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Spectral analysis of localization in nonlocal and over-nonlocal materials with softening plasticity or damage

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Abstract

The paper shows that spectral wave propagation analysis reveals in a simple and clear manner the effectiveness of various regularization techniques for softening materials, i.e., materials for which the yield limits soften as a function of the total strain. Both plasticity and damage models are considered. It is verified analytically in a simple way that the nonlocal integral-type model with degrading yield limit depending on the total strain works correctly if and only one adopts an unconventional nonlocal formulation introduced in 1994 by Vermeer and Brinkgreve (and in 1996 by Planas, and by Strömberg and Ristinmaa), which is here called, for the sake of brevity, 'over-nonlocal' because it uses a linear combination of local and nonlocal variables in which a negative weight imposed on the local variable is compensated by assigning to the nonlocal variable weight greater than 1 (this is equivalent to a nonlocal variable with a smooth positive weight function of total weight greater than 1, normalized by superposing a negative delta-function spike at the center). The spectral approach readily confirms that the nonlocal integral-type generalization of softening plasticity with an additive format gives correct localization properties only if an over-nonlocal formulation is adopted. By contrast, the nonlocal integral-type generalization of softening plasticity with a multiplicative format provides realistic localization behavior, just like the nonlocal integral-type damage model, and thus does not necessitate an over-nonlocal formulation. The localization behavior of explicit and implicit gradient-type models is also analyzed. A simple analysis shows that plasticity and damage models with gradient-type localization limiter, whether explicit or implicit, have very different localization behaviors.

Keywords: Softening; Plasticity; Damage; Nonlocal models; Gradient models; Spectral analysis; Localization; Harmonic waves

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1. Introduction

Already in the mid 1970s it was recognized that a continuum formulation for softening materials leads to serious problems: the boundary value problem becomes ill-posed, and the numerical calculations cease to be objective, exhibiting pathological spurious mesh sensitivity, incorrect size effect, and excessive damage localization as the mesh is refined (Bažant, 1976). In dynamics, the differential equations of motion become parabolic or elliptic when the material softens. Thus, the initial value problem becomes ill-posed as the wave velocity becomes imaginary (Hadamard, 1903). By contrast, in statics the elliptic partial differential equations of equilibrium become hyperbolic or parabolic. This fact was formally demonstrated for the linearized rate equilibrium problem by using the method of linear comparison solid (Hill, 1958). To recover a well-posed problem and to prevent the damage from localizing into a zone of zero volume, the continuum theory must be complemented by certain conditions called 'localization limiters', involving a characteristic length of the material (Bažant, 1976; Bažant and Oh, 1983; Bažant et al., 1984; Bažant and Belytschko, 1985; Lasry and Belytschko, 1988).

A broad class of localization limiters is based on the concept of a *nonlocal continuum*. The nonlocal concept was introduced in the 1960s (Eringen, 1966; Kröner, 1967; etc.) for elastic deformations and in the 1980s expanded to hardening plasticity. A nonlocal continuum is a continuum in which the stress at a point depends not only on the strain at that point but also on the strain field in the neighborhood of that point. Bažant (1984) and Bažant et al. (1984) introduced the nonlocal concept as a localization limiter for a strain-softening material. This formulation was later improved in the form of the nonlocal damage theory (Pijaudier-Cabot and Bažant, 1987) and was applied in real-life problems (Bažant and Lin, 1988; Bažant and Lin, 1989; Saouridis and Mazars, 1992). To improve the performance in large post-peak deformations, Vermeer and Brinkgreve (1994), Planas et al. (1996) and Strömberg and Ristinmaa (1996) proposed a novel formulation in which a nonlocal averaging integral with a weight function everywhere positive is modified by multiplying the weight function by a factor greater than 1 and compensating for it by adding at the center a negative delta-function spike, equivalent to subtracting the local variable (this is unlike applying the common rule of mixture because the nonlocal component in the 'binary mixture' of the local and nonlocal variables has a weight greater than 1, compensated by a negative weight for the other component). For the sake of brevity, this formulation is here called *over-nonlocal*.

