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Fishnet model for failure probability of nacre-like imbricated lamellar materials and Monte Carlo verification

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ABSTRACT: The conference paper reviews recent studies at Northwestern University, in which the connectivity in nacreous staggered lamellar systems is, for probabilistic analysis, represented as a fishnet pulled diagonally. The probability distributions of nacre, including its tail at 10^{-6} probability, turns out to be analytically tractable. The fishnet distribution is intermediate between those corresponding to the weakest link chain (series coupling) and fiber-bundle (parallel coupling). Millions of Monte Carlo simulations are presented to verify the analytical distribution, including its tail.

1 INTRODUCTION

In spite of their weak brittle constituents, nacre-like imbricated (staggered) lamellar structures can attain very high strength and fracture energy, exceeding by one to two orders of magnitude the strength or the constituents. The reasons have been clarified in a host of studies of the mechanics of failure [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, e.g.].

These studies, however, were mostly deterministic and provided only the mean behavior. For nacreous structures, no realistic probability distribution of the strength with the far left fail seems to exist at present, yet this is where the 'devil' resides. To capture the tail is the goal of this study (whose main ideas were compactly presented in [12] and developed in full detail in [13]).

To design safe structures with nacre-mimetic materials typically requires knowing their strength distribution up to the tail with failure probability of about $P_f = 10^{-6}$ (per lifetime), which requires determining the extreme value distribution [14, 15]. This is generally the level of safety required for engineering structures such as bridges, aircraft, MEMS, etc. It ensures the risks of engineering structures to be three orders magnitude lower than other risks that people willingly or inevitably take (e.g., car driving), and to be of about the same level as the risk of being killed, e.g., by a lightning of falling tree. Such low tail probabilities can hardly be determined by histogram testing of strength of many identical specimens of structures.

Consequently, one needs a realistic mathematical model for the strength distribution, to be verified only indirectly, by predictions depending on the tail. Here a diagonally pulled fishnet is proposed as the basis of such a model, providing a sufficiently realistic simplification of the connectivity of nacre's microstructure, for which the probability distribution is analytically tractable.

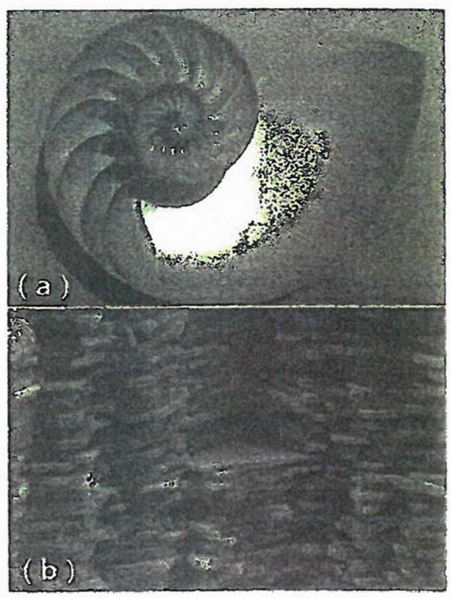

Figure 1. a) Nacre inside a nautilus shell; b) Electron microscopy image of a fractured surface of nacre (both images are from Wikipedia; https://en.wikipedia.org/wiki/Nacre).

The new idea of this article is to model the tail probability of strength of nacre-like structures by a square fishnet pulled along one of the diagonals. Same as the weakest-link model, the failure probability of fishnet, P_f, is obtained by calculating its counterpart—the survival probability, $1 - P_f$. As will be shown, these additional survival probabilities greatly enhance the strength for $P_f < 10^{-6}$, compared to the the weakest-link model.

The analytical predictions of failure probability are here verified by millions of Monte-Carlo simulations. Monte Carlo simulations of nacreous structures have previously been conducted with the random fuse model (RFM) [16, 5], in which the brittle bonds in the structure are simplified as a lattice of resisters with random burnout thresholds. The RFM simulates the gradual failure of resister network under increasing voltage. This is similar to the failure process of quasibrittle elastic material under controlled uniaxial load.

To calculate the maximum loads of the system of fishnet links, a simple finite element (FE) program for a pin-jointed truss is developed (in MatLab). For each of many shapes and sizes of the fishnet, the maximum loads are calculated for about 1 million input samples of randomly generated strengths of the links, based on the assumption that the link strength follows the grafter Gauss-Weibull distribution (see [15]). Running each set of about 1 million FE solutions takes a few days. With such a large number of random samples, the resulting strength histograms become visually indistinguishable from the theoretical cumulative probability density function (cdf) of failure probability P_f, derived in [13].

For the purpose of statistical analysis, the longitudinal load transmission must be realistically simplified. Almost no load gets transmitted between the ends of adjacent lamellae in one row, and virtually all the load gets transmitted by shear resistance of ultra-thin biopolymer layers between parallel lamellae. The links of the lamellae in adjacent rows many be imagined as the lines connectin lamellae centroids, as marked in Fig. 2a.

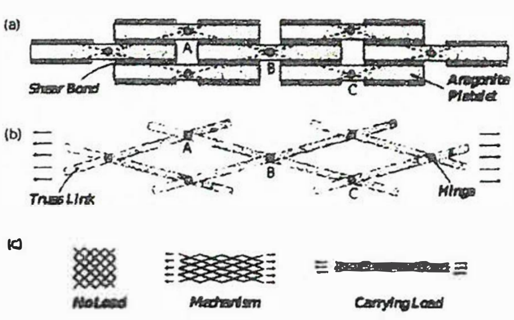

Figure 2. a) Microstructure of nacre; b) Equivalent fishnet structure with similar topology; c) Deformation mechanism of transversely unconstrained fishnet.

2 LOAD TRANSMISSION AND REDISTRIBUTION

The essence of load transmission may thus characterized by a system of diagonal tensile lin (Fig. 1b, which looks like a fishnet loaded in th diagonal direction and can be simulated by a finit element program for pin-jointed trusses. The transverse stiffness is found to be statistically unimportant, and is neglected. Thus the fishnet model is initially a mechanism in which all the links immediately collapse under longitudinal load into a single line (Fig. 2c) while retaining, crucially, the imbricated (or staggered) connections.

3 FAILURE PROBABILITY OF FISHNET MODEL

We consider the case of load control, for which the failure load is the maximum load, σ_{max}. We analyze rectangular fishnets with k rows and n columns, containing $N = k \times n$ links (Fig. 2c), loaded uniformly by uniaxial stress σ imposed at the ends of rows. Let $P_f(\sigma)$ be the failure probability of fishnet loaded by σ, and $X(\sigma)$ the total number of links failed at the end of experiment under constant load σ. This means that $X(\sigma)$ is measured when no more damages occur. The failed links may be contiguous or scattered discontinuously. The events $\{X(\sigma) = r\}, r = 1,2,3,\ldots$ are mutually exclusive (or disjoint). So, to obtain the survival probability of the whole fishnet, the corresponding survival probabilities, $P_{S_r}(\sigma)$, must be summed;

$$1 - P_f(\sigma) = P_{S_0}(\sigma) + P_{S_1}(\sigma) + P_{S_2}(\sigma) + \cdots \quad (1)$$

$$+P_{S_{k-1}}(\sigma) + \text{Prob}(X(\sigma) \geq \text{k and structure still safe}) \quad (2)$$

where $P_f(\sigma) = \text{Prob}(\sigma_{max} \leq \sigma)$; σ_{max} = nominal strength of structure; and $P_{S_r}(\sigma) = \text{Prob}(X(\sigma) = r)$, $r = 0,1,2,\ldots$.

4 TWO-TERM FISHNET STATISTICS

To get a better upper bound, we now include the second term in Eq.(1), i.e., $1 - P_f(\sigma) = P_{S_0}(\sigma) + P_{S_1}(\sigma)$ where σ = average longitudinal stress in the cross section, the same in every section. For the sake of simplicity, we further assume that: 1) the stress redistribution affects only a finite number, ν_1, of links in a finite neighborhood of the first failed link in which $\lambda_i > 1.1$, and 2) factor λ_i is treated as constant, $\lambda_i = \eta_o^{(1)}$ (> 1) within this neighborhood, taken either as the weighted average of all redistribution factors (to get the best estimate), or as the maximum of these factors (to preserve an upper bound on P_f). With this simplification,

$$P_{S_1}(\sigma) = NP_1(\sigma)[1 - P_1(\sigma)]^{N-\nu_1-1}[1 - P_1(\eta_o^{(1)}\sigma)]^{\nu_1} \quad (3)$$

Here N means that failure can start in any one of the N links, which gives N mutually exclusive cases. The two bracketed terms mean that the failure of one of the N links must occur jointly with the survival of: (i) each of the remaining ($N - \nu_1 - 1$) links with stress σ, and of (ii) each of the remaining ν_1 links with redistributed stress $\eta_o^{(1)}$. Analysis shows that the second term of fishnet statistics P_{S_1} increases the terminal slope of strength probability distribution in Weibull scale by the factor of 2. Particularly important are the implications for structural safety. In Fig. 3b, the horizontal line for $P_f = 10^{-6}$ marks the maximum failure probability that is tolerable for engineering design.

