Computational Modelling of Concrete Structures – Bićanić et al. (Eds)
© 2014 Taylor & Francis Group, London, ISBN 978-1-138-00145-9

Statistical distribution and size effect of residual strength after a period of sustained load

M. Salviato, K. Kirane & Z.P. Bažant
Northwestern University, Evanston, IL, USA

ABSTRACT: The present paper formulates the statistics of the residual strength of a quasibrittle structure after it has been subjected to a period of sustained loading. Here, quasibrittle structures (of positive geometry) are modeled by a finite (rather than infinite) chain of the weakest-link model. A strength degradation equation is derived based on the static crack propagation law which shows that the rate of strength degradation is not constant but continuously increasing. The cdf of residual strength of one RVE, representing one link in the chain, is shown to be closely approximated by a graft of Weibull and Gaussian (normal) distributions. In the left tail, the cdf is a three-parameter Weibull distribution consisting of the $(n + 1)$th power of the residual strength, where n is the exponent of the crack propagation law and the threshold is a function of the applied load and the load duration. The finiteness of the threshold, which is typically small, is a new feature of quasibrittle residual strength statistics, contrasting with the previously established absence of a threshold for strength and lifetime. Its cause is that there is a non-zero probability that some specimens fail during the static preloading, and thus are excluded from the statistics of the overload. The predictions of the theory are validated by available test data on glass-epoxy composites and on borosilicate glasses. The size effect on the cdf of residual strength is also determined. The size effect on the mean residual strength is found to be as strong as the size effect on the mean initial strength.

1 INTRODUCTION

In most engineering applications such as bridges, dams, ships, aircraft and microelectronic components, it is essential for the design to ensure a very low failure probability such as 10^{-6} throughout the lifetime (Nkb, 1978; Bažant & Pang, 2006). Therefore, the cumulative probability distribution function (cdf) of the structure must be known up to the very remote tail region. It must be established theoretically since such small probabilities are beyond direct experimental verification. This is especially true for quasibrittle materials, which represent heterogeneous materials characterized by brittle constituents and inhomogeneities that are not negligible compared to structural dimensions; e.g. concrete, fiber composites, tough ceramics, rocks, and many more (Bažant & Planas, 1998).

The type of cdf of strength for perfectly ductile structures must be Gaussian (based on the central limit theorem), whereas for perfectly brittle structures, it must be Weibullian (based the weakest link model with an infinite number of chain links). For quasibrittle structures, which behave as ductile when small and brittle when large, the type of cdf of strength and of static lifetime, was mathematically derived from atomistic scale arguments based on nano-scale cracks propagating by many small, activation energy-controlled, random breaks of atomic bonds in the nanostructure (Bažant & Pang, 2006, 2007; Le et al., 2011). It was shown that a quasibrittle structure (of positive geometry) must be modeled by a finite (rather than infinite) weakest-link model, and that the cdf of structural strength as well as lifetime varies from nearly Gaussian to Weibullian as a function of structure size and shape. Excellent agreement with experimentally obtained distributions was demonstrated.

In this paper, the theory is extended to the probabilistic distributions of residual strength after a period of sustained load. Knowing the statistics of residual strength is important for meaningful estimates of safety factors by taking into account the strength degradation of the structure depending on the load history and duration. It is also important to obtain better estimates of the remaining service life of structures, for which maintenance design is a primary concern, as it is for modern large aircraft made of load bearing quasibrittle composites.

2 THEORETICAL FORMULATION

The nano-mechanical derivation of the cdf of RVE strength as well as lifetime under static and cyclic loads is based on the fact that failure probability can be exactly predicted only on the atomic scale because the bond breakage process is quasi-stationary, which means that the probability must be exactly equal to the frequency (Kramer's rule). To derive the statistics of residual strength of an RVE, it is first noted that the crack growth rate on the atomic scale must follow a power law of applied stress with the exponent of 2. Equating the time rates of energy dissipations on the RVE and on the atomic level explains provides derivation of Evans' law for subcritical macrocrack growth and explains why it has a much higher exponent, typically about 10 for concrete and 30 for tough ceramics (Evans,1972; Thouless et al., 1983; Evans & Fu, 1984). Using Evans' law to integrate the failure probability contributions over time yielded a simple relation between the strength and static lifetime statistics (Le et al., 2011)—assuming the mechanisms of crack growth in a strength test and in a static lifetime test are the same. The argument is extended here to the statistics of residual strength.

