Proposal of m-Index for Rating Fracture and Damage Models by Their Ability to Represent a Set of Distinctive Experiments

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Abstract: A recent comparative study revealed that the commonly offered experimental validations of peridynamics and phase-field fracture models have been insufficient because they involved only nondistinctive experiments, i.e., experiments that can be closely fitted by different models that at the same time give very different predictions in important practical applications. The comparisons showed that the peridynamic and phase-field models are incapable of simulating a set of 11 distinctive experiments—experiments that are critical for assessing the accuracy of different models and are representative of fracture behavior of engineering structures. Practical applications would be helped by common adoption of a model index that would compare the predictive capability of various fracture models quantitatively. Proposed here for further discussion is an example of a possible numerical index, the m-Index, which attempts to characterize how well the optimal calibration of model parameters can match the experimental evidence, such as the fracture patterns, measured response curves, size effect, and crack-parallel stress effect. Included are only the distinctive experiments. As an example, the m-Index is here calculated for a set of seven fracture models whose performance was previously compared with 11 distinctive experiments. This previous comparison of seven models is here extended to an eighth model, proposed as a fresh improvement of peridynamics. The choice of distinctive experiments is one of the subjects calling for further discussion. Despite inevitable imperfections, a widely adopted index for appraising new material models would mitigate waste of researchers' effort and grant funds, as well as the space in scientific journals and conference programs. **DOI:** 10.1061/JENMDT.EMENG-6887.

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The Problem Faced

At present, there exist many different models for fracture and damage of quasi-brittle materials, particularly concrete. The main ones can be categorized as follows:

- Discrete line crack models, in which crack is considered to be a line (i.e., it has a zero width, even at the tip), and the crack initiation and propagation follows either (1) the linear-elastic fracture mechanics (LEFM) (Griffith 1921) in which the basic concept is a pointwise fracture energy; or (2) the cohesive crack model (CCM) (also called fictitious) (Barenblatt 1959, 1962; Chorin and Bažant 2022), which is defined by a softening scalar stress-separation relation, active within a softening zone of finite length (cf. Appendix I).
- The crack band model (CB) proposed in 1983 recognized that the fracture process zone (FPZ) at the crack tip has a finite width, within which the material must be characterized by a tensorial damage constitutive law with progressive postpeak softening damage (Bažant and Oh 1983) (cf. Appendix I).
- Peridynamics (PD) (Silling 2000; Silling and Askari 2005; Silling and Bobaru 2005; Silling et al. 2007; Foster et al. 2011) has some common features with the nonlocal model and represents a generalization of the 1977 network model of Burt and Dougill (1977).
- Phase-field (PF) models, in which a line crack is anchored in the finite-element mesh by a fictitious nonuniform scalar damage band and a pointwise fracture energy is used (Francfort and Marigo 1998; Bourdin et al. 2000, 2008; Amor et al. 2009; Lancioni and Royer-Carfagni 2009; Miehe et al. 2010; Lee et al. 2016; Borden et al. 2016; Nguyen and Wu 2018).
- Nonlocal and gradient models, which are categorized as strongly (Bažant et al. 1984; Pijaudier-Cabot and Bažant 1987) and weakly (Aifantis 1984; Bažant 1984a; Peerlings et al. 1998) nonlocal, and both use a tensorial damage constitutive law. Their critical evaluation is beyond the scope of this paper, although they share some of the problematic features of peridynamics discussed by Bažant et al. (2022a), e.g., those of boundary and crack face conditions.

Both the PD and PF models, described in detail and examined by Bažant et al. (2022a), are relatively new models developed within the last two decades. They have received enormous attention, attracted huge funding, were featured in hundreds of journal and conference papers [e.g., 168 papers at World Congress on Computation Mechanics (WCCM) 2018]. They have been heavily favored in the computational mechanics community, where realistically looking computer movies of structural failures have often been offered as their sole verifications. The crack band model (Bažant and Oh 1983; Červenka et al. 2005; Nguyen et al. 2021; Bažant and Planas 1998; Bažant et al. 2021; Nguyen et al. 2020c), having strong experimental support, has been popular in structural

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and geotechnical engineering and has become the mainstay in failure analysis of large composite airframes. Variants of the foregoing models have been used for damage and failure of fiber composites, ceramics, rocks, sea ice, wood, bone, and others.

Due to this cornucopia of models and continuing emergence of new ones, a method that could evaluate the performance of these models through experimental data became a pressing issue. But which experimental data should be used?

Two Kinds of Experimental Data

In view of the recent data comparison by Bažant et al. (2022a), two kinds of experimental data should be distinguished:

- Nondistinctive experiments, which are those that can be closely fitted by many different material models and thus are insufficient to validate or invalidate any particular model.
- Distinctive experiments, which can be closely reproduced by only one model and reveal various characteristic, practically important, features of damage and failure behavior.

An example of a distinctive experiment is the size-effect test (Bažant 1984b; Bažant and Planas 1998). The peak load of geometrically scaled structures made of the same material can only be captured if the size and shape of the fracture process zone is represented realistically, remaining almost fixed across structure sizes. Another compelling example is the gap test.

In the case of quasi-brittle materials, particularly concrete, a few examples of nondistinctive experiments, typically conducted at only one size, are shown in Fig. 1. These include the following:

- Case a: The uniaxial-load deflection and the load-crack length curves with postpeak softening of the compact-tension specimens (of one size) (Narayan and Anand 2021) [Fig. 1(a)].
- Case b: The diagram of crack-mouth opening displacement (COD) versus crack length and the crack path in a compact tension specimen (of one size), pierced or not by a hole on the side of notch extension line (Pham et al. 2017; Zhang 2017) [Fig. 1(b)].
- Case c: Nooru-Mohamed (1992)'s of four-point-loaded double-edge-notched beam (of one size), which leads to two curved cracks diverging from each other [not exhibited here; this test was omitted by Bažant et al. (2022a) because it is too similar to the one shown in Fig. 1(b)—if one gets matched, the other will, too] [Fig. 1(c)].
- Case d: The fracture of a tensile dog-bone specimen (of one size), with holes at different places (Behzadinasab and Foster 2020a) (used in Sandia Fracture Challenge) [Fig. 1(d)].
- Case e: The dynamic branching of a fast-propagating crack (Nguyen and Wu 2018; Ha and Bobaru 2010) [Fig. 1(e)].

