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Computational structural reliability – a major challenge and opportunity for concrete and other quasibrittle structures

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ABSTRACT: An overview of several recent advances in reliability of quasibrittle structures, identifying the main challenges and pointing out opportunities for further progress, is presented. The paramount importance of reliability analysis is obvious if one notes that the load factors and understrength (capacity reduction) factors, still essentially empirical and physically poorly understood, are far larger than the typical errors of modern computer analysis of structures. An effect of particular interest for structural reliability is the transition from quasi-plastic to brittle response with increasing structure size and the effect of structure type and geometry (or brittleness). To simulate this effect in the sense of extreme value statistics, a hierarchical model of nano-scale interatomic bonds is proposed, simulating one representative volume element (RVE) of the heterogenous material. A chain-of-RVEs is used to model structures larger than one RVE size and the number of RVEs in the chain is related to structure size (as well as geometry). The model shows that the distribution of structural strength exhibits, at increasing size (or brittleness), a gradual transition from Gaussian distribution (except in far-out tails) to Weibull distribution. The fact that the distribution of interatomic bond strength must be governed by Maxwell-Boltzmann distribution of atomic energies and that the activation energy barriers are modified by applied stress leads to a physical proof that perfectly brittle failures must follow Weibull distribution. It also reveals a new physical meaning of Weibull modulus-the number of dominant bonds that must fail simultaneously to cause an RVE to fail. Observing that, for Weibull distribution of typical variance, the point of failure probability such as 10^{-6} (a value tolerable in design) is about twice as far from the mean than it is for Gaussian distribution of the same mean and variance, one must conclude that the safety factor cannot be size-independent (as in current codes) but must be approximately doubled as the structure size changes from very small to very large. A way to capture it through reliability indices and through understrength factors applied to computational results is suggested. The brittleness factor, which needs to be made size dependent (contrary to current design codes), the material randomness factor, and the model error factor (both covertly implied by code provisions), need to be used as random variates in multi-variate Freudenthal-type reliability integral. At the same time, the size effect hidden in excessive self-weight factors currently imposed by codes has to be eliminated. The objective is to supplant realistic reliability estimates on deterministic computational mechanics. The paper concludes with examples of probabilistic structural analysis of some major structural disasters.

1 INTRODUCTION

Recent developments in probabilistic modeling of quasibrittle fracture with size effect, and in statistical data bases for failure loads, reveal problems whose resolution will require a major overhaul of the existing design codes and practices (Ellingwood et al. 1980, 1982), especially those for concrete, as well as generalization of the basic concepts of structural reliability, important especially for extrapolation of laboratory experimental evidence to very large structures failing in a non-ductile manner. From the statistical viewpoint, the problem is that the current practice is to model the strength variability by normal (Gaussian) or lognormal distribution. But this cannot apply to large structures obeying the weakest-link chain model because the stability postulate of extreme value statistics is violated. The reliability problems associated with brittleness have come to light as a result of theoretical modeling of the size dependence of the cumulative probability distribution function (cdf) of load capacity of quasibrittle structures (Bažant 2004a, b) based on nonlocal Weibull theory (Bažant and Xi 1991; Bažant and Novák 2000a, b) and asymptotic matching (Bažant 1997, 2004a, b), and also as a result of Monte Carlo structural simulations and of statistical studies of extensive databases. Quasibrittle materials, which include concretes, rocks, tough ceramics, fiber composites, concrete structure strengthened or retrofitted by composites, sea ice, stiff cohesive soils, wood, paper, foams, etc., typically exhibit a transitional size effect. So do modern tough metallic structures. Structures made of such materials exhibit not only the Weibull-type statistical size effect but also an energetic size effect on the nominal structure strength, σ_N, due to stress re-distribution caused by a large fracture process zone (FPZ) or by large fracture growth before the maximum load or load parameter, P_{max}, is reached (σ_N is defined as P_{max}/bD where D is the characteristic size of structure and b its thickness in the third dimension). The size effects are basically of two types: Type 1 occurs for structural geometry that permits stable growth of large fractures prior to reaching P_{max}, and type 2 for structures failing at macro-fracture initiation from an initial cracking zone (there exists also a type 3 size effect, but it is very similar to type 2); Bažant (2002). For type 2, material randomness affects only the scatter of σ_N but not its mean, while for type 1 it affects both and thus is more important. This study will deal only with type 1, for which the combined probabilistic-energetic size effect on the mean of σ_N can be approximately described as (Bažant 2004a):

$$\sigma_N = A\left(\vartheta^{rn_d/m} + r\kappa\vartheta\right)^{1/r}, \quad \vartheta = B\left(1 + D/\eta l_0\right)^{-1} \tag{1}$$

where n_d, m, r, κ, η, A, B, l_0 = constants. Eq. (1) was derived by asymptotic matching of the first two terms of the small-size and large-size asymptotic expansions of σ_N based on the cohesive crack model and the nonlocal Weibull theory. By asymptotic approximations it was also shown that, for large D, the Weibull distribution must be approached with an error second-order small in $1/D$ (Bažant 2002). The present brief article will (1) formulate a statistical series-parallel coupling model for the size effect on cdf in quasi-brittle failure; (2) propose a re-form of the standard reliability indices used in the first-order reliability method (FORM) to take into account the cdf tail; and (3) point out the reliability consequences of covert understrength factors and excessive load factor for self-weight.

2 HIERARCHICAL MODEL FOR CDF OF STRENGTH OF ONE RVE

Structures containing a cohesive crack, scaled down to vanishing size (D0), approach the case of an elastic body with a crack filled by a perfectly plastic "glue" (Bažant 2002). In this limit case, all the bonds along the failure surface are failing simultaneously. Therefore, the cdf of failure load should obey the Gaussian (or normal) distribution. This is implied by Daniel's (1945) fiber-bundle model (or parallel coupling of elastic-brittle fibers) in which the load is shared equally by all the unbroken fibers (Fig. 1a). The

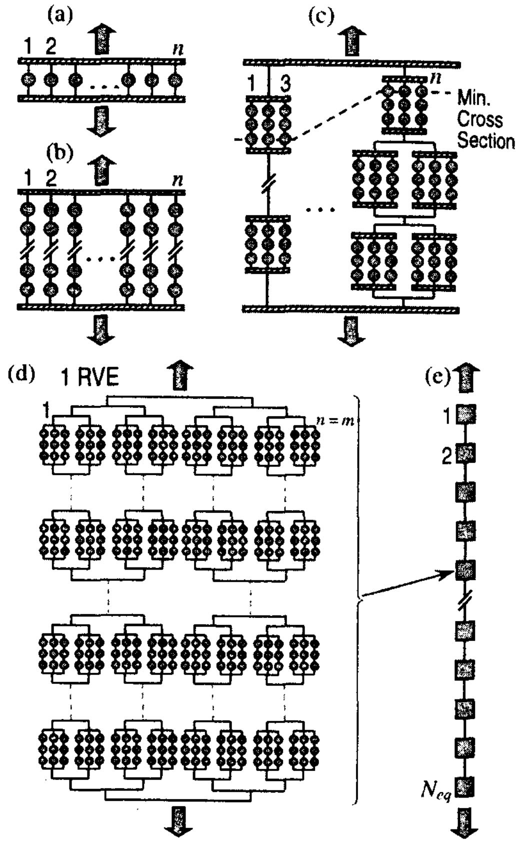

Figure 1. Models of series and parallel couplings a) fiber bundle; b) bundle of chains; c) example of a complex bundle with sub-chains and sub-bundles; d) hierarchy of sub-chains and sub-bundles; e) a chain model with each link representing a RVE.

cdf of the strength, σ, of a fiber-bundle with n elastic-brittle fibers is given exactly by Daniels' recursive formula:

$$G_n(\sigma) = \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} F^k(\sigma) G_{n-k}\left(\frac{n\sigma}{n-k}\right) \tag{2}$$

where $G_0(\sigma) = 1$ and $F(\sigma)$ = cdf of fiber strength.