Another well-known regularizing technique is based on an explicit second-order gradient model, which is weakly nonlocal (Aifantis, 1984; Zbib and Aifantis, 1988; Lasry and Belytschko, 1988; Mühlhaus and Aifantis, 1991; Vardoulakis and Aifantis, 1991; de Borst and Mühlhaus, 1992; Pamin, 1994). More recently, Peerlings et al., 1996a,b developed an effective (strongly nonlocal) modification of the gradient approach in which the nonlocal strain is solved from a Helmholtz differential equation.

The present study will first focus on the localization problem for a constitutive model in which the softening law is a function of the total strain. Behavior of this model in one-dimensional wave propagation will be studied analytically considering this kind of strain-softening material. It will be shown that, if an over-nonlocal model of integral type is used, a complete regularization of the softening problem is achieved only if an "over-nonlocal" formulation is adopted. This result will also be demonstrated by numerical analysis of the localization problem in a one-dimensional bar. After that, the over-nonlocal model of integral type will be applied to nonlocal plasticity and to nonlocal damage. In closing, the explicit and implicit gradient enhancement techniques will be applied to these constitutive laws to clarify the relation between the nonlocal models and the gradient models. Dispersion analysis will be used to show that the nonlocal integral-type generalization of classical softening plasticity model with an additive format of the softening law exhibits realistic localization properties only if an over-nonlocal formulation is considered. By contrast, the nonlocal integral-type generalization of softening plasticity with a softening–hardening law written in the multiplicative format (ductile damage model of Geers et al., 2001; Engelen et al., 2003) provides realistic localization behavior, just like the nonlocal integral-type damage model, and thus does not necessitate an over-nonlocal formulation.

The approach followed in this study has been pursued in several previous studies. For a gradient plasticity model, wave propagation and dispersion were investigated by Sluys (1992) and de Borst et al. (1995). For the damage model regularized by a nonlocal formulation or by gradient-type enhancement, a dispersion analysis was conducted by Peerlings et al., 1996a,b, Askes et al. (2000), Comi and Rizzi (2000), Peerlings et al. (2001), Askes and Sluys (2002), and Peerlings et al. (2002). Other noteworthy studies on the nonlocal plasticity model have been presented by Bažant and Jirásek (2002), Jirásek and Rolsoven (2003), and Rolshoven (2003).

Even though this study of softening constitutive models taken from the literature is focused on the localization problem in one-dimension, it does capture the essential aspects of localization in the direction across a damage band, as shown in many previous studies limited also to one-dimensional analysis. The simplicity of the one-dimensional approach makes it possible to compare the softening behavior of plasticity and damage models regularized by nonlocal averaging or by gradient enhancement. The transparency of such an approach is very useful for understanding what kind of softening model describes in an objective and realistic way the initial bifurcation and the subsequent response up to complete failure.

2. Basic concept of nonlocal formulation

The nonlocal model, in general, consists in replacing a certain local variable $f(\mathbf{x})$, characterizing the softening damage of material, by its nonlocal counterpart $\bar{f}(\mathbf{x})$. The nonlocal variable is defined as

$$\bar{f}(\mathbf{x}) = \int_V \alpha^*(\mathbf{x}, \xi) f(\xi) \mathrm{d}V(\xi) \tag{1}$$

where V is the volume of the structure, $\mathbf{x}$, ξ are the coordinates vectors, and $\alpha^*(\mathbf{x}, \xi)$ is the normalized nonlocal weight function defined as

$$\alpha^*(\mathbf{x}, \xi) = \frac{\alpha(\xi - \mathbf{x})}{\int_V \alpha(\xi - \mathbf{x}) \mathrm{d}V(\xi)} \tag{2}$$

in which $\alpha(\mathbf{x} - \xi)$ is the basic nonlocal weight function for an unbