a body with many cracks. This problem has been studied extensively, beginning with Budianski and O'Connell (1976), Hoenig (1979) and Kachanov (1992). However, these classical models, based on the self-consistent method for composites, predict only the stiffness of the material under the assumption that the crack sizes are fixed. In reality, cracks grow during loading, and the crack growth causes strain softening. An extension of the composite models for a material with many cracks that are allowed to grow during loading has recently been developed by Bažant and Prat (1997). Such models for fracture pressure zone bear probably the greatest promise at present, but much work remains to be done.

3 Generalized Size Effect Analysis by J-Integral

In connection with the present survey, some new results on the application of Rice's J-integral to the analysis of size effect will now

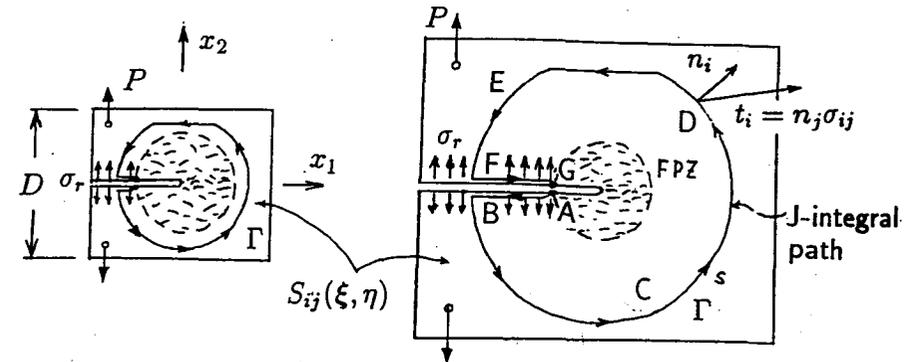


Fig. 1 Path of J-integral for size effect analysis

be presented. This approach provides the most general derivation of the size effect law and lends itself naturally to a generalization for compressive fracture in which normal stresses are transmitted across the cracking band.

We consider geometrically similar structures scaled in two dimensions (the treatment for three dimensions, however, would be analogous). We introduce dimensionless cartesian coordinates (Fig. 1) $\xi_i = x_i/D$ and dimensionless displacements $\zeta_i = u_i/D$ where $i = 1, 2$. For two-dimensional similarity, the elastic material compliances scale as $C_{ijkl} = c_{ijkl}/E$ where c_{ijkl} are constant. If c_f were zero, the stresses would scale as $\sigma_{ij} = \sigma_N S_{ij}(\xi)$ where $\xi =$ coordinate vector of ξ_i , and S_{ij} are size-independent functions. However, the presence of nonzero material length c_f will influence the stress distributions. Based on the principles of dimensional analysis, this influence and the influence on the displacement field must have the form:

$$\sigma_{ij} = \sigma_N S_{ij}(\xi, \theta), \quad u_i = (\sigma_N/E) D \zeta_i(\xi, \theta) \quad (12)$$

where $\theta = c_f/D$ and ζ_i are dimensionless functions. The flux of energy into a fracture process zone advancing in the direction of x_1 (Fig. 1) can be calculated by Rice's J-integral:

$$\begin{aligned} J &= \int_{\Gamma} (W n_1 - n_j \sigma_{ij} u_{i,1}) ds \\ &= \int_{\Gamma} \left(\frac{1}{2} C_{ijkl} \sigma_{ij} \sigma_{kl} n_1 - n_j \sigma_{ij} \frac{\partial u_i}{\partial x_1} \right) ds \\ &= \int_{\Gamma} \left(\frac{\sigma_N^2}{2E} c_{ijkl} S_{ij} S_{kl} n_1 - n_j \sigma_N S_{ij} \frac{\sigma_N D}{E} \frac{\partial \zeta_i}{\partial \xi_1} \right) D d\bar{s} \\ &= \frac{\sigma_N^2 D}{E} \mathcal{J}(\theta), \end{aligned} \quad (13)$$

$$\mathcal{J}(\theta) = \int_{\Gamma} \left(\frac{1}{2} c_{ijkl} S_{ij}(\xi, \theta) S_{kl}(\xi, \theta) - n_j S_{ij}(\xi, \theta) \frac{\partial \zeta_i(\xi, \theta)}{\partial \xi_1} \right) d\bar{s} \quad (14)$$

