MICROMECHANICS OF FAILURE OF QUASI-BRITTLE MATERIALS

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RECENT ADVANCES IN FAILURE LOCALIZATION AND NONLOCAL MODELS

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ABSTRACT

The lecture presents an apercu of recent advances in the modeling of damage and fracture of micro-inhomogeneous materials by nonlocal continuum. After reciting the basic problems with the classical smeared cracking model, the concept of nonlocal continuum is described and the manner in which it serves as a localization limiters is discussed. Arguments for adopting the nonlocal approach are then reviewed, and a particularly effective form of the nonlocal approach, called the nonlocal continuum with local strain (or nonlocal damage) is described in some detail. Various examples of large-scale finite element analysis of strain-softening structural failures are presented. Finally, the lecture presents in greater detail a new nonlocal form of the microplane model for concrete, which is particularly versatile and powerful in its material modeling capability, being able to simulate both tensile and compressive splitting fractures as well as general nonlinear triaxial behavior of concrete. The microstructural mechanisms, stability aspects of damage localization and their implications for size effect, which are also discussed in the lecture, are beyond the scope of the present article, which is focused only on finite element modeling.

INTRODUCTION

The nonlocal approach is an effective and general way to deal with the problems of localization and spurious mesh sensitivity (or inobjectivity) inherent to distributed damage in general and smeared cracking in particular. The purpose of the present lecture is to review recent advances in this approach and attempt an appraisal of the current research directions. In addition, the lectures will present in more detail a new versatile constitutive-fracture model for concrete (Bažant and Ožbolt [1,2]) which represents a nonlocal generalization of the microplane model and can describe both tensile a compression splitting fractures as well as general nonlinear triaxial behavior. Although the lecture will also discuss the stability aspects, microstructural mechanisms and size effects associated with damage localization, space limitations demand that the present article be limited to finite element modeling.

SMEARED CRACKING: REASONS FOR ITS USE AMD PROBLEMS

If the smeared cracking approach is adopted, and if at the same time localization of cracking into a zone of arbitrarily small width is a possibility, one must use some form of localization limiter, which prevents the smeared cracking zone from becoming narrower than a certain minimum width w_c that is a material property. (This limitation applies only for the actual material properties; if these properties are modified, a smaller width can of course be permitted; see ACI [3]. The smeared cracking, introduced be Rashid [4] has become the most widely used approach in practice. One may give four reasons for adopting this approach:

1. Computational convenience.
2. The fact that distributed damage in general and densely distributed parallel cracks in particular are indeed often observed in structures (the best evidence to date of a wide zone of distributed damage in front of a fracture is provided my measurements of the locations of sound emission sources).
3. The fact that a crack in concrete is not straight but highly tortuous, and such a crack is represented by a smeared crack band probably better than by a line crack.
4. In case that parallel cracks in concrete are densely and uniformly distributed, the fact such cracks are well represented by smeared cracking (with the minimum crack band width equal of the actual spacing of the parallel cracks), while the line crack model is for this case unobjective unless it is generalized by imposing a minimum spacing of the cracks (equal to the crack band width).

There are, however, serious problems with the classical smeared cracking models. They are in principle unobjective as they exhibit spurious mesh sensitivity (Bažant [5]), i.e. the results may depend significantly on the subjective choice of the mesh size (element size) by the analyst. For example, in a tensioned rectangular plain concrete panel with a rectangular finite element mesh, the cracking localizes into a band of single-element width. The stress in the element just ahead of the crack band front increases as the mesh is refined, and tends to infinity as the element size tends to zero (Bažant and Cedolin [6,7,8]; Darwin [9], Rots et al. [10]). Consequently, the load needed to extend the crack band into the next element is less for a finer mesh (it decreases roughly by the factor $1/\sqrt{2}$ if the element size is halved, and tends to zero as the element size tends to zero). In structures where the crack band grows at increasing load (e.g. diagonal shear failure of beams), the maximum load decreases as the mesh is refined (Bažant and Cedolin, discussion of [2]). Furthermore, the energy that is consumed (and dissipated) during structural failure depends of the mesh size, and tends to zero as the mesh size tends to zero. Such behavior, which is encountered not only for cracking with a sudden stress drop but also for gradual crack formation with a finite slope of the post-peak tensile strain-softening stress-strain diagram (Bažant and Oh [11]), is unobjective. It makes the classical smeared cracking unacceptable as a general approach, although in some structures such inobjectivity might be mild or even negligible (this occurs especially when the failure is due to yield of reinforcement rather than cracking of concrete).

To avoid the inobjectivity or spurious mesh sensitivity, some form of a mathematical device called localization limiter must be introduced. A general form of localization limiter is implied in nonlocal averaging.

NONLOCAL CONTINUUM CONCEPT AND SPATIAL AVERAGING

Nonlocal continuum is a continuum in which at least some field variables are subjected to spatial averaging over a certain finite neighborhood of a point. For example, the average (nonlocal) strain is defined as

$$\bar{\varepsilon} = \frac{1}{V_r(x)} \int_V \alpha(x - s)\, \varepsilon(s)\, dV(s) = \int_V \alpha'(x, s)\, \varepsilon(s)\, dV(s) \qquad (1)$$

in which $V_r(x) = \int_V \alpha(x - s)\, dV(s)$ and $\alpha'(x, s) = \alpha(x - s)/V_r(x)$; $\varepsilon(x)$ is the usual (local) strain at a point of coordinate vector x; V is the volume of the structure; V_r for an infinite solid represents the representative volume of the material (*Fig. 1*), understood as the smallest volume for which the heterogeneous material can be treated as a continuum (the size of V_r is determined by the characteristic length ℓ, which introduces the localization limiter); α is the given weight function, which decays with the distance from point x and is zero or nearly zero at points sufficiently remote from x; and the superimposed bar denotes the averaging operator.

As the simplest form of the weight function, one may consider $\alpha = 1$ within a certain representative volume V_0 centered at point x and $\alpha = 0$ outside this volume. Convergence of numerical solutions, however , is better if α is a smooth bell-shaped function. An effective choice is (*Fig. 1c*)

$$\alpha = [1 - (r/\rho_0\ell)^2]^2 \quad \text{if} \quad |r| < \rho_0\ell, \qquad \alpha = 0 \quad \text{if} \quad |r| \geq \rho_0\ell \qquad (2)$$

where $r = |x - s|$ = distance from point x, ℓ = characteristic length (material property, *Fig. 1*), and ρ_0 = coefficient chosen in such a manner that the volume under function α given by Eq. 2 be equal to the volume under function $\alpha = 1$ (*Fig. 1c*) for $r \leq \ell/2$ and $\alpha = 0$ for $r > \ell/2$ (which represents a line segment in 1D, a circle in 2D, and a sphere in 3D). From this requirement, $\rho_0 = 15/16 = 0.9375$ for one dimension, $\rho_0 = \sqrt[3]{3/4} = 0.9086$ for two dimensions, and $\rho_0 = \sqrt[3]{105/192} = 0.8178$ for three dimensions. (Alternatively, the normal distribution function has also been used instead of Eq. 2 and was found to work well enough, although its values are nowhere exactly zero.) Note that the limit of nonlocal continuum for $\ell \to 0$ is the local continuum (because $\bar{\varepsilon} \to \varepsilon$).

For points whose distance from all the boundaries is larger than $\rho_0\ell$, $V_r(x) = 1/\ell$ = constant; otherwise the averaging volume protrudes outside the body, and then $V_r(x)$ is variable because the averaging domain is not constant (*Fig. 1b*).

In finite element computation, the spatial averaging integrals are evaluated by finite sums over all the integration points of all the finite elements of the structure. For this purpose, the matrix of the values of α' for all the integration points of all the elements is computed and stored in advance of finite element analysis.

The concept of nonlocal continuum was originally conceived and over a long time studied for elastic materials with a randomly heterogeneous microstructure (Kröner [12], Krumhansl [13], Levin [14], Kunin [15], Beran and McCoy, Eringen and Edelen [16], Eringen and Ari [17]). The macroscopic continuum stresses $\sigma(x)$ and strains $\varepsilon(x)$ in such a material are defined as the statistical averages of the randomly scattered microstresses $\sigma_M(x)$ and microstrains $\sigma_M(x)$, taken over a suitable representative volume around point x.

