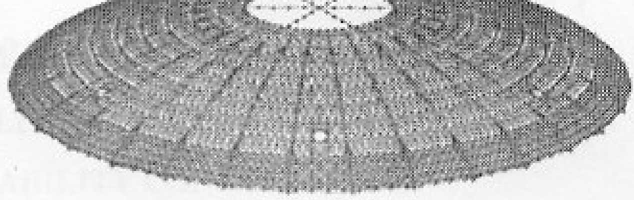
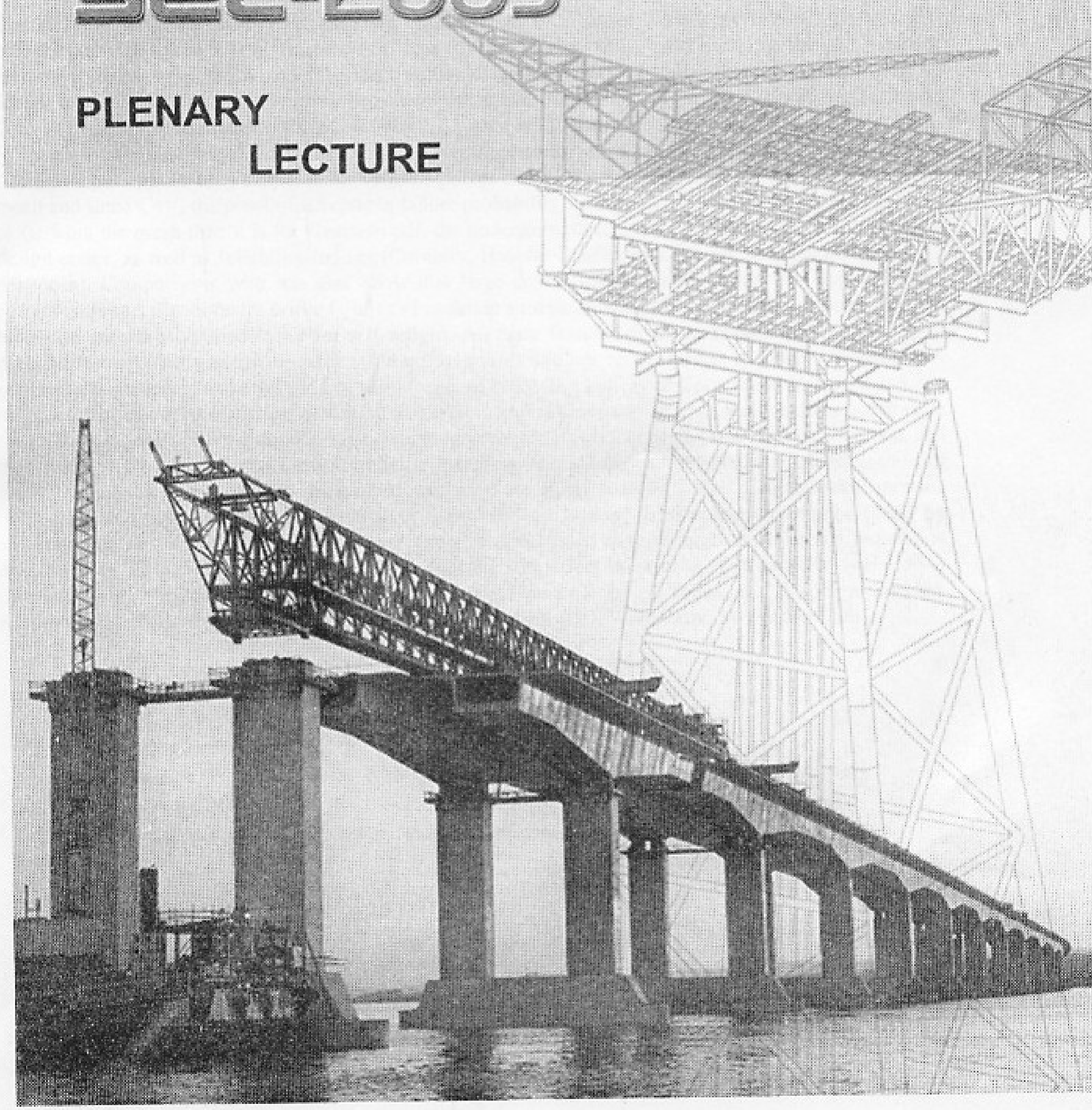


SEC-2005

PLENARY LECTURE



THE STRUCTURAL ENGINEERING CONVENTION (SEC 2005)
INDIAN INSTITUTE OF SCIENCE, BANGALORE 560 012, INDIA (DEC 14-16, 2005)
D. ROY AND J. M. CHANDRA KISHEN, EDS.

EFFECT OF SIZE ON SAFETY FACTORS AND STRENGTH OF QUASIBRITTLE STRUCTURES: BECKONING REFORM OF RELIABILITY CONCEPTS

Zdeněk P. Bažant[1] and Sze-Dai Pang[2]

ABSTRACT

The effect of structure size D on structural strength has two causes: 1) energy release due to stress redistribution caused by a large fracture or a large fracture process zone, and 2) extreme value statistics of random material strength. While, in perfectly ductile failures, the probability density function (pdf) of failure load is Gaussian, with a size-independent mean and coefficient of variation (CoV) decreasing as $D^{-1/2}$, in perfectly brittle failures the pdf is Weibull, with a size-dependent mean but size-independent CoV. In quasibrittle structures, there is a continuous size-dependent transition, which can be described by a composite pdf approaching Gaussian for vanishing sizes and Weibull for infinite sizes. Since, for the same mean and same CoV, the point of acceptable failure probability such as 10^{-7} is for Weibull pdf about twice as far from the mean than it is for Gaussian pdf, the understrength (capacity reduction) factors in concrete design codes, as well as reliability indices (Cornell's, Hasofer-Lind's), must be revised, making them size dependent. Comparisons with test data show that large covert understrength factors are implied in the current code specifications for brittle failures of concrete structures, and also that an irrational size effect is hidden in an excessive load factor for self-weight. All these features preclude at present any meaningful probabilistic reliability estimates when failure is quasibrittle. A restructuring of safety provisions in concrete design codes, and a revision of the concept of reliability indices of Cornell and Hasofer-Lind, are proposed. An energetic-statistical theory of size effect, applicable to concrete, as well as other quasibrittle materials (e.g., rocks, fiber composites, sea ice, tough ceramics, wood, cohesive soils, etc.), is outlined, and then justified by experimental data, asymptotic matching considerations, and nonlocal generalization of Weibull statistical theory. A new method of stochastic structural analysis which combines deterministic nonlocal finite element analysis with analytical probabilistic scaling is sketched and demonstrated by showing that, in the Malpasset Dam disaster, the size effect must have reduced the tolerable foundation movement by 77%. Finally, it is pointed out that the size effect must have contributed to many other disasters (e.g., bridges in Northridge, Loma Prieta and Kobe earthquakes, Schoharie Bridge, Koror-Babeldaob Bridge in Palau, Sleipner Oil Platform, Shelby AF Warehouse, St. Francis Dam).

[1] McCormick Institute Professor and W.P. Murphy Professor of Civil Engineering and Materials Science, Northwestern University, 2145 Sheridan Road, CEE, Evanston, Illinois 60208: z-bazant@northwestern.edu

[2] Graduate Research Assistant and Doctoral Candidate, Northwestern University.

Keywords: Structural reliability, Brittleness, Capacity reduction, Probability distribution, Size effect, Quasibrittle materials, Extreme value statistics, Weibull theory, Scale bridging, Fracture mechanics.

INTRODUCTION

The basic theory of energetic size effect and energetic-statistical size effect in quasibrittle structures evolved during the last two decades, and significant support by experiments on concrete and other quasibrittle materials has been provided. Computer simulations based on nonlocal finite element method and nonlocal Weibull theory further strengthened this support. A review of this theory, computer simulations and experimental evidence is the subject of the first part of the conference lecture. However, since several recent comprehensive articles, surveys, and textbooks are available (Bažant 2004a,b, 2002, 1997, Bažant and Planas 1998, RILEM 2004, Bažant and Jirásek 2002), the reader is referred for the background to these works, and this conference article will, therefore, deal only with the latest developments combining the energetic and statistical size effects in quasibrittle materials, and concrete in particular. Significant parts of what follows are reprinted, with updates, from the authors' article in recent CD proceedings (Bažant and Pang 2005a), with the kind permission of the editors of these proceedings.