If the fibers are elastic-perfectly plastic, the strength of the bundle is a sum of the individual fiber strengths (i.e., load redistribution is ignored), and Daniels' result is similar to the central limit theorem (CLT), which states that the sum of a large number of independent random variables approaches the Gaussian (or normal) distribution, irrespective of the individual distribu tions (unless they have infinite variances, in which case another stable distribution, Levy's, is approached). The cdf of the strength, σ, of a fiber-bundle with n

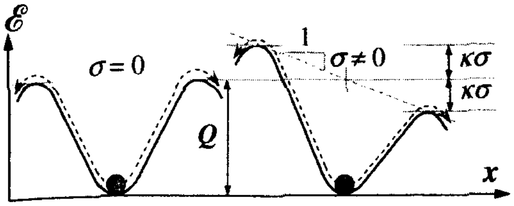

Figure 2. Interatomic potential profile and change of activation energy Q caused by applied stress σ.

elastic-plastic fibers is given exactly by the following convolution expression:

$$H_n(\sigma) = \frac{1}{n}\int_0^{n\sigma}\int_0^{x_n}\dots\int_0^{x_n-(x_2+\dots x_{n-1})} f(x_1)\dots f(x_n-(x_1+\dots x_{n-1}))\,dx_1\dots dx_n \quad (3)$$

The fibers in the fiber bundle model are a collection of interatomic bonds on the nano-scale within a RVE of the material. The RVE size depends on its purpose. Here the RVE is considered to be about three aggregates in size and is defined as the smallest material volume whose failure suffices to make a structure of positive geometry fail (this definition differs from that used in the homogenization theory for heterogeneous elastic-plastic materials). The failure probability of each interatomic bond can be deduced from statistical thermodynamics. The Maxwell-Boltzmann distribution of thermal energies of atoms (Hill 1956; Cottrell 1964; McClintock and Argon 1966) states that the fraction of atoms exceeding atomic energy level ε at absolute temperature T is:

$$\Phi(\mathcal{E}) = e^{-\mathcal{E}/kT} \quad (4)$$

where k = Boltzmann constant. If the energy of an atom exceeds its activation energy Q, the bond is broken. So the fraction (or frequency) of interatomic bond breaks is $\Phi(Q) = e^{Q/kT}$.

Application of stress, σ, causes the activation energy barrier in Figure 2 to change from Q to $Q - \kappa\sigma$ for bond breaking, and from Q to $Q + \kappa\sigma$ for bond restoration (where factor κ depends on microstructure geometry and randomness, and is assumed to be independent of σ). From this, it can be deduced that the net frequency, F_b, of permanent bond breaks under stress σ at temperature T is $e^{-(Q-\kappa\sigma)/kT} - e^{-(Q+\kappa\sigma)/kT}$, which may be rewritten as:

$$F_b = 2\,e^{-Q/kT}\sinh(\kappa\sigma/kT) \quad (5)$$

The reason that the transfer of load from the broken to the unbroken bonds is not taken into account within each fiber is that it is approximately captured by load sharing within the bundle itself.

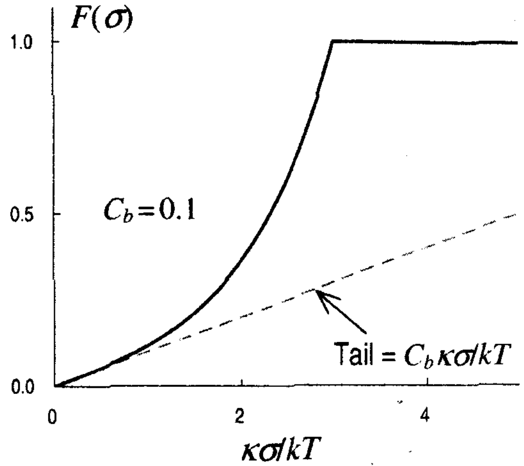

Figure 3. Effect of temperature and stresses on the cdf of interatomic bonds.

A continuous crack within an RVE will occur when the broken bonds create a contiguous surface of atomic bond breaks separating the atomic lattice into two parts. This will occur when the fraction of broken bonds in the lattice reaches a certain critical value ϕ_b (the calculation of which is a problem of percolation theory). That critical value will be attained after a certain critical duration τ of exposure of the RVE to stress σ, and so $\phi_b = \phi_b(\tau)$. So, the cumulative probability of creating a continuous crack may be expressed as (Fig. 3):

$$F(\sigma) = \min[\,C_b\sinh(\kappa\sigma/kT),\ 1\,] \quad \text{for } \sigma \geq 0 \quad (6)$$

where $C_b = 2\phi_b(\tau)\,e^{-Q/kT}$; the "min" serves to ensure that $F(\sigma)$ terminate at 1. The stress at which $F(\sigma) = 1$ is $\sigma_1 = (kT/\kappa)\ \sinh^{-1}(1/C_b)$, which is a function of temperature T as well as the stress duration, τ (thus the time or rate dependence of strength is included in the formulation). The corresponding pdf is: $p_b = (C_b\kappa/kT)\ \cosh(\kappa\sigma/kT)$ for $\sigma \in (0, \sigma_1)$; else $p_b = 0$. Of main interest here is the left tail of $F(\sigma)$. Since $\sinh x \approx x$ for small x, the cdf tail is a power law with exponent 1 and a zero threshold;

$$F(\sigma)_{\sigma\to 0} \approx (C_b\kappa/kT)\,\sigma \propto \sigma^p \quad \text{with } p = 1 \quad (7)$$

The important point to note is that the tail is a power law with exponent 1 (thus, for example, the strength of a chain, or a series coupling, of many potential break surfaces would have a Weibull cdf with $m = 1$, which is the exponential distribution, i.e. $F(\sigma) = 1 - e^{-c\sigma}$ in which $c = C_b\kappa/kT$; however, the exponential distribution is not needed for our purpose and probably would anyway be an oversimplification, because the break surfaces surely do not interact as simply as the links in a chain).