The statistical theory of elastic heterogeneous material developed during the 1960's showed that if $\varepsilon(x)$ is nonuniform, the macroscopic constitutive relation is not exactly of the form $\sigma(x)$ = function of $\varepsilon(x)$ but $\sigma(x)$ also depends on the the average macroscopic strain $\bar{\varepsilon}(x)$ from a certain

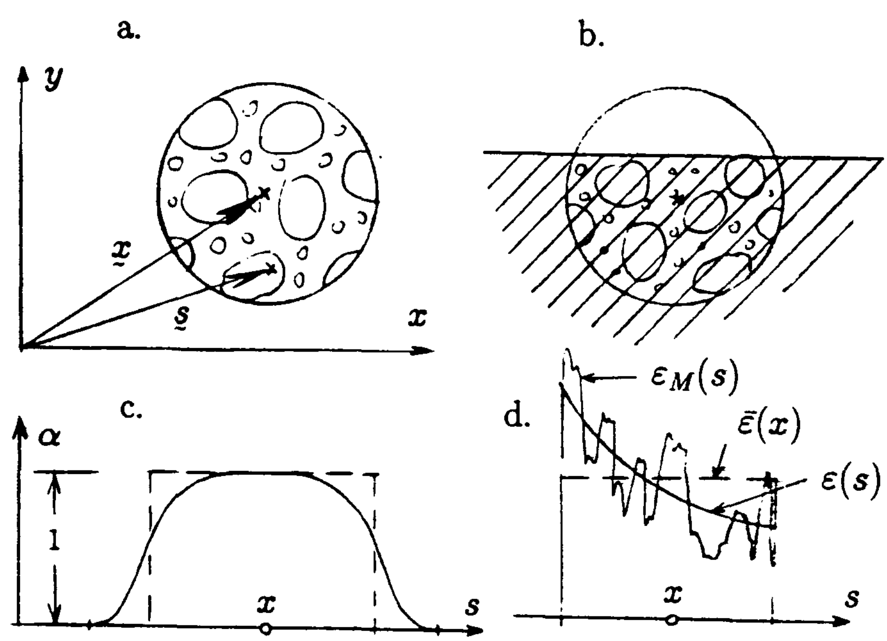

Figure 1. Spatial averaging.

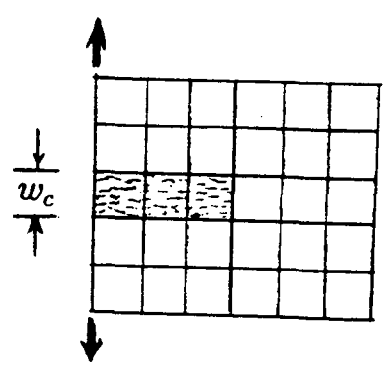

Figure 2. Crack band model.

characteristic volume centered at point x, even though $\varepsilon(x)$ itself is and average of the microstrain field $\varepsilon_M(x)$ from the representative volume (*Fig. 1d*). This important result provided the original impetus for the development of nonlocal elasticity.

During the 1980's, the nonlocal concept was established as an effective means to impose a limitation on the minimum width of a region in which cracking damage can localize (Bažant, Belytschko and Chang [18], Bažant [19]; cf. a review in Bažant [20]).

TYPES OF LOCALIZATION LIMITERS

Several types of localization limiters have been proposed:

1. *Crack Band Model.* The simplest localization limiter is to impose a minimum admissible finite element size, w_c, which can be used in conjunction with the actual material properties (if the post-peak declining slope of the stress-strain diagram is decreased, however, this minimum size can be reduced). This is the essence of the crack band model (Bažant [21]; Bažant and Oh [11]); *Fig. 2*.

Practically the most important feature of the crack band model is that it can represent the effect of structure size on: (1) the maximum load of the structure, in agreement with the (approximate) size effect law (proposed in Bažant [22]), and (2) on the slope of the post-peak load-deflection diagram. The classical smeared cracking, on the other hand, cannot represent the size effect. (We mean the deterministic size effect due to the release of stored energy of the structure, not the statistical Weibull-type size effect, which plays only a minor role in structures that do not fail at the crack initiation but only after extensive cracking, as is normally the case for concrete structures; cf. ACI [3]).

The limitation by w_c, however, has drawbacks, too. The finite element system is an approximation to a continuous body, and the results must converge to the exact continuum solution as the element mesh is refined. For the crack band model, however, convergence cannot be checked because it is prohibited to decrease the element size below w_c. From the physical viewpoint, another problem is that the cracking zone at the front of a continuous fracture (i.e. the fracture process zone) is represented by a band single element wide and cannot be subdivided into many elements; consequently, possible variations in the process zone size, which cause variation of effective fracture energy (i.e. R-curve behavior) cannot be captured and the stress and strain states throughout the fracture process zone cannot be resolved.

2. *Nonlocal Continuum.* A general localization limiter is provided by the nonlocal continuum. The nonlocal approach makes it possible to refine the element mesh arbitrarily, and so convergence to an exact continuum solution becomes meaningful and the stress and strain distributions throughout the fracture process zone can be resolved (although, due to random inhomogeneity of the microstructure, these have merely a statistical meaning, representing the averages taken over many macroscopically identical structures).

Practically the most important feature of a nonlocal finite element model is that it can correctly represent the effect of structure size on the maximum load, as well as on the post-peak slope of the load-deflection diagram.

3. *Gradient Models.* Another general way to introduce a localization limiter is to use a constitutive relation in which the stress is a function (or functional) of not only the strain but also the first or second spatial

derivatives (or gradients) of strain. This idea appeared originally in the theory of elasticity. A special form of this idea, which is called Cosserat continuum and is characterized by the presence of couple stresses, was introduced in the work of Cosserats [23] as a continuum approximation of the the behavior of crystal lattices on a small scale. A generalization of Cosserat continuum, involving rotations of material points, is the micropolar continuum of Eringen [24].

Recently (Bažant [19], Schreyer et al. [25] , Vardoulakis et al. [26] it has been realized that the idea of using spatial gradients of strains or strain-related variables can be co-opted to serve as a localization limiter. It has been shown (Bažant [19]) that expansion of the averaging integral (Eq. 1) into Taylor series generally yields a constitutive relation in which the stress depends on the second spatial derivatives of strain, provided the averaging domain is symmetric and does not protrude outside the body, and also on the first spatial derivatives (gradients) of strain, provided this domain is unsymmetric or protrudes outside, as it happens for points near the boundary (*Fig. 1b*). Note also that, since the dimension of a gradient is (1/length)-times that of the differentiated variable, introduction of a gradient into the constitutive equation inevitably requires a characteristic length, ℓ, as a material property. Thus, the use of spatial gradients can be regarded as an approximation to nonlocal continuum, or as its special case.

Schreyer et al. [25] proposed a constitutive relation which is a modification of plasticity which uses not only the yield limit but also the spatial gradient of the yield limit. They applied this formulation to dynamic and static one-dimensional strain-softening problems and showed the use of the spatial gradient does indeed prevent localization of strain-softening into a point, thus serving as a localization limiter. The Cosserat continuum concepts have been applied by Vardoulakis et al. [26,27] to strain-softening shear bands in soils, a problem mathematically analogous to tensile cracking of concrete. Again they demonstrated that localization into a zone of zero volume is prevented. A similar idea, consisting of the application of the diffusion operator to strain, was proposed in continuum mechanics by Aifantis and Triantafyllidis [28,29].

In material research of concrete, the idea of a spatial gradient appeared in the work of L'Hermite et al. [30], who found that, in order to describe differences between observations on small and large specimens, the formation of shrinkage cracks needs to be assumed to depend not only on the shrinkage stress but also on its spatial gradient (see also L'Hermite [30] and a discussion in Bažant [31]). From the present viewpoint, this was probably the first appearance of the nonlocal concept in fracture. The idea that the stress gradient (or equivalently the strain gradient) influences the material response has also been advanced by Karsan and Jirsa [32] and others in order to describe the measured strength of concrete beams under flexure and axial loading. The use of strain gradient was also proposed for describing some test results on metal fatigue in beams.