2.1 *Relation between structural strength and static residual strength*

Evans's law for subcritical crack growth under sustained load reads:

$$\dot{a} = Ae^{-Q_0/kT} K_1^{\,n} \tag{1}$$

where a is the crack length, $\dot{a} = da/dt$ (t = time), A = material constant, Q_0 = activation energy, k = Boltzmann constant and T = absolute temperature. The stress intensity factor is denoted as K_I where the subscript 1 indicates the RVE level. So, we have $K_1 = \sigma\sqrt{l_0}k_1(\alpha)$ where $\sigma = F/l_0^2$ = nominal stress, l_0 = RVE size, $\alpha = a_1/l_0$ = relative crack length and k_1 = dimensionless stress intensity factor. Accordingly, the above equation becomes:

$$\dot{a} = Ae^{-Q_0/kT}\sigma^n l_0^{n/2} k_1^{\,n}(\alpha) \tag{2}$$

Consider now the different load histories illustrated in Figure 1. The load history O-A corresponds to the strength test, O-B-C to a static lifetime test and O-B-D-E to a residual strength test. Integration over load history O-B-D-E provides:

$$\frac{1}{r}\int_0^{\sigma_0}\sigma^n\sigma + \int_{t_0}^{t_R}\sigma_0^{\,n}t + \frac{1}{r}\int_{\sigma_0}^{\sigma_R}\sigma^n\sigma = e^{Q_0/kT}\int_{\alpha_0}^{\alpha_c}\frac{1}{Al_0^{\frac{n-2}{2}}k_1^{\,n}(\alpha)}d\alpha \tag{3}$$

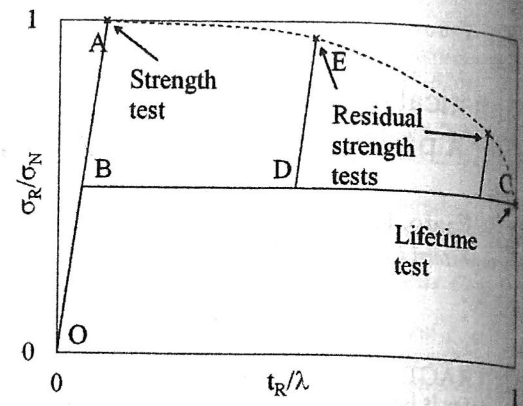

Figure 1. Schematic of various load histories.

By a similar integration of load histories O-A and O-B-C and appropriate substitution, one gets a very simple relation between σ_N, λ, and σ_R as:

$$\sigma_R = \left[\sigma_N^{\,n+1} - \sigma_0^n(n+1)(rt_R - \sigma_0)\right]^{\frac{1}{n+1}} \tag{4}$$

This is the equation for the degradation of the residual strength as a function of two independent (deterministic) variables, applied load and time of sustained load application. This equation also represents a link between the short-time strength and the residual strength.

2.2 *Analysis of residual strength degradation for one RVE*

We now proceed to analyze the effect of the exponent of the crack propagation law (Eq. 1) on the residual strength degradation. Figure 2 shows the degradation in strength of one RVE under static load for various values of n for applied load $\sigma_0 = 0.5\sigma_N$. The time of load application, normalized with respect to the lifetime, is shown on the horizontal axis. σ_N is assumed to be unity and the loading rate is taken as 0.5 MPa/second. It is seen that the rate of strength degradation is negligible initially but progressively increases and the most rapid degradation is seen in the end. This effect is seen to be more pronounced for higher values of n. Based on this observation, the degradation curve could be roughly divided in two regions, one of relatively slow degradation and one of rapid degradation—the distinction being more pronounced for higher values of n. This study reveals the usefulness of Eq. 4 since for given load parameters and crack growth exponent, one may determine the portion of lifetime for which the strength degradation is negligible.

