

# Comparison of main models for size effect on shear strength of reinforced and prestressed concrete beams

*This paper presents a critical comparison of the existing code provisions for the shear strength of concrete beams. The comparison is based on the computerized filtering-out of the inevitable statistical bias from the available multivariate database on shear strength, on an examination of the predicted size effects on shear strength and their underlying hypotheses and on the results of recent high-fidelity numerical simulations of shear failure. In addition to examining the existing models, the present comparison also provides several key considerations for testing the scientific soundness of any model of shear failure in concrete beams, which is necessary for future revisions to the design code provisions.*

**Keywords:** shear strength, size effect, reinforced concrete, fracture energy, multivariate database

## 1 Introduction

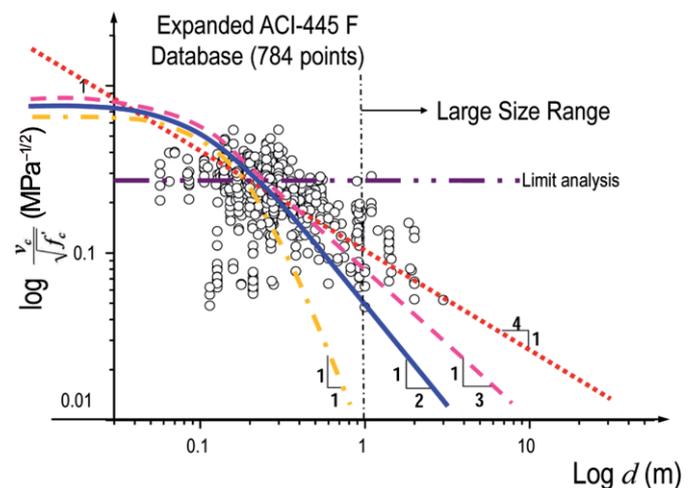
Among the three major engineering societies, ACI, *fib* and JSCE [1]–[3], in the early 1980s the JSCE was the first to introduce a size effect for the shear strength of beams, albeit in a statistical form now known to be inapplicable. The CEB, the predecessor of the *fib*, was the first to recognize that the size effect was not statistical, but the empirical form introduced is now known to be unrealistic. Among these three societies, the ACI remains the only one that does not yet take into account the size effect on the shear strength of concrete beams. Yet the ACI is also the only one among the three without a size effect in its code which is questionable from today’s perspective, and thus has a chance of being the first society to introduce a scientifically sound form. (Note, though, that all three societies have already introduced the size effect, and in a correct, fracture mechanics-based, form, into the specifications for anchor pullout [1], [2], [4], [5], which is essentially a shear failure.)

The size effect is a general property of failure of all quasi-brittle materials [6]–[9], among which concrete is an archetypical case. Currently, a debate about introducing the size effect into the specifications of the ACI-318 de-

sign code [1] for the shear strength of reinforced and prestressed concrete beams is underway in committees ACI-318, ACI-445 and ACI-446. A similar debate about the *fib* Model Code for Concrete Structures 2010 was initiated at the *fib* Congress in Prague in 2011 [10]. The question in all societies is no longer whether the size effect should be taken into account, but what is the proper form of size effect to adopt. The aim of this brief paper is to aid this debate by explaining and critically comparing the main options and the methods of their evaluation.

## 2 Extracting statistical evidence from a biased heteroscedastic database

Fig. 1 shows the newly compiled database (ACI-445d/DfStb), an expansion of the previous ACI-445F database [11]. The shear strength  $v_c$  contributed by the concrete is plotted against beam depth  $d$  in a double logarithmic plot. Each point represents one laboratory test of a reinforced concrete beam without stirrups which failed due to shear. Highly scattered though the data are, they nevertheless show a downward trend. This trend tempted some engineers in ACI-445D to fit a regression line, which happens to have a slope of about  $-1/3$  in the log-log plot and thus gives a size effect factor of the type  $\theta = (d/d_0)^{-1/3}$ , where  $d$  is the beam depth (measured from top face to longitudinal



**Fig. 1.** Expanded ACI-445F database of 784 shear beam tests and various size effect asymptotes

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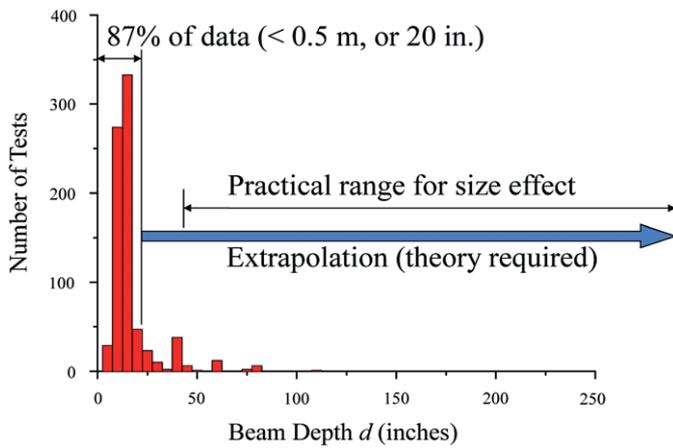


Fig. 2. Histogram of number of beams tested versus beam depth in inches

steel centroid) and  $d_0$  is a constant to be determined from tests (to compare the ACI specifications with those of *fib* and JSCE, note that 1 in. = 25.4 mm and 1 psi = 0.00689 MPa).

However, such a simplistic regression analysis is incorrect. One reason is that, inevitably, the database available is heavily biased statistically. The collected data points do not cover the size range uniformly (i.e. the database is not homoscedastic). In the existing database, plotted in Fig. 2, about 87% of the data points pertain to a beam depth  $d < 508$  mm (20 in.) and 97% to beam depth  $d < 1.27$  m (50 in.), while nowadays the outriggers of tall buildings have a depth of about 6 m (19.7 ft.) and the KB bridge girder in Palau, which collapsed due to shear with compression, had a depth of 14.2 m (46 ft.) [12], [13].

Another source of bias, in fact a major one, is a simultaneous variation in the means of steel ratio  $\rho_w$ , shear span ratio  $a/d$  and other secondary variables as  $d$  increases. This is due to the fact that the database has not been (and could not have been) generated according

to a proper statistical sampling scheme. For example, if the size range is subdivided into several intervals of constant width on a logarithmic scale (five in Fig. 3), the average of the other influencing parameters such as steel ratio  $\rho_w$  and shear span ratio  $a/d$  (as well as concrete strength  $f'_c$  and maximum aggregate size  $d_a$ ), calculated separately for each size interval, should be about the same for all the size intervals [14]. However, as can be seen in Fig. 3 (top), both the averages of  $\rho_w$  and  $a/d$  in the subsequent size intervals decrease with increasing size. For  $\rho_w$ , they decrease by an order of magnitude, and for  $a/d$  by about 30% (as it happens, for large beams the testers generally chose a smaller  $\rho_w$  and a smaller  $a/d$  than they did for small beams). The same kind of statistical bias is found for  $f'_c$  and  $d_a$ , see Fig. 3 (bottom) (in which  $a/d = M/Vd$  where  $M$  and  $V$  are the bending moment and shear force at the moment- and shear-critical cross-section respectively).