The load factors and understrength (or capacity reduction) factors are currently considered to be independent of structure size and geometry, and separate from structural analysis. In what follows, it is argued that this is incorrect for quasibrittle failures, characterized by post-peak softening (typical of concrete as well as rocks, fiber composites, tough ceramics, sea ice, wood, cohesive soils, etc.). Why? – Whereas the strength of perfectly ductile structures necessarily follows the Gaussian (normal) probability density function (pdf), for quasibrittle structures of increasing size D there is a continuous transition from Gaussian to Weibull pdf, in which a Weibull tail, expanding with D, is grafted onto a shrinking Gaussian core. The pdf is calculated from the equivalent number of representative material volumes (RVE) in the structure, in which the pdf of each RVE is deduced from a statistical model with hierarchy of parallel and series couplings. The pdf form, with its dependence on temperature and load duration, is deduced from Maxwell-Boltzmann distribution of atomic energies and stress dependence of the activation energy barriers of interatomic potential. The practical consequences are serious because, for the same mean and the same typical coefficient of variation of failure load, the point of tolerable failure probability such as 10^{-7} is for Weibull pdf about twice as far from the mean than it is for Gaussian pdf. To capture it, the reliability index (of Cornell or Hasofer-Lind, used in FORM) must be made to depend on D. The size effect on pdf form is superposed on the mean energetic-statistical size effect, which is most generally derived by dimensional analysis coupled with asymptotic matching based on asymptotic properties of the cohesive crack model. Related problems calling for reform are an irrational size effect currently hidden in excessive load factor for self-weight acting alone, as well as covert understrength factors hidden in using reduced (rather than mean) concrete strength in design, and in setting the load capacity formulas that have been set at the lower margin rather than the mean of the experimental database. For computer analysis, the stochastic finite element method must be modified so as to exhibit the energetic-probabilistic size effect with a pdf of correct tail, and the reliability integral must be generalized for multiple random understrength factors. A practical method of calculating the size effect on failure probability, which necessitates only deterministic nonlocal finite element analysis, is outlined and demonstrated by calculating the tolerable abutment displacement of the ill-fated Malpasset Dam. It is shown that its mean value should have been about 4x smaller, and its value for 10^{-7} failure probability about 8x smaller, than considered in 1950. Finally, it must be emphasized that the size effect, previously unrecognized, must have contributed to many other disasters (e.g., bridges in Northridge, Loma Prieta and Kobe earthquakes, Schoharie Creek Bridge, Koror Bridge in Palau, Sleipner Oil Platform, Shelby AF Warehouse, St. Francis Dam).

Recent developments in probabilistic modeling of quasibrittle fracture with size effect, and in statistical databases for failure loads, reveal problems whose resolution will require a major overhaul of the existing design codes and practices (Ellingwood et al. 1980, 1982), especially those for concrete, as well as generalization of the basic concepts of structural reliability, important especially for extrapolation of laboratory experimental evidence to very large structures failing in a non-ductile manner. From the statistical viewpoint, the problem is that the current practice is to model the strength variability by normal (Gaussian) or lognormal distribution. But this cannot apply to large structures obeying the weakest-link chain model because the stability postulate of extreme value statistics is violated.

The reliability problems associated with brittleness have come to light as a result of theoretical modeling of the size dependence of the cumulative probability distribution function (cdf) of load capacity of quasibrittle structures (Bažant 2004a, b) based on nonlocal Weibull theory (Bažant and Xi 1991; Bažant and Novák 2000a, b) and asymptotic matching (Bažant 1997, 2004a, b), and also as a result of Monte Carlo structural simulations and of statistical studies of extensive databases.

Quasibrittle materials, which include concretes, rocks, tough ceramics, fiber composites, concrete structure strengthened or retrofitted by composites, sea ice, stiff cohesive soils, wood, paper, foams, etc., typically exhibit a transitional size effect. So do modern tough metallic structures. Structures made of such materials exhibit not only the Weibull-type statistical size effect but also an energetic size effect on the nominal structure strength, σ_N, due to stress redistribution caused by a large fracture process zone (FPZ) or by large fracture growth before the maximum load or load parameter, P_{max}, is reached (σ_N is defined as P_{max}/bD where D is the characteristic size of structure and b its thickness in the third dimension).

The size effects are basically of two types: Type 1 occurs for structural geometry that permits stable growth of large fractures prior to reaching P_{max}, and type 2 for structures failing at macro-fracture initiation from an initial cracking zone (there exists also a type 3 size effect, but it is very similar to type 2); Bažant (2002). For type 2, material randomness affects only the scatter of σ_N but not its mean, while for type 1 it affects both and thus is more important.

This study will deal only with type 1, for which the combined probabilistic-energetic size effect on the mean of σ_N can be approximately described as (Bažant 2004a)

$$\sigma_N = A\left(\vartheta^{rn/m} + rx\vartheta\right)^{1/r}, \quad \vartheta = B\left(1 + D/\eta l\right)^{-1} \tag{1}$$

where n, m, r, x, η, A, B, l = constants. Eq. (1) was derived by asymptotic matching of the first two terms of the small-size and large-size asymptotic expansions of σ_N based on the cohesive crack model and the nonlocal Weibull theory. By asymptotic approximations it was also shown that, for large D, the Weibull distribution must be approached with an error second-order small in $1/D$ (Bažant, 2002).

The present brief article will (1) formulate a statistical series-parallel coupling model for the size effect on cdf in quasi-brittle failure; (2) propose a reform of the standard reliability indices used in the first-order reliability method (FORM) to take into account the cdf tail; and (3) point out the reliability consequences of covert understrength factors and excessive load factor for self-weight.

HIERARCHICAL MODEL FOR CDF OF STRENGTH OF ONE RVE

Structures containing a cohesive crack, scaled down to vanishing size ($D \rightarrow 0$), approach the case of an elastic body with a crack filled by a perfectly plastic 'glue' (Bažant, 2002). In this limit case, all the micro-bonds along the failure surface are failing simultaneously. Therefore, the cdf of failure load should obey the Gaussian (or normal) distribution. This is implied by Daniel's (1945) fiber-bundle model (or parallel coupling of elastic-brittle fibers) in which the load is shared equally by all the unbroken fibers (Fig. 1a). The cdf of the strength, , of a fiber-bundle with n fibers is given exactly by Daniels' recursive formula:

$$G_n(\sigma) = \sum_{i=1}^{n} (-1)^{i+1} \binom{n}{i} [F(\sigma)]^i \, G_{n-i}\left(\frac{n\sigma}{n-i}\right) \tag{2}$$

where $G_0(\sigma) = 1$ and $F(\sigma)$ = cdf of fiber strength.

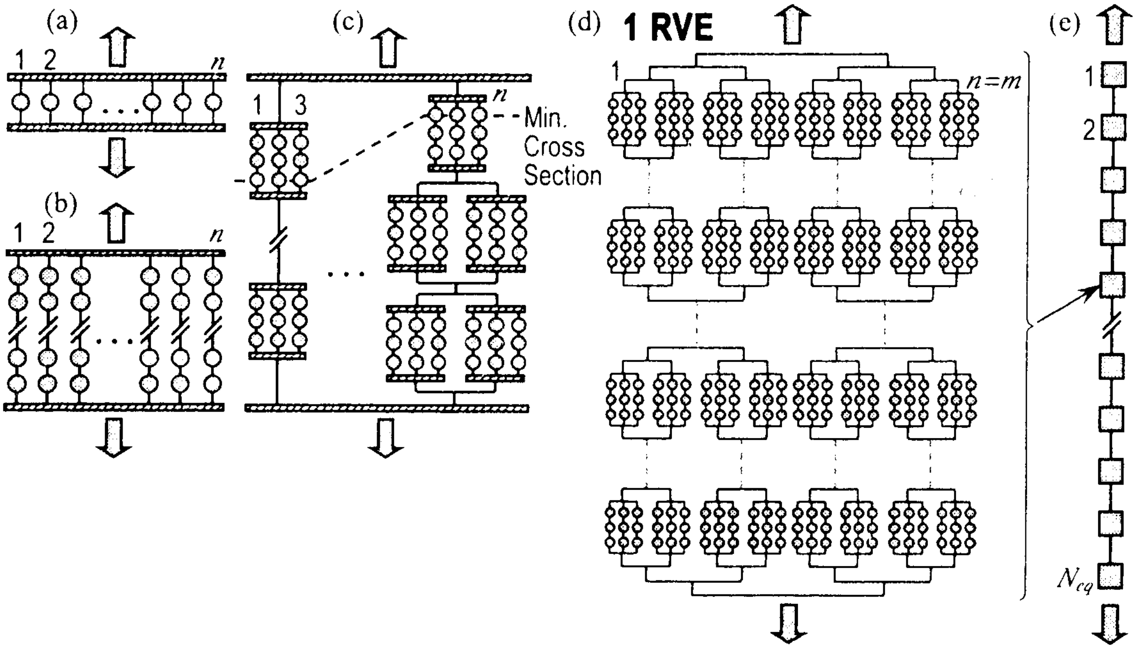


FIG. 1. Models of series and parallel couplings a) fiber bundle; b) bundle of chains; c) example of a complex bundle with sub-chains and sub-bundles; d) hierarchy of sub-chains and sub-bundles; e) a chain model with each link representing a RVE.