In Cases a–d, it suffices for different constitutive laws to capture the crack propagation normal to the maximum principal stress direction. Many do, and what matters is that the constitutive law would lead to the same scalar cohesive softening stress-separation curve. Many models can produce the same cohesive curve, although they give vastly different responses when crack-parallel stresses and multiaxial damage behaviors in the FPZ matter.

Similar to Cases (a)–(d), Case (e) is not a distinctive check on the material damage and fracture model per se either because, for all material models, the crack branching is dominated by material inertia forces. At high enough speed approaching the Rayleigh wave speed, the maximum tensile stress occurs at directions inclined from the crack extension line [Fig. 1(e)]. So this test checks mainly the capability of capturing properly the material inertia forces. The crack path follows the direction perpendicular to the maximum circumferential tensile stress near the moving crack tip, and this direction is not significantly affected by the material model.

Needless to elaborate on, if some model cannot fit even the nondistinctive experiments, its further consideration makes no sense.

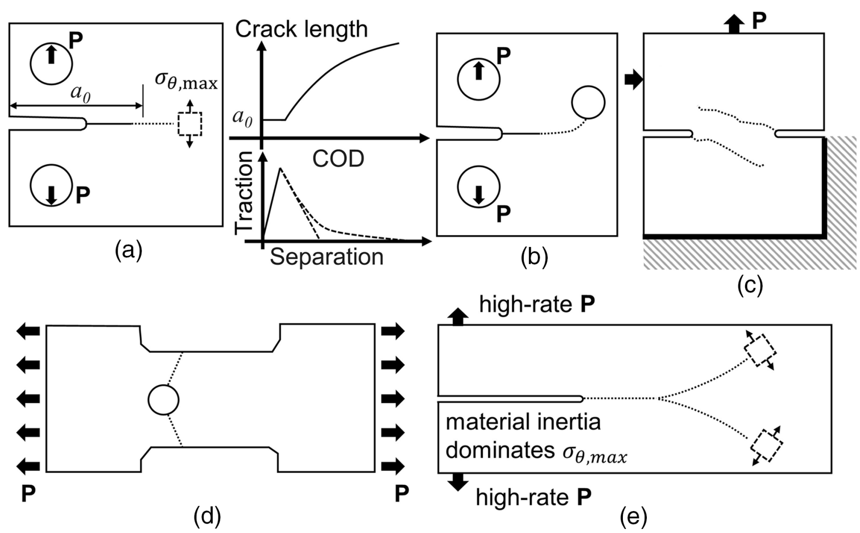

Fig. 1. Examples of nondistinctive experiments (schematic crack paths and plots were sketched from experimental data of Nooru-Mohamed (1992), Behzadinasab and Foster (2019), and Pham et al. (2017): (a) load-deflection and load-crack length curves of compact tension specimen; (b) crack paths in compact tension specimen with a hole on the side of the notch extension line; (c) Nooru-Mohamed (1992)'s experiment to investigate the effect of mix-mode loading on concrete; (d) uniaxial test of dog-bone specimen with holes; and (e) branching of dynamic fast-running crack in double cantilever specimen, in which the location of $\sigma_{\theta,\max}$ points is dominated by material inertia forces rather than constitutive law. COD = crack opening displacement. Detailed results of how different models can simulate these tests have been given by Nguyen and Wu (2018), Bažant et al. (2022a), Pham et al. (2017), Bazilevs et al. (2022), Behzadinasab and Foster (2019), and Ha and Bobaru (2010).

Proposed Criteria to Evaluate Performance of Various Models

For each experiment, it is proposed that its m-Index may be obtained on the basis of the following criteria:

1. How well the simulated crack pattern at the maximum load matches the experimental one, and how closely the observed crack growth process is captured (it is, of course, debatable how to quantify it by a number).
2. How well the simulations match the measured load-displacement diagram and possibly other measured histories of displacement and strain fields [instead of the visually intuitive characterization of Bažant et al. (2022a), one could, for instance, consider using the statistical nonlinear regression characteristics from the Levenberg-Marquardt nonlinear optimization algorithm].
3. How well the simulations match the measured size effect, i.e., the curve of nominal strength versus structure size—plotted, of course, in the log-log (or double logarithmic) scale, because the linear scale is misleading (again, instead of intuitive visual characterization, one might better use regression statistics).
4. How well the general features of numerical simulations, and especially their asymptotic trends in the log-log size-effect plot, match the general features of the experimental data. For example, the simulated Type 2 size-effect curve must decrease, rather than increase, with the structure size. Also, the curve must be concave rather then convex in the log-log plot, and must approach the terminal asymptotic slope of $-1/2$ (again, the way to capture such mismatches statistically by one number is debatable).

Some criteria may be more important than the others, which must be satisfied simultaneously. For example, when the simulated fracture pattern or load-displacement curve of some model is completely wrong, the other criteria need not even be checked. Vice versa, when these are correct while the size effect and its asymptotics are wrong, it is sufficient to invalidate a model.