## 2.1 Filtering of database to remove bias in secondary variables

In previous work [14], a computer program was developed to delete gradually, and in an unbiased way (without human intervention), the extreme points in each size interval in order to create data subsets with a nearly uniform variation in the secondary parameters throughout the size intervals. This program deletes outlier points one by one, selecting each point deletion candidate so as to achieve the greatest possible reduction in the variance of the averages in all the intervals until the coefficient of variation of the averages drops below a specified value (such as 5%). Here, this program is used to handle the enlarged database of 782 data points. The filtering process deals here only with the secondary parameters that have a large effect, which are  $\rho_w$ ,  $a/d$  and  $d_a$  (but if other parameters are

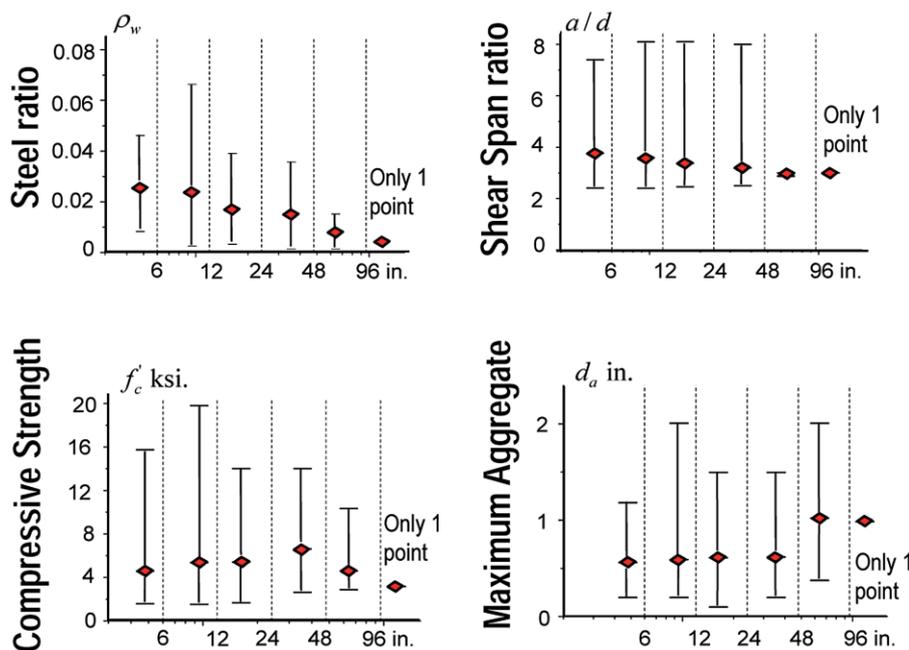
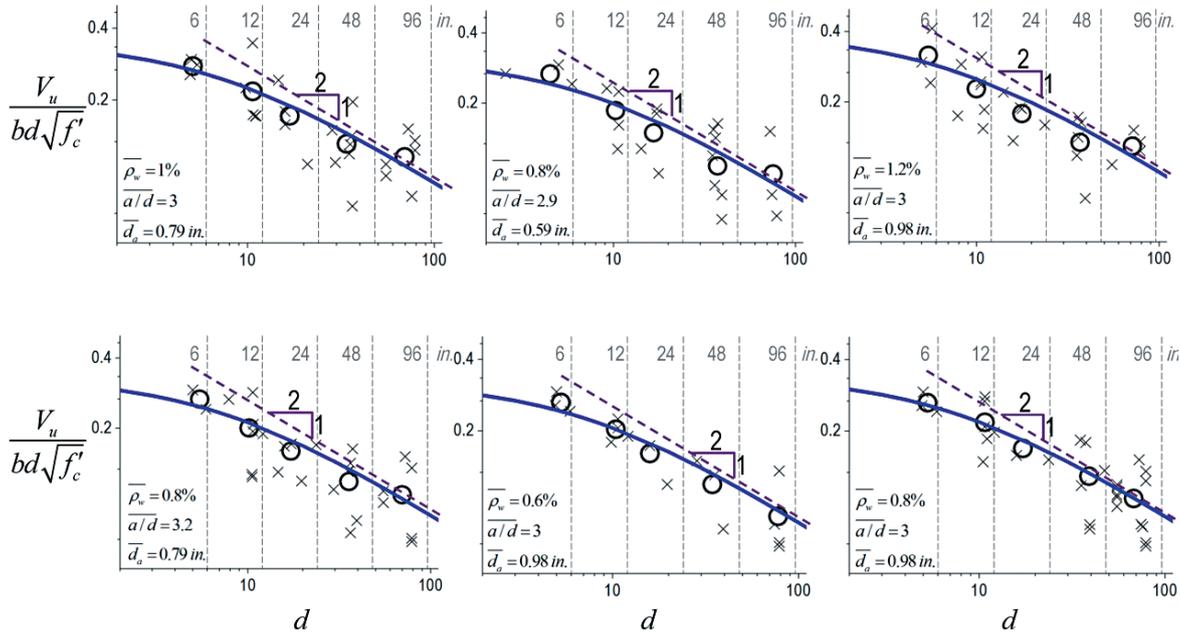


Fig. 3. Variation of secondary parameters in different size intervals: a) longitudinal steel ratio  $\rho_w$ ; b) shear span ratio  $a/d$ ; c) concrete strength; and d) maximum aggregate size  $d_a$



**Fig. 4.** Data filtering to achieve uniform distribution of secondary parameters throughout size intervals and the fitting based on the mean values of remaining test points

deemed important, too, they could easily be included in the same filtering procedure).

The database subset filtered in this way, with the number of points reduced by computer filtering, shows the values of the  $\bar{\rho}_w$ ,  $\bar{a}/\bar{d}$ , and  $\bar{d}_a$  averages to be almost uniform throughout the size intervals. To counter the statistical effect of deletions, a dozen different unbiased filtered data subsets were easily generated by the computer, with different sets of  $\bar{\rho}_w$ ,  $\bar{a}/\bar{d}$ , and  $\bar{d}_a$  averages and with different sets of undeleted points. The averages were selected at different levels so that different data points would be included in different filtered database subsets. When a selected parameter average such as  $\bar{\rho}_w$  is high enough, hardly any data are deleted near the upper margin, whereas considerable data are deleted near the lower margin, and vice versa when the selected average is low enough. Thus, different data points appear in at least one filtered database, suppressing the human bias in choosing the interval means level.

To save space, Fig. 4 shows only six of these filtered databases, along with the corresponding regressions of the optimum fit of the average values of shear strength (circles) of the remaining data points (crosses) in each size interval. The trends in Fig. 4 are seen to be very different from the regression of the unfiltered database in Fig. 1. The averages of the data in the size intervals now agree closely with the ACI-446 size effect formula, which terminates with the size effect factor

$$\theta_{LEFM} \approx \sqrt{\frac{d_0}{d}} \quad (1)$$

where  $d_0$  is a constant for geometrically similar beams. This size effect factor is characteristic of the fracture mechanics of sharp cracks (i.e. linear elastic fracture mechanics, LEFM). In the log-log plot, the terminal slope is  $-1/2$ , see Fig. 4. Further, note that if the average of data in each interval is taken with the same weight, the bias due

to unequal numbers of data points in subsequent intervals is also eliminated.