The nonlocal averaging integral is meaningful only if the finite elements are not larger than about one third of the representative volume (Droz and Bažant [33], Bažant [34]). The gradient approach, on the other hand, offers the possibility of using finite elements as large as the representative volume (i.e. roughly equal to ℓ). Thus, the gradient approach offers the possibility of using a smaller number of finite elements in the analysis. Finite element implementation of the gradient approach for concrete has been worked out by Belytschko and Lasry [35]. It appears, however, that the programming may be more complicated and less versatile than for the spatial averaging integrals. The problem is that one must enforce interelement continuity of not only the displacements but also the strains. This requires the use of higher-order elements, or alternatively, if the first-order elements are preferred, the use of an independent strain field with separate first-order finite elements. Anyhow, the gradient approach deserves deeper study.

4. *Artificial viscosity.* The problems of localization into a zone of zero volume and the associated spurious mesh sensitivity are related to a change of type of the differential equation, caused by the loss of positive definiteness of the tangential stiffness matrix of the material due to strain softening. In space, the type of the partial differential equation changes from elliptic to hyperbolic and in space-time it changes from hyperbolic to elliptic, in which case one gets what is called in mathematics an 'ill-posed' problem (Courant and Hilbert [36]). The consequence of the latter is that the material in a strain-softening state cannot propagate loading waves (it can still propagate unloading waves, though) (Hadamard [37], Thomas, and Sandler [38].

This consequence, which is from the mathematical viewpoint at the root of all the problems with strain-softening, can be eliminated by introducing some sort of viscosity (or creep) into the constitutive equation (which is usually done according to Maxwell viscoelastic model). The partial differential equation of motion of the continuum (i.e. the wave equation) then remains of hyperbolic type (and the problem remains 'well posed'). This approach to overcoming the problems of strain-softening was proposed for damage in metals by Needleman et al.[39]. It works, but only to a limited extent. Instead of localizing instantly, the strain softening localizes into a zone of zero volume gradually. Thus a realistic response can be obtained in this manner only for a range of loading rates or time delays. But for loading that is too fast or too slow, or times that are too short or too long (compared to the relaxation time associated with material viscosity), localization problems and spurious mesh sensitivity are again manifested.

THEORETICAL ARGUMENTS FOR NONLOCAL APPROACH

1. *Statistical Theory of Heterogeneous Materials.* As already mentioned, statistical theory of heterogeneous elastic materials has shown that the macroscopic (continuum) stress at a point depends not only on the macroscopic strain at that point but also on the spatial average (or gradient) of this strain over the representative volume. One may reasonably expect the same to be true for the inelastic part of macroscopic strain, e.g. for the cracking strain, although a proof has not been given.

2. *Characteristic Length.* Whether the body is large or small must have no influence on the material properties, if defined properly. If the body is large, its failure is governed by linear elastic fracture mechanics, for which the basic material property is the fracture energy, G_f, whose dimension is J/m^2. If the body is small, the response depends only on the stress-strain relation, whose basic characteristic is the area under the complete tensile uniaxial stress-strain diagram with strain-softening. This area represents the dissipated energy per unit volume, W_s, whose dimension is J/m^3. The ratio G_f/W_s has the dimension of length, and happens to be approximately equal to the characteristic length of nonlocal continuum, ℓ (Bažant and Pijaudier-Cabot [40]). Mathematically, the way a material parameter of the dimension of length can get manifested in the constitutive relation is either a spatial averaging integral or a spatial gradient – each a basic attribute of nonlocal continuum.

The relation $\ell = W_s / G_f$ provides the basis of a simple method of measuring the characteristic length (Bažant and Pijaudier-Cabot [40]), as described in ACI (1990). For one particular concrete, this method indicated that $\ell \cong 2.7\, d_a$ where d_a = maximum aggregate size.

3. *Homogenization.* It would be nice if homogenization theory could provide the continuum approximation to a solid consisting of an elastic matrix and elastic inclusion, with an array of microcracks. But this seems too difficult. So far, such a result has been obtained only for the one-dimensional problem of a quasiperiodic parallel cubic array of either small penny-shaped cracks or small circular ligaments in an infinite homogeneous elastic body, subjected to uniaxial stress in the direction normal to the cracks (Bažant [41]). In these two simple cases, the conditions of macro-micro compatibility and of macro-micro work equivalence have been satisfied exactly, and the result of continuum homogenization was a nonlocal continuum, with a characteristic length ℓ equal to the spacing of cracks.

Since, in an analytical form, the continuum homogenization problem for the general case of a matrix with random inclusion and random cracks appears unsurmountable, one must turn to numerical modeling.

4. *Random Particle Model.* Indirect evidence for the nonlocal character of homogenizing continuum can be obtained numerically, using the random particle model, which approximately simulates the basic properties of the microstructure of concrete (Bažant, Tabbara, Kazemi and Pijaudier-Cabot [42]; Bažant and Tabbara [43]). In similarity to Cundall's [44] particle model for sand. The aggregate pieces in concrete are modeled as rigid particles separated by thin contact layers of matrix. The particles are assumed to interact only by center-to-center forces, and shear interactions between particles are neglected (they could be taken into account, but have been found unessential). The particle interaction is elastic up to the given interparticle strength limit, after which the stress-displacement diagram descends linearly, with a slope determined from the the given interparticle fracture energy.

While the general constitutive law of the homogenizing continuum appears impossible to find, one can calculate from this model the properties which distinguish the local and nonlocal continua. The basic one is the size effect on nominal strength. If the continuum were local, this effect would have to be the same as in linear elastic fracture mechanics. If it is nonlocal, the size effect must be transitional between plasticity (no size effect) and linear elastic fracture mechanics. Now, calculations (Bažant, Tabbara, Kazemi and Pijaudier-Cabot [42]) showed that the latter is the case. Furthermore, the fracture energy of the continuum that homogenizes the random particle system, G_f, and the energy dissipated per unit volume, W_s, have been calculated numerically, from which the characteristic length of the homogenizing continuum has been determined as $\ell = G_f / W_s$.

NONLOCAL CONTINUUM WITH LOCAL STRAIN

The original nonlocal continuum model for strain-softening (Bažant, Belytschko and Chang [18]; Bažant [19]) involved the nonlocal (averaged) strain $\bar{\epsilon}$ as the basic kinematic variable. This was shown to correspond to a system of imbricated (i.e. overlapping in a regular manner, like roof tiles) finite elements, overlaid by a regular finite element system. Although this imbricate model was found to indeed limit localization of strain softening and guarantee mesh insensitivity, especially in terms of energy dissipated at failure (Bažant and Chang [45]), the programming was complicated, due to nonstandard form of the differential equations of equilibrium and boundary conditions.

The property that gives rise to this nonstandard form may be explained by considering the derivation from the virtual work relation

$$\delta W = \int_V \sigma_{ij}(\bar{\varepsilon}) \, \delta\bar{\varepsilon}_{ij} \, dV - \int_V f_i \, \delta u_i \, dV - \int_S p_i \, \delta u_i \, dS = 0 \tag{3}$$

where V, S = volume and surface of the structure; f_i, p_i = given volume and surface forces; u_i = displacements and σ_{ij} = stresses. The fact that the strain in the variation $\delta\bar{\varepsilon}_{ij}$ is nonlocal poses difficulties, however. For a local continuum one can set $\delta\varepsilon_{ij} = \delta u_{i,j}$ and then use Gauss integral theorem to obtain the differential equations and boundary conditions. But here this standard procedure is impossible since $\delta\bar{\varepsilon}_{ij}$ involves a spatial integral or spatial gradient. A much more complex procedure had to be devised (Bažant [19]). This experience led to the idea of a partially nonlocal continuum in which $\delta\bar{\varepsilon}_{ij}$ in Eq. 3 is replaced by $\delta\varepsilon_{ij}$ but σ_{ij} is still determined from $\bar{\varepsilon}_{ij}$ (and the elastic strains are taken as local). After this modification, Gauss integral theorem can be applied to the first integral in Eq. 3, and then the resulting differential equations of equilibrium and the boundary conditions are of standard form.