The cdf of a simple fiber bundle (Fig. 1b) approaches a Gaussian distribution with the asymptotic mean and coefficient of variation (CoV) given by (Daniels 1945):

$$\mu = \sigma^*[1 - F(\sigma^*)] \tag{8}$$

$$\omega = n^{-1/2}\sqrt{F(\sigma^*)[1 - F(\sigma^*)]} \tag{9}$$

where σ^* is the point for which $\mu(\sigma)$ reaches its maximum. Obviously, for $n \to \infty$ one has $\omega \to 0$, i.e., an infinite bundle is deterministic, and so only a finite n make sense. The Gaussian approximation of cdf reveals nothing about the far-out cdf tail. The Gaussian distribution is only valid for the central core expanding in proportion to $n^{1/2}$ whereas the left tail of cdf has the cdf of the fibers (Harlow et al. 1983). The probability of failure for each fiber has a tail cdf σ^p (for $\sigma \to 0$) where $p = 1$ given in Eq. (7). If the tail cdf of each fiber has the same power law, it can be proven from Eq. (2) by mathematical induction that the tail of the cdf of the strength of a bundle with n fibers is a power law $\propto \sigma^{np}$ (Bažant and Pang 2005b). This leads to an interesting conclusion–the Weibull modulus m of the tail distribution of strength σ of a RVE is equal to the number n of fibers in the bundle. If the fibers are elastic-plastic, the same conclusion could also be reached by induction, on the basis of the joint probability theorem given by Eq. (3) (Bažant and Pang 2005), and may be assumed to hold for bundles with softening fibers as well.

A simple fiber bundle (Fig. 1b), with $n = m$ fibers, would have a Gaussian cdf with Weibull tail so short (Table 1) that it would be virtually nonexistent and no effect on structural behavior (Bažant and Pang 2005b). If the interatomic bonds were coupled in series in each fiber (Fig. 1b) to extend the length of the Weibull tail, it would still remain so short that it could never be manifested in experiments on brittle materials such as ceramics (Weibull 1939; Bansal et al. 1976a, b; Quinn and Morrell 1991).

To obtain a long enough Weibull tail, such that structures with >1000 RVE would be have an almost entirelyWeibull cdf, it is necessary and more realistic to model a RVE by a hierarchical model, in which a RVE is statistically described by a bundle of n_1 fibers, each of which consists of a long chain of bundles with n_2 sub-fibers, with each of the sub-fibers having the cdf in Eq. (6). In this case the cdf of RVE would have a power-law tail of exponent $m = n_1 n_2$, which is again equal to the number of all sub-fibers coupled in parallel. The sub-fibers of secondary bundles are refined on a still lower-scale of microstructure as chains of tertiary sub-sub-bundles consisting of long sub-sub-chains, until the nano-scale of interatomic bonds are reached. The detailed probabilistic micro-structural model is surely non-unique, but the number of parallel fibers in cuts separating the hierarchical model into two parts at the lowest level must be equal to the exponent m of the tail cdf of a RVE (Fig. 1d). The actual behavior of a RVE will, of course, correspond an irregular hierarchical model, such as that shown in Figure 1c. In that case, according to the aforementioned basic properties, the exponent of the power-law tail for the RVE, and thus the Weibull modulus of a large structure, is defined as the minimum number of cuts of elementary serial bonds that are needed to separate the model into two halves.

Consequently, the detailed parallel and series coupling of the hierarchical model for the RVE does not matter for our purpose because we seek only qualitative information-the type of cdf, while a quantitative analytical prediction from atomic microstructure is beyond reach. What matters is that (i) the cdf of a RVE consists, in any case, of a Gaussian core with a Weibull-type lower tail, whose Weibull modulus m is equal to the number n of parallel fibers across the weakest cross section, and that (ii) the lower tail of cdf for each micro-bond is a Weibull cdf (or a power law, if short enough) with exponent 1 (i.e., a linear function of stress), as in Eq. (5).

An enormous advantage of anchoring the theory in statistical thermodynamics, particularly the Maxwell-Boltzmann distribution, is that the dependence of cdf of failure load on temperature T and load duration τ is captured automatically (in practice, though, the temperature range can be limited because of interplay of several different activation energies for different atoms). Because the activation potential barrier is affected by the presence of water molecules, it is, in principle, possible to capture also the effects of the content of moisture or various corrosive agents in porous hydrophilic solids such as concrete.

3 GRAFTED WEIBULL-GAUSSIAN DISTRIBUTION FOR ONE RVE

From experimental data on brittle materials, such as ceramics (e.g. Weibull 1939; Bansal et al. 1976a, b; Quinn and Morrell 1991), Weibull size effect is often clearly evident for equivalent number of RVE's $N_{eq} > 500$. But this Weibull cdf is unobtainable if a simple fiber bundle with $n = 24$ fibers (typical of concrete, Bažant and Novák 2000a, b) were used to model a RVE, the Weibull tail would be extremely short, reaching only up to $P_f = 10^{-45}$ (Table 1); which would the structure to be about 10^{47} times larger than the RVE for its strength to exhibit Weibull cdf. This is, of course, impossible. A feasible statistical model for a RVE is a hierarchical model of the kind shown in Figure 1d. This model can provide a cdf whose Weibull tail extends up to about $P_f = 0.003$ or 0.0003 when its elements (or fibers) are brittle (Fig. 4a) or plastic (Fig. 4b). More realistic doubtless are softening

Table 1. Extent of power-law tail probabilities for elastic -brittle and -plastic bundles with a tail exponent of 24.

p	n	Elastic d_n/d_1	Elastic P_{tn}	Plastic d_n/d_1	Plastic P_{tn}
24	1	1.00	3.0×10^{-01}	1.00	3.0×10^{-01}
12	2	0.50	5.5×10^{-05}	0.79	3.0×10^{-03}
8	3	0.33	1.3×10^{-07}	0.67	7.2×10^{-05}
6	4	0.25	7.9×10^{-10}	0.58	2.3×10^{-06}
4	6	0.16	3.8×10^{-13}	0.44	3.0×10^{-09}
3	8	0.11	4.0×10^{-16}	0.33	2.2×10^{-12}
2	12	0.06	2.8×10^{-22}	0.18	2.5×10^{-19}
1	24	0.01	3.6×10^{-45}	0.02	7.2×10^{-44}

d_1, d_n = length of power-law tail for 1 and n fibers respectively; P_{tn} = extent of power-law probabilities.

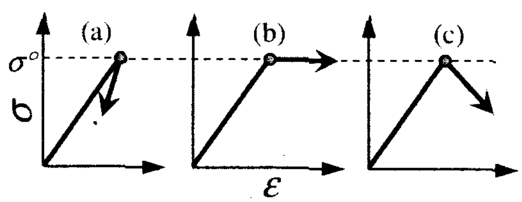

Figure 4. Post-peak behavior of fibers: a) Elastic-Brittle; b) Elastic-Plastic; c) Elastic-Softening.

elements of the model (Fig. 4c), which are harder to analyze but may be expected to exhibit intermediate behavior. A parallel coupling at any scale of hierarchy tends to build up a Gaussian core of cdf and drastically shorten Weibull tail while raising its exponent. A series coupling at any scale of hierarchy tends to shorten the Gaussian core and extend theWeibull tail while keeping its exponent unchanged (Bažant and Pang 2005b).

For increasing D, the Gaussian core shrinks and the Weibull tail spreads toward higher probabilities (Bažant 2004a, b) until, for infinite D, the entire cdf becomes Weibull. To describe such behavior, we introduce a Gaussian distribution with a truncated lower (i.e. left) tail, onto which we graft a Weibull tail. The upper tail of cdf of strength is irrelevant for larger structures, according to the weakest-link model. So, a one-sided lower tail graft will suffice. The grafted pdf can be mathematically described as follows:

$$p_1(\sigma_N) = r_f \phi_W = r_f \left(\frac{m}{s_1}\right)\left(\frac{\sigma_N}{s_1}\right)^{m-1} e^{-(\sigma_N/s_1)^m} \quad \text{for } \sigma_N < \sigma_{N,gr} \quad (10)$$

$$p_1(\sigma_N) = r_f \phi_G = \frac{r_f}{\delta_G\sqrt{2\pi}} e^{-(\sigma_N-\mu_G)^2/2\delta_G^2} \quad \text{for } \sigma_N \geq \sigma_{N,gr} \quad (11)$$

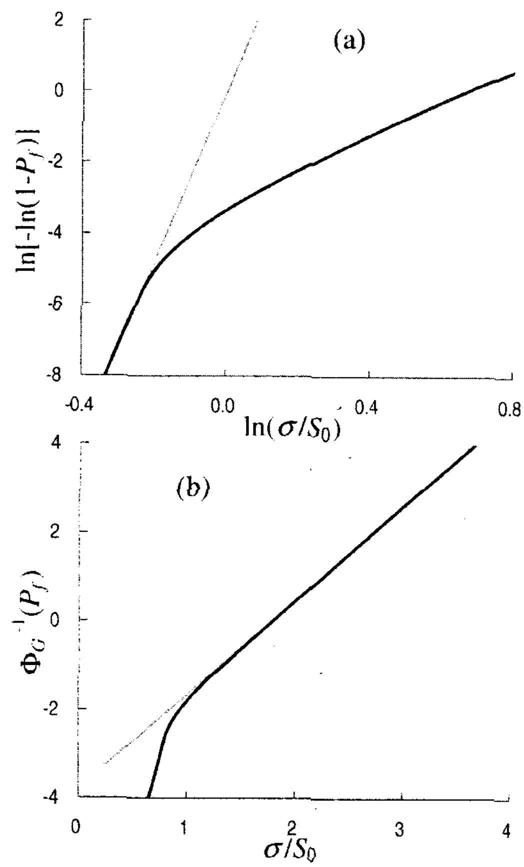

Figure 5. cdf of a RVE with $P_{gr} = 0.003$ and CoV = 25% in a) Weibull probability paper; b) normal probability paper.

where μ_G, s_G = mean and standard deviation of Gaussian core; m, s_1 = shape and scale parameters of Weibull tails; σ_{gr} is the grafting point; r_f is a scaling factor ensuring that the combined cdf of the Weibull-Gaussian graft is normalized.

$$r_f = [1 - \Phi_G(\sigma_{gr}) + \Phi_W(\sigma_{gr})]^{-1} \quad (12)$$

Both pdf's, as defined in Eqs. (10) and (11), are matched to be continuous at the grafting point, which leads to the following compatibility condition

$$\mu_{Gn} = \varsigma_{gr} - \delta_{Gn}\sqrt{-2\ln[\sqrt{2\pi}\, m\, \delta_{Gn}\, \varsigma_{gr}^{m-1}\, e^{-\varsigma_{gr}^{m}}]} \quad (13)$$

where $\mu_{Gn} = \mu_G/s_1$; $\delta_{Gn} = \delta_G/s_1$; ς_{gr}/s_1. If the standard deviation of the Gaussian core δ_G, and the scale parameter s_1 of the Weibull tail, are known, μ_{Gn} can be calculated from Eq. (13) and the Weibull-Gaussian graft for cdf of one RVE (Fig. 5) can be expressed

Table 2. Mean μ, standard deviation δ, and CoV ω_0 of the grafted distribution for one RVE with various grafting probabilities.

P_{gr}	r_f	μ_{Gn}	δ_{Gn}	μ/s_1	ω_0
0.001	1.009	2.723	0.850	2.746	0.30
0.001	1.003	1.649	0.334	1.653	0.20
0.001	1.000	1.084	0.109	1.085	0.10
0.003	1.024	2.112	0.686	2.151	0.30
0.003	1.008	1.497	0.309	1.504	0.20
0.003	1.000	1.079	0.108	1.080	0.10
0.005	1.040	1.894	0.638	1.951	0.30
0.005	1.013	1.432	0.301	1.443	0.20
0.005	1.001	1.074	0.107	1.075	0.10
0.010	1.086	1.632	0.600	1.730	0.30
0.010	1.029	1.344	0.292	1.363	0.20
0.010	1.002	1.069	0.108	1.070	0.10

explicitly as:

$$P_1(\varsigma) = r_f\left(1 - \mathrm{e}^{-\varsigma^m}\right) \qquad \text{for } \varsigma < \varsigma_{gr}$$

$$P_1(\varsigma) = r_f\Phi(\varsigma_{gr}) + \frac{r_f}{\delta_{Gn}\sqrt{2\pi}}\int_{\varsigma_{gr}}^{\varsigma} \mathrm{e}^{-\frac{1}{2}\left(\frac{\varsigma'-\mu_{Gn}}{\delta_{Gn}}\right)^2}\mathrm{d}\varsigma' \qquad \text{for } \varsigma \geq \varsigma_{gr} \quad (14)$$

where $\Phi(\varsigma_{gr}) = r_f(1 - \mathrm{e}^{-\varsigma_{gr}^m})$. The scale parameter s_1 used in the grafting method is related to s_0 of Weibull cdf by $s_0 = r_f^{1/m} s_1$. Typical values for r_f range between 1.00–1.14, which means that s_0 differs from s_1 by less than 0.5%. In practice, $r_f^{1/m}$ can be taken as 1, and $s_0 = s_1$ but r_f should remain in the formulation of P_1 else an error of up to 12% in the cdf of one RVE is likely.