## 2.2 Evidence from weighted multivariate regression of complete database

Another way to cope with database bias (or heteroscedasticity) is to attach weights inversely proportional to the data density in transformed variables (i.e., in  $\log d$ ) to the data points and then use multivariate non-linear optimization of the data fits. This can be done by a standard computer subroutine, such as the *Levenberg-Marquardt* algorithm, which optimizes all the variables simultaneously [15], [16] and also delivers the coefficients of variation of the optimized parameters. This is a standard procedure for dealing with a heteroscedastic dataset [17], which is the case here.

The bias due to the size dependence of data density can be characterized by the numbers  $N_i$  of data points in the subsequent size intervals, which are here found to decrease sharply with increasing size  $d$ . If this bias were to be ignored, the resulting optimum fit of data would be dominated by small beams, particularly those with  $d < 508$  mm (20 in.), while the data for beams with  $d > 1.27$  m (50 in.) would have hardly any effect on the optimum fit. This statistical bias must be countered by attaching weights to the data points.

To this end, it is necessary to subdivide the  $\log d$  scale into several equal intervals  $i = 1, 2, \dots, n_i$ , then count the number  $N_i$  of the data points in each size interval and finally apply the weight  $1/N_i$  [15,16] to all the data points in interval  $i$ . (Similarly, further weights could be attached to the data points in multi-dimensional boxes in the space of all secondary variables to counter the non-uniformity of the number of points in the boxes.) An optimum fit of the shear database obtained in this manner is shown in Fig. 5, in which the solid curve is the optimum fit using the formula of ACI committee 446 and, for better clarity, the ar-

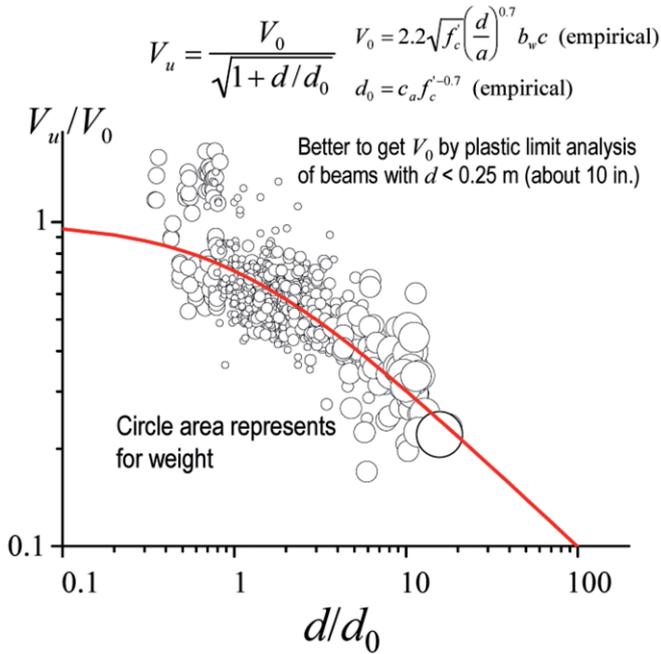


Fig. 5. Fitting of the expanded ACI-445D database of 784 weighted points

as of the circular data points are shown proportional to their weights.

Normally, multivariate regression with proper weighting would be sufficient for analysing the database. However, it should be noted that, to succeed, the form of the dependence of  $v_c$  on the concrete strength and the other secondary parameters of the beam must be known first. Unfortunately, that is not the case here. This is a disadvantage of the regression approach and this is why creating filtered unbiased subsets of the database is useful.

### 3 Load capacity in shear

In elasticity with strength limit as well as in plastic limit analysis, the characteristic (or nominal) stress at maximum load (or at failure under load control), the nominal strength (e.g.  $v_u$  in the case of beam shear), is independent of the structure size  $d$  when geometrically similar situations are compared. The dependence of this strength on the structure size  $d$  cannot be captured by elasticity or plasticity and came to be called the size effect. The shear

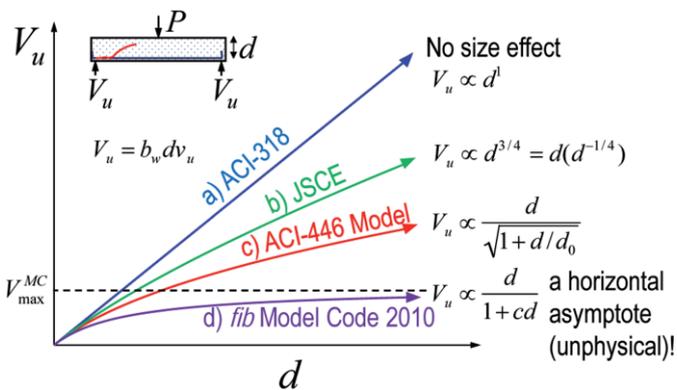
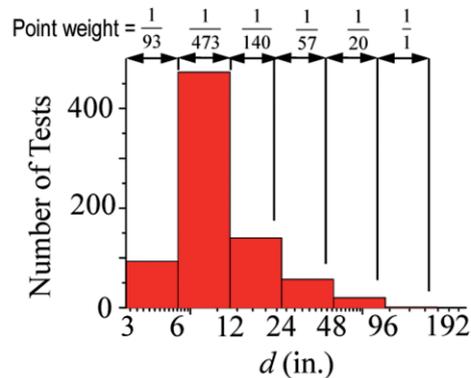


Fig. 6. Comparison of the load capacity asymptotes based on different size effects



force  $V_u$  carried at maximum load by the concrete is (for a constant beam width) proportional to  $v_u d$  and so  $V_u$  increases linearly with the structure size when the size effect is absent, but slower than linearly when it is present (for example, such a size effect has long been embodied in the ACI specifications for anchor pullout [1], [4], [5]).

Fig. 6 shows four plots of  $V_u$  versus beam depth  $d$  corresponding to the ACI code (ACI-318-14 [1]), *fib* Model Code 2010 [2], the JSCE code (Japan Society of Civil Engineers) [3] and the 2007 code proposal of ACI committee 446 [18]. For comparison, all the curves must be scaled to the same initial slope. All the asymptotic size effects on  $V_u$  are power laws of  $d$ , as indicated in the figure, and the size effect formulas are also listed. The differences between the size effect curves are certainly striking and are discussed next.