The choice of these experiments is, of course, debatable. In the recent comprehensive study (Bažant et al. 2022a), the distinctive experiments, all except one conducted on concrete, have been chosen as follows:

1. The new gap test (Nguyen et al. 2020a, b) conducted on geometrically scaled specimens at several different sizes, which reveals the effect of crack-parallel stresses on the fracture energy and the FPZ width.
2. Type 2 size effect in notched three-point bend specimens.
3. Type 1 size effect in unnotched three-point bend specimens.
4. Concentrated shear in four-point loaded double-notched specimen.
5. Compression-torsion fracture of a notched cylinder, at various axial forces.
6. Uniaxial compression of a cylinder, unconfined, with postpeak.
7. Confined compression of a slab, with various lateral stresses.
8. Compression of a cylinder confined up to very high stress, with pore collapse.
9. Vertex effect in shear generated by torsion superposed on axial compression of a cylinder.
10. Diagonal shear failure of reinforced concrete beams, with size effect.
11. Axial double-punch of cylinder, with a broad range of size effect.

These experiments were chosen so that each reflected a unique aspect of concrete that the others in the list could not.

An example of model evaluation is given in Table 1. It was adapted from the tables by Bažant et al. (2022a, b) and Bazilevs et al. (2022). A brief discussion of this model is mentioned in Appendix II, in which qualifications labeled as OK, fair, poor, or wrong were used instead of the present numbers, and each experiment was discussed in detail.

Table 1 makes the following comparisons:

- Two versions of the crack band model (CB) are compared, of which CB-M7 employs the microplane material model M7 (Caner and Bažant 2013a, b), and CB-Gr employs the tensorial damage constitutive model of concrete by Grassl et al. (2013).
- Two versions of the phase-field model are compared. Specifically, bPF is the basic model by Francfort and Marigo (1998), and PF-Wu is a model modified to account for cohesive stresses and fracture of both Modes I and II (Wu 2017; Wu et al. 2021).
- Four versions of peridynamics are compared: bPD is the basic ordinary state-based model with the critical-stretch damage law developed by Silling and Askari (2005) as downloaded from Sandia National Laboratory's website (Parks et al. 2012); PD-Gr is the nonordinary (correspondence) state-based model in which the constitutive law was replaced by that of Grassl, and PDba-Gr and PDba-M7 are Bazilevs' nonordinary (bond-associated) peridynamic models, which were revised to yield the correct deformation gradient (Behzadinasab and Foster 2020a, b; Behzadinasab et al. 2021; Bazilevs et al. 2022). The description and the main ideas and weaknesses of PF models and PD models are schematically illustrated in Fig. 2.

The coded material model and all input files for these simulations can be freely downloaded, as detailed in the "Data Availability Statement."

Table 1. Model indexes (m-Index) of CB, PF, and PD models on 11 types of distinctive experiments

Distinctive test type	CB-M7	CB-Gr	bPF	PF-Wu	bPD	PD-Gr	PDba-Gr	PDba-M7
1. Gap test	3	2	0	0	0	0	0	0
2. Size effect, Type 1	4	4	0	1	0	0	0	3
3. Size effect, Type 2	4	4	0	0	0	0	2	3
4. Concentrated shear fracture (Mode II)	4	4	3	4	3	0	4	4
5. Compression-torsion fracture (Mode III)	4	3	0	0	0	0	0	0
6. Uniaxial compression	4	3	0	0	0	0	2	4
7. Confined compression of slab	4	3	0	0	0	0	0	2
8. Confined compression of cylinder	4	3	0	0	0	0	0	3
9. Vertex effect	4	4	0	0	0	0	4	4
10. RC beam shear failure with size effect	4	3	0	0	0	0	0	0
11. Double punch with size effect	4	3	0	0	0	0	0	4
m-Index (average)	3.9	3.3	0.3	0.5	0.3	0.0	1.1	2.5

Note: The simulation results associating with this assessment have been given by Bažant et al. (2022a, b) and Bazilevs et al. (2022). CB = crack band; PF = phase field; and PD = peridynamic.

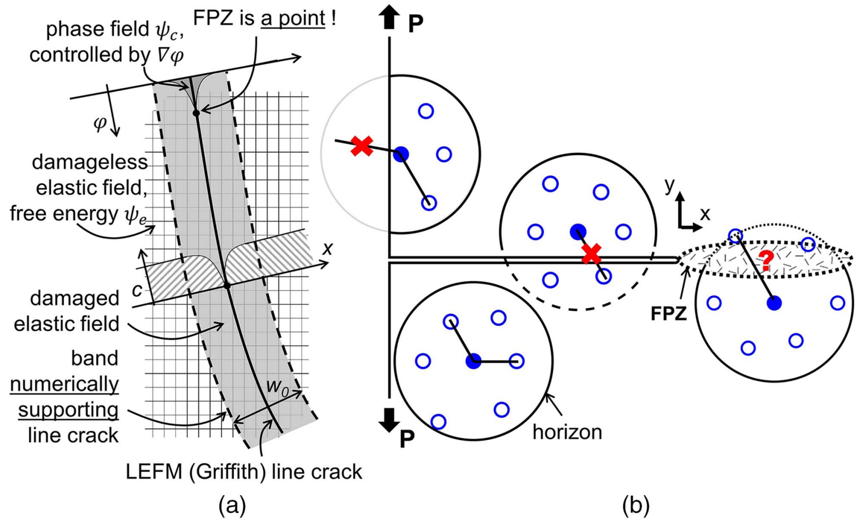

Fig. 2. Schematic of (a) phase-field; and (b) peridynamic models, with simple illustration of their main merits and weaknesses as discussed by Bažant et al. (2022a). (Reprinted from Bažant et al. 2022a, with permission.)

Here, we grade each model using the proposed criteria. This process is exemplified by the evaluation (Bažant et al. 2022a) of the performance of the afore-mentioned models against Experiment 10, RC beam shear failure with size effect. We assign an integer grade to a model depending on whether it satisfies the following criteria, and the grade is zeroed if a criterion considered esssential is not satisfied:

- The failure mechanism is consistent among specimens of different sizes, in agreement with experiments. If this criterion is not satisfied, the comparison of structure strengths for different specimen sizes is nonsensical. Because all PF and PD models (except for PDba-M7) could not meet this requirement, they were graded zero without further considerations (Bazilevs et al. 2022, Fig. 8; Bažant et al. 2022a, Figs. 17–19).
- The development of cracks is consistent among different sizes, reflecting the experiments. Because there were discrepancies in the cracking patterns and their development between the experiments and the results of the CB-Gr model (Bažant et al. 2022a, Fig. 18), its performance on this aspect was downgraded.
- The trend of the size-effect curve must be correct and must correspond to the size-effect type, which is Type 2 in this case. Because PDba-M7 showed a nonmonotonic trend (Bazilevs et al. 2022, Fig. 8), it was also graded zero.
- The prediction of the nominal strength for each specimen size approximately agrees with the test data.

The last column in Table 1 did not appear in the previous comparison in JAM (Bažant et al. 2022a). It is now added to compare a recent improvement of peridynamics by Bazilevs et al. (2022); Appendixes II and III provide a description and discussion of this model.

Discussion and Issues Calling for Broader Debate in Research Community

In Table 1, integral numbers were assigned to m-Index values for individual tests and real numbers with one decimal place were assigned to averaged values. The Grade 4 is, of course, debatable. It must not be understood as perfect, but merely as essentially correct [or OK, as labeled by Bažant et al. (2022a)] in a general sense, i.e., the model can be used in practice with a high confidence. Grade 0 means that the prediction is likely both useless and misleading for practice.

It should be kept in mind that in a highly heterogeneous material such as concrete, the observations of local response in fracture tests are often scattered within $\pm 10\%$ of the mean, which makes perfect predictions virtually impossible. A perfectionist, looking at the scatter in the extensive figures in Bažant et al. (2022a), might replace all the Grade 4 by Grade 3, but then all the other grades would have to be scaled down in proportion, and the distinctions among models would get blurred.

Another debatable issue is the relative importance of various tests. Of prime importance is the size effect. Indeed the scaling is the most important aspect of every physical theory. If the scaling is incorrect, the theory itself is incorrect. Thus the size-effect tests, Types 2 and 1, are the most important.

In this light, the size-effect test might perhaps deserve an increased weight in the present list of 11 distinctive experiments. So might the gap test, the vertex effect test, and the RC beam shear test, because of their practical importance.

However, to elude suspicion of bias, the choice of statistical weights is an issue that should better be left to a committee. This is why no weights (i.e., equal weights) have been used here and by Bažant et al. (2022a).

In future deliberations, the foregoing list might be modified or expanded by adding new experiments which, however, must first be evaluated for distinctiveness, and also for similarity to other tests on the list. If an experiments proposed to be added is too similar to another one already on the list, with a similar failure mode, it would add nothing except for doubling the statistical weight of its failure mode in the overall comparison of all tests, and that would introduce bias for that failure mode.

Two candidates for added experiments might be the inelastic (plastic or damage) strain development in the indentation test, and the extra damping due to comminution of concrete in tests of projectile impact and penetration, which affects the exit velocities and depths of penetration. Both are features not covered by the present 11 experiments. Should the rate effects be included, several more experiments would have to be added. Another candidate of the anchor pullout.

On the other hand, some other distinctive experiments need not, and should not, be added to the list if their behaviors approximately duplicate one or some experiments already on the list. For example, the compact tension, edge-notched tension, and eccentric notched compression tests essentially duplicate the information from the notched three-point-bend test. Partly, this also applies to the wedge-splitting test (Tschegg et al. 1995), which is more difficult to interpret because there is a crack-parallel field of low compression that is rather nonuniform and does not have a maximum on the crack line. The Brazilian split-cylinder test partly duplicates the double punch test, i.e., Case 11, and is more difficult to evaluate because the contact stresses depend on the loading strip, which is not scaled, and also on the cylinder diameter, which disturbs similarity among different cylinder sizes. This causes a complex size effect, sometimes with a trend reversal (Bažant and Mazars 1990; Bažant et al. 1991), which is why the Brazilian test is not on the list. The size effect in torsion of a long prism or hollow cylinder, without or with a reinforcement, is another simple test that might be considered. Anyway, these are issues with subjective overtones. A broader discussion, e.g., in a committee or workshop, is called for.

The problem of model validation is not confined to concrete. A similar validation problem exists, for example, in plasticity and fracture of metals, where models are often validated and calibrated only by uniaxial stress–strain curves, and by fracture tests with the single-edge notched specimen (SENT) and double-edge notched specimen (DENT). However, some important distinctive tests have been ignored in the simulation of metals. This includes the vertex effect, i.e., the variation of the initial torsional stiffness of an axially compressed specimen in the plastic range. The vertex effect is important for all practical situations (typical in earthquakes or projectile impacts) in which the shear follows high axial compression or tension, or when deformation under high axial compression localizes into an inclined shear band. The currently used tensorial plasticity models based on invariants violate this test, although the microplane plasticity model (Brocca and Bažant 2000) reproduces it correctly. Likewise, a similar index should also be developed for fiber composites and other quasi-brittle materials.

In general, the m-Index could, and should, be adapted to evaluate the performance of a model dedicated to any material. We posit that, especially, newly created models aiming to capture the inelastic behavior of a material should be assigned an m-value.

Also note that a model that receives a low m-Index value need not be of no use. It may still yield valid results for situations that are very similar to the experiment in which the model received a high grade. A similar argument can be found for interatomic potentials in molecular dynamics simulations (Zhang et al. 2021).

Further debate may address the optimal choice of distinctive experiments, the proper weights to be assigned to them, and the best choice of a number grade capturing the errors in observed fracture patterns and in measured response characteristics. Hopefully, this study would stimulate such debate.