If the fibers are elastic-perfectly plastic, the strength of the bundle is a sum of the individual fiber strengths (i.e., load redistribution is ignored), and Daniels' result is similar to the central limit theorem (CLT), which states that the sum of a large number of independent random variables approaches the Gaussian (or normal) distribution, irrespective of the individual distributions (unless they have infinite variances, in which case another stable distribution, Levy's, is approached).

The fibers in the fiber-bundle model are a collection of interatomic bonds on the nano-scale within a representative volume element (RVE) of the material. The RVE size depends on its purpose. Here the RVE is considered to be about three aggregates in size and is defined as the smallest material volume whose failure suffices to make a structure of positive geometry fail (this definition differs from that used in the homogenization theory for heterogeneous elastic-plastic materials). The failure probability of each micro-bond can be deduced from statistical thermodynamics. The Maxwell-Boltzmann distribution of thermal energies of atoms (Hill, 1956; Cottrell 1964; McClintock and Argon 1966) states that the fraction of atoms exceeding atomic energy level $\mathscr{E}$ at absolute temperature T is:

$$\Phi(\mathscr{E}) = \exp(-\mathscr{E}/kT) \tag{3}$$

where k = Boltzmann constant. If the energy of an atom exceeds its activation energy Q, the bond is broken. So the fraction (or frequency) of interatomic bond breaks is $\Phi = e^{-Q/kT}$.

Application of stress, σ, causes the activation energy barrier to change from Q to $Q-\kappa\sigma$ for bond breaking, and from Q to $Q+\kappa\sigma$ for bond restoration (where factor κ depends on microstructure geometry and randomness, and is assumed to be independent of σ). From this, it can be deduced that the pdf of failure of one fiber, simulating an interatomic bond in an RVE, at temperature T under stress σ, may be expressed as

$$f_b(\sigma) = 2e^{-Q/kT}\sinh(\kappa\sigma/kT) \tag{4}$$

The RVE is made up of trillions of interatomic bonds. They may be idealized as coupled either in parallel (as described by the fiber bundle model), or in series (chain model). A crack is formed when the critical fraction ν_{cr} of interatomic bonds, which must be broken in order to form a continuous surface, is reached (the value of ν_{cr} is a problem of statistical percolation theory). Because the critical fraction ν_{cr} can be reached by breaks of many different interatomic bonds, each of which may lead to failure of the fiber, each fiber may be more precisely described by the weakest-link model simulating a lower-scale microstructure—a chain of very many bonds. Its overall cdf depends only on the tail of $F_b(\sigma)$ given by integral of (4) with respect to σ. Since this cdf tail is $\propto \sigma^p$ where p=1, it can be shown that, in this case, the cdf of each fiber under stress σ for duration τ_σ, must be the exponential distribution:

$$F_b(\sigma) = 1 - \exp\left[-\phi_b(\tau_\sigma)\frac{\kappa\sigma}{kT}e^{-Q/kT}\right] \tag{5}$$

where ϕ_b = function of time τ_σ, derived from the critical fraction ν_{cr} of atomic bonds. The reason that the transfer of load from the broken to the unbroken bonds is not taken into account within each fiber is that it is approximately captured by load sharing within the bundle itself.

The cdf of a simple fiber bundle (Fig 1b) approaches a Gaussian distribution with the asymptotic mean and coefficient of variation (CoV) given by (Daniels 1945):

$$\mu = \sigma^*\left[1 - F(\sigma^*)\right], \quad \omega = n^{-1/2}\sqrt{F(\sigma^*)/\left[1 - F(\sigma^*)\right]} \tag{6}$$

where σ^* is the point for which $\mu(\sigma)$ reaches its maximum. Obviously, for $n \to \infty$ one has $\omega \to 0$, i.e., an infinite bundle is deterministic, and so only a finite n make sense. The Gaussian approximation of cdf reveals nothing about the far-out cdf tail. The Gaussian distribution is only valid for the central core expanding in proportion to $n^{1/2}$ whereas the left tail of cdf has the cdf of the fibers (Harlow et al. 1983). The probability of failure for each fiber given in Eq. (5) has a tail cdf $\propto \sigma^p$ (for $\sigma \to 0$) where $p=1$. If the tail cdf of each fiber has the same power law, it can be proven from Eq. (2) by mathematical induction that the tail of the cdf of the strength of a bundle with n fibers is a power law $\propto \sigma^{np}$ (Bažant and Pang 2005b). This leads to an interesting conclusion—the Weibull modulus m of the tail distribution of strength σ of a RVE is equal to the number n of fibers in the bundle. If the fibers are elastic-plastic, the same conclusion could also be reached on the basis of the joint probability theorem (Bažant and Pang 2005), and may be assumed to hold for bundles with softening fibers as well.

A simple fiber bundle (Fig 1b), with $n=m$ fibers, would have a Gaussian cdf with Weibull tail so short that it would be virtually nonexistent and no effect on structural behavior. If the interatomic bonds wer coupled in series in each fiber (Fig 1b) to extend the length of the Weibull tail, it would still remain so short that it could never be manifested in experiments on brittle materials such as ceramics (Weibull 1939; Bansal *et al.* 1976a, b; Quinn and Morrell 1991).

To obtain a long enough Weibull tail, such that structures with >1000 RVE would be have an almost entirely Weibull cdf, it is necessary and more realistic to model a RVE by a hierarchical model, in which a RVE is statistically described by a bundle of n_1 fibers, each of which consists of a long chain of bundles with n_2 sub-fibers, with each of the sub-fibers having the cdf in Eq. (5). In this case the cdf of RVE would have a power-law tail of exponent $m = n_1 n_2$, which is again equal to the number of all sub-fibers coupled in parallel. The sub-fibers of secondary bundles are refined on a still lower-scale of microstructure as chains of tertiary sub-sub-bundles consisting of long sub-sub-chains, until the nano-scale of interatomic bonds are reached. The detailed probabilistic micro-structural model is surely non-unique, but the number of parallel fibers in cuts separating the hierarchical model into two parts at the lowest level must be equal to the exponent m of the tail cdf of a RVE (Fig 1d). The actual behavior of a RVE will, of course, correspond an irregular hierarchical model, such as that shown in Fig. 1c. In that case, according to the aforementioned basic properties, the exponent of the power-law tail for the RVE, and thus the Weibull modulus of a large structure, is defined as the minimum number of cuts of elementary serial bonds that are needed to separate the model into two halves.

Consequently, the detailed parallel and series coupling of the hierarchical model for the RVE does not matter for our purpose because we seek only qualitative information—the type of cdf, while a quantitative analytical prediction from atomic microstructure is beyond reach. What matters is that (i) the cdf of a RVE consists, in any case, of a Gaussian core with a Weibull-type lower tail, whose Weibull modulus m is equal to the number n of parallel fibers across the weakest cross section, and that (ii) the lower tail of cdf for each micro-bond is a Weibull cdf (or a power law, if short enough) with exponent 1 (i.e., a linear function of stress), as in Eq. (5).