#### 3.1 Size effect of JSCE

Let us consider the JSCE curve first. JSCE was the first society to introduce the size effect into its design code (in the early 1980s) based on the vision of *H. Okamura* [19], [20]. It was a revolutionary step. Even though it is now known that the JSCE formula for size effect is not the correct one, it nevertheless provides significantly better structural safety than ignoring the size effect altogether. Of course, in the early 1980s, JSCE could not introduce a better formula because the quasi-brittle fracture mechanics required for concrete had not yet been developed. The only size effect theory that existed in the early 1980s was the *Weibull* statistical theory [21], according to which

$$v_u = v_0 \theta_{JSCE} \quad (2)$$

$$\theta_{JSCE} = \left(\frac{d_0}{d}\right)^n$$

where  $d_0$  is an experimental constant for beams of similar shape and here may be taken as 1 m (39.4 in.). Based on

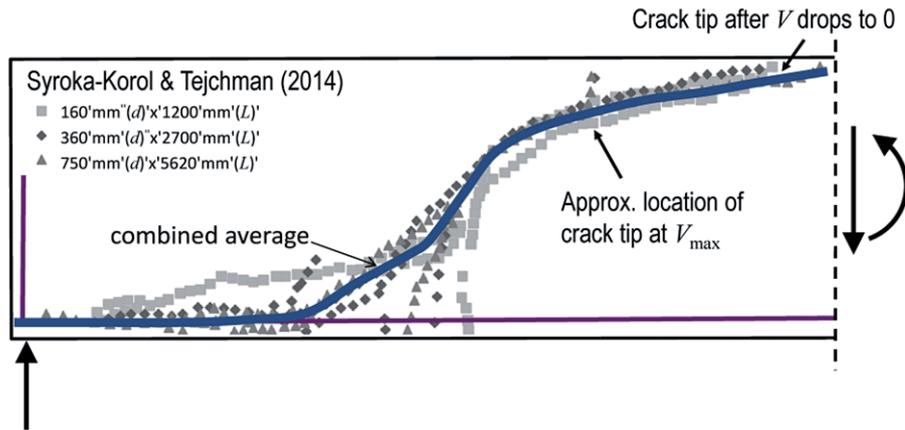


Fig. 7. Similarity of crack pattern documented in scaled beams of different sizes

some experiments for *Weibull* theory prior to 1980, exponent  $n$  was thought to be about 1/4 for concrete, although based on the current extensive calibration of the *Weibull* theory for concrete,  $n$  should be about 1/12, e.g. [30].

Today, however, it is clear that the *Weibull* statistics, implied by JSCE, do not apply to beam shear, and so the JSCE size effect has no valid theoretical foundation. The applicability of the *Weibull* size effect rests on the weakest-link model, which assumes the existence of many material elements (or representative volume elements, RVEs) such that the failure of only one of them triggers failure of the whole structure. The larger the structure, the more failure-triggering elements exist, and since the material strength is a random field, the strength of the weakest element decreases with the number of elements, and thus also with the size of the structure.

Such a situation exists in some unreinforced concrete structures, e.g. arch dams, but not in the shear failure of beams. The reason for this is that the location of the point of failure initiation is predetermined by mechanics. The failure initiates at the tip of a long diagonal shear crack that forms before maximum load. The relative location of this tip is almost fixed by mechanics, which dictates the crack path. In beams of different sizes, the path is almost geometrically similar. This fact was established by finite element fracture analysis and is confirmed by many experiments. *Iguero et al.* [20] in particular, whose tests included beams up to 3 m deep, observed that the crack patterns were similar in beams of all sizes. One clear experimental confirmation is furnished by the crack paths in Fig. 7 recently observed in geometrically scaled tests of beams of very different sizes [23].

The randomness of material strength cannot be the cause of size effect on the mean shear strength (although it surely affects the scatter). During the last decade, many reports appeared on the limits of the applicability of *Weibull* theory to quasi-brittle materials, e.g. [22]. For type 1 failures, which are those that occur upon macrocrack initiation from a smooth surface, the correct statistical distribution is a *Gauss-Weibull* graft, which is nearly Gaussian for small structures [22]. However, the shear failure of beams is of type 2, which is the failure case that occurs only after large stable crack growth. Since the crack path is determined by mechanics, the relative location of the dominant crack tip is almost fixed (Fig. 7), and

so the crack tip cannot randomly sample a large enough volume with many points of different random strength realizations. Therefore, the statistical randomness of material can have no effect on the mean strength, although it controls the coefficient of variation.

### 3.2 Size effect of *fib* Model Code 2010

Another kind of size effect curve was introduced into *fib* Model Code 2010 [2]. It replaced an earlier purely empirical curve introduced into the CEB code in 1990. The *fib* Model Code 2010 size effect curve (Eq. 7.3-19) has the following form:

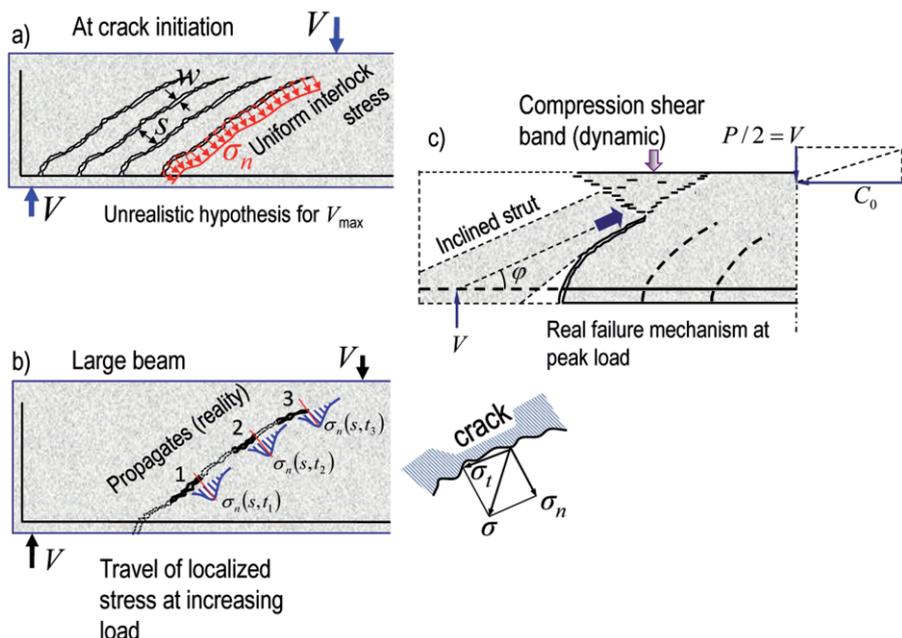
$$v_u = v_0 \theta_{MC} \quad (3)$$

$$\theta_{MC} = \frac{1}{1 + d/d_0}$$

where  $v_0$  and  $d_0$  are constants and  $\theta_{MC}$  is the size effect factor of *fib* Model Code 2010 (note that  $180/(1000 + 1.25z) = 0.18 \times [1/(1 + z/800)]$ ). The size dependence of  $\theta_{MC}$  is plotted in Fig. 6. Among all the formulae proposed so far, this curve gives the extreme size effect in relative terms.

The *fib* Model Code 2010 size effect factor (Eq. (3) above or Eq. 7.3-19 in the code) is unrealistic. This can be simply demonstrated, for example, by the asymptote of the curve of  $V_u = b_w d v_u$  versus  $d$  (here  $b_w$  = beam width). In contrast to all the other size effect curves, this asymptote is horizontal, featuring the only upper bound on  $V_u$  among all the size effect curves proposed so far. Thus, if a beam is sufficiently deep, the doubling of its depth, for example, would not lead to any increase in load capacity. Although such beam sizes are never reached in practice, this feature defies common sense, and also is questionable from the theoretical viewpoint.

The proper theoretical viewpoint is that of asymptotic matching, which is a sort of “interpolation” between the opposite asymptotes of a curve, in our case the curve of  $V_c$  as a function of  $d$  (the ACI-446 curve rests on such matching). Interpolation is always better than extrapolation, in this case from the initial asymptotic the curve of  $V_c(d)$ . The point is that the opposite asymptotes are obvious (in this case for plastic limit analysis and for fracture mechanics), but the transition between them, which is



**Fig. 8.** a) Interlock stress distribution assumed in MCFT; b) localization of bridging stress as diagonal cracks propagate; and c) shear stress transmitted through the inclined compression strut

what is needed for practice, is next to impossible to calculate (analytically, FEM excepted). That is why it is necessary to “interpolate” and why both the small-size and the large-size asymptotes ought to be respected.