A perfect index will never be found, but an imperfect but useful index is better than none. This is attested by the general adoption of the Dow Jones, NASDAQ, and Standard and Poor indexes in financial investing and economy. The success of these indices is what stimulated the present proposal.

Proposal

Scientific journals, computational societies such as WCCM, US National Congress on Computational Mechanics (USNCCM), and European Community on Computational Methods in Applied Sciences (ECCOMAS), and engineering societies such as ASCE, ASME, American Concrete Institute (ACI), Japan Society of Civil Engineers (JSCE), and fib-Model Code for Concrete Structures (*fib*) should adopt a universal m-Index to compare material fracture and damage models according to their ability to fit all of the data from an agreed upon set of distinctive fracture and damage experiments.

Conclusion

Adoption of this or similar proposal would avoid wasting effort, grant funds, and space in journals and conference programs on fictitious material models of computational mechanics that give misleading predictions and have a negligible hope of practical applicability.

So far, however, this proposal is just that—a proposal.

Appendix I. Further Notes on the Models Compared in Table 1

Discrete Crack Models

A popular computational version that allows the crack to follow any path through the finite-element mesh is the extended finite-element method, or XFEM (Moës et al. 1999). The recent gap test (Nguyen et al. 2020a, b) restricts the applicability of all line crack models to the case of negligible crack-parallel stress, which is rare in practical applications. Clough (1960) analyzed a cracked by two-dimensional (2D) finite elements with an interelement crack but without fracture mechanics.

Crack Band Model

An effective adjustment of crack band model that ensured a constant fracture energy when the element size was increased and was formulated in 2005 for complex tensorial constitutive laws such as microplane models (Červenka et al. 2005). Furthermore, a generalization of microplane constitutive damage model from explicit to implicit was presented in 2020 (Nguyen et al. 2021). The earliest, primitive version of the crack band model (in 1978) used a sudden stress drop in which the material strength was varied with the crack band width so as to ensure constant fracture energy dissipation (Bažant and Cedolin 1979). However, the softening of entire elements by element deletion was first introduced in 1967 (Rashid 1998; Ngo and Scordelis 1967).

Appendix II. Latest Improvement of Peridynamics with Microplane Model M7

The last version of peridynamics in Table 1, PDba-M7, which was published after the previous comparative study (Bažant et al. 2022a), was a significant improvement developed by Bazilevs et al. (2022) [after discussing with the present authors the performance of peridynamic models evaluated by Bažant et al. (2022a)]. This model used Bazilevs' bond-associated version of peridynamics, which featured a corrected deformation gradient (and was already used in PDba-Gr). The improvement was achieved by two significant changes:

- incorporation of microplane model M7 into peridynamics, and
- avoidance of particle skipping interactions by restricting the horizon to encompass only the nearest neighbors of a material point, which made the model partly similar to the crack band model.

However, this improvement is still far from perfect, as confirmed by a lower m-Index than CB-M7, indicated in Table 1. The main reason is that the horizon for material points at or near the boundaries, crack faces, and the fracture process zone border still protrudes through these borders (Fig. 2). This requires that (even for the shortened horizon, and in contrast to the crack band model) some nonphysical intuitive adjustment must still be made to compensate for the part of horizon that does not interact physically.

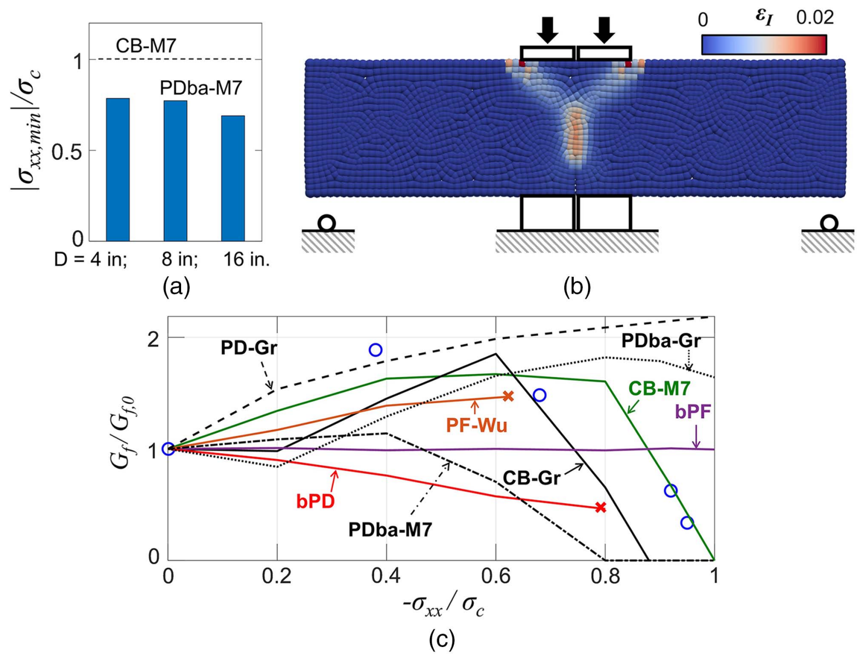

Fig. 3. (a and b) Premature failure of the beams after the compressing stage; and (c) gap test results showing the relative change of measured concrete fracture energy Gf versus crack- parallel compression σ_{xx}, normalized by uniaxial compression strength σc resulting from CB, PF, and PD models [PDba-M7 results were simulated with code provided by Bazilevs et al. (2022)] (reprinted from Bažant et al. 2022a, with permission).

The adjustment can preserve the scalar measures of material strength, or initial elastic stiffness, or incremental stiffness, but not all of them, and not tensorially. This problem does not arise in the crack band model.