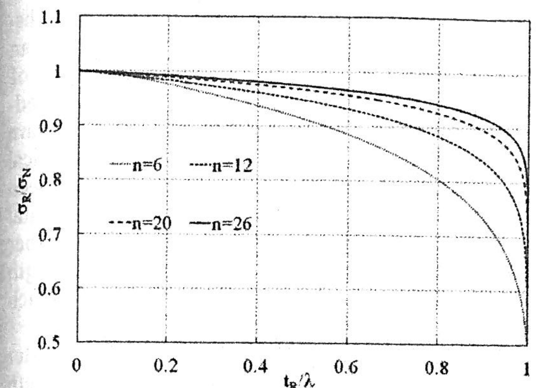

Figure 2. Predicted degradation curves for various values of static crack growth exponent n using Eq. (4).

2.3 *Formulation of statistics of residual strength for one RVE*

The analysis of interatomic bond breaks and multiscale transitions to the RVE has shown that the strength of one RVE must have a Gaussian distribution transitioning to a power law in the tail of probability within the range of 10^{-4} to 10^{-3} (Le et al., 2011). Starting from the cdf of strength, it is now possible to determine the cdf of residual strength for one RVE by means of Eq. (4). This yields (Salviato et al., *in press*):

$$P(\sigma_R) = 1 - \exp\left[-\left(\langle \sigma_R^{n+1} + \sigma_A \rangle / s_R\right)^{\bar{m}}\right] \quad (5)$$

for $\sigma_0 \le \sigma_R < \sigma_{R,gr}$ and:

$$P_{1,R} = P_{gr} + \frac{rf}{\sqrt{2\pi}\delta_G}\int_{\sigma_{R,gr}}^{\left(\sigma_R^{n+1}+\sigma_A\right)^{\frac{1}{n+1}}} e^{-(\sigma'-\mu_G)^2/2\delta_G^2} d\sigma' \quad (6)$$

for $\sigma_R \ge \sigma_{R,gr} > \sigma_0$.

Note that in Eqs (5,6) $\sigma_A = \sigma_0^n(n+1)(rt_R - \sigma_0)$, $\sigma_{R,gr} = (\sigma_{N,gr}^{n+1} - \sigma_A)^{1/(n+1)}$, while for the parameters $s_R = s_0^{n+1}$, $\bar{m} = m/(n+1)$; $P_{1,R}$ represents the probability of failure of one RVE under an overload, and $P_{1,R}(\sigma_0)$ represents the probability of failure of one RVE before the overload is applied. Note that only the part of the cdf where the residual strength is defined, i.e. where $\sigma_R \ge \sigma_0$, is considered.

Unlike the strength distribution, the residual strength cdf of one RVE does not have a pure Weibull tail. It is noteworthy that Eq. (5) describes a three parameter Weibull distribution in the variable σ_R^{n+1}, which has a finite threshold. Although it was proved that there can be no finite threshold in the distribution of strength (Le et al., 2011), the same does not hold true for the residual strength. The existence of a threshold value, σ_A in the cdf stems from the fact that some specimens could fail already during the period of sustained preload, which excludes them from the statistics of the overload. These are the specimens for which $\lambda < t_R$ or $\sigma_N < \sigma_0$.

2.4 *Formulation of residual strength cdf for structures of any size*

Once the cdf of residual strength related to one RVE is found, the cdf of failure of a structure of any size and geometry can be determined by means of the weakest link theory. The general applicability of this theory for brittle, ductile or quasi-brittle structures is guaranteed by the definition of RVE itself and the fact that failure is considered to occur at macro-crack initiation. One RVE is defined as the smallest part of the structure whose failure causes the failure of the entire structure. Thus, the RVE statistically represents a link (the failing RVE is the weakest link) and the structure can be statistically treated as a chain.