An enormous advantage of anchoring the theory in statistical thermodynamics, particularly the Maxwell-Boltzmann distribution, is that the dependence of cdf of failure load on temperature T and load duration τ_σ is captured automatically (in practice, though, the temperature range can be limited because of interplay of several different activation energies for different atoms). Because the activation potential barrier is affected by the presence of water molecules, it is, in principle, possible to capture also the effects of the content of moisture or various corrosive agents in porous hydrophilic solids such as concrete.

CDF OF ONE RVE

From experimental data on brittle materials, such as ceramics (e.g. Weibull 1939; Bansal *et al.* 1976a, b; Quinn and Morrell 1991), Weibull size effect is often clearly evident for equivalent number of RVE's $N_{eq} > 500$. But this Weibull cdf is unobtainable if a simple fiber bundle with n=24 fibers (typical of concrete, Bažant and Novák, 2000a, b) were used to model a RVE, the Weibull tail would be extremely short, reaching only up to $P_f = 10^{-45}$; which would the structure to be about 10^{47} times larger than the RVE for its strength to exhibit Weibull cdf. This is, of course, impossible. A feasible statistical model for a RVE is a hierarchical model of the kind shown in Fig. 1d. This model can provide a cdf whose Weibull tail extends up to about $P_f = 0.003$ or 0.0003 when its elements (or fibers) are brittle (Fig. 2a) or plastic (Fig. 2b). More realistic doubtless are softening elements of the model (Fig. 2c), which are harder to analyze but may be expected to exhibit intermediate behavior. A parallel coupling at any scale of hierarchy tends to build up a Gaussian core of cdf and drastically shorten Weibull tail while raising its exponent. A series coupling at any scale of hierarchy tends to shorten the Gaussian core and extend the Weibull tail while keeping its exponent unchanged.

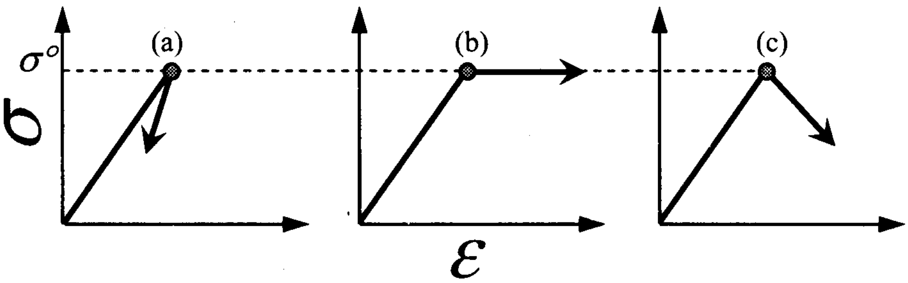


FIG. 2. Post-peak behaviour of fibers a) Brittle; b) Plastic; c) Elastic Softening

For increasing D, the Gaussian core shrinks and the Weibull tail spreads toward higher probabilities (Bažant, 2004a, b) until, for infinite D, the entire cdf becomes Weibull. To describe such behavior, we introduce a Gaussian distribution with a truncated lower (i.e. left) tail, onto which we graft a Weibull tail. The upper tail of cdf of strength is irrelevant for larger structures, according to the weakest-link model. So, a one-sided lower tail graft will suffice. The grafted pdf can be mathematically described as follows:

$$\phi_W(\sigma_N) = r_f (m/r_0)(\sigma_N/r_0)^{m-1}\exp\{-(\sigma_N/r_0)^m\} \qquad \text{for } \sigma_N < \sigma_{gr} \tag{7}$$

$$\phi_G(\sigma_N) = r_f \exp\{-0.5[(\sigma_N - \mu_G)/\delta_G]^2\}/(\delta_G\sqrt{2\pi}) \qquad \text{for } \sigma_N \geq \sigma_{gr} \tag{8}$$

where μ_G, s_G = mean and standard deviation of Gaussian core; m, r_0 = shape and scale parameters of Weibull tails; σ_{gr} is the grafting point; $r_f = [1-\Phi_G(\sigma_{gr})+\Phi_W(\sigma_{gr})]^{-1}$, is a scaling factor ensuring that the combined cdf of the Weibull-Gaussian graft is normalized. Both pdf's, as defined in Eqs. 7 and 8, are matched to be continuous at the grafting point, which leads to the following compatibility condition:

$$\eta = \alpha_{gr} + \beta\sqrt{-2\ln(\beta\sqrt{2\pi}\phi_W(\sigma_{gr}))} \tag{9}$$

where $\eta = \mu_G/r_0$; $\beta = \delta_G/r_0$; $\alpha_{gr} = \sigma_{gr}/r_0$. If the standard deviation of the Gaussian core δ_G, and the scale parameter r_0 of the Weibull tail, are known, η can be calculated from Eq. 9 and the Weibull-Gaussian graft for cdf of one RVE (Fig 3) can be expressed explicitly as:

$$P_1(\alpha) = r_f[1 - e^{-(\sigma_N/r_0)^m}] \qquad \text{for } \alpha_N < \alpha_{gr} \tag{10}$$

$$P_1(\alpha) = r_f[1 - e^{-(\sigma_{gr}/r_0)^m}] + \frac{r_f}{\beta\sqrt{2\pi}} \int_{\alpha_{gr}}^{\alpha} e^{-0.5[(\alpha'-\eta)/\beta]^2} d\alpha' \qquad \text{for } \alpha_N \geq \alpha_{gr} \tag{11}$$

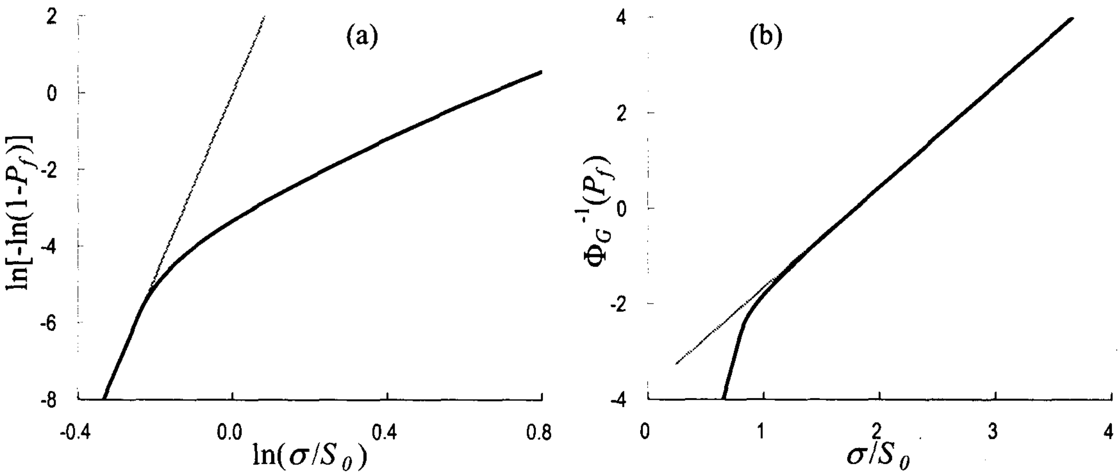


FIG. 3. cdf of a RVE with P_{gr} = 0.003 and CoV = 25% in a) Weibull probability paper; b) normal probability paper

The normalized mean and CoV of a RVE, which characterizes small-size quasibrittle structures, which fail in a nearly ductile manner, is tabulated in Table 1 for some typical parameter values.

Table 1. Mean and CoV of Gaussian-Weibull graft for a RVE

$\Phi_W(\sigma_{gr})$	r_f	η	β	μ/r_0	ω
	1.009	2.723	0.850	2.746	0.30
0.001	1.003	1.649	0.334	1.653	0.20
	1.000	1.084	0.109	1.085	0.10
	1.024	2.112	0.686	2.151	0.30
0.003	1.008	1.497	0.309	1.504	0.20
	1.000	1.079	0.108	1.080	0.10
	1.040	1.894	0.638	1.951	0.30
0.005	1.013	1.432	0.301	1.443	0.20
	1.001	1.074	0.107	1.075	0.10

The grafted power-law tail for each link in the chain is crucial for getting large-size asymptotic distribution of the Weibull type (if the links had Gaussian tails, the cdf of the chain would approach Gumbel's (1958) cdf, which would be physically unacceptable).