The normalized mean and CoV of a RVE, which characterizes small-size quasibrittle structures, which fail in a nearly ductile manner, can be calculated from the following procedure:

1. Locate P_{gr} from Weibull plots of cdf.
2. Calculate relative length of Weibull tail $\varsigma_{gr} = [\ln(1 - \Phi_W(\varsigma_{gr}))]^{1/m}$ where $P_{gr} = r_f\Phi_W(\varsigma_{gr})$. For initial guess, assume $r_f = 1$ such that $P_{gr} = \Phi_W(\varsigma gr)$.
3. Estimate relative standard deviation for the Gaussian cdf with empirical equation (with error $< 5\%$) $\delta_{Gn} = \exp[-3.25 + 11.6\,\omega_0 - h(\varsigma_{gr})\omega_0^2]$ where $h(\varsigma_{gr}) = 1000\Phi_W(\varsigma_{gr})/[109\Phi_W(\varsigma_{gr}) + 0.133]$.
4. Calculate the relative mean for Gaussian cdf μ_{Gn} with the compatibility equation (13).
5. Check if $r_f \approx 1$, else iterate by substituting back into Step 2.

Alternatively, the normalized mean and CoV of a RVE can be read from Table 3 for some typical parameter values. The grafted power-law tail for

Table 3. Determination of B of Eq. (32) for $m = 24$.

s_F/μ_F	B	s_F/μ_F	B	s_F/μ_F	B
0.050	20.55	0.175	7.19	0.300	5.25
0.075	14.13	0.200	6.59	0.325	5.05
0.100	11.00	0.225	6.13	0.350	4.88
0.125	9.19	0.250	5.77	0.375	4.74
0.150	8.01	0.275	5.48	0.400	4.62

each link in the chain is crucial for getting large-size asymptotic distribution of the Weibull type (if the links had Gaussian tails, the cdf of the chain would approach Gumbel's (1958) cdf, which would be physically unacceptable).

4 STATISTICS OF BRITTLE FAILURE

We consider structures of positive geometry (i.e., a geometry for which K_I for unit load increases with crack length), which fail at fracture initiation. With increasing D, the weakest-link model (Fig. 1e) gives a cdf quickly approaching the Weibull cdf [with an error of only $O(D^{-2})$; Bažant, 2004a] because the tails of each link (each RVE) are Weibull and thus satisfy the stability postulate of extreme value statistics (Fisher and Tippett 1928).

$$P_f(\sigma_N) = 1 - \mathrm{e}^{-(\sigma_N/S_0)^m} = 1 - \mathrm{e}^{-N_{eq}(\sigma_N/s_0)^m} \quad (15)$$

where N_{eq} is the equivalent number of RVEs which accounts for non-uniform stress field

$$N_{eq} = (D/l_0)^{n_d}\int_V S^m(\xi)\mathrm{d}V(\xi) \quad (16)$$

where $\xi = x/D$, is the dimensionless coordinate vector; $S(\xi)$ is the dimensionless stress field which depends on structure geometry but not on structure size D; l_0 = material characteristic length (roughly the RVE size); n_d = number of dimensions in which the structural failure is scaled. For narrow beams in flexure which fail at macro-crack initiation, the number of links (or RVE's) in the chain scales as D^{nd} with $n_d = 2$, because widening of a narrow beam has no effect.

Note that, aside from Weibull cdf, there exist only two other extreme value cdf's–Gumbel's (derived by Fisher and Tippett 1928) and Fréchet's (1927); but they are excluded, not only because of lacking a power-law tail for each link but also because their far-out tails reach into physically meaningless negative strength values. The mean of nominal strength, $\bar{\sigma}_N$, is scaled by the number of links as follows:

$$\bar{\sigma}_N = s_0 N_{eq}^{-1/m}\Gamma(1 + 1/m) \quad (17)$$

where s_0 = Weibull scale parameter of each link; m = shape parameter (or Weibull modulus) common to all links and the whole chain.

The CoV of nominal strength is independent of the number of links and is given by:

$$\omega_N = \sqrt{\frac{\Gamma(1+2/m)}{\Gamma^2(1+1/m)} - 1} \tag{18}$$

Note that when ω_N depends on D, it means that Weibull statistical theory does not apply and that the size effect is caused, at least in part, by energy release due to stress redistribution, as captured by the nonlocal Weibull theory (Bažant and Xi 1991). The links in the chain (weakest link-model) (Fig. 1e) correspond to individual RVEs, having roughly the size of a FPZ dictated by material heterogeneity (considered identical to the nonlocal averaging domain; Bažant and Xi 1991). For very large structures dwarfing the FPZ (or the zone of localized distributed cracking), a positive-geometry structure fails as soon as the full FPZ, capable of dissipating energy at the rate equal to the fracture energy of material, develops (in the case of notches or structures of negative geometry failing after large macro-crack growth, the cdf is predominantly Gaussian, with a short Weibull tail).

According to Eqs. (7) and (15), means that the scaling parameter S_0 of Weibull cdf of structural strength must depend on absolute temperature T and on load duration τ (or loading rate $1/\tau$), and that the dependence must have the form:

$$S_0 = S_{0r} \frac{T}{T_0} \frac{\phi_b(\tau_0)}{\phi_b(\tau)} e^{(\frac{1}{T}-\frac{1}{T_0})\frac{Q}{k}} \tag{19}$$

where T_0 = reference absolute temperature (e.g., room temperature 298°K), τ_0 = reference load duration (or time to reach failure of the specimen, e.g., 1 min.), and S_{0r} = reference value of S_0 corresponding to T_0 and τ_0. The corresponding Weibull cdf of structural strength at any temperature and load duration may be written as:

$$P_f = 1 - \exp\left\{ - \left[\frac{\sigma_N}{S_{0r}} \frac{T_0}{T} \frac{\phi_b(\tau)}{\phi_b(\tau_0)} e^{(\frac{1}{T_0}-\frac{1}{T})Q/k} \right]^m \right\} \tag{20}$$

Note, however, that this simple dependence on T is expected to apply only through a limited range of temperatures and load durations. The reason is that the interatomic potential surface typically exhibits not one but many different barriers Q and coefficients κ for different atoms and bonds, with different Q and κ dominating in different temperature ranges.

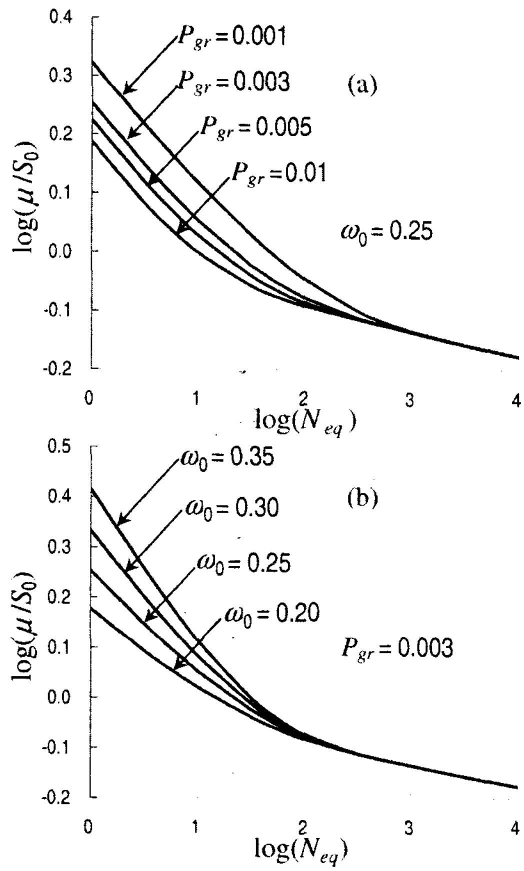

Figure 6. Mean size effect curves for a) different P_{gr}; b) different CoV of one RVE ω_0.