Such a nonlocal model, called the nonlocal continuum with local strain (Bažant and Pijaudier-Cabot [46,47], Pijaudier-Cabot and Bažant [48,49, Bažant and Lin [50,51]), has proven to be quite simple and effective for finite element programming. In this formulation, the usual constitutive relation for strain softening is simply modified so that all the state variables that characterize strain softening are calculated from the nonlocal rather than local strains. Then, all that is necessary to change in a local finite element program is to provide a subroutine which delivers (at each integration point of each element, and in each iteration of each loading step) the value of $\bar{\varepsilon}_{ij}$.

The nonlocal continuum with local strain can be of various kinds. In nonlocal damage theory (Pijaudier-Cabot and Bažant [48]), one needs to either calculate the nonlocal damage Ω from $\bar{\varepsilon}_{ij}$ or replace the usual (local) damage ω with its spatial average Ω. If strain-softening is considered to be the consequence of degradation of the yield limit τ (Bažant and Lin [50]), either τ must be replaced by its spatial average $\bar{\tau}$ or the value of the effective plastic strain γ^p from which τ is calculated must be replaced by its spatial average $\bar{\gamma}^p$. If strain softening is described through the fracturing strain $\underset{\sim}{\varepsilon}^{fr}$, one needs to either replace $\underset{\sim}{\varepsilon}^{fr}$ with its spatial average $\underset{\sim}{\bar{\varepsilon}}^{fr}$ or calculate nonlocal $\underset{\sim}{\bar{\varepsilon}}^{fr}$ from $\underset{\sim}{\bar{\varepsilon}}$ rather than $\underset{\sim}{\varepsilon}$.

The essential property of all these variants is that the energy dissipation density rate due to damage must be nonlocal. Then it can be mathematically proven that the energy dissipation density rate cannot localize into a vanishing volume.

Other mathematical arguments for a nonlocal continuum with a local elastic strain and nonlocal energy dissipation were advanced by Simo [52], from the viewpoint of regularization.

The localization instability in a bar made of the nonlocal continuum withlocal strain has been studied by numerical solution of a certain integral equation (Bažant and Pijaudier-Cabot [47]). It was found that the width of the localization segment is roughly 1.88 ℓ. This gives the approximate thickness of the cracking bands.

While the original imbricate (fully nonlocal) continuum model for strain softening was fully symmetric, the recent nonlocal continuum with local strain is not always symmetric. The nonsymmetry, which is manifested in a lack of symmetry of the tangential stiffness matrix of the structure, has two causes: (1) the averaging domain included both loading and unloading regions, and (2) the averaging domain protrudes outside the body. Without these two causes, the tangential stiffness matrix would be symmetric. The lack of symmetry has so far been found to cause no problem in finite element computations. These computations, based on the initial stiffness method, used for the iterations in each loading step the initial elastic stiffness matrix, which is always symmetric.

NUMERICAL EXAMPLES

Example 1: Stability of a Tunnel (Bažant and Lin [50]. A subway tunnel (*Fig. 3*) is excavated in a strain-softening soil. To make possible excavation by a tunneling machine, the soil is first stabilized by injecting cement grout from the surface, which converts the soil into a sort of low-strength concrete, which is described by the Mohr-Coulomb plasticity model whose yield limit decreases linearly as a function of the nonlocal (spatially averaged) effective plastic strain. The excavation (without temporary supports) leads to development of strain-softening zones at the sides of the tunnel, and there is a tendency toward cave-in failure. The excavation process is simulated in two dimensions by gradually reducing the stresses acting on the outline of the tunnel to zero. The problem is solved by finite elements, and in order to assess convergence, four meshes shown in *Fig. 3* are used, with 218, 608, 1640 and 3248 unknown displacements. Based on the observed spacing of the inhomogeneities in the stabilized soil (about 0.6 m), the characteristic length is assumed to be $\ell = 1.8$ m.

Fig. 4a shows the boundary of the strain-softening zone calculated for each mesh, and it is seen that the results for the four meshes agree well, indicating good convergence. *Fig. 4b* shows the calculated histories of the lateral displacement at the side of the tunnel as a function of the degree of excavation, for both local and nonlocal analysis. The nonlocal results are seen to converge but the local results for the two finest meshes still differ significantly. It is also interesting to note that the running time for the largest mesh (on Cray-II supercomputer) has been shorter for the nonlocal program that for the corresponding local program. Apparently, the stabilizing effect of the nonlocal formulation engenders a faster convergence of iterations, which outweighs the time needed for calculating the spatial averages.

Example 2. Nonlocal Smeared Cracking Analysis of a Beam (Bažant and Lin [51]). Geometrically similar notched beams of three different sizes of ratio 1:2:4 are analyzed using the meshes of four-node quadrilaterals shown in *Fig. 5a*. The concrete is described by the nonlocal smeared cracking model, which is the same as the classical smeared cracking except that the cracking strain is calculated from the maximum principal value of the spatially averaged nonlocal strain $\bar{\underset{\sim}{\varepsilon}}$ rather than local strain $\underset{\sim}{\varepsilon}$ (the nonlocal constitutive model was described in ACI [3].The stress is assumed to decrease with the maximum nonlocal principal strain either linearly or exponentially. The characteristic length is taken in the first case as 2.3 in. (58.4 mm) and in the second case as 3.2 in. (81.2 mm), while the maximum aggregate size is $d_a = 0.5$ in. (12.7 mm).

The maximum values of the nominal stress σ_N (load divided by beam depth and thickness) obtained by nonlocal finite element analysis with step-by-step loading are plotted in *Fig. 5b* (in logarithmic scales) as a function of the

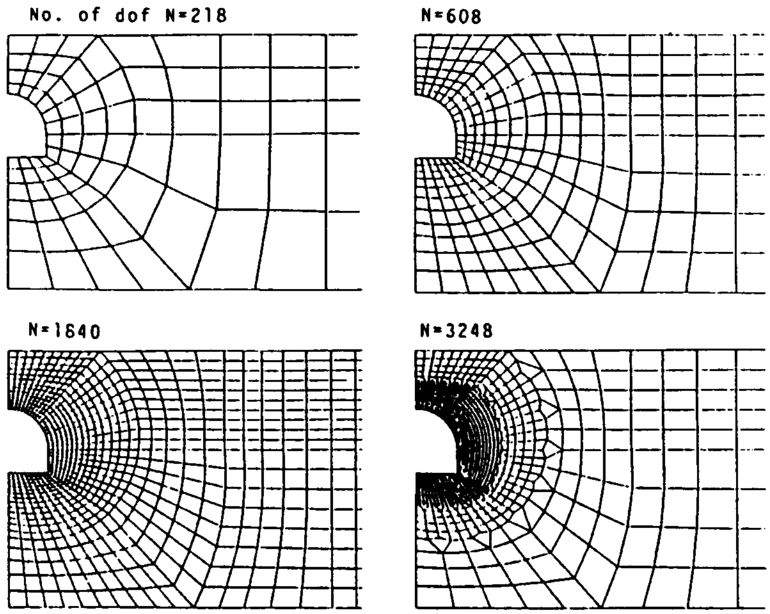

Figure 3. Example of meshes used for tunnel cave-in analysis (after Bažant and Lin, 1988).