Similar to the definition of nominal strength, we define the nominal applied stress, $\sigma_0 = c_n P/bD$ or $c_n P/D^2$ for two- or three-dimensional scaling, where P = applied load. Then, by applying the joint probability theorem to the survival probabilities, the residual strength distribution of the structure can be expressed as:

$$P_{f,R} = 1 - \prod_{i=1}^{N}\left\{1 - P_{1,R}\left[\langle \sigma_0 s(\mathbf{x}_i)\rangle, t_R, \sigma_R\right]\right\} \quad (7)$$

where $s(\mathbf{x})$ dimensionless stress field $\mathbf{x}$ is the position vector and N is the number of RVEs. Similar to the chain model for the cdf of structural strength, the residual strength of the i-th RVE is here assumed to be governed by the maximum average principal stress $\sigma_0 s(\mathbf{x}_i)$ within the RVE, which is valid provided that the other principal stresses are fully statistically correlated.

3 RESULTS AND DISCUSSION

3.1 *Optimum fits of strength and residual strength histograms of borosilicate glass*

In this section, we determine the parameters of the distribution by fitting strength histograms and then we use them to predict the cdf of residual strength of borosilicate glasses. The predictions are compared to experiments by Sglavo & Renzi (1999). Figures 3a to d show the experimentally observed strength and residual strength histograms plotted in the Weibull scale. All the data considered were determined by conducting, in deionized water, four-point bend tests of borosilicate glass

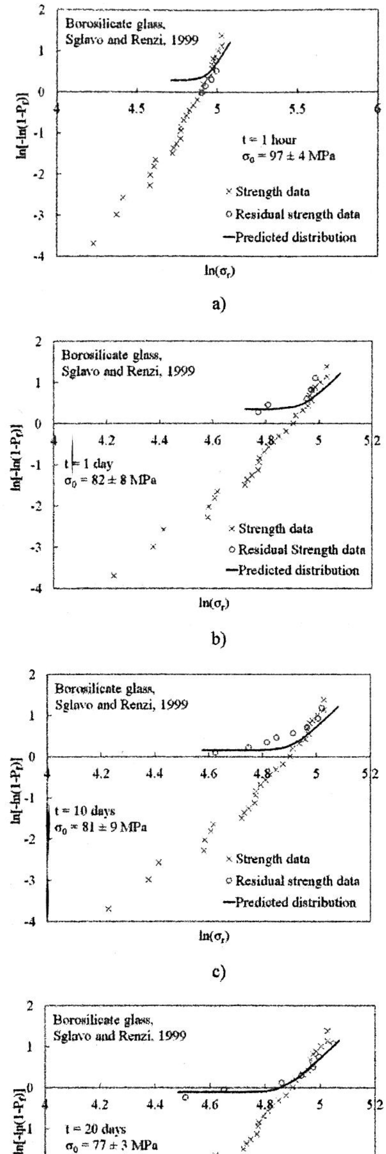

Figure 3. Predictions of residual strength histograms for borosilicate glass Hold times: (a) 1 hour (b) 1 day (c) 10 days and (d) 20 days. Data from Sglavo & Renzi, (1999).

rods with a nominal diameter of 3 mm and length of 100 mm. The loading rate was set to about 60 Mpa/s and different sustained load durations were used. Since glass is a brittle material and its RVE size is very small compared to the tested specimen size, the distribution of strength is virtually undistinguishable from the Weibull distribution, as can be seen in Figures 3a to d. By optimum fitting of the strength and residual strength, a Weibull modulus m of about 6 and a value of n of about 30 have been estimated. The fit predicted by the statistical formulation, shown by the solid line curves, is seen to be in good agreement with the experimental results. Except for the one hour case, all the other plots show the deviation of the residual strength distribution from the strength distribution to reach the probability value $P_{f,R}(\sigma_0)$.

It should be emphasized that, despite the scatter and a low number of data, all the residual strength distributions are predicted using the same set of parameters.

3.2 *Optimum fit of strength histograms and prediction of lifetime and mean residual strength for unidirectional glass/epoxy composites*

The methodology of the previous section is now pursued for the strength and residual strength data on unidirectional glass-epoxy composites reported by Hahn & Kim, (1975). Each specimen analyzed consisted of 8 unidirectional plies. 71 specimens were tested to obtain the strength and lifetime distributions. A constant sustained load σ_0 = 758 MPa was applied for all the lifetime tests. Figure 4a shows the fit of strength histograms by means of the grafted Gauss-Weibull distribution in the Weibull scale.