This improvement also brings to light a questionable inherent feature of all peridynamics not emphasized previously. The basic concept of continuum homogenization of a heterogeneous material is the representative volume element (RVE). In the finite-element method, the RVE properties are the properties of its fundamental building block, the finite element, as in the crack band model. Peridynamics, questionably, is not based on the concept of an RVE, although in the state-based peridynamics the tensorial constitutive relation is implanted *ex* post facto, based on estimating the deformation gradient from the interparticle bond forces.

Appendix III. Gap Test Results by Bazilevs' New Model

In this improvement, Bazilevs et al. (2022) did not simulate Test 1 in Table 1 (i.e., gap test). This test is here simulated to complete the last column of Table 1. Figs. 3(a and b) show that this model predicted a premature failure under the compression when the crack-parallel stress σ_{xx} only reached $\approx 80\% \sigma_c$. At intermediate levels of σ_{xx}, the increase of G_f was much smaller than in the experiment. At higher σ_{xx}, the model showed a more significantly brittle behavior. This model shared the same parameters with CB-M7, and the uniform horizon had a size comparable to that of the size of the crack band width. Note that, the ranking of CB-M7 on the gap test is 3 rather than 4 because the measured maximum of fracture energy is underestimated by about 30% (the cause is that the crack band width cannot be increased in CBM).

Data Availability Statement

The data that support the findings of this study are available from the corresponding author upon reasoned request. The coding and all of the corrresponding input files for included simulations of the present 11 distinctive experiments is made available for free download at the authors' websites: http://www.civil.northwestern.edu/people/bazant/andgithub.com/htn403.

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References

Aifantis, E. C. 1984. "On the microstructural origin of certain inelastic models." *J. Eng. Mater. Technol.* 106 (4): 326–330. https://doi.org/10.1115/1.3225725.

Amor, H., J.-J. Marigo, and C. Maurini. 2009. "Regularized formulation of the variational brittle fracture with unilateral contact: Numerical experiments." *J. Mech. Phys. Solids* 57 (8): 1209–1229. https://doi.org/10.1016/j.jmps.2009.04.011.

Barenblatt, G. I. 1959. "The formation of equilibrium cracks during brittle fracture. General ideas and hypotheses. Axially-symmetric cracks." *J. Appl. Math. Mech.* 23 (3): 622–636. https://doi.org/10.1016/0021-8928(59)90157-1.

Barenblatt, G. I. 1962. "The mathematical theory of equilibrium cracks in brittle fracture." *Adv. Appl. Mech.* 7 (1): 55–129. https://doi.org/10.1016/S0065-2156(08)70121-2.

Bažant, Z. P. 1984a. "Imbricate continuum and progressive fracturing of concrete and geomaterials." Supplement, *Meccanica* 19 (S1): 86–93. https://doi.org/10.1007/BF01558458.

Bažant, Z. P. 1984b. "Size effect in blunt fracture: Concrete, rock, metal." *J. Eng. Mech.* 110 (4): 518–535. https://doi.org/10.1061/(ASCE)0733-9399(1984)110:4(518).

Bažant, Z. P., T. B. Belytschko, and T.-P. Chang. 1984. "Continuum theory for strain-softening." *J. Eng. Mech.* 110 (12): 1666–1692. https://doi.org/10.1061/(ASCE)0733-9399(1984)110:12(1666).

Bažant, Z. P., and L. Cedolin. 1979. “Blunt crack band propagation in finite element analysis.” *J. Eng. Mech. Div.* 105 (2): 297–315. https://doi.org/10.1061/JMCEA3.0002467.

Bažant, Z. P., M. T. Kazemi, T. Hasegawa, and J. Mazars. 1991. “Size effect in Brazilian split-cylinder tests. measurements and fracture analysis.” *ACI Mater. J.* 88 (3): 325–332. https://doi.org/10.14359/1987.

Bažant, Z. P., J.-L. Le, and M. Salviato. 2021. *Quasibrittle fracture mechanics and size effect: A first course*. Oxford, UK: Oxford University Press.

Bažant, Z. P., and J. Mazars. 1990. “France-US workshop on strain localization and size effect due to cracking and damage.” *J. Eng. Mech.* 116 (6): 1412–1424. https://doi.org/10.1061/(ASCE)0733-9399(1990)116:6(1412).

Bažant, Z. P., H. T. Nguyen, and A. Abdullah Dönmez. 2022a. “Critical comparison of phase-field, peridynamics, and crack band model M7 in light of gap test and classical fracture tests.” *J. Appl. Mech.* 89 (6): 061008. https://doi.org/10.1115/1.4054221.

Bažant, Z. P., H. T. Nguyen, and A. A. Dönmez. 2022b. “Reappraisal of phase-field, peridynamics and other fracture models in light of classical fracture tests and new gap test.” In *Computational modelling of concrete and concrete structures*, 3–7. Boca Raton, FL: CRC Press.

Bažant, Z. P., and B. H. Oh. 1983. “Crack band theory for fracture of concrete.” *Matér. Constr.* 16 (3): 155–177. https://doi.org/10.1007/BF02486267.

Bažant, Z. P., and J. Planas. 1998. *Fracture and size effect in concrete and other quasibrittle materials*. Boca Raton, FL: CRC Press.

Bazilevs, Y., M. Behzadinasab, and J. T. Foster. 2022. “Simulating concrete failure using the microplane (M7) constitutive model in correspondence-based peridynamics: Validation for classical fracture tests and extension to discrete fracture.” *J. Mech. Phys. Solids* 166 (Sep): 104947. https://doi.org/10.1016/j.jmps.2022.104947.

Behzadinasab, M., and J. T. Foster. 2019. “The third Sandia Fracture Challenge: Peridynamic blind prediction of ductile fracture characterization in additively manufactured metal.” *Int. J. Fract.Mech.* 218 (Jul): 97–109. https://doi.org/10.1007/s10704-019-00363-z.