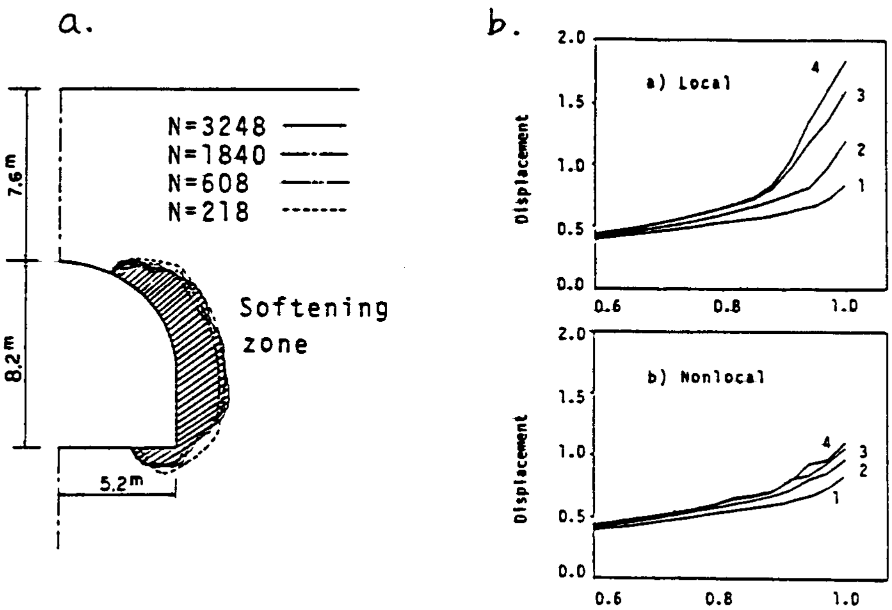

Figure 4. Strain-softening zone and displacement history for the example of tunnel cave-in (after Bažant and Lin, 1988)

relative beam depth d/d_a, both linear and exponential softening laws. For comparison, the figure also shows the test results of Bažant and Pfeiffer [53] and their optimum fit by the size effect law (reviewed in ACI report [3]. We see that the agreement is quite good (for both softening laws). This demonstrates that the nonlocal finite element model can represent the size effect quite well. The usual, local finite element programs, by contrast, give no size effect, and linear elastic fracture mechanics gives too strong a size effect, represented in the figure by the straight line of slope -1/2. *Fig. 6a* shows the calculated strain-softening zones (cross-hatched areas) and fully fractured zones (black areas); it is interested to note that there are mild differences among the zones for specimens of various sizes (which is a source of the R-curve behavior).

Fig. 6b shows the results obtained for slanted square meshes (on the right) compared to aligned meshes (on the left). This illustrates another important advantage of the nonlocal approach: If the element size is less than about $^1/_3$ of the the characteristic length ℓ, there is no mesh bias with regard to the crack direction. With the crack band model and other local approaches, it is impossible to get the fractures in the specimens with the slanted meshes (on the right) run vertically, as they should.

Example 3. Tensile and Compression Failures (Bažant and Ožbolt [1], Droz and Bažant [33]). A rectangular concrete specimen in plane strain, loaded by prescribed uniform displacement increment at the top boundary (*Fig. 7*), is analyzed using the two meshes shown. The material is described by the nonlocal microplane model (see ACI, 1990), taken from Bažant and Prat [54], which can represent tensile cracking in multiple directions as well as nonlinear triaxial behavior in compression and shear, including post-peak response. *Fig. 7* shows the calculated load-displacement diagrams (for free lateral sliding at top and bottom boundaries) obtained by local and nonlocal analysis for the two meshes. We see that while the local results differ substantially (spurious mesh sensitivity), the nonlocal results are nearly mesh independent (except for a small numerical error).

Furthermore, notched three-point-bend specimens of three sizes in the ratio 1:2:4 are analyzed using the three meshes shown in *Fig.8*. The calculated values on the nominal stress σ_N (maximum load divided by specimen depth d and thickness) are plotted in log-scales as a function of the relative size d/d_a. We see that the results exhibit the correct size effect, in good agreement with the test results of Bažant and Pfeiffer [53] and with the size effect law. The figure also shows the contours of the strain-softening zone (black), nonlinear hardening zone (densely cross-hatched) and elastic zone where the difference from the linear elastic asymptotic near-tip field does not exceed 5% of the tensile strength of concrete (white).

Finally, *Fig. 9* shows a specimen loaded in compression whose material is described by Drucker-Prager plasticity with a nonlocal degrading yield limit. The finite elements that are strain-softening are marked by black dots, and those that started to strain-soften but subsequently unloaded by asterisks. The finite element results reveal that the failure mode consists of a shear band. Note that the shear band running obliquely through the square mesh can be modeled by the nonlocal approach without having to resort to special finite elements with discontinuous strain or displacement field, which were recently developed for shear band analysis by the usual (local) finite element codes (Leroy and Ortiz [55]; Dvorkin, Coutinho and Gioia , private comm.). This is made possible by the fact that the finite element size is not more than $^1/_3$ of the characteristic length ℓ.

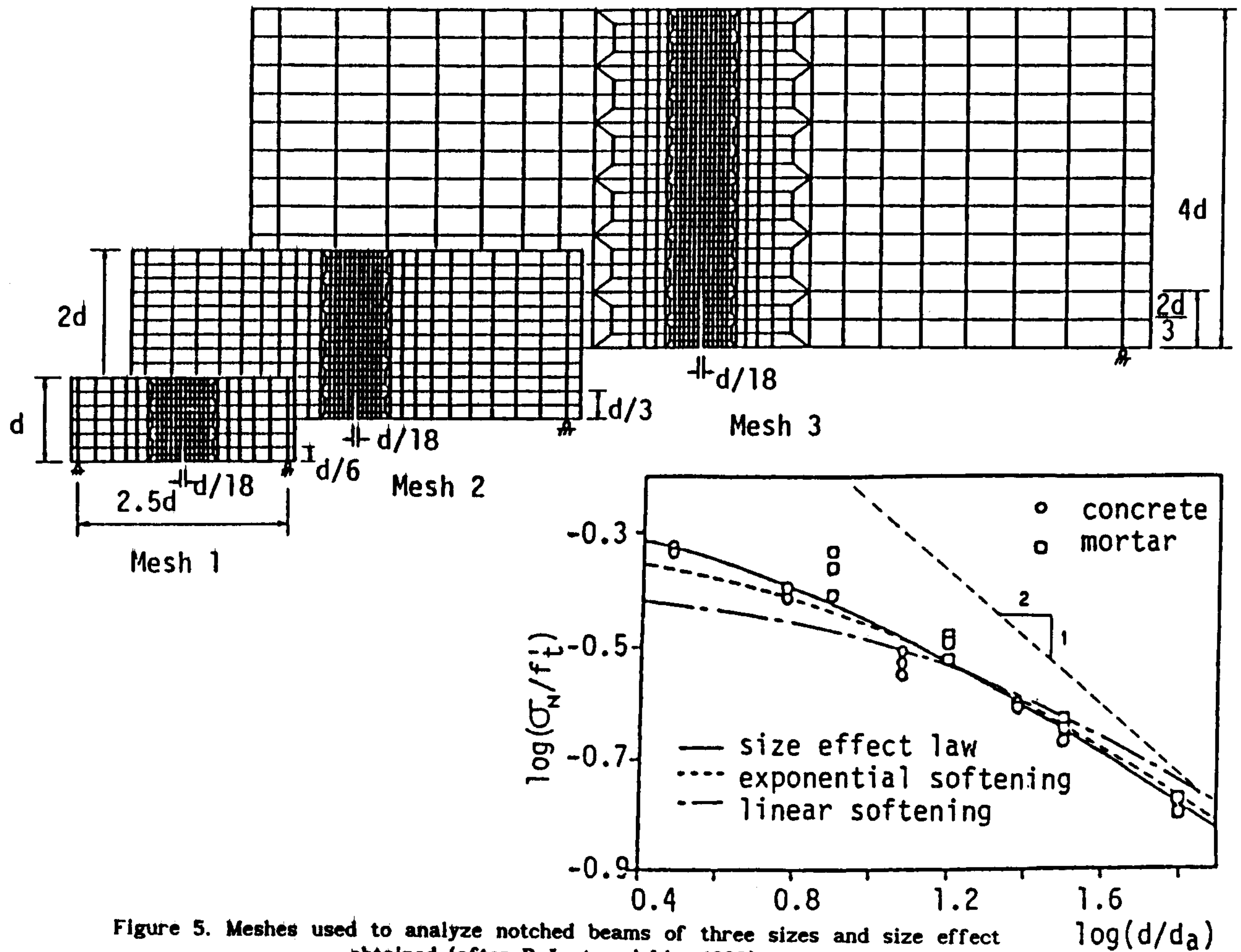

Figure 5. Meshes used to analyze notched beams of three sizes and size effect obtained (after Bažant and Lin, 1988).