This fit shows a kink in the curve corresponding to the transition from Weibull to Gaussian distribution. A value of m equal to 56 and a value of n equal to 27 are estimated by least-square optimum fitting. Now that the required parameters of the distribution have been identified, the theory is applied to predict the mean residual strength and compare it to the experimental data. The comparison is made only for the mean since the number of available data is not sufficient to study the entire cdf. The resulting cdf of residual strength is then used to compute the mean values. The results are shown in Figure 4b for the different initial overloads and durations considered. Note that the predictions agree with the experiments, the difference being always less than 7%. The agreement provides another support for the present theory.

3.3 *Size effect on mean residual strength*

A more severe check on the theory would be to test the size effect on the mean lifetime and residual strength. However, no such test data seem to be available in the literature. It is nevertheless interesting to predict the size effect on the mean residual

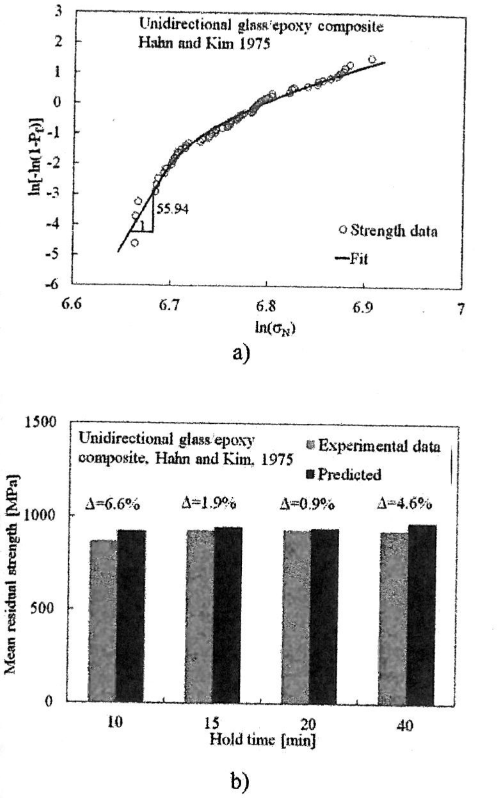

Figure 4. (a) Optimum Gauss Weibull fit of strength histogram. (b) Comparison of predicted mean residual strength for unidirectional glass/epoxy composite (Hahn and Kim, 1975).

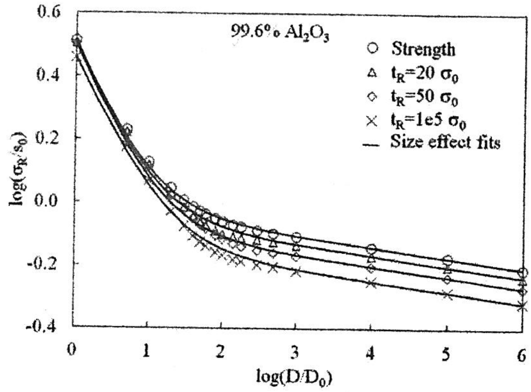

Figure 5. Calculated size effect curves on the mean strength residual strength at different hold times for 99.6% Al_2O_3.

strength integrating Eq. (7). Figure 5 shows the calculated size effect on the mean residual strength of 99.6% Al_2O_3. The set of parameters of the distribution is determined from the strength and lifetime histograms reported in (Fett & Munz, 1991). An applied load $\sigma_0 = 0.78\bar{\sigma}_N$—being $\bar{\sigma}_N$ the mean strength- is considered. Different times of load application are used, as reported in the figure, depending on the mean strength, i.e., $rt_R = \beta\bar{\sigma}_N$. Note that, for a given t_R, the mean residual strength shows a similar trend as the strength for the large size limit. In fact, the means tend to a straight line with the same slope as the mean strength.

It is impossible to obtain closed-form analytical expressions for the mean residual strength. However, sufficiently accurate analytical formulas can be derived by asymptotic matching. The size effect on residual strength can reasonably be approximated by the equation:

$$\bar{\sigma}_R = \left[\frac{M_a}{D} + \left(\frac{M_b}{D}\right)^{\eta/m}\right]^{1/\eta} \quad (8)$$

where m is the Weibull modulus of the cdf of strength and M_a, M_b and η can be derived by matching three asymptotic conditions:

1) $\left[\bar{\sigma}_R\right]_{D\to l_0}$,
2) $\left[d\bar{\sigma}_R/dD\right]_{D\to l_0}$, and
3) $\left[\bar{\sigma}_R D^{1/m}\right]_{D\to\infty}$.