5 TRANSITION OF CDF BETWEEN SMALL AND LARGE STRUCTURE SIZES

The transition of cdf from small sizes to large structures can be calculated, according to the weakest-link (or chain) model (Fréchet 1927), as follows :

$$P_f(\sigma_N) = 1 - [1 - P_1(\sigma_N))]^{N_{eq}} \tag{21}$$

where $P_1(\sigma_N)$ is the failure probability of each RVE (or each link) computed from Eqs. (11) and (10). The number of links (Fig. 1e) equals the number of RVEs in the structure if the stress is uniform or calculated from Eq. (16) if the stress field is non-uniform.

The transition from Gaussian to Weibull cdf when passing from small to large sizes is evident in Figure 6. On a linear scale of σ_N and P_f (Fig. 6a), in which the CoV is proportional to the maximum slope of the cdf curve, the CoV first decreases with increasing N_{eq}, which is typical of parallel coupling statistics, and then stabilizes at a constant value, which is typical of weakest-link statistics. For $N_{eq} > 1000$ and grafting probability $P_{gr} = 0.001$ of one RVE, the cdf is visually indistinguishable from Weibull cdf and an increase in size causes merely a leftward shift of the cdf curve as a rigid body.

On the Weibull scale (i.e., Weibull probability paper), on which the Weibull cdf of σ_N is a straight line, the straight segment lengthens with increasing N_{eq} (Fig. 6b) while the Gaussian core, appearing as a concave curve, shifts upwards. The transition is a distinctive link (Fig. 6b), the location of which may be precisely defined as the intersection between the extensions of the Gaussian core and the Weibull tail.

On the normal probability paper, on which the Gaussian (normal) cdf of σ_N is a straight line the straight segment shortens with increasing N_{eq} (Fig. 6c), while the tail segment of Weibull cdf appears as a curve, shifting and deforming with N_{eq}.

The mean nominal strength for any number of RVEs can be determined as follows:

$$\bar{\sigma}_N = \int_0^\infty \sigma_N N_{eq}[1 - P_1(\sigma_N)]^{N_{eq}-1} p_1(\sigma_N) \mathrm{d}\sigma_N \quad (22)$$

Analytical evaluation of Eq. (22), with $p_1(\sigma N)$ and $P_1(\sigma N)$ defined in Eqs. (10)–(11) and (14) respectively, is impossible but the asymptotes can be determined. For large size, the asymptote must be the Weibull size effect, due to the power-law tail of the cdf of each RVE (Hypotheses I and II), which is given by the following expression on a logarithmic scale:

$$\log(\bar{\sigma}_{N,W}) = -\frac{1}{m}\log(N_{eq}) + \log[s_0\Gamma(1 + 1/m)] \quad (23)$$

For a short chain, the deviation of the mean nominal strength from Eq. 23 may then be expressed as:

$$\log(\bar{\sigma}_{N,G}) = \mu_G[1 - \chi(N_{eq})(\omega_0/\sqrt{\pi})] \quad (24)$$

where $\chi(N_{eq}) = 1$, 3/2 for $N_{eq} = 2$, 3 respectively. For intermediate values, $3 < N_{eq} < 10$, an analytical expression for parameter μ_s is unavailable (Ang and Tang 1984). Since the Weibull tails of RVEs spoil approach to Gumbel distribution even for small N_{eq}, it is necessary that the mean size effect curve in Eq. (22) be obtained by nonlinear regression.

When the asymptotic Weibull strength of Eq. (23) is subtracted from the mean nominal strength Eq. (22) in logarithmic scale, the difference fits very well with the following empirical function:

$$\log\bar{\sigma}_N - \log(\bar{\sigma}_{N,W}) = \exp[-\exp(a_1 - a_2 \log N_{eq})] \quad (25)$$

$$\text{where } a_1 = \ln\left[-\ln\left(\log\frac{\mu_0}{s_0\Gamma(1 + 1/m)}\right)\right] \quad (26)$$

Parameter a_1 anchors the size effect curve for one RVE; μ_0 is the mean for one RVE; a_2 controls the rate of transition to Weibull size effect and depends on the length of the Weibull tail for one RVE. Calibrating the mean size effect curve requires a minimum of 4 parameters: μ_0, P_{gr}, m, and s_0.

6 RELIABILITY BASED DESIGN

The reliability-based design requires consistent evaluation of failure probability risk using probability theory. Structures are designed for very low probability, typically of the order of 10^{-7} (Allen 1968; CIRIA 1977), which is totally dominated by far-off tail distributions of the load and the resistance. The probability of failure is the integral of the bivariate probability density over the domain where the resistance is less than the load (Freudenthal et al. 1966; Ang and Tang 1984; Melchers, 1987). This integral can be rearranged as:

$$P_f(\sigma_N) = \int_{-\infty}^{\infty} f(\sigma_N)R(\sigma_N)\mathrm{d}\sigma_N \quad (27)$$

where $R(\sigma_N)$ = cdf of structural resistance and $l(\sigma_N)$ = probability density function (pdf) of the load. If the distributions of random load variable L and resistance variable R are Gaussian, the safety margin, $M = R - F$, is also Gaussian, and its mean $\mu_M = \mu_R - \mu_F$ and variance $s_M^2 = s_R^2 + s_F^2$ give

$$P_f = \Phi_G\left(-\frac{\mu_R - \mu_F}{\sqrt{s_R^2 + s_F^2}}\right) \quad (28)$$

To avoid dealing with small probabilities, it is often more convenient to adopt reliability index β, which is for Gaussian distributions simply defined as:

$$\beta = \frac{\mu_R - \mu_F}{\sqrt{s_R^2 + s_F^2}} = \frac{\omega_R(s_R/s_F) - (\mu_L/s_F)}{\sqrt{1 + (s_R/s_F)^2}} \quad (29)$$

as proposed by Cornell (1969). In the space of normalized differences of load and resistance from their means, has the geometrical meaning of the distance from the origin to the closest point (called the design point) on the boundary of the safe region ($L < R$) (e.g. Haldar and Mahadevan 2000). Eq. (18) assumes a linear failure surface (a hyperplane) and it belongs to the first-order second-moment method (FOSM) (Madsen, et. al, 1986; Melchers 1987). If the variables are non-normal, or if the failure surface is nonlinear while the first-order approximation of the failure surface at the design point is still used, one may use an improved Hasofer-Lind reliability index (FORM), calculated by an iterative procedure.