STATISTICS OF BRITTLE FAILURE

We consider structures of positive geometry (i.e., a geometry for which K_I for unit load increases with crack length), which fail at fracture initiation. With increasing D, the weakest-link model (Fig. 1e) gives a cdf quickly approaching the Weibull cdf [with an error of only $O(D^{-2})$; Bažant, 2004a] because the tails of each link (each RVE) are Weibull and thus satisfy the stability postulate of extreme value statistics (Fisher and Tippett, 1928). Note that, aside from Weibull cdf, there exist only two other extreme value cdf's—Gumbel's (derived by Fisher and Tippett 1928) and Fréchet's (1927); but they are excluded, not only because of lacking a power-law tail for each link but also because their far-out tails reach into physically meaningless negative strength values. The mean of nominal strength, $\bar{\sigma}_N$, is scaled by the number of links as follows:

$$\bar{\sigma}_N = \sigma_0 N^{-1/m} \Gamma\left(1 + m^{-1}\right) \tag{12}$$

where σ_0 = Weibull scale parameter of each link; m = shape parameter (or Weibull modulus) common to all links and the whole chain. The CoV of nominal strength is independent of the number of links and is given by:

$$\omega_N = \sqrt{\Gamma\left(1+\tfrac{2}{m}\right)\Gamma^{-2}\left(1+\tfrac{1}{m}\right) - 1} \tag{13}$$

Note that when tests seem to suggest ω_N depending on D, it means that Weibull statistical theory does not apply and that the size effect is caused, at least in part, by energy release due to stress redistribution, as captured by the nonlocal Weibull theory (Bažant and Xi 1991).

The links in the chain (weakest link model) (Fig. 1e) correspond to individual RVEs, having roughly the size of a FPZ dictated by material heterogeneity (considered identical to the nonlocal averaging domain; Bažant and Xi 1991). For very large structures dwarfing the FPZ (or the zone of localized distributed cracking), a positive-geometry structure fails as soon as the full FPZ, capable of dissipating energy at the rate equal to the fracture energy of material, develops (in the case of notches or structures of negative geometry failing after large macro-crack growth, the cdf is predominantly Gaussian, with a short Weibull tail).

TRANSITION OF CDF BETWEEN SMALL AND LARGE STRUCTURE SIZES

The transition of cdf from small sizes to large structures can be calculated, according to the weakest-link (or chain) model (Fréchet 1927), as follows:

$$P_f(\sigma_N) = 1 - \left[1 - P_1(\sigma_N)\right]^{N_{eq}} \tag{14}$$

where $P_1(\sigma_N)$ is the failure probability of each RVE (or each link) computed from Eqs 10 and 11, which ceases being size dependent for very large structures. The number of links (Fig. 1e) equals the number of RVEs in the structure if the stress is uniform. It the stress field is non-uniform,

$$N_{eq} = (D/l_0)^{n_d} \int_V S^m(\xi) dV(\xi) \tag{15}$$

where $\xi = x/D$, is the dimensionless coordinate vector; $S(\xi)$ is the dimensionless stress field which depends on structure geometry but not on structure size D; l_0 = material characteristic length (roughly the RVE size); n_d = number of dimensions in which the structural failure is scaled. For narrow beams in flexure which fail at macro-crack initiation, the number of links (or RVE's) in the chain scales as D^{n_d} with n_d=2, because widening of a narrow beam has no effect.

The transition of the mean size effect curve from the small-size to the large-size asymptote, calculated with the failure probability in Eq. 14, is shown in Fig 4 (which is similar to the curve ensuing from nonlocal Weibull theory; Bažant, 2004a).

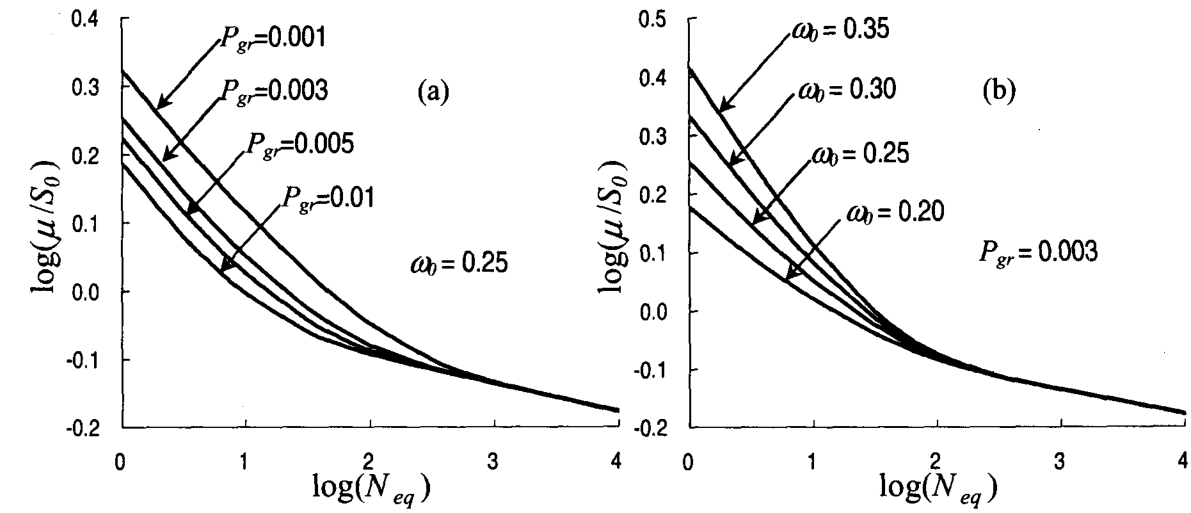


FIG. 4. Mean size effect curves for a) different P_{gr}; b) different CoV of one RVE

RELIABILITY-BASED DESIGN

The reliability-based design requires consistent evaluation of failure risk using probability theory. Structures must be designed for very low failure probability, typically of the order of 10^{-7} (Allen, 1968; CIRIA, 1977), which is totally dominated by far-off tail distributions of the load and structure resistance. The probability of failure equals the integral of the bivariate probability density over the domain where the resistance is less than the load (Freudenthal et al. 1966; Ang and Tang 1984; Melchers, 1987). This integral can be rearranged as:

$$P_f(\sigma_N) = \int_{-\infty}^{\infty} l(\sigma_N) R(\sigma_N)\, dx \tag{16}$$

where $R(\sigma_N)$ = cdf of structural resistance and $l(\sigma_N)$ = probability density function (pdf) of the load.

If the distributions of random load variable L and structural resistance variable R are Gaussian, the safety margin, $M = R-L$, is also Gaussian, and its mean $\mu_M = \mu_R - \mu_L$ and variance $s_M^2 = s_R^2 + s_L^2$ give

$$P_f = \Phi_G\left(-(\mu_R - \mu_L)\Big/\sqrt{s_R^2 + s_L^2}\right) \tag{17}$$

To avoid dealing with small probabilities, it is usually more convenient to adopt reliability index β, which is for Gaussian distributions simply defined as

$$\beta = (\mu_R - \mu_L)\Big/\sqrt{s_R^2 + s_L^2} \tag{18}$$

as proposed by Cornell (1969). In the space of normalized differences of load and resistance from their means, β has the geometrical meaning of the distance from the origin to the closest point (called the design point) on the boundary of the safe region ($L<R$) (e.g. Haldar and Mahade 2000). Eq. (18) assumes a linear failure surface (a hyperplane) and it belongs to the first-or second-moment method (FOSM) (Madsen, et. al, 1986; Melchers, 1987). If the variables are

non-normal, or if the failure surface is nonlinear while the first-order approximation of the failure surface at the design point is still used, one may use an improved Hasofer-Lind reliability index (FORM), calculated by an iterative procedure.

REVISION OF RELIABILITY INDICES AS FUNCTIONS OF BRITTLENESS OR SIZE

The aforementioned reliability indices, however, are based only on second-moment statistics, utilizing only the mean and standard deviation. Unfortunately, they cannot distinguish between different cdf tails governing very small failure probability. The seriousness of this point for quasibrittle structures has not been properly appreciated so far. For such structures, the far-off tail probability of failure depends strongly on structure brittleness, which varies with the size (as well as geometry) of the structure. Most reliability-based design codes have been based on Cornell's reliability index (Eq. 18). For quasibrittle failures, however, there is a huge size effect on the tail of probability distribution. Therefore, the design code provisions for quasibrittle failures need to be overhauled.