Behzadinasab, M., and J. T. Foster. 2020a. “Revisiting the third Sandia Fracture Challenge: A bond-associated, semi-Lagrangian peridynamic approach to modeling large deformation and ductile fracture.” *Int. J. Fract.* 224 (2): 261–267. https://doi.org/10.1007/s10704-020-00455-1.

Behzadinasab, M., and J. T. Foster. 2020b. “A semi-Lagrangian constitutive correspondence framework for peridynamics.” *J. Mech. Phys. Solids* 137 (Apr): 103862. https://doi.org/10.1016/j.jmps.2019.103862.

Behzadinasab, M., N. Trask, and Y. Bazilevs. 2021. “A unified, stable and accurate meshfree framework for peridynamic correspondence modeling—Part I: Core methods.” *J. Peridyn. Nonlocal Model.* 3 (1): 24–45. https://doi.org/10.1007/s42102-020-00040-z.

Borden, M. J., T. J. Hughes, C. M. Landis, A. Anvari, and I. J. Lee. 2016. “A phase-field formulation for fracture in ductile materials: Finite deformation balance law derivation, plastic degradation, and stress triaxiality effects.” *Comput. Methods Appl. Mech. Eng.* 312 (Dec): 130–166. https://doi.org/10.1016/j.cma.2016.09.005.

Bourdin, B., G. A. Francfort, and J.-J. Marigo. 2000. “Numerical experiments in revisited brittle fracture.” *J. Mech. Phys. Solids* 48 (4): 797–826. https://doi.org/10.1016/S0022-5096(99)00028-9.

Bourdin, B., G. A. Francfort, and J.-J. Marigo. 2008. “The variational approach to fracture.” *J. Elast.* 91 (1): 5–148. https://doi.org/10.1007/s10659-007-9107-3.

Brocca, M., and Z. P. Bažant. 2000. “Microplane constitutive model and metal plasticity.” *Appl. Mech. Rev.* 53 (10): 265. https://doi.org/10.1115/1.3097329.

Burt, N. J., and J. W. Dougill. 1977. “Progressive failure in a model heterogeneous medium.” *J. Eng. Mech. Div.* 103 (3): 365–376. https://doi.org/10.1061/JMCEA3.0002234.

Caner, F. C., and Z. P. Bažant. 2013a. “Microplane model M7 for plain concrete. I: Formulation.” *J. Eng. Mech.* 139 (12): 1714–1723. https://doi.org/10.1061/(ASCE)EM.1943-7889.0000570.

Caner, F. C., and Z. P. Bažant. 2013b. “Microplane model M7 for plain concrete. II: Calibration and verification.” *J. Eng. Mech.* 139 (12): 1724–1735. https://doi.org/10.1061/(ASCE)EM.1943-7889.0000571.

Červenka, J., Z. P. Bažant, and M. Wierer. 2005. “Equivalent localization element for crack band approach to mesh-sensitivity in microplane model.” *Int. J. Numer. Methods Eng.* 62 (5): 700–726. https://doi.org/10.1002/nme.1216.

Chorin, A. J., and Z. P. Bažant. 2022. *Grigory Isaakovich Barenblatt. 10 July 1927—22 June 2018*. London: Royal Society.

Clough, R. 1960. *The finite element method in plane stress analysis*. Reston, VA: ASCE.

Foster, J. T., S. A. Silling, and W. Chen. 2011. “An energy based failure criterion for use with peridynamic states.” *Int. J. Multiscale Comput. Eng.* 9 (6): 675–688. https://doi.org/10.1615/IntJMultCompEng.2011002407.

Francfort, G. A., and J.-J. Marigo. 1998. “Revisiting brittle fracture as an energy minimization problem.” *J. Mech. Phys. Solids* 46 (8): 1319–1342. https://doi.org/10.1016/S0022-5096(98)00034-9.

Grassl, P., D. Xenos, U. Nyström, R. Rempling, and K. Gylltoft. 2013. “CDPM2: A damage-plasticity approach to modelling the failure of concrete.” *Int. J. Solids Struct.* 50 (24): 3805–3816. https://doi.org/10.1016/j.ijsolstr.2013.07.008.

Griffith, A. 1921. “The phenomena of rupture and flow in solids.” *Philos. Trans. R. Soc. London, Ser. A, Containing Pap. Math. or Phys. Character* 221 (Jan): 163–198. https://doi.org/10.1098/rsta.1921.0006.

Ha, Y. D., and F. Bobaru. 2010. “Studies of dynamic crack propagation and crack branching with peridynamics.” *Int. J. Fract.* 162 (1): 229–244. https://doi.org/10.1007/s10704-010-9442-4.

Lancioni, G., and G. Royer-Carfagni. 2009. “The variational approach to fracture mechanics. A practical application to the French Panthéon in Paris.” *J. Elast.* 95 (1): 1–30. https://doi.org/10.1007/s10659-009-9189-1.

Lee, S., M. F. Wheeler, and T. Wick. 2016. “Pressure and fluid-driven fracture propagation in porous media using an adaptive finite element phase field model.” *Comput. Methods Appl. Mech. Eng.* 305 (Jun): 111–132. https://doi.org/10.1016/j.cma.2016.02.037.

Miehe, C., F. Welschinger, and M. Hofacker. 2010. “Thermodynamically consistent phase-field models of fracture: Variational principles and multi-field FE implementations.” *Int. J. Numer. Methods Eng.* 83 (10): 1273–1311. https://doi.org/10.1002/nme.2861.

Moës, N., J. Dolbow, and T. Belytschko. 1999. “A finite element method for crack growth without remeshing.” *Int. J. Numer. Methods Eng.* 46 (1): 131–150. https://doi.org/10.1002/(SICI)1097-0207(19990910)46:1%3C131::AID-NME726%3E3.0.CO;2-J.

Narayan, S., and L. Anand. 2021. “Fracture of amorphous polymers: A gradient-damage theory.” *J. Mech. Phys. Solids* 146 (Jan): 104164. https://doi.org/10.1016/j.jmps.2020.104164.

Ngo, D., and A. C. Scordelis. 1967. “Finite element analysis of reinforced concrete beams.” *J. Proc.* 64 (3): 152–163. https://doi.org/10.14359/7551.

Nguyen, H., M. Pathirage, G. Cusatis, and Z. P. Bažant. 2020a. “Gap test of crack-parallel stress effect on quasibrittle fracture and its consequences.” *J. Appl. Mech.* 87 (7): 071012. https://doi.org/10.1115/1.4047215.

Nguyen, H., M. Pathirage, M. Rezaei, M. Issa, G. Cusatis, and Z. P. Bažant. 2020b. “New perspective of fracture mechanics inspired by gap test with crack-parallel compression.” *Proc. Natl. Acad. Sci.* 117 (25): 14015–14020. https://doi.org/10.1073/pnas.2005646117.

Nguyen, H., S. Rahimi-Aghdam, and Z. P. Bažant. 2020c. “Unsaturated nanoporomechanics.” *Proc. Natl. Acad. Sci.* 117 (7): 3440–3445. https://doi.org/10.1073/pnas.1919337117.

Nguyen, H. T., F. C. Caner, and Z. P. Bažant. 2021. “Conversion of explicit microplane model with boundaries to a constitutive subroutine for implicit finite element programs.” *Int. J. Numer. Methods Eng.* 122 (6): 1563–1577. https://doi.org/10.1002/nme.6590.

Nguyen, V. P., and J.-Y. Wu. 2018. “Modeling dynamic fracture of solids with a phase-field regularized cohesive zone model.” *Comput. Methods Appl. Mech. Eng.* 340 (Oct): 1000–1022. https://doi.org/10.1016/j.cma.2018.06.015.

Nooru-Mohamed, M. B. 1992. “Mixed-mode fracture of concrete: An experimental approach.” Ph.D. thesis, Dept. of Civil Engineering and Geosciences, Delft Univ. of Technology.

Parks, M. L., D. J. Littlewood, J. A. Mitchel, and S. A. Silling. 2012. *Peridigm Users’ Guide*. Tech. Rep. No. SAND2012-7800. Albuquerque, NM: Sandia National Laboratories.

Peerlings, R. H., R. de Borst, W. Brekelmans, and M. G. Geers. 1998. “Gradient-enhanced damage modelling of concrete fracture.” *Mech. Cohesive-Frict. Mater.* 3 (4): 323–342. https://doi.org/10.1002/(SICI)1099-1484(1998100)3:4<323::AID-CFM51>3.0.CO;2-Z.

Pham, K., K. Ravi-Chandar, and C. Landis. 2017. “Experimental validation of a phase-field model for fracture.” *Int. J. Fract.* 205 (1): 83–101. https://doi.org/10.1007/s10704-017-0185-3.

Pijaudier-Cabot, G., and Z. Bažant. 1987. “Nonlocal damage theory.” *J. Eng. Mech.* 113 (10): 1512–1533. https://doi.org/10.1061/(ASCE)0733-9399(1987)113:10(1512).

Rashid, M. 1998. “The arbitrary local mesh replacement method: An alternative to remeshing for crack propagation analysis.” *Comput. Methods Appl. Mech. Eng.* 154 (1–2): 133–150. https://doi.org/10.1016/S0045-7825(97)00068-6.

Silling, S. A. 2000. “Reformulation of elasticity theory for discontinuities and long-range forces.” *J. Mech. Phys. Solids* 48 (1): 175–209. https://doi.org/10.1016/S0022-5096(99)00029-0.

Silling, S. A., and E. Askari. 2005. “A meshfree method based on the peridynamic model of solid mechanics.” *Comput. Struct.* 83 (17–18): 1526–1535. https://doi.org/10.1016/j.compstruc.2004.11.026.

Silling, S. A., and F. Bobaru. 2005. “Peridynamic modeling of membranes and fibers.” *Int. J. Non-Linear Mech.* 40 (2–3): 395–409. https://doi.org/10.1016/j.ijnonlinmec.2004.08.004.

Silling, S. A., M. Epton, O. Weckner, J. Xu, and E. Askari. 2007. “Peridynamic states and constitutive modeling.” *J. Elast.* 88 (2): 151–184. https://doi.org/10.1007/s10659-007-9125-1.

Tschegg, E., M. Elser, and S. Stanzl-Tschegg. 1995. “Biaxial fracture tests on concrete—Development and experience.” *Cem. Concr. Compos.* 17 (1): 57–75. https://doi.org/10.1016/0958-9465(95)95760-W.

Wu, J.-Y. 2017. “A unified phase-field theory for the mechanics of damage and quasi-brittle failure.” *J. Mech. Phys. Solids* 103 (Jun): 72–99. https://doi.org/10.1016/j.jmps.2017.03.015.

Wu, J.-Y., Y. Huang, H. Zhou, and V. P. Nguyen. 2021. “Three-dimensional phase-field modeling of Mode I+ II/III failure in solids.” *Comput. Methods Appl. Mech. Eng.* 373 (Jan): 113537. https://doi.org/10.1016/j.cma.2020.113537.

Zhang, G. 2017. “Peridynamic models for fatigue and fracture in isotropic and in polycrystalline materials.” Ph.D. thesis, Dept. of Mechanical and Materials Engineering, Univ. of Nebraska-Lincoln.

Zhang, X., H. Nguyen, J. T. Paci, S. K. Sankaranarayanan, J. L. Mendoza-Cortes, and H. D. Espinosa. 2021. “Multi-objective parametrization of interatomic potentials for large deformation pathways and fracture of two-dimensional materials.” *NPJ Comput. Mater.* 7 (1): 1–11. https://doi.org/10.1038/s41524-021-00573-x.