Why has this basic problem with *fib* Model Code 2010 been overlooked? The reasons are twofold:

- 1) the size effect is normally plotted as the dependence of  $v_u$ , rather than  $V_u$ , on the size  $d$ , and
- 2) the plot of  $v_u$  versus  $d$  has habitually been presented in the linear (rather than logarithmic) scale of  $d$ , which cannot visually reveal the asymptotic behaviour and cannot show the quality of fit outside a narrow central range of sizes.

The size effect suggested by *fib* Model Code 2010 was conceived as a generalization of the so-called modified compression field theory (MCFT) which gives  $v_0$ . This is one theory that works well for small beams with  $d < 254$  mm (10 in.), for which the size effect is negligible ( $\theta_{MC} \approx 1$ ), and is acceptable for  $d < 508$  mm (20 in.).

But why did the generalization of MCFT lead to an incorrect size effect for larger beams? – Because the generalization rested on three oversimplified hypotheses [16]:

1. The first hypothesis was that the size effect was controlled by the spacing  $s_e$  of multiple parallel diagonal shear cracks. However, these cracks do not control failure; they form at crack initiation, well before reaching the maximum load and before their openings localize into one dominant diagonal crack.
2. The second hypothesis was a softening law for the parallel diagonal cracks [24], which was arbitrarily chosen on the basis of limited and, by now outdated, 1981 tests, in disregard of the softening law, now amply verified, for the shear and normal displacements in mixed-mode concrete fracture.
3. The third hypothesis was that the stress transmission across the diagonal shear cracks determines the shear

load capacity due to the concrete, in the manner pictured in Fig. 8a.

Later, finite element failure analysis [16], based on a realistic triaxial constitutive model for damage in concrete (the microplane model), contradicted these hypotheses. It revealed that, for large beams, the stress transmission across the diagonal crack plays a minor role at best. More specifically, it revealed that:

- a) The uniformity of distribution of the aggregate interlock stresses along the diagonal shear crack, implied in the MCFT generalization, is only true for very small beams ( $d < 254$  mm, 10 in.). For large beams, these stresses localize, and the localized peak travels along the diagonal crack as the crack is being opened under increasing load (Fig. 8b).
- b) In small beams the stresses transmitted across the diagonal crack at maximum load contribute about 40% of the shear force  $V_c$  carried by the concrete, and in deep beams only a small fraction of  $V_c$  – only 23% of  $V_c$  in the beam 1.89 m (74.4 in.) deep tested in Toronto [16]. This contribution drops below 10% for yet larger beams.
- c) Most of the shear force due to the concrete at maximum load is carried by an (imagined) inclined compression strut above the diagonal crack, loaded by inclined compression stresses transmitted across the ligament above the tip of the diagonal crack (Fig. 8c).

Ideally, of course, hypotheses 2 and 3 should be checked by experiments. Unfortunately, direct measurements of stresses across the crack are impossible. Therefore, the study in [14] resorted to finite element simulations. Although the results cannot be regarded as a perfect proof, their credibility rests on sound principles of fracture and damage mechanics, on calibrations with extensive material tests and on success in matching other measurable re-

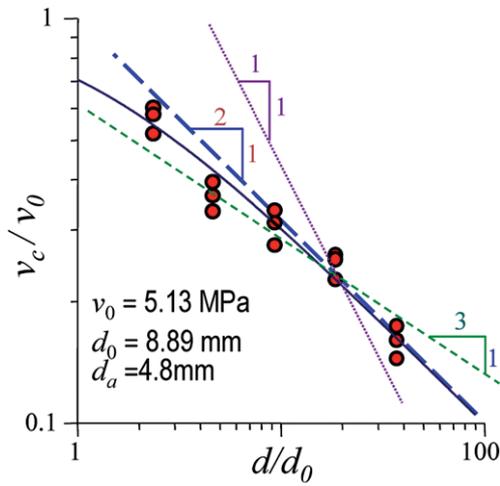


Fig. 9. Size effect tests of beam shear, meticulously scaled, conducted at Northwestern University and the fittings based on different size effects

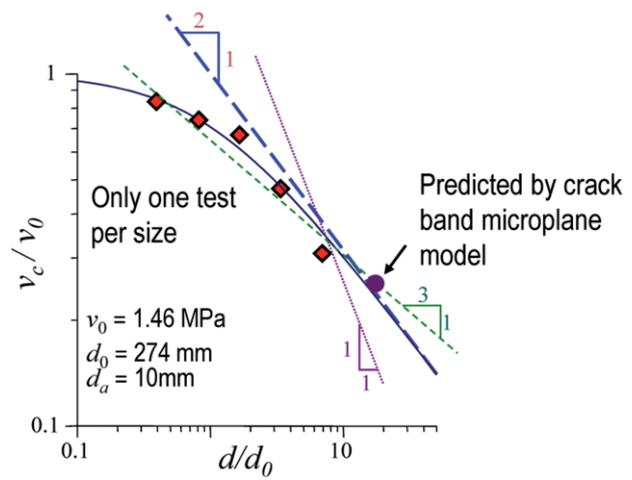


Fig. 10. Size effect tests of beam shear conducted at the University of Toronto and the fittings based on different size effects

sponses such as the peak loads and load-deflection diagrams.

### 3.3 Roles of stresses across main diagonal shear crack and along compression strut

Figs. 9 and 10 show comparisons of ACI-445 model (Fig. 6) with the main published data on the size effect on the shear strength due to concrete in reinforced concrete beams without stirrups [25]–[27]. The optimum values of  $D_0 = d_0$  are indicated and the best fits of other size effect curves are shown (note that, for beams with stirrups, the optimum values of  $d_0$  obtained from individual size effect tests are, according to a recent study [28], about 10 times larger, which pushes the size effect into extremely deep beams). These data have been used to calibrate a finite element program featuring microplane model M7 for concrete, an advanced and realistic constitutive law for fracturing damage in concrete (which is embedded in the commercial software ATENA). M7 was implemented in the UMAT user’s subroutine of ABAQUS and then used to check some assumptions.

For example, Fig. 11 shows the stress distributions calculated for small and large beams very similar to those tested at the University of Toronto. They confirm that in small beams the compressive strength of concrete at maximum load is mobilized throughout the whole ligament above the tip of the main diagonal crack, whereas in large

beams the stress is localized into only a part of that ligament. This provides a simple explanation of one source of the size effect. Also, in Fig. 10, the program with model M7, calibrated by the Toronto tests for five beam sizes, is used to compute an additional, sixth, point for a beam 5 m deep, as shown in Fig. 10. Note that this calculated point agrees well with the terminal slope of  $-1/2$  of the ACI-446 size effect at the large-size limit, Eq. (7), and differs from the terminal slope  $-1$  of the large size asymptote implied by *fib* Model Code 2010.