The urgency of overhaul is clear from Fig. 5 where T_G and T_W are the distances from the mean to the tail point of specified failure probability P_f. The tail offset ratio $\theta = T_W / T_G$ can be in realistic situations as large as about 2 if the tolerable failure probability P_f is tiny and CoV small (Fig. 5) (and can be arbitrarily large if P_f and CoV are small enough). As the cdf of quasibrittle structure gradually changes, at increasing size D, from Gaussian to Weibull and the grafting point gradually moves towards the mean, θ grows to about 2 (depending on the CoV of resistance of structure, as well as its geometry).

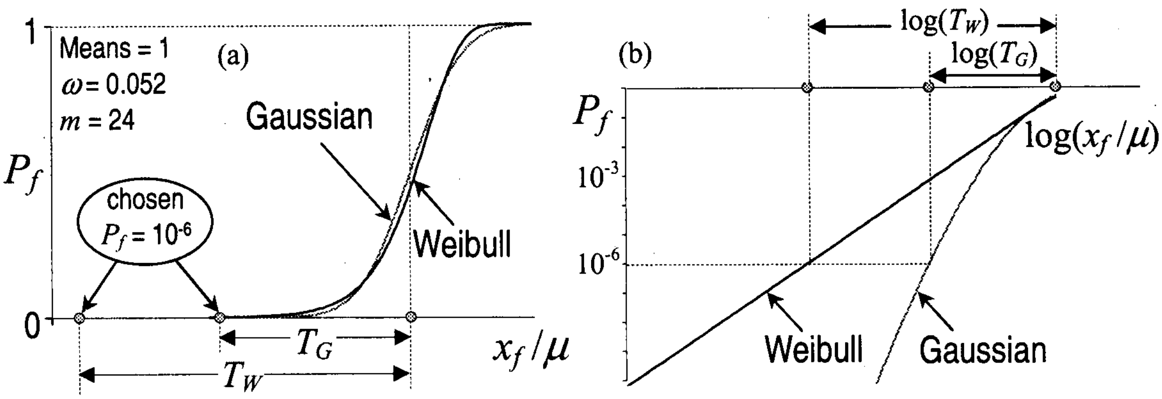


FIG. 5. Large difference between points of failure probability 10^{-6} for Gaussian and Weibull distributions with mean 1 and CoV = 5.2% in (a) linear scale; (b) log scale.

To relate the reliability index to its value for purely ductile behavior with Gaussian distribution of resistance, the Cornell reliability index (Eq. 18) may be generalized by introducing the tail offset ratio θ as follows:

$$\beta = (\mu_R - \mu_L) \Big/ \sqrt{\theta^2 s_R^2 + s_L^2} \tag{19}$$

θ as a function of s_R (which is a function of size D) can be calculated from the grafted Weibull-Gaussian distribution based on the hierarchical statistical model, the mean size effect law, and the values of s_L, and the ratio μ_R/μ_L (see Table 2).

Table 2. Determination of reliability correction factor θ

N_{eq}	s_L	s_R	μ_R/μ_L	$P_{f,old}$	$P_{f,new}$	θ
10	0.10	0.069	1.62	1.84×10^{-5}	1.08×10^{-4}	1.199
100	0.10	0.057	1.62	2.65×10^{-6}	5.75×10^{-5}	1.360
1000	0.10	0.052	1.62	1.09×10^{-6}	5.13×10^{-5}	1.470
10	0.10	0.156	1.62	1.12×10^{-2}	9.26×10^{-3}	0.965
100	0.10	0.108	1.62	1.02×10^{-3}	2.98×10^{-4}	0.862
1000	0.10	0.057	1.62	1.04×10^{-6}	5.60×10^{-5}	1.365
10	0.20	0.069	2.08	5.97×10^{-6}	2.00×10^{-5}	1.183
100	0.20	0.057	2.08	1.74×10^{-6}	1.08×10^{-5}	1.319
1000	0.20	0.052	2.08	1.05×10^{-6}	9.69×10^{-6}	1.420
10	0.20	0.156	2.08	2.30×10^{-3}	1.37×10^{-3}	0.925
100	0.20	0.108	2.08	1.63×10^{-4}	5.42×10^{-5}	0.867
1000	0.20	0.057	2.08	1.67×10^{-6}	1.06×10^{-5}	1.323

The limiting small-size Gaussian probability distribution becomes, of course, irrelevant to the reliability index in the rare situation where the given P_f or CoV is so small that the P_f point lies outside the Gaussian core. Note in Table 2 (and Fig. 5) that an increase in μ_R/μ_L, or a decrease in s_R or s_L, moves the overlapping failure domain in the integral of Eq. (16) farther out into the Weibull tail of resistance cdf, creating a smaller P_f.

As discussed later in more detail, in addition to the usual understrength (capacity reduction) factor, which essentially accounts for the brittleness of failure, code provisions for brittle failures of concrete structures tacitly imply covert understrength factors for the error of theory or formula and for randomness of material strength (Bažant and Yu 2006). Thus there are in fact three random variables for structural resistance, which all affect θ. In that case, the calculation of θ will be cumbersome (and impossible if all three are not known). It will be desirable to reduce the number of variables. The probability of structural failure in Eq. (17), which is easily calculated and determined for a given structure, takes into account the ratio and variation of the load and of the resistance variables. Choosing the unmodified probability of failure $P_{f,old}$, s_L, and s_{R0}, which is the CoV of one RVE, θ can be determined as shown in Fig. 6.

For large sizes, the entire cdf becomes Weibull. This alters the reliability index profoundly. The difference made by Weibull distribution arises from the tail, which is approximately a power law, contrasting with the exponential tail of the Gaussian distribution (Fig. 5). For tail offset ratio θ, the following, highly accurate (<0.5% error), approximation, has been formulated to correct the current failure probability for brittle failures:

$$\theta = (2.61s_L^2 - 1.93s_L + 1.16)[-\log(P_{f,old})]^{(-1.46s_L^2+0.97s_L+0.14)} \qquad (20)$$

The Hasofer-Lind reliability index (FORM) must be revised similarly. As can be shown, it can be computed in the usual manner, except that the so-called reduced variable for resistance must be modified as:

$$R_i' = \left(R_i - \mu_{R_i}\right)/\theta_i s_{R_i} \qquad (21)$$

When $\theta_i = 1$, this reduces to the classical form (e.g., Eq. 7.48 in Haldar and Mahadevan 2000).

The need for a reform of the existing reliability concepts is evident from the discrepancy between the theoretical and actual failure probabilities. The observed frequency of catastrophic failures of large structures has been about an order of magnitude higher than the theoretical probabilities of failure computed in the classical way, assuming fully Gaussian distributions only (Allen 1968, 1975; CIRIA 1977; Livingstone 1989). This discrepancy may be largely due to ignoring the long power-law tails characterizing brittle failures of large structures which is shown in Table 2.

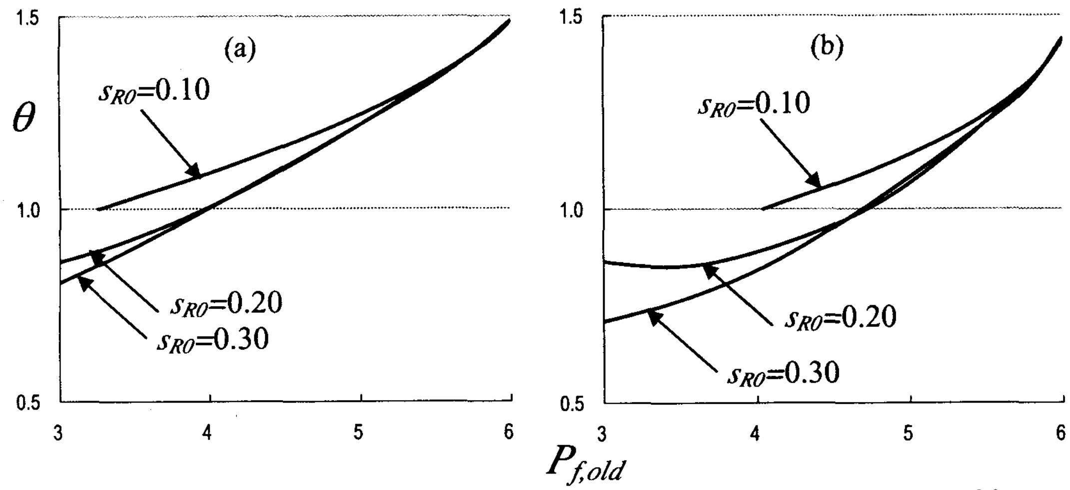


Fig. 6. Ratios θ calculated for coefficient of variation of load: a) s_L=0.10; b) s_L=0.20.