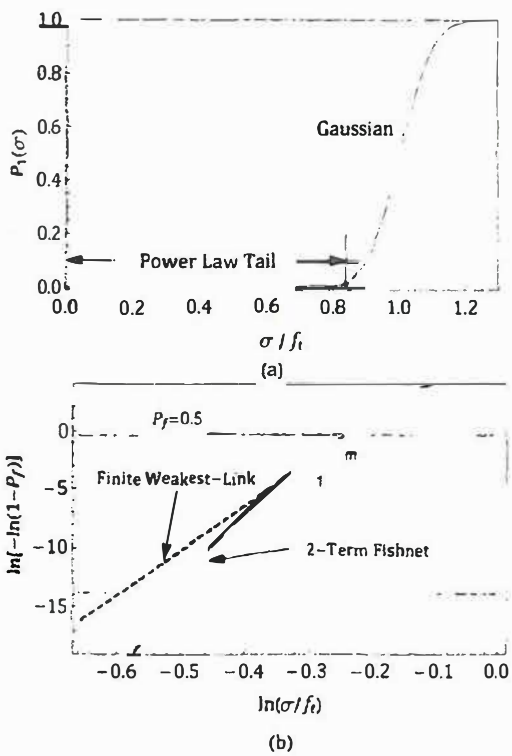

Figure 3. a) Cumulative distribution function (cdf) of failure for a single link with mean f_t = 10.016 MPa and CoV = 7.8%); b) Comparison of P_f (in Weibull scale) between the finite weakest-link model and the fishnet model with first 2 terms in the expansion of Eq. 1.

In this typical case, for constant N, the strength for $P_f = 10^{-6}$ is seen to increase by 10.5% when passing from the weakest-link failures to fishnet failures, while, at fixed strength, the P_f is seen to decrease about 25-times. The P_f decrease depends on the fishnet configurations and on P_1, but is generally more than 10-times greater. This is an enormous safety advantage of the imbricated lamellar microstructure, which comes in addition to the advantages previously identified by deterministic studies.

5 THREE-TERM FISHNET STATISTICS

Further improvement can be obtained by including the third term of the sum in Eq.(1). This term may be split into two parts, $P_{S_2} = P_{S_{21}} + P_{S_{22}}$, which are mutually exclusive, and thus additive. They represent the survival probabilities when the next failed link is, or is not, adjacent to the previously failed link. For detailed derivation, see [13].

6 MONTE CARLO FAILURE SIMULATIONS

A rectangular fishnet truss, with k rows and n columns of identical links, has been simulated by a finite element program (in MatLab). For computational stability, the fishnet is loaded under displacement control, by incrementing equal longitudinal displacements u_0 at the right boundary. At the left boundary, the horizontal displacement is zero. The boundary nodes slide freely in the transverse direction.

According to the arguments in [15, 17, 18, 19], based on nano-mechanics and scale transitions, the cumulative distribution function (cdf) of strength of each link, $P_1(\sigma)$, is assumed to be a Gaussian (or normal) distribution with a Weibull tail of exponent m grafted on the left at failure probability P_g (for $\sigma \to 0$, the cdf $\propto \sigma^m$). The strength of each of $N = k \times n$ links is generated randomly according to $P_1(\sigma)$. The autocorrelation length of the link strength field is assumed to be equal to the link size and, therefore, is not considered.

To verify the analytical two- or three-term statistics, respectively, the cases in which more than one, or two, links failed prior to the maximum load have been deleted from the set of about 1 million simulations of a fishnet having 16 × 32 links, CoV = 0.987 of P_1, and grafting point at P_g = 0.09. This is equivalent to omitting in Eq.(1) all the terms except the first two or three, respectively.

The remaining histograms ($\sigma_{max}^{(1)}$ and $\sigma_{max}^{(2)}$) are compared with the analytical cdf in Fig. 4b (Fig. 4a shows, for all simulations of σ_{max}, only the histogram). Despite simplifications, such as using a uniform redistribution ratio η and not