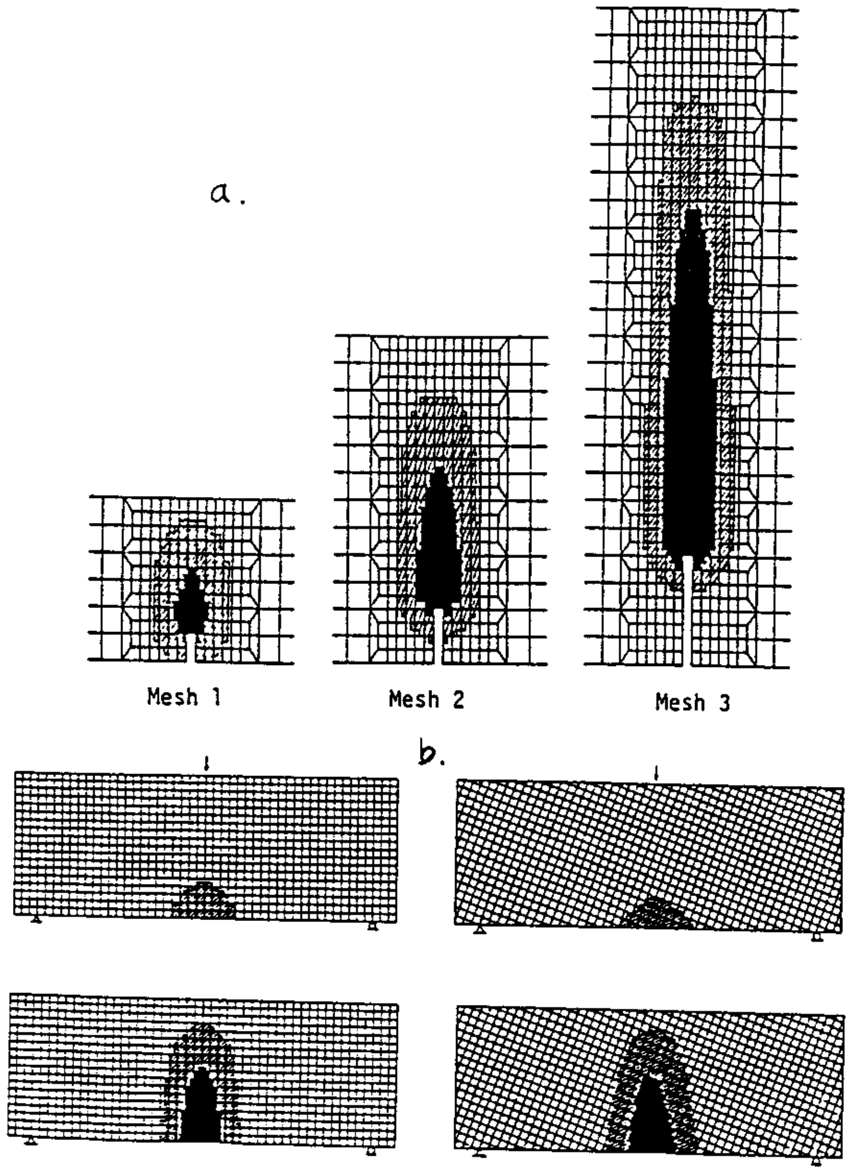

Figure 6. Nonlinear and strain-softening zones in notched beam (after Bažant and Lin, 1988).

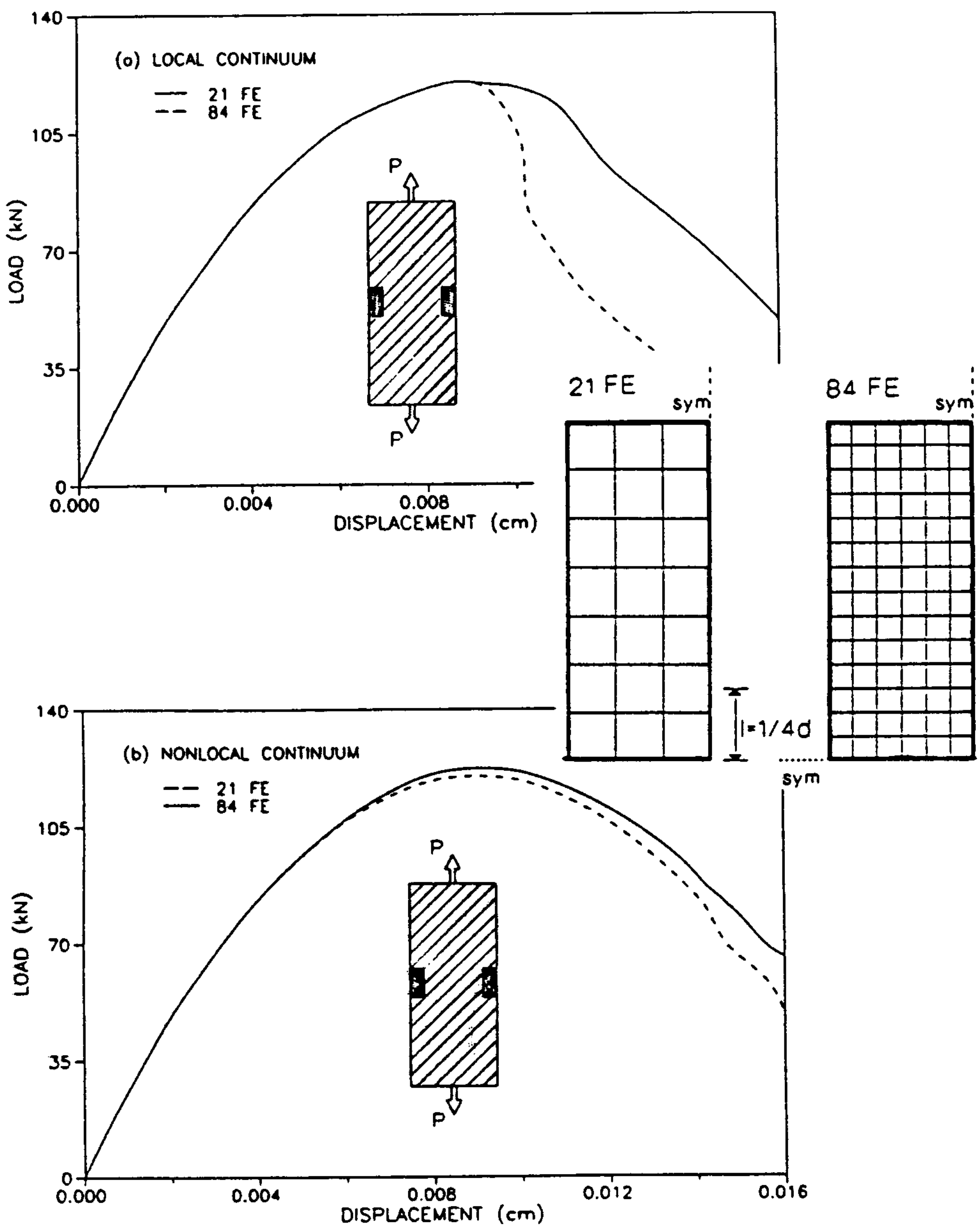

Figure 7. Local and nonlocal analyses of tensile specimens by smeared cracking model with different meshes (after Bažant and Ožbolt, 1989).