The need for introducing the effect of the tail-offset ratio θ means that reliability methods, taking into account only the first and second order statistical moments (FOSM, SOSM), must be abandoned for quasibrittle structures. But it does not mean that the reliability methods (FORM, SORM), taking into account the non-normal variables by equivalent Gaussian variables, should be used. Rather, the use of θ could be called 'EVRM'—the extreme-value reliability method.

COVERT UNDERSTRENGTH FACTORS

Another related and inseparable problem, already alluded to, is caused by the fact that, for brittle failures, concrete design codes unfortunately specify not mean prediction formulas but "fringe formulas", i.e. formulas that have been set at the margin (or fringe), rather than the mean, of the scatter band of test data. The use of fringe formulas implies a hidden presence of "covert" understrength factors (or capacity reduction factors) (Bažant and Yu 2006), which are not evident to the user of the code. Their determination requires tedious examination of the databases used by the code-writing committees, which are usually hard to access. The covert understrength factors in ACI code represent great strength reductions—in shear failure of concrete beams they are about 0.65 for formula scatter (caused primarily by error of theory and randomness of cracking) and 0.70 for randomness of concrete strength (due to the fact that the design is based on a reduced, rather than mean, strength from cylinder tests). The usual ("overt") understrength factor, which distinguishes diverse failure modes and is the only one evident to the user, is now 0.75 for beam shear in the ACI Standard 318.

Bažant and Yu (2006) proposed to undertake a major revision of concrete design codes, making the covert factors overt in all the code specifications for brittle failures, and specifying both the probability cut-off and the CoV associated with each covert factor. Until this is done, structural reliability assessments will remain a mathematical exercise with no real meaning.

The problem of multiple understrength factors also affects the reliability indices. Fundamentally, it means generalizing the standard Freudenthal's reliability integral in Eq. (16) to multiple pdf's associated with the individual understrength factors (which leads to a multiple integral) and then translating this integral in a suitably simplified manner into the reliability index of Cornell or Hasofer-Lind type. The Hasofer-Lind type index will have to be considered in a multidimensional space (four-dimensional for beam shear failure). Because the multiple, simultaneously applicable, understrength factors should properly be considered as functions of brittleness (as affected by structure size), the scaling properties, fracture mechanics, and reliability concepts are inseparable.

The current code thus implies the concrete design formulas for various types of brittle failure (shear, torsion, punching, column crushing, etc.) to have the form

$$\sigma_N = \varphi\psi F(\varsigma, f_c) \tag{22}$$

where F = function, φ = reduction factor applied to mean material strength f_c; ψ = overt and ς = covert understrength factors, taking into account brittleness and formula error. Let $r_\varphi(\varphi)$, $r_\psi(\psi)$, $r(\varsigma)$ be the corresponding pdf's of RVEs, which may be assumed to be statistically independent. Then one can show that Eq. (16) must be replaced by (Bažant 2004a, Bažant and Yu 2006):

$$\begin{aligned} P_f &= 1 - \int\int\int\int_{\Re} g(P) r_\varphi(\varphi) r_\psi(\psi) F'(\varsigma, f_c) d\varsigma d\psi d\varphi dP \\ &= 1 - \int\int\int_{\Re} G(P) r_\varphi(\varphi) r_\psi(\psi) F'(\varsigma, f_c) d\varsigma d\psi d\varphi \end{aligned} \tag{23}$$

where P = applied random load whose pdf is $g(P)$ and cdf is $G(P)$; $F'(\varsigma, f_c) = \partial F(\varsigma, f_c)/\partial\varsigma$; and integration is performed over hyper-region $\Re$ in which $\varphi\psi F(\varsigma, f_c) > P/bD$.

IRRATIONAL SIZE EFFECT HIDDEN IN LOAD FACTOR FOR SELF WEIGHT

ACI Standard 318 (2005) imposes the load factor of 1.4 for dead load acting alone. In a very large structure, the self-weight may represent 95% of the total load or more. But an error of 40% in the self-weight (i.e., in mass density and structural dimensions) is inconceivable; at most 3% to 5% could be justified. This means that large structures are systematically overdesigned, compared to small ones in which the self-weight contributes a negligible part of loading.

This implies a hidden size effect of about 30% (Bažant and Frangopol 2002), and partly compensates for the lack of size effect in structural resistance formulas of ACI code, but is irrational because it does not distinguish among various types of failure. For shear or torsion of very large beams, this hidden size effect is far too small, while for flexural failure of unreinforced beams it should vanish. For prestressed concrete or high-strength concrete it is smaller than for normal concrete because such structures are lighter, yet it should be greater because they are more brittle, etc. This further implies that probabilistic calculations predict incorrect reliability for structures of different sizes (Bažant and Frangopol 2002).

However, elimination of the excessive and irrational dead load factor would be dangerous unless the size effect is introduced at the same time into the code provisions for all brittle failures. Reliability experts and fracture experts will have to collaborate on this task.

DEFINITION OF BRITTLENESS NUMBER

Practical application of the present theory necessitates defining a brittleness number, $\square$, as a shape-independent characteristic that allows determining the size effect for any structural geometry if it has been calibrated in the laboratory for one structure geometry. For type 2 size effect (due to large cracks or notches), such brittleness number has been expressed as $\square = D/D_0$ where D_0 is the transitional size defined in terms of the energy release rate function of LEFM (e.g. Bažant 2002). But here the focus is on type 1 size effect, for which $\square$ has not yet been defined. In the transition from plastic to brittle response, $\square$ should characterize the proximity to brittle response, i.e., to LEFM. This has nothing to do with strength randomness and should be defined strictly on the basis of the energetic part of size effect law, which is (for type 1) the special case of Eq. (1) for $m\to\infty$. From this equation, it can be shown that $\square = (D/D_l)^{-1/2}$. Geometrically, this number has the meaning $\square = (c/b)^{-1/2}$ where c and b are the distances in linear scale of $\square_N$ between the circled points marked on the size effect curves in Fig. 7. This geometrical definition of   is universal, valid for not only type 1 but also type 2.

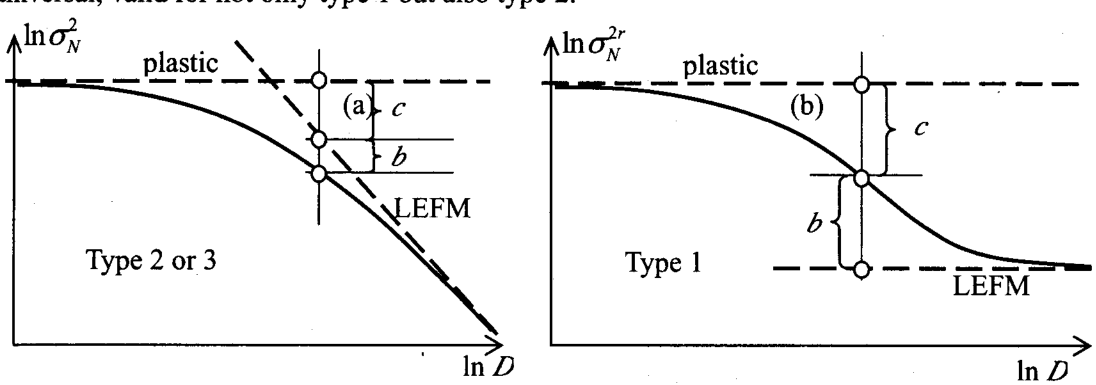


Fig. 7. Geometrical definition of   for (a). Type 2 or 3 structures (b). Type 1 structures.

CLOSING OBSERVATIONS

For more than two decades, the structural reliability theory seemed to be understood almost perfectly. However, this has been true only as long as the theory of limit states, anchored in plasticity, is applicable. In that case, the failure proceeds simultaneously along the whole failure surface, the material strength is mobilized at all the points of the surface, the size effect is nil, the scatter follows Daniels fiber-bundle model, and the structural strength distribution is, for any structure size, necessarily Gaussian (or normal, but never log-normal).