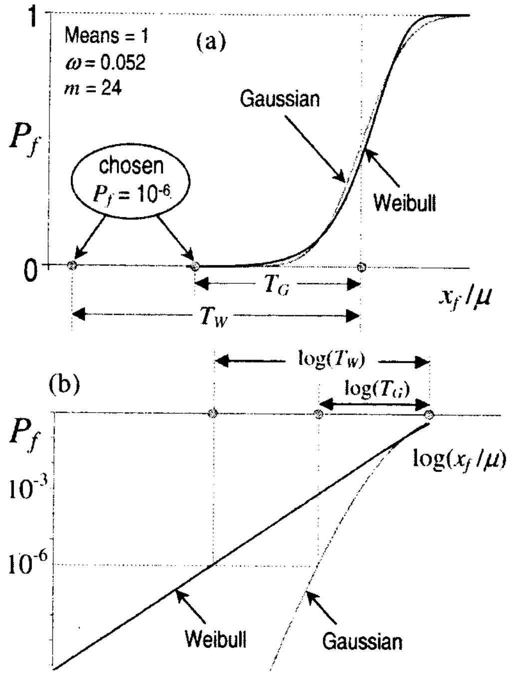

Figure 7. Large difference between points of failure probability 10^{-6} for Gaussian and Weibull distributions with mean 1 and CoV = 5.2% in (a) linear scale; (b) log scale.

7 REVISION OF RELIABILITY INDICES AS FUNCTIONS OF BRITTLENESS OR SIZE

The aforementioned reliability indices, however, are based only on second-moment statistics, utilizing only the mean and standard deviation. Unfortunately, they cannot distinguish between different cdf tails governing very small failure probability. The seriousness of this point for quasibrittle structures has not been properly appreciated so far. For such structures, the far-off tail probability of failure depends strongly on structure brittleness, which varies with the size (as well as geometry) of the structure. Most reliability-based design codes have been based on Cornell's reliability index in Eq. (29). For quasibrittle failures, however, there is a huge size effect on the tail of probability distribution. Therefore, the design code provisions for quasibrittle failures need to be overhauled.

The urgency of overhaul is clear from Figure 7 where T_G and T_W are the distances from the mean to the tail point of specified failure probability P_f. The tail offset ratio $\theta = T_W/T_G$ can be in realistic situations as large as about 2 if the tolerable failure probability P_f is tiny and CoV small (Fig. 7) (and can be arbitrarily large if P_f and CoV are small enough). As the cdf of quasi brittle structure gradually changes, at increasing size D, from Gaussian to Weibull and the grafting point gradually moves towards the mean, grows to about 2 (depending on the CoV of resistance of structure, as well as its geometry). For large sizes, the entire cdf becomesWeibull. This alters the reliability index profoundly. The difference made by Weibull distribution arises from the tail, which is approximately a power law, contrasting with the exponential tail of the Gaussian distribution (Fig. 7). For small design failure probabilities of $10^{-5} - 10^{-7}$, the reliability integral of Eq. 27 only relies on the far-off tail of $R(\sigma_N)$, leading to a simplified integral

$$P_{f,EV} = \int_0^\infty f(\sigma_N)(\sigma_N/s_R)^m \mathrm{d}\sigma_N \tag{30}$$

where $(\sigma_N/s_R)^m$ is the far-off power-law tail of $R(\sigma_N)$ and s_R is a size dependent scale parameter defined as follows:

$$s_R = s_0(l_0/D)^{n_d/m}\Psi^{-1/m} \tag{31}$$

The load variable pdf $f(\sigma_N)$ is often taken as Gaussian, except for extreme load conditions where $f(\sigma_N)$ is more accurately modelled as Gumbel distribution, and Eq. 30 can be expressed as:

$$P_{f,EV} = [(s_F/s_R)B(\mu_F/s_F, m)]^m \tag{32}$$

If the power-law tail of the resistance cdf is extrapolated to a cdf, the tail exponent m is related to the mean of the resistance variable by the expression given in Eq. 17. B is displayed in Table ?? for different values of μ_F/s_F and $m = 24$ for concrete.

The equivalent Cornell reliability index for the grafted cdf with power-law tail to account for the extreme value statistics of resistance variable is thus given by:

$$\bar{\beta} = \Phi_G^{-1}(P_{f,EV}) = \psi(s_F/\mu_F, s_R/s_F, m)\beta \tag{33}$$

As discussed later in more detail, in addition to the usual understrength (capacity reduction) factor, which essentially accounts for the brittleness of failure, code provisions for brittle failures of concrete structures tacitly imply covert understrength factors for the error of theory or formula and for randomness of material strength (Bažant and Yu 2006). Thus there are in fact three random variables for structural resistance, which all affect ψ:

The Hasofer-Lind reliability index (FORM) must be revised similarly. As can be shown, it can be computed in the usual manner, except that the so-called reduced variable must be replaced by the equivalent reduced variables:

$$X_i' = (X_i - \mu_{X_i})/\psi s_{X_i} \tag{34}$$

When $\psi = 1$, this reduces to the classical form (e.g., Eq. 7.48 in Haldar and Mahadevan 2000).

The need for a reformof the existing reliability concepts is evident from the discrepancy between the theoretical and actual failure probabilities. The observed frequency of catastrophic failures of large structures has been about an order of magnitude higher than the theoretical probabilities of failure computed in the classical way, assuming fully Gaussian distributions only (Allen 1968, 1975; CIRIA 1977; Livingstone 1989). This discrepancy may be largely due to ignoring the long power-law tails characterizing brittle failures of large structures. The need for introducing the effect of the tail-overlap ratio ψ means that reliability methods, taking into account only the first and second order statistical moments (FOSM, SOSM), must be abandoned for quasibrittle structures. But it does not mean that the reliability methods (FORM, SORM), taking into account the non-normal variables by equivalent Gaussian variables, should be used. Rather, the use of ψ could be called "EVRM"– the extreme-value reliability method

8 COVERT UNDERSTRENGTH FACTORS

Another related and inseparable problem, already alluded to, is caused by the fact that, for brittle failures, concrete design codes unfortunately specify not mean prediction formulas but "fringe formulas", i.e. formulas that have been set at the margin (or fringe), rather than the mean, of the scatter band of test data. The use of fringe formulas implies a hidden presence of "covert" understrength factors (or capacity reduction factors) (Bažant and Yu 2006), which are not evident to the user of the code. Their determination requires tedious examination of the databases used by the code-writing committees, which are usually hard to access. The covert understrength factors in ACI code represent great strength reductions-in shear failure of concrete beams they are about 0.65 for formula scatter (caused primarily by error of theory and randomness of cracking) and 0.70 for randomness of concrete strength (due to the fact that the design is based on a reduced, rather than mean, strength from cylinder tests). The usual ("overt") understrength factor, which distinguishes diverse failure modes and is the only one evident to the user, is now 0.75 for beam shear in the ACI Standard 318.