As can be noted from Figure 5, the approximation given by Eq. (8) is rather good for all the different times of load application. In deriving the foregoing result, the two ratios, i.e., the applied load to strength and the hold time to lifetime, were kept constant across the sizes. It is trivial to note however that if the absolute value of the applied load or the hold time, or both, are kept constant, the size effect will of course be much stronger. However, in this case, the mean residual strength does not resemble the strength curve and it cannot be described by Eq. (8).

4 CONCLUSIONS

A theory for predicting the probabilistic distributions of residual strength after a period of static load has been developed and validated against test data. An important practical merit of the present theory compared with predecessors (Bažant & Pang, 2006, 2007; Le et al., 2011) is that it provides a way to determine the strength, residual strength and lifetime distributions without any histogram testing.

The rate of degradation of strength under a constant static load is not constant. Initially it is very slow and in the end very rapid. This effect is more pronounced for higher static crack growth exponents.

The cdf of residual strength of quasibrittle materials may closely be approximated by a graft of Gaussian and Weibull distributions. In the left tail, the distribution is a three parameter Weibull distribution in the variable σ_R^{n+1}. Unlike the cdf's of strength and lifetime, the cdf of residual strength has a finite threshold, albeit often very small.

The finiteness of the threshold is explained by the fact some specimens may fail during the sustained static preload and are thus excluded from the statistics of overload.

An expression for the size effect on the residual strength is derived using asymptotic matching. It is shown that the size effect on the residual strength is as strong as the size effect on strength.

Good agreement with the existing test data on glass-epoxy composites and on borosilicate and soda-lime silicate glasses is demonstrated.

ACKNOWLEDGMENT

Financial support under U.S. National Science Foundation Grant CMMI-1129449 to Northwestern University is gratefully acknowledged.

REFERENCES

Bažant, Z.P., Pang, S.-D., 2006. Mechanics based statistics of failure risk of quasibrittle structures and size effect on safety factors. Proc. Natl. Acad. Sci. USA 103 (25), 9434–9439.

Bažant, Z.P. & Pang, S.-D., 2007. Activation energy based extreme value statistics and size effect in brittle and quasibrittle fracture. J. Mech. Phys. Solids 55, 91–134.

Bažant, Z.P., Planas, J., 1998. Fracture and Size Effect in Concrete and Other Quasibrittle materials. CRC Press.

Evans, A.G. 1972. A method for evaluating the time-dependent failure characteristics of brittle materials and its application to polycrystalline alumina. J. Mater. Sci. 7, 1146–1173.

Evans, A.G., Fu, Y. 1984. The mechanical behaviour of alumina. In: Fracture in Ceramic Materials. Noyes Publications, Park Ridge, NJ, 56–88.

Fett, T., Munz, D., 1991 Static and cyclic fatigue of ceramic materials. In: Vincenzini, P. (Ed.), Ceramics Today—Tomorrow's Ceramics. Elsevier Science Publisher B.V. 1827–1835.

Hahn, H.T., Kim, R.Y., 1975. Proof testing of composite materials. J. Compos. Mater. 9, 297–311.

Le, J.-L., Bažant, Z.P., Bažant, M.Z., 2011. Unified nano-mechanics based probabilistic theory of quasibrittle and brittle structures: I. Strength, static crack growth, lifetime and scaling. J Mech Phys Solids 59, 1291–1321.

Salviato, M., Kirane K., Bažant Z.P. *in press* Statistical Distribution and Size Effect of Residual Strength After a Period of Constant Load.

Sglavo, V.M., Renzi, S. 1999. Fatigue limit in borosilicate glasses by interrupted static fatigue test. Phys. Chem. Glasses 40 (2), 79–84.

Thouless, M.D., Hsueh, C.H., Evans, A.G., 1983. A damage model of creep crack growth in polycrystals. Acta Metal. 31 (10), 1675–1687.