In recent years, though, it gradually transpired that this classical theory does not apply to quasibrittle structures, such as large concrete structures failing due to concrete fracture rather than yielding of steel reinforcement, load-bearing fiber-composite parts of large ships or aircraft, sea ice plates; etc., in which the failure is progressive, propagating along the failure surface. Often failure occurs as soon as the full FPZ is formed, and then there is a strong energetic-probabilistic size effect, with a structural strength distribution that has a Gaussian core and Weibull tail, the tail expanding and the core shrinking as the size increases. This has enormous effect on the structural design satisfying the typical tolerable failure probability (about one out of ten million). To preserve a constant safety margin for such a small failure probability, the required understrength factor depends on the structure size, and with increasing size almost doubles. This, as well as related problems due to covert understrength factors in codes and to excessive load factor for self weight, will necessitate profound modification of reliability analysis and a major overhaul of design codes and practices for quasibrittle structures.

ACKNOWLEDGMENT

Partially supports under ONR grant N00014-10-I-0622 to Northwestern University, and und a DoT grant to the Infrastructure Technology Institute at Northwestern University, are grateful acknowledged.

REFERENCES

ACI Standard 318-05 (2005). Building Code and Commentary. Am. Concrete Institut Farmington Hills, MI.

Allen, D.E. (1968). Discussion of T.J. Turkstra, 'Choice of Failure Probabilities.' J. Struct. Div., ASCE, Vol 94: 2169-2173.

Allen, D.E. (1975). Limit States Design – A Probabilistic Study. Canadian Journal of Civil Engrg. 2 (1): 36-49.

Ang, A.H.-S., and Tang, W.H. (1984). Probability concepts in engineering planning and design. Vol II. Decision, risk and reliability. J. Wiley, New York.

Bansal, G.K., Duckworth, W.H. and Niesz, D.E. (1976a). Strength-size relations in ceramic materials: Investigation of an alumina ceramic. Journal of American Ceramic Society, 59, 472-478.

Bansal, G.K., Duckworth, W.H. and Niesz, D.E. (1976b). Strength analysis of brittle materials. Battelle-Report, Columbus.

Bažant, Z.P. (1997). Scaling of quasibrittle fracture: Asymptotic Analysis. Int. J. of Fracture 83 (1): 19-40.

Bažant, Z.P. (2002). Scaling of Structural Strength. London, UK: Hermes Penton Science (Kogan Page) (also 2nd ed. Elsevier 2005; and French translation, Hermes, Paris 2004).

Bažant, Z.P. (2004a). Probability Distribution of Energetic-Statistical Size Effect in Quasibrittle Fracture. Probabilistic Engineering Mechanics 19: 307-319.

Bažant, Z.P. (2004b). Scaling theory for quasibrittle structural failure. Proc., National Academy of Sciences 101 (37), 13397-13399.

Bažant, Z.P., and Chen, E.-P. (1997). "Scaling of structural failure." Applied Mechanics Reviews ASME} 50 (10), 593--627; transl. in Advances in Mechanics (China) 29 (3), 383-433.

Bažant, Z.P., and Frangopol, D.M. (2002). Size effect hidden in excessive dead load factor. J. of Structural Engrg. ASCE 128 (1), 80-86.

Bažant, Z.P., and Jirásek, M. (2002). "Nonlocal integral formulations of plasticity and damage: Survey of progress". ASCE J. of Engrg. Mechanics 128 (11), 1119-1149 (ASCE 150th anniversary article).

Bažant, Z.P., and Novák, D. (2000a). Probabilistic nonlocal theory for quasibrittle fracture initiation and size effect. I. Theory, and II. Application. J. of Engrg. Mechanics ASCE 126 (2):

166-174 and 175-185.

Bažant, Z.P., and Novák, D. (2000b). Energetic-statistical size effect in quasibrittle failure at crack initiation. ACI Materials Journal 97 (3): 381-392.

Bažant, Z.P., and Pang, S.-D. (2005a). "Revision of Reliability Concepts for Quasibrittle Structures and Size Effect on Probability Distribution of Structural Strength." Safety and Reliability of Engrg. Systems and Structures} (CD) (Proc., 8th Int. Conf. on Structural Safety and Reliability, ICOSSAR 2005, held in Rome), G. Augusti, G.I. Schueller and M. Ciampoli, eds., Millpress, Rotterdam, pp. 377--386.

Bažant, Z.P., and Pang, S.-D. (2005b). Activation energy based extreme value statistics and size effect in brittle and quasibrittle fracture. Journal of the Mechanics and Physics of Solids, submitted to.

Bažant, Z.P., and Planas, J. (1998). Fracture and Size Effect in Concrete and Other Quasibrittle Materials}. CRC Press, Boca Raton and London (textbook and reference volume, 616 pp.).

Bažant, Z.P., and Xi, Y., (1991). Statistical size effect in quasibrittle structures: II. Nonlocal theory. ASCE J. of Engineering Mechanics 117 (11), 2623--2640.

Bažant, Z.P., and Yu, Q. (2006). Reliability, brittleness and fringe formulas in concrete design codes. J. of Structural Engineering ASCE 132 (1), in press.

CIRIA (1977). Rationalization of Safety and Serviceability Factors in Structural Codes. Construction Industry Research and Information Association, Report No. 63. London.

Cornell, C.A. (1969). A Probability Based Structural Code, ACI Journal 66: 974-985.

Cottrell, A.H. (1964). The Mechanical Properties of Matter. J. Wiley, New York.

Daniels, H.E. (1945). The statistical theory of the strength of bundles and threads." Proc. Royal Soc. A183, London: 405-435.

Ellingwood, B.R., Galambos, T.V., McGregor, J.G., and Cornell, C.A. (1980). Development of probability based load criterion for American National Standard A58. NBS Special Publication 577, U.S. Department of Commerce, Washington, D.C.

Ellingwood, B.R., McGregor, J.G., Galambos, T.V., and Cornell, C.A. (1982). Probability based load criteria: Load factors and load combinations. Journal of Structural Engineering ASCE 108 (ST5): 978-997

Fréchet, M., (1927). Sur la loi de probabilité de l' écart maximum. Ann. Soc. Polon. Math. 6, 93.

Fisher, R.A., Tippett, L.H.C., (1928). Limiting forms of the frequency distribution of the largest and smallest member of a sample. Proc. Cambridge Philosophical Society 24: 180-190.

Freudenthal, A.M., Garrelts, J.M., and Shinozuka, M. (1966). The analysis of structural safety. J. of the Structural Division ASCE 92 (ST1), 619--677.

Gumbel, E.J. (1958). Statistics of Extremes. Columbia University Press, New York.

Halder, A. and Mahadevan, S. (2000). Probability, Reliability and Statistical Methods in Engineering Design. J. Wiley & Sons, New York.

Harlow, D.L., Smith, R.L., Taylor, H.M. (1983). Lower tail analysis of the distribution of the strength of load-sharing systems. J. of Applied Probability, 20, 358-367.

Hill, T.L. (1956). Statistical Mechanics: Principles and Selected Applications. McGraw-Hill, New York.

Livingstone, W.R. (1989). CSA Code for Design, Con-struction & Installation of Fixed Offshore Structures. Symposium on Limit States Design in Foundation En-gineering, Canadian Geotechnical Society, Toronto: 77-89.

Madsen, H.O., Krenk, S., and Lind, N.C. (1986). Methods of Structural Safety. Prentice Hall. Englewoods Cliff, NJ.

McClintock, F.A., and Argon, A.S. (1966). Mechanical Behaviour of Materials. Addison-Wesley, Reading, Mass.

Melchers, R.E. (1987). Structural Reliability, Analysis & Prediction. Wiley, New York.

Quinn, G.D. and Morrell, R. (1991). Design data for engineering ceramics: a review of the flexure test. Journal of American Ceramic Society, 74 (9), 2037-2066.

RILEM TC QFS (Z.P. Bažant, chair) (2004). "Quasibrittle fracture scaling and size effect—Final report." Materials and Structures (Paris) 37 (No. 272), 547--586.

Weibull, W. (1951). A statistical distribution function of wide applicability. Journal of Applied Mechanics, ASME 153, Stockholm, 18, 293-297.