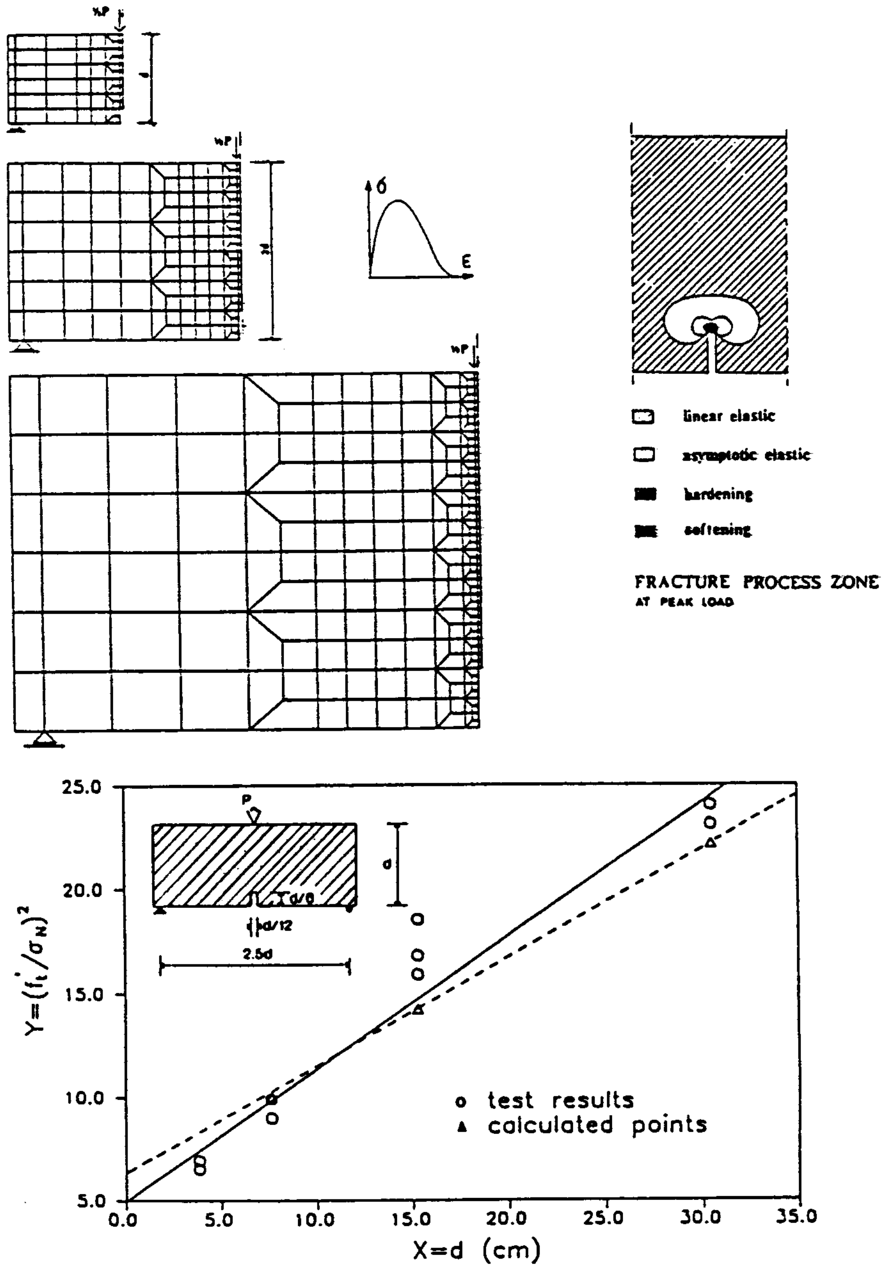

Figure 8. Analysis of notched specimens by nonlocal microplane model and size effect (after Bažant and Ožbolt, 1989).

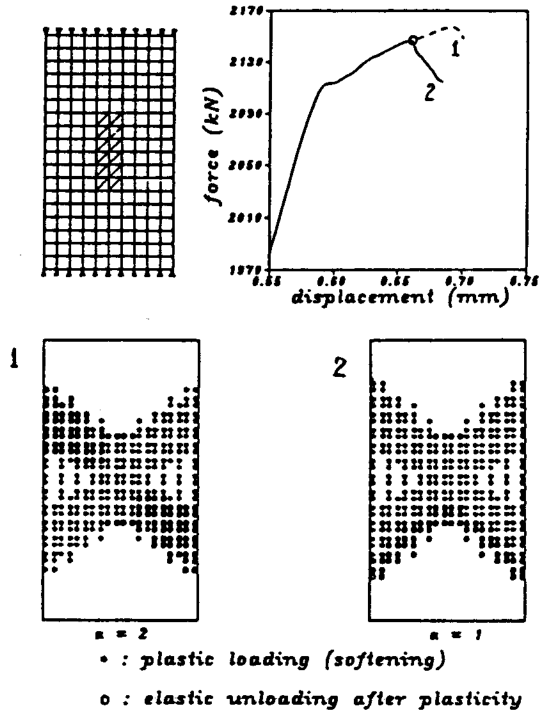

Figure 9. Nonlocal analysis of shear bands in compression specimen (after Droz and Bažant, 1988).

It may also be noted that the latest studies of compression failure of concrete with the nonlocal microplane model have shown that the axial splitting failure can be obtained if there is large enough volume dilatancy due to deviatoric straining in the fractures process zone (Bažant and Ožbolt [2].

CONCLUSION

As the preceding exposition has demonstrated, the nonlocal finite element analysis of concrete fracture gives results which agree with all the basic theoretical requirements as well as experimental results available at present. It is free of spurious mesh sensitivity, yet can describe zones of smeared cracking (distributed damage). It is also free of bias due to mesh line direction. It correctly represents the size effect on the nominal strength, which (in agreement with the size effect law) exhibits a transition between plasticity (no size effect) and linear elastic fracture mechanics (the strongest possible size effect). It also gives the size effect on the post-peak descending slope of the load-deflection diagram. The nonlocal finite element analysis appears ready for practical applications.

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REFERENCES

1. Bažant, Z.P. and Ožbolt, J., Nonlocal microplane model for fracture, damage and size effect in structures. Report 89-10/498n, Northwestern University, Chicago, 1989; also Journal of Engineering Mechanics ASCE 116, 1990 in press.

2. Bažant, Z.P. and Ožbolt, J., Nonlocal microplane model: tensile and compression fracture and triaxial damage. Proceedings: Second International Conference on Computer Aided Design of R. C. Structures, held in Zell am See, Austria, April 1990, ed. H. Mang and N. Bićanić, Pineridge Press, Swansea, U.K.

3. ACI Committee 448, Fracture mechanics, Fracture mechanics of concrete: concepts, models and determination of material properties. American Concrete Institute, Detroit, under review.

4. Rashid, Y.R., Analysis of prestressed concrete pressure vessels. Nuclear Engineering Design 7(44), 334-355.

5. Bažant, Z.P., Instability, ductility and size effect in strain-softening concrete. J. Eng. Mech. Div., ASCE, 1976, 102 (EM2), pp. 331-344; Discussion, 103, 257-358, 775-777; 104, 501-502.

6. Bažant, Z.P. and Cedolin, L., Blunt crack band propagation in finite element analysis. J. Eng. Mech. Div. ASCE, 1979, 105, pp. 297-315.

7. Bažant, Z.P. and Cedolin, L., Fracture mechanics of reinforced concrete. J. Eng. Mech. Div. ASCE, 1980, 106, 1287-1306; Discussion and closure, 1982, 108, pp. 464-471.

8. Bažant, Z.P. and Cedolin, L., Finite element modeling of crack band propagation. J. Eng. Mech. Div. ASCE 109 (2), pp. 69-92.

9. Darwin, D., Crack propagation in concrete-study of model parameters, in Proceedings, Seminar on Finite Element Analysis of Reinforced Concrete Structures, Tokyo, 1985.

10. Rots, J.G., Kusters, G.M.A. and Blaauwendraad, J., The need for fracture mechanics options in finite element models for concrete structures, in Proceedings, International Conference on Computer Aided Analysis and Design of Concrete Structures, Split, Yugoslavia, Sept. F. Damjanić, E Hinton et al. eds. Pineridge Press, Swansea, U.K., pp. 19-32.

11. Bažant, Z.P. and Oh, B.H., Crack band theory for fracture of concrete. Matériaux et Constructions (Materials and Structures), RILEM, Paris, 1983, 16 (93), pp. 155-177.

12. Kröner, E., Interrelations between various branches of continuum mechanics, in Mechanics of Generalized Continua, E. Kröner, ed., Springer, W. Berlin (1968) pp. 330-340.

13. Krumhansl, J.A., Some considerations of the relations between solid state physics and generalized continuum mechanics, in Mechanics of Generalized Continua, E. Kroner, ed., Springer, West Berlin, 1968 pp. 298-331.

14. Levin, V.M., The relation between mathematical expectation of stress and strain tensors in elastic microheterogenous media. Prikl. Mat. Mekh. 1971 (in Russian), 35 pp. 694-701.

15. Kunin, I.A., The theory of elastic media with microstructure and the theory of dislocations, in Mechanics of Generalized Continua, E. Kröner, ed., Springer-Verlag, West Berlin, Germany, pp. 321-328.