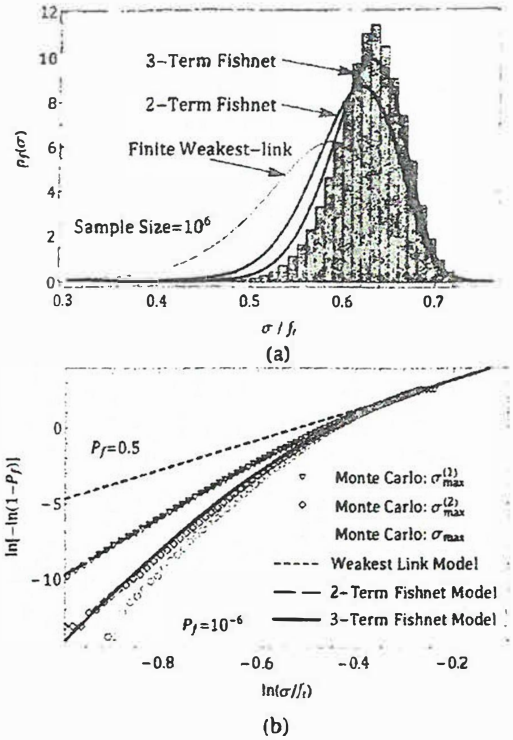

Figure 4. (a) Normalized histogram of 10^6 Monte Carlo realizations (σ_{max}) compared with the probability density functions of the weakest-link, 2-term fishnet and 3-term fishnet models; (b) The same data as well as the histogram of $\sigma^{(1)}_{max}$ and $\sigma^{(2)}_{max}$ converted into cumulative probability distribution and plotted on the Weibull paper. f_t = 9.87 MPa is the mean strength of one link and CoV = 9.87%.

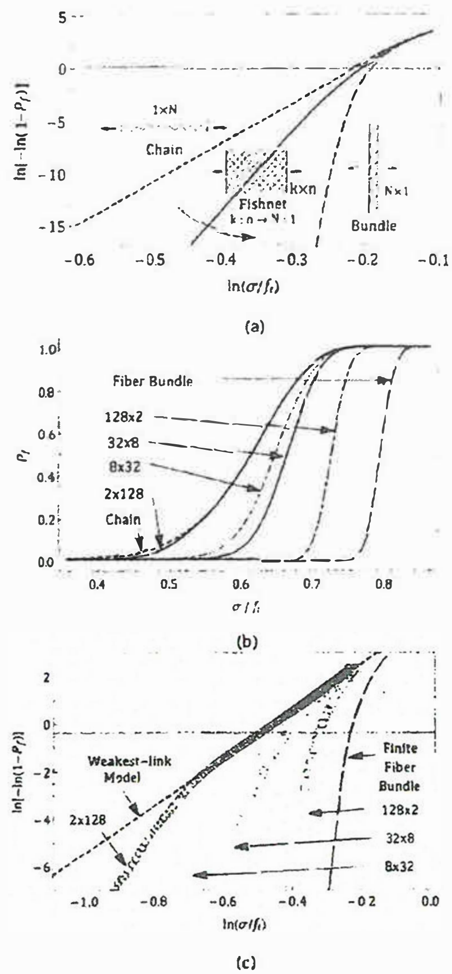

Figure 5. a) Change of failure probability of a fishnet pulled horizontally caused by varying the aspect ratio *k/n* gradually from 1:*N* to *N*:1 at constant number of links (Weibull scale); b) Monte Carlo simulations showing the transition of P_f as the aspect ratio of fishnet is changed from 1 × *N* to *N* × 1 (*N* = 256); c) The same data re-plotted on Weibull paper. f_t = 9.87 MPa is the mean strength of one link and CoV = 9.87%.

distinguishing link failures at the boundary from those in the interior, the agreement is excellent. This validates the analytical solution.

Fig. 4 shows, for comparison, also the histograms of all the Monte Carlo simulations, which correspond to the complete sum in Eq.(1). Note that, in this case, the three-term model, and even the two-term model, give a satisfactory estimate of fishnet cdf.

Shape Effect: Consider now the effect of the fishnet shape, or aspect ratio *k/n*. Fig. 5 shows the histograms obtained by random simulations (again about a million each) for fishnets with *N* = 256 links when their dimensions *k* × *n* are varied from 128 × 2, which represents the weakest-link chain (or series coupling), to 2 × 128, which represents the fiber bundle (or parallel coupling, with mechanics-based load sharing, i.e., equal extensions of all fibers). Obviously, the shape effect is very strong. However, fishnets with $k \gg n$ and rigid-body boundary displacements are not relevant to practical situations.

Fig. 4 shows the transition of P_f as the aspect ratio of fishnet is changed from 1 × *N* to *N* × 1 (*N* = 256). As expected, P_f gradually transforms from a Weibull distribution to Gaussian distribution as the shape of a fishnet changes from a chain to a bundle. Further note that this transformation from a chain to a bundle makes the fishnet stronger. Evidently, the weakest-link model and fiber bundle model give the upper and lower bound of P_f of all fishnets respectively.