Here Γ are geometrically scaled closed integration contours BCDE (Fig. 1) with length coordinate s , starting and ending on the crack and passing outside the fracture process zone, $\bar{\Gamma}$ = chosen fixed contour in dimensionless coordinates, with length coordinate \bar{s} ($ds = Dd\bar{s}$), n_i = unit normal to the contour (which does not change with scaling), W = strain energy density; $\mathcal{J}(\theta)$ is the dimensionless J-integral. This integral may be expanded in Taylor series, providing

$$\mathcal{J}(\theta) = \mathcal{J}_0 + \mathcal{J}_1\theta + \mathcal{J}_2\theta^2 + \dots \quad (15)$$

$$\mathcal{J}_0 = \int_{\bar{\Gamma}} [c_{ijkl} S_{ij}^0 S_{kl}^0 / 2 - n_j S_{ij}^0 \zeta_{i,1}^0] d\bar{s}, \quad (16)$$

$$\mathcal{J}_1 = \int_{\bar{\Gamma}} [c_{ijkl} (S_{ij}^0 S_{kl,\theta}^0 + S_{kl}^0 S_{ij,\theta}^0) / 2 - n_j (S_{ij}^0 \zeta_{i,1\theta}^0 + S_{ij,\theta}^0 \zeta_{i,1}^0)] d\bar{s} \quad (17)$$

Superscript 0 labels the values or fields evaluated for $\theta = 0$ (which is the case of LEFM). Substituting this into (13) and truncating the series after the second term, one gets again the formula of size effect law:

$$\sigma_N = \sqrt{\frac{2EG_f}{[\mathcal{J}_0 + \mathcal{J}_1(c_f/D) + \mathcal{J}_2(c_f/D)^2 + \dots]D}} \approx \frac{Bf/\lambda}{\sqrt{1 + (D/D_0)}} \quad (18)$$

The foregoing derivation has been simplified in the sense that the length parameter influencing J has been considered as a known constant. Although this seems a good approximation, one could more generally consider J to depend on c/D instead of c_f/D , where c is an unknown crack length. One could then also introduce a variable fracture resistance in the form of an R -curve, and impose the maximum load condition as the condition of the tangency of the R -curve to the energy release curve, in the same manner as used in equivalent LEFM analysis by Bažant (1996). The size effect law ensuing from such refined analysis is the same.

The foregoing derivation can be generalized to the case of fracture with a known residual crack-bridging stress σ_r applied on the crack faces, as considered for compression fracture (Fig. 1). In that case the stress distributions for various sizes are written as

$$\sigma_{ij} = \sigma_N S_{ij}(\xi, \theta) + \sigma_r T_{ij}(\xi, \theta) \quad (19)$$

The J-integral must in this case be generalized by extending its path along the crack surfaces along which the work is non-zero (Fig. 1). As it transpires, the path must begin and end at points on the crack

surfaces lying at the boundary of the fracture process zone (points A and B in Fig. 1), i.e., the integration path must be ABCDEF because the contribution from the path segments AB and EF along the crack surface is not zero. The subsequent procedure is analogous, leading again to equations of the form of (9) and (10).

Closing Comment

In the community of concrete fracture researchers, it has been clear since the 1980's that the design code formulae for checking various types of brittle failures of reinforced and plain concrete structures should include the quasibrittle size effect due to energy release and be based on fracture mechanics rather than plastic limit analysis. However, relatively little progress has been achieved so far. Insufficient education of civil engineers in fracture mechanics concepts has been one major obstacle, inducing a similar reluctance as the introduction of plastic limit analysis concepts did half a century ago. The relative scarcity of properly scaled and unambiguously interpretable tests of full-size structures has been another major obstacle. The researchers and educators in this field should strive to overcome both.

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