16. Eringen, A.C. and Edelen, D.G.D., On nonlocal elasticity. Int. J. of Engng. Sci. 1972, 10, pp. 233-248.4.

17. Eringen, A.C., and Ari, N., Nonlocal stress field at Griffith crack. Cryst. Latt. Def. Amorph. Mat. 10, pp. 33-38.

18. Bažant, Z.P., Belytschko, T.B., and Chang, T.P., Continuum model for strain-softening. ASCE J. of Engng. Mech. 1984, 110 (12), pp.1693-1712.

19. Bažant, Z.P., Imbricate continuum and its variational derivation. ASCE J. of Engng. Mech., 1984, 110 (12), pp. 1693-1712.

20. Bažant, Z.P., Mechanics of distributed cracking. ASME Applied Mechanics Reviews, 1985, 39 (5), pp. 675-705.

21. Bažant, Z.P., and Oh, B.H., Crack band theory for fracture of concrete. Materiaux et Constructions (Materials and Structures), RILEM, Paris, 1983, 16 (93), pp. 155-177.

22. Bažant, Z.P., Instability, ductility and size effect in strain-softening concrete. ASCE J. of Engng. Mech., 1976, 102 (2), pp. 331-334; Discussions Vol. 103, pp. 357-358, 775-777, Vol. 104, pp.501-502.

23. Cosserat, E. and Cosserat, F., Sur la théorie de l'elasticité. Annales de la Faculté des Sciences de l'Université de Toulouse 10, 1896.

24. Eringen, C.E., Linear theory of micropolar elasticity. J. Math. Mech. 15, 1966, pp. 909-923.

25. Schreyer, H.L., and Chen, Z., One-dimensional softening with localization. J. Appl. Mech. ASME, 1986, 53, pp. 791-797.

26. Sulem, J. and Vardoulakis, I., Bifurcation analysis of the triaxial test on rock specimens, in Cracking and Damage, ed. by J. Mazars and Z.P. Bažant, Elsevier, London, 1989, pp. 308-322 (Proc. of France-U.S. Workshop held at E.N.S., Cachan, France, Sept. 1988)

27. Mühlhaus, H.B., and Vardoulakis, I., The thickness of shear bands in granular materials. Géotechnique, 1987, 37, pp. 271-283.

28. Triantafyllidis, N. and Aifantis, E., A gradient approach to localization of deformation. I. Hyperelastic materials, J. Elasticity, 1986, 16, pp. 225-237.

29. Aifantis, E.C., On the microstructural origin of certain inelastic models. J. Eng. Mat. Tech., 1984, 106, pp. 326-330.

30. L'Hermite, R. and Grieu, J.J., Etudes experimentales recentes sur le retrait des ciments et des betons. Annales ITBTP, 1952, 5 (52-3), pp. 494-514.

31. Bažant, Z.P., Impact of Robert L'Hermite on the evolution of creep and shrinkage theory, in Mathematical Modeling of Creep and Shrinkage of Concrete, ed. by Z.P. Bazant, J. Wiley, Chichester and New York, 1988, pp. 7-40.

32. Karsan, I.D. and Jirsa, J.O., Behaviour of concrete under compressive loadings. J. Struct. Eng. Div. ASCE 95 (12), 1969, pp. 2543-2563.

33. Droz, P., and Bažant, Z.P., Nonlocal analysis of stable states and stable paths of propagation of damage shear bands, in Cracking and Damage, ed. by J. Mazars and Z.P. Bazant, Elsevier Appl. Science, London 1989, pp. 415-429 (Proc. of France-U.S. Workshop held at E.N.S., Cachan, in Sept. 1988).

34. Bažant, Z.P., Bifurcations and thermodynamic criteria of stable paths of structures exhibiting plasticity and damage propagation, in Computational Plasticity, Proc. of Conf. held in Barcelona, Sept. 1989, ed. D.R.J. Owen et al, Pineridge Press, Swansea, U.K., 1989, pp. 1-25.

35. Lasry, D., and Belytschko, T., Gradient-type localization limiters for strain-softening problems. Report, Northwestern University, Evanston, IL, 1987.

36. Courant, R., and Hilbert, D., "Methods of mathematical physics", Interscience Publ. - J. Wiley, New York 1953.

37. Hadamard, J., Leçons sur la propagation des ondes. Hermann, Paris, chap. VI, 1903.

38. Sandler, I.S., Strain-softening for static and dynamic problems, in Proc. Symp. Constit. Equations: Micro, Macro and Computational Aspects, ASME, Winter Annual Meeting, New Orleans, ed. K.J. Willam, ASME, New York, 1984, pp. 217-231.

39. Needleman, A., Material rate dependence and mesh sensitivity in localization problems. Comp. Methods Appl. Mech. Eng., 1988, 67, pp. 69-85.

40. Bažant, Z.P., and Pijaudier-Cabot, G., Measurement of characteristic length of nonlocal continuum. J. of Engng. Mech. ASCE, 1989, 115 (4), pp. 755-767.

41. Bažant, Z.P., Why continuum damage is nonlocal: justification by quasiperiodic microcrack array. Mech. Res. Comm., 1987, 14, pp. 407-419.

42. Bažant, Z.P., Tabbara, M.T., Kazemi, M.T. and Pijaudier-Cabot, G., Random particle model for damage and fracture in aggregate or fiber composites. Report, Northwestdern University, 1989; also J. of Engng. Mech. ASCE, 116, 1990 in press.

43. Bažant, Z.P., and Tabbara, M.T., Random particle simulation of fracture, Proc., Int. Conf. on Num. Meth. in Engng. (NUMETA), held in Swansea, Jan. 1990, Pineridge Press, Swansea.

44. Cundall, P.A. and Strack, O.D.L., A discrete numerical model for granular assemblies. Geotechnique, 1979, 29, pp. 47-65.

45. Bažant, Z.P. and Chang, T.P., Continuum Theory for Strain-softening solids. J. of Engng. Mech. ASCE, 1987, 113 (1), pp. 89-105.

46. Bažant, Z.P. and Pijaudier-Cabot, G., Modeling of distributed damage by nonlocal continuum with local strain. Preprints, Fourth Int. Conf. on Numerical Methods in Fracture Mechanics, held in San Antonio, Texas, March 1987, publ. by Pineridge Press, Swansea, U.K., ed. by A.R. Luxmore et al., 1987, pp. 441-432.

47. Bažant, Z.P. and Pijaudier-Cabot, G., Nonlocal continuum damage, localization instability and convergence. ASME J. of Applied Mechanics 1988, 55, pp. 287-293.

48. Pijaudier-Cabot, G., and Bazant, Z.P., Nonlocal damage theory. J. of Engng. Mech. ASCE 113, 1987, pp. 1512-1533.

49. Pijaudier-Cabot, G. and Bazant, Z.P., Comparison of various models for strain-softening. Engng. Computations, 1988 (June), 5, pp. 141-150.

50. Bažant, Z.P. and Lin, F.-B., Nonlocal yield-limit degradation. Int. J. for Numerical Methods in Engineering, 1988, 55, pp. 287-293.

51. Bažant, Z.P. and Lin, F.-B., Nonlocal smeared cracking model for concrete fracture. J. of Struct. Eng. ASCE, 114 (1), pp. 2493-2510.

52. Simo, J.C., Strain-softening and dissipation: a unification of approaches, in Cracking and Damage, ed. by J. Mazars and Z.P. Bažant, Elsevier, London, 1989, pp. 440-461. (Proc. of France-U.S. Workshop held at E.T.S. Cachan, France, Sept. 1988)

53. Bažant, Z.P. and Pfeiffer, Determination of fracture energy from size effect and brittleness number. ACI Materials Journal, 1987, 84 (Nov.), pp. 463-480.

54. Bažant, Z.P. and Prat, P.C., Microplane model for brittle plastic material: I. Theory, II. Verification. ASCE J. of Engng. Mech., 1988, 114, pp. 1672-1702.

55. Bažant, Z.P., Impact of Robert L'Hermite on the evolution of creep and shrinkage theory, in Mathematical Modeling of Creep and Shrinkage of Concrete. ed. by Z.P. Bažant, J.Wiley, Chichester-New York, 1988, pp. 7-40.