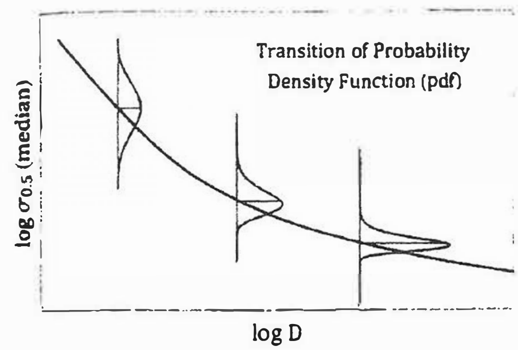

Figure 6. Statistical size effect on the median strength of quasi-brittle fishnet.

7 SIZE EFFECT

For simplicity, the effect of fishnet size D (chosen either as k or n) at constant shape k/n is here studied only for the median strength, $\sigma_{0.5}$, rather than the mean strength, $\bar{\sigma}$. Both analytical considerations and computer simulations show that the size effect curve in the plot of log $\sigma_{0.5}$ vs. log D is not a straight line, as in Weibull theory. Rather, the size effect curve descends at decreasing slope. Also, the CoV of $\sigma_{0.5}$ decreases with size D; see Fig. 6. This is all similar, but not identical, to the Type 1 size effect in fracture of concrete, rock, tough ceramics, fiber composites and other quasibrittle materials [20, 15].

8 CONCLUSIONS

1. The failure statistics of nacre-like material with imbricated (or staggered) lamellar microstructure under longitudinal tension can be approximately modelled by square fishnets pulled diagonally.
2. The probability distribution of fishnet strength, including the far-out left tail, can be calculated as a series of failure probabilities for maximum load occurring after the failure of one, two, three, etc., links. The series converges rapidly—the faster the greater the coefficient of variation (CoV) of scatter of each link.
3. The terms of this series represent various combinations of joint probabilities of survival and additive probabilities of failure for disjoint events. Near the zone of failed links, the link survival probabilities must be modified according to the mechanical stress redistribution due to previously failed links.
4. Compared to probability distribution for the finite weakest-link model developed for particulate materials and fiber composites, the strength at the failure probability level $P_f = 10^{-6}$ is about one to two orders of magnitudes higher, in terms of the ratio of strength to the mean strength. This ratio increases with increasing CoV of strength scatter of each link, but at the same time the mean strength decreases. Thus the combined effect at the level of $P_f = 10^{-6}$ can be strength decrease or increase.
5. There is no fixed-size representative volume element of material (RVE), in contrast to the weakest-link model for Type 1 quasibrittle failures of particulate materials. The size of the zone of failed links at maximum load grows with the CoV of link strength.
6. The size effect law is similar, though not the same, as in quasibrittle Type 1 finite weakest-link model. The nominal strength of fishnet at the same width-to-length ratio decreases significantly with the fishnet size.
7. The fishnet shape, i.e., the width-to-length aspect ratio, has a major effect on the probability distribution of strength, which contrasts with to finite weakest-link model for Type 1. The greater this ratio, the higher is the safety margin, i.e., the greater is the strength at the failure probability level $P_f = 10^{-6}$. As the aspect ratio is increased from 0 to ∞, the fishnet gradually transits for the weakest-link chain to the fiber bundle as the limit cases.
8. The fishnet model exhibits a strong size effect, similar to, though different from, the finite weakest-link model for Type 1 quasibrittle size effect characterizing particulate or granular materials and fiber composites. The evolution of cdf curves shows that, with increasing structure size, the cdf curves in Weibull scale get progressively steeper and cross each other. This is a qualitative difference from quasibrittle particulate materials or composites.
9. The fishnet model is verified by about a million Monte Carlo simulations of failure. The simulations were run for each of many different aspect ratios, link strength CoVs and fishnet sizes.
10. There now exist three basic, analytically tractable, statistical models for the strength of materials and structures:
    - the fiber bundle model (parallel coupling),
    - the weakest-link chain model (series coupling), and,
    - the fishnet model (mixed, or imbricated coupling).

    The third case includes the first two as the limit cases.
11. A similar steepening of the distribution slope at the lower end of Weibull scale plot can also

be achieved by the chain-of-bundles model, but only if a convenient intuitive non-mechanical load-sharing rule is empirically postulated for each bundle, and if the specimen length is subdivided by chosen cross sections into statistically independent segments of suitable length, corresponding to each bundle. However, the imbricated (staggered) lamellar connectivity cannot be captured.

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