Bažant and Yu (2006) proposed to undertake a major revision of concrete design codes, making the covert factors overt in all the code specifications for brittle failures, and specifying both the probability cut-off and the CoV associated with each covert factor. Until this is done, structural reliability assessments will remain a mathematical exercise with no real meaning. The problem of multiple understrength factors also affects the reliability indices. Fundamentally, it means generalizing the standard Freudenthal's reliability integral in Eq. (27) to multiple pdf's associated with the individual understrength factors (which leads to a multiple integral) and then translating this integral in a suitably simplified manner into the reliability index of Cornell or Hasofer-Lind type. The Hasofer-Lind type index will have to be considered in a multidimensional space (four-dimensional for beam shear failure). Because the multiple, simultaneously applicable, understrength factors should properly be considered as functions of brittleness (as affected by structure size), the scaling properties, fracture mechanics, and reliability concepts are in-separable.

The current code thus implies the concrete design formulas for various types of brittle failure (shear, torsion, punching, column crushing, etc.) to have the form

$$\sigma_N = \varphi\psi F(\zeta, f_c) \tag{35}$$

where F = function, ζ = reduction factor applied to mean material strength f_c; φ = overt and ψ = covert understrength factors, taking into account brittleness and formula error. Let $r_\varphi(\varphi)$, $r_\psi(\psi)$, $r_\zeta(\zeta)$ be the corresponding pdf's of RVEs, which may be assumed to be statistically independent. Then one can show that Eq. (27) must be replaced by (Bažant 2004a, Bažant and Yu 2006):

$$P_f = 1 - \int\int\int_{\Re} G(P) r_\varphi(\varphi) r_\psi(\psi) F'(\zeta, f_c) \mathrm{d}\zeta \mathrm{d}\varphi \mathrm{d}\psi \tag{36}$$

where P = applied random load whose pdf is $g(P)$ and cdf is $G(P)$; $F'(\zeta, f_c) = \partial F(\partial, f_c)/\partial\zeta$; and integration is performed over hyper-region $\Re$ in which $\varphi\psi F(\zeta, f_c) > P/bD$.

9 IRRATIONAL SIZE EFFECT HIDDEN IN LOAD FOR SELF WEIGHT

ACI Standard 318 (2005) imposes the load factor of 1.4 for dead load acting alone. In a very large structure, the self-weight may represent 95% of the total load or more. But an error of 40% in the self-weight (i.e., in mass density and structural dimensions) is inconceivable; at most 3% to 5% could be justified. This means that large structures are systematically overdesigned, compared to small ones in which the selfweight contributes a negligible part of loading.

This implies a hidden size effect of about 30% (Bažant and Frangopol 2002), and partly compensates for the lack of size effect in structural resistance formulas of ACI code, but is irrational because it does not distinguish among various types of failure. For shear or torsion of very large beams, this hidden size effect is far too small, while for flexural failure of unreinforced beams it should vanish. For prestressed concrete or high-strength concrete it is smaller than for normal concrete because such structures are lighter, yet it should be greater because they are more brittle, etc. This further implies that probabilistic calculations predict incorrect reliability for structures of different sizes (Bažant and Frangopol 2002).

However, elimination of the excessive and irrational dead load factor would be dangerous unless the size effect is introduced at the same time into the code provisions for all brittle failures. Reliability experts and fracture experts will have to collaborate on this task.

10 DEFINITION OF BRITTLENESS NUMBER

Practical application of the present theory necessitates defining a brittleness number, β_n, as a shape-independent characteristic that allows determining the size effect for any structural geometry if it has been calibrated in the laboratory for one structure geometry. For type 2 size effect (due to large cracks or notches), such brittleness number has been expressed as $\beta_n = D/D_0$ where D_0 is the transitional size defined in terms of the energy release rate function of LEFM (e.g. Bažant 2002). But here the focus is on type 1 size effect, for which β_n has not yet been defined. In the transition from plastic to brittle response, β_n should characterize the proximity to brittle response, i.e., to LEFM. This has nothing to do with strength randomness and should be defined strictly on the basis of the energetic part of size effect law, which is (for type 1) the special case of Eq. (1) for $m \rightarrow \infty$. From this equation, it can be shown that $\beta_n = (D/D_1)^{-1/2}$. Geometrically, this number has the meaning $\beta_n = (c/b)^{-1/2}$ where c and b are the distances in linear scale of σ_N between the circled points marked on the size effect curves in Figure 8. This geometrical definition of β_n is universal, valid for not only type 1 but also type 2.

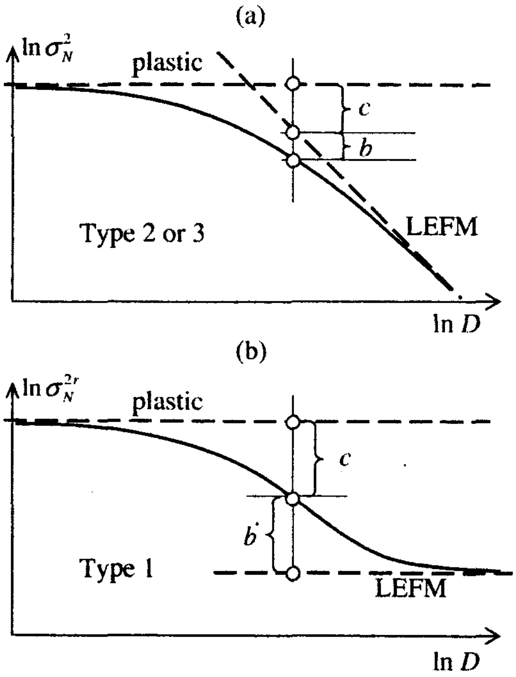

Figure 8. Geometrical definition of β for a) Type 2 or 3 structures; b) Type 1 structures.

11 CONCLUSIONS

For more than two decades, the structural reliability theory seemed to be understood almost perfectly. However, this has been true only as long as the theory of limit states, anchored in plasticity, is applicable. In that case, the failure proceeds simultaneously along the whole failure surface, the material strength is mobilized at all the points of the surface, the size effect is nil, the scatter follows Daniels fiber-bundle model, and the structural strength distribution is, for any structure size, necessarily Gaussian (or normal, but never log-normal).

In recent years, though, it gradually transpired that this classical theory does not apply to quasibrittle structures, such as large concrete structures failing due to concrete fracture rather than yielding of steel reinforcement, load-bearing fiber-composite parts of large ships or aircraft, sea ice plates; etc., in which the failure is progressive, propagating along the failure surface. Often failure occurs as soon as the full FPZ is formed, and then there is a strong energetic-probabilistic size effect, with a structural strength distribution that has a Gaussian core andWeibull tail, the tail expanding and the core shrinking as the size increases. This has enormous effect on the structural design satisfying the typical tolerable failure probability (about one out of ten million). To preserve a constant safety margin for such a small failure probability, the required understrength factor depends on the structure size, and with increasing size almost doubles. This, as well as related problems due to covert understrength factors in codes and to excessive load factor for self weight, will necessitate profound modification of reliability analysis and a major overhaul of design codes and practices for quasibrittle structures.

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