It might seem surprising that beams with stirrups should exhibit any size effect. However, the size effect must be expected whenever the peak stress (or peak load) is followed by progressive softening of the concrete (or structure) rather than a plastic plateau. Indeed, this must occur for normal stirrups because it was experimentally demonstrated in [29] that, to suppress the softening completely, concrete would have to be triaxially confined by steel with a volume equal to about 16% of the concrete’s volume, which is for design unacceptable.

### 4 ACI-446 size effect as a consequence of energy conservation and dimensional analysis

This size effect, labeled as ACI-446 Model (Fig. 6), was unanimously endorsed in 2007 by ACI committee 446 “Fracture Mechanics”. It is deterministic and is caused by the release of the strain energy in the structure during fail-

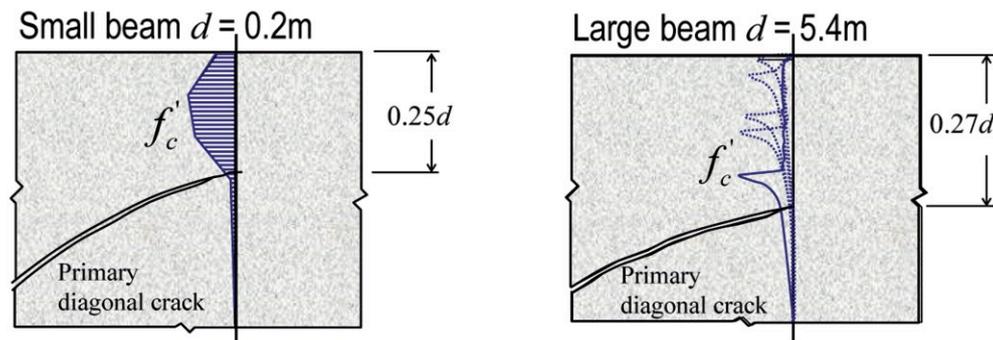


Fig. 11. Stress distributions at  $V_{max}$  along the ligament above the primary crack tip in small and large beams

ure (thus, it is also called the energetic size effect). It has been derived as a general consequence of fracture mechanics of quasi-brittle materials for the so-called Type 2 failures, in which the maximum load is reached only after long stable crack growth [7] (note that the Type 1 failures, which are those occurring immediately upon macrocrack initiation from a smooth surface, exhibit a different type of deterministic size effect and, for large sizes, they transit to the Weibull statistical size effect). Quasi-brittle failures are not only typical of reinforced concrete, but also of sea ice, fibre composites, tough ceramics, wood, rigid foams, rock masses and all other quasi-brittle materials. These are brittle heterogeneous materials in which the magnitude of the inhomogeneity and the fracture process zone size are not negligible compared with the structure dimensions in engineering applications. This is a salient characteristic of concrete structures and, in particular, of beam shear.

The simplest, yet completely general, derivation of the size effect in quasi-brittle fracture can be based solely on *dimensional analysis* and *energy conservation* (i.e., the *First Law* of thermodynamics) [6]. The total release of (complementary) strain energy  $W$  caused by fracture is a function of length  $a_0$  of the opened crack (or crack band) augmented by the size,  $w_c$ , of the fracture process zone (FPZ) in front of the crack. The fully developed FPZ size is essentially constant (a material constant), independent of  $d$ . The augmented (or effective) crack (or crack band) length is  $a = a_0 + w_c$ . Here  $w_c$  is a material constant approximately proportional to the maximum aggregate size  $d_a$  and depending also on the strength and stiffness of aggregates and on the interface properties. Geometric similarity means that the relative length of crack or crack band  $\alpha_0 = a_0/D = \text{constant}$  (dimensionless, size-independent).

According to dimensional analysis, the total (complementary) strain energy of the structure must have the form:

$$W = \frac{1}{2E} \left( \frac{P}{bD} \right)^2 bD^2 f \left( \alpha_0 + \frac{w_c}{D} \right) \quad (4)$$

where  $P$  is the maximum load (or ultimate load);  $f(\alpha)$  is a smooth dimensionless function of dimensionless variable  $\alpha = \alpha_0 + w_c/D$ , and  $b$  = width of fracture front (normally equal to beam width). Size  $D$  can approximately be taken either as the total beam depth  $h$  or as the depth  $d$  from the compression face to the tensile reinforcement centroid. The nominal strength  $\sigma_N$  is here represented by the average shear strength  $v_u = P/bD$ .

Regardless of the way by which the size effect equation is obtained (whether by fracture mechanics, strut-and-tie model, crack spacing hypotheses, concrete teeth, etc.), the energy conservation (or First Law of thermodynamics) during crack (or crack band) extension  $\Delta a$  requires that

$$\begin{aligned} \Delta W &= G_f b \Delta a \text{ or} \\ \frac{v_u^2}{2E} bD^2 \Delta f(\alpha) &= G_f b \Delta a, \Delta f(\alpha) = g(\alpha) \Delta \alpha = g(\alpha) \frac{\Delta a}{D} \end{aligned} \quad (5)$$

where

$$g(\alpha) = \left[ \frac{df(\alpha)}{d\alpha} \right]_{\alpha=\alpha_0+w_c/D} = g(\alpha_0 + w_c/D) \approx g_0 + g_1 \frac{w_c}{D} \quad (6)$$

Here  $g_0 = g(\alpha_0)$  and  $g_1 = [df(\alpha)/d\alpha]_{\alpha=\alpha_0}$  are constants if the structures of different sizes are geometrically similar; and  $G_f$  = fracture energy concrete (energy dissipated by fracture per unit length and unit width of crack or crack band, which is a fixed material property). Solving Eq. (5) for  $v_u$  and substituting Eq. (6), one obtains the deterministic (or energetic) size effect of ACI-446:

$$\begin{aligned} v_u &= v_0 \theta \\ \theta &= \frac{1}{\sqrt{1 + D/D_0}} \end{aligned} \quad (7)$$

where  $D_0 = w_c g_1 / g_0$ ,  $v_0 = \sqrt{2EG_f / g_1 w_c}$ . For geometrically similar structures,  $D_0$  is a constant (independent of structure size), called the transitional structure size. According to the fitting of the ACI 445-D beam shear database,  $D_0 \approx 200$  mm (7.9 in.). Factor  $\theta$  represents the ACI-446 size effect factor. The value of  $v_0$  is obtained by data fitting.

Note that one basic assumption of the foregoing derivation is that, approximately, the cracks at maximum load must be geometrically similar. The recent tests of *Syroka-Korol* and *Tejchman* [23] give an excellent confirmation of such similarity, see Fig. 7. Other confirmations are provided by realistic finite element simulations.

Eq. (7) represents an approximate size effect law that has been experimentally verified for many kinds of brittle failures of reinforced concrete structures as well as fibre composites, tough ceramics, rocks, sea ice, wood, foam, etc. In 2007, ACI-446 unanimously endorsed Eq. (7) for the shear capacity of concrete in the shear failure of beams.

Note that when  $D \ll D_0$ , there is virtually no size effect. This is the case of small laboratory beams ( $d \approx 0.25$  m), which can be treated by plastic limit analysis, for which there is no size effect. For the asymptotic case of very large beams (i.e.  $D \gg D_0$ ), Eq. (6) indicates that  $\sigma_N \propto 1/\sqrt{D}$ , which represents a straight-line of slope  $-1/2$  in the plot of  $\log \sigma_N$  (or  $\log v_u$ ) versus  $\log D$ . This asymptote gives the size effect of geometrically similar sharp cracks according to linear elastic fracture mechanics (LEFM), in which the FPZ is negligible compared to beam depth ( $w_c = 0$ ). This case lies generally beyond the range of most practical applications.

Since the fracture energy,  $G_f$ , is not relevant to the strength of very small beams (cca  $D \leq 0.25$  m),  $v_c$  should be evaluated by the best known method based on plastic limit analysis. The best choice of this method is not yet completely settled, but a point to note is that the size effect factor  $\theta$  can be applied to the  $v_c$  value calculated according to any such method (e.g. those of *Frosch*, *Hsu*, or MCFT). For example, according to the latest calibration in

ACI-446,  $V_0 = 12 b_w c \sqrt{f'_c}$ , where  $f'_c$  is in psi and  $V_0$  in lb.,

and  $b_w$  is the width of a rectangular cross-section or the web. Based on the idea of *Frosch* and *Wolf* [30],  $c$  is the depth to the neutral axis according to elastic analysis, i.e.

$c = d \sqrt{2\rho n + (\rho n)^2} - \rho n$ , which would replace the  $d$  value

used previously to calculate the shear force in the concrete. This replacement has the advantage that the  $c$  takes into account (approximately) the effect of axial force or prestress force (note that the flanges on the sides of the web must be ignored when calculating  $c$  [30]).

### 5 Meaningful and misleading statistical evaluations from a large database encompassing many concretes and laboratories

Committees of concrete societies nowadays try to compare various models for beam shear by using the statistics of prediction errors compared with a large database representing a collection of strength tests of beams made of all kinds of concrete tested in different laboratories. Meaningful comparisons, however, are a complex problem, with different overlapping trends and many random variables of widely different magnitudes of scatter, while the database is drawn from separate studies that had to be conducted without any coordinated scientific sampling scheme. In this kind of problem, the statistical inferences are tricky and can easily lead to misleading conclusions.

There has been a tendency to apply the statistics of data points to a problem that requires the statistics of trends, in our case the trend with respect to beam size (although the problem is the same for the trends with respect to shear span, steel ratio, etc.). What makes the comparisons particularly difficult is that (aside from relatively small experimental errors), random scatter types of very different magnitudes must be distinguished:

- 1) scatter due to differences among concretes,
- 2) scatter due to variations in beam size, and
- 3) scatter due to variations in shear span, steel ratio, concrete strength, aggregate size, type of prestress (if any), rate of loading, concrete age, etc.

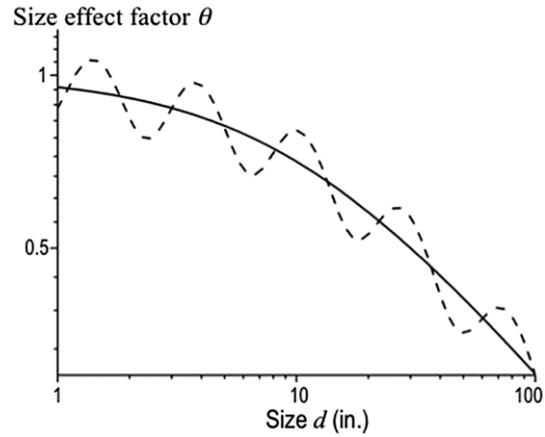


Fig. 12. Unperturbed size effect factor  $\theta$  of ACI-446 (solid curve) and size effect factor perturbed by nonsensical sinusoidal oscillation (dashed curve) (in the current ACI-318, the size effect factor is 1 for shear design)

Type 1 scatter happens to be an order of magnitude higher than the other scatters and thus covers them up, including the statistical trend of the size effect factor. Furthermore, the fact that the averages of  $a/d$ ,  $\rho_w$ , etc. vary significantly through the subsequent size intervals obfuscates the size effect trend. To make the cover-up conspicuous, Figs. 12 and 13 present an example comparing the latest joint database (beams without stirrups) of ACI committee 445D and the Deutscher Ausschuss für Stahlbeton (DAfStb) with a size effect factor that is deliberately made to be er-

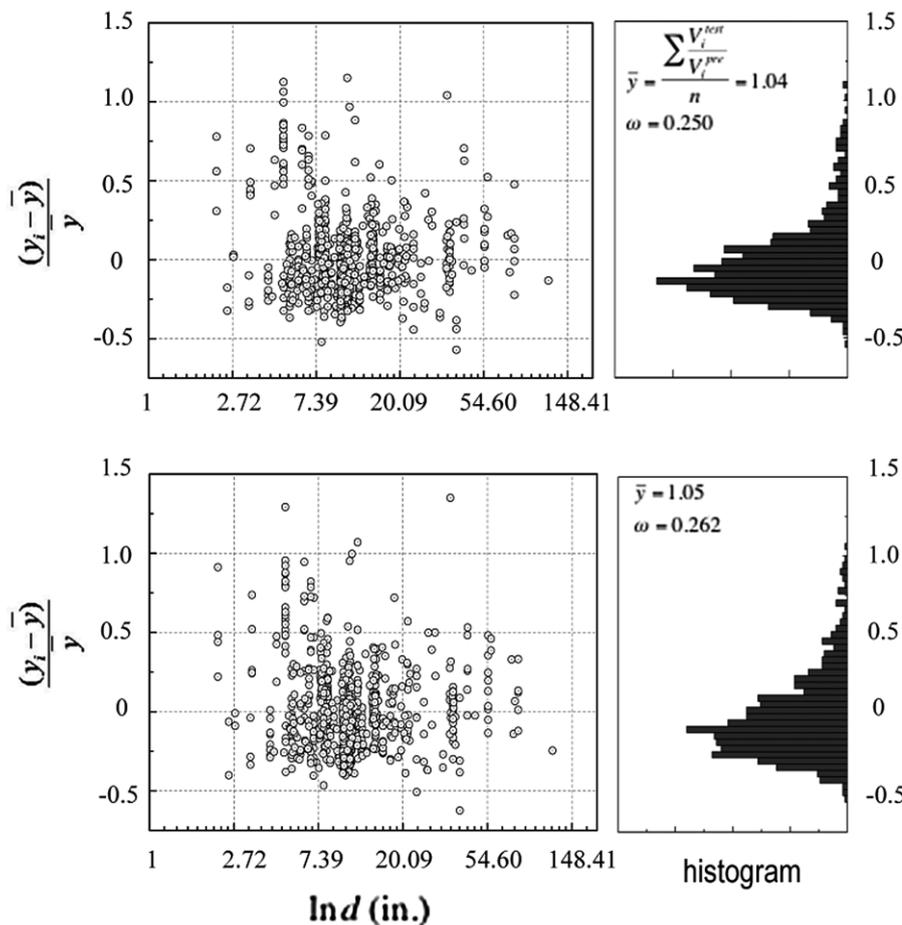


Fig. 13. Residuals (or errors) and their histograms of unperturbed (top) and perturbed (bottom) ACI-446 size effect factor compared to the ACI-445D database.

roneously perturbed. It was checked whether this error would be detected by the usual point-based statistics. Fig. 13 compares two kinds of predictions:

1. Fig. 13 (top) shows the plot of errors (or residuals) of the predictions according to the beam shear equation of Hsu et al., which uses the ACI-446 size effect factor  $\theta$ , Eq. (7). The statistics are calculated using the ACI-445D method (which uses uniform weights).
2. Fig. 13 (bottom) shows the plot of residuals calculated when the ACI-446 size effect factor  $\theta$  is deliberately perturbed by  $\theta + \Delta\theta$ , where  $\Delta\theta$  is a periodic perturbation expressed as

$$\Delta\theta = \frac{0.14}{\sqrt{1 + d/d_0}} \cos[2\pi(\ln_{e1} d - s)] \quad (8)$$

see the curves in Fig. 12, in which  $d_0 = 254$  mm (10 in.). Parameter  $s$  is a random phase shift that must be introduced to prevent bias due to the fact that the inflexion points of the cosine curve, which are placed arbitrarily, have no perturbation while all the others do. The phase shift  $s$  varies randomly with uniform probability between  $-0.5$  and  $+0.5$  (as obtained from a random number generator). These random shifts eliminate any evaluator's bias so that no point on the curve of size effect factor  $\theta$  would be favoured over the others. It should be noted, though, that if the shifts  $s$  were omitted, i.e. if  $s$  were fixed as zero, the resulting change in the coefficient of variation (C.o.V.) of errors would be about the same. The oscillating perturbation is considered to have amplitude of 0.14, which means that the spread between maxima and minima caused by this intentional error is 0.28.

The question is whether the ACI-445D statistics can detect this deliberately erroneous perturbation.

In the current ACI-445 statistics, the prediction is evaluated based on the ratio of the test value  $V_{test}$  to the predicted one  $V_{pre}$ . The errors (or residuals) of the predictions of data with the unperturbed and perturbed size effect factors are plotted as functions of beam size on a semi-logarithmic scale ( $\ln d$ ) in Fig. 13 for all the data points in the ACI-445D database. The histograms of the errors are plotted on the right, again for both the unperturbed and perturbed cases. The perturbation is found to change the C.o.V. of the errors (root-mean-square error divided by the data mean) from 0.250 to 0.262, which is  $< 5\%$  of the C.o.V.

It is now obvious that both plots of the residuals, both histograms and both C.o.V.s are almost the same. Evidently, the ACI-445D statistical method cannot distinguish an erroneously perturbed model from a realistic one. How can it then be trusted to rank various proposed models?

The assessment and ranking of various models clearly requires that the scatters of type 2 and 3 be taken into account. The shape of the curve of  $\log v_u$  versus  $\log D$  must be checked by individual sets of data for the same concrete and the same laboratory before anything else. It is, likewise, important to check separately the trends of the type 3 effects.

Consequently, the proposed size effect factor must first be shown to be capable of fitting closely the individual test series on the same concrete and from the same lab-

oratory. Subsequently, the proposed equation for beam shear strength needs to be compared with the individual tests and the C.o.V. computed separately for each. All these coefficients of variation then need to be combined (in a root-mean-square manner) into one overall coefficient of variation or errors of the proposed beam shear equation. Alternatively, the overall coefficient of variation can be extracted from the multivariate optimization of the database fitted with an algorithm such as that of *Levenberg-Marquardt*, in which all the parameters are varied simultaneously.

As the available data cover only parts of the size range, it is important to ensure that the size effect trend is supported by a sound theory and that the theory is validated by experiments. This is crucial for extrapolating the size effect to large sizes. Currently, for beams without stirrups, 87% of the available tests pertain to  $d < 508$  mm (20 in.), 97% to  $d < 1.27$  m (50 in.) and none to  $d > 3048$  mm (10 ft.), while, in practice, beams of depths up to 14.2 m (46 ft.) have been built.

## 6 Conclusions and closing remarks

1. For small beam sizes, the plastic limit analysis is applicable and yields a small-size estimate of the shear strength  $v_u$  due to the concrete. The size effect factor  $\theta$  should be applied to the  $v_u$  value obtained by the best plastic limit analysis (or strut-and-tie) model.
2. Unfortunately, the statistical evaluation of a beam shear formula must deal with a heteroscedastic database that has various kinds of strong bias, mainly:
  - a) the data are crowded for small sizes, very scant for large sizes and non-existent for the largest sizes used in practice, and
  - b) the average shear span and reinforcement ratio of the tests in the database systematically decrease through subsequent size intervals.
 The bias must be filtered by statistical weights. The filtering demonstrates that, for large sizes, the size effect factor is  $\sigma_N \propto 1/\sqrt{d}$  (which represents the main resolution of ACI-446).
3. The JSCE *Weibull*-type power law (adopted when no other size effect theory was known) is theoretically unjustified and does not agree with the data subsequently accumulated.
4. The size effect in *fib* Model Code 2010 is an extreme case that is theoretically and experimentally unjustified for two reasons:
  - a) the increase in the shear force  $V_u$  with beam size terminates with a horizontal asymptote, which is an overlooked unphysical property contradicted by tests (and even common sense), and
  - b) its derivation rests on oversimplified hypotheses based on the crack initiation state rather than the peak load state.

The shear force is assumed to be transmitted by aggregate interlock distributed uniformly along uniformly spaced diagonal cracks, rather than localizing into a single dominant crack and also within the length of this crack as the size increases. Ignored here is the fact that, in large beams, most of the shear force is transmitted by an inclined compression strut above the main

crack in which the failure localizes as the size increases, thus causing the deterministic size effect. The form of the softening law for mixed-mode shear and normal displacements across the parallel cracks, formed well below the peak load, is unjustified. It conflicts with the law known from extensive fracture studies.

5. Dimensional analysis and the requirement of energy conservation during crack formation inevitably lead to the size effect factor of ACI-446.
6. Evaluating different beam shear equations according to the coefficient of variation of the scatter of prediction errors (residuals) compared with the database points is fruitless and misleading. The reasons are that the scatter due to differences between different concretes and testing laboratories, and that the systematic variation in the average values of  $a/d$ ,  $\rho_w$  in subsequent size intervals, cover up the size effect trend. The former kind of cover-up is demonstrated by insensitivity to a large intentional sinusoidal perturbation of the size effect factor. If the statistical method cannot discern such an unrealistic perturbation, it cannot be used to compare different beam shear models.
7. It should also be noted that the deterministic size effect can, of course, be taken into account automatically when the structure is analysed by finite elements based on a realistic constitutive model and damage fracture concepts with a localization limiter (such a limiter is, for example, automatically featured in the widely used crack band model that is embedded in programs such as ATENA, SBETA, OOFEM and DIANA, and is also easily implemented in the user's material subroutine of other programs such as UMAT or VUMAT in ABAQUS). The designs of large sensitive structures are increasingly subjected to checks by such finite element analysis. In such cases, a high degree of safety and design efficiency is likely to be achieved even if the design code features a wrong size effect or none. Nevertheless, this is not yet the standard practice, and so embedding the size effect (of the correct form) in the design code is important.
8. Even if the design safety is checked by finite element analysis taking into account material uncertainty, embedment of the correct size effect in the design code is important for two other reasons:
  - a) to achieve an economic design, and
  - b) to allow a creative designer to exploit freely the true capacity of the material in daring new structural forms.

As long as the size effect is incorrect, the code will, a priori, exclude some innovative, large and daring structural designs from consideration even if they are safe and can pass the detailed safety check by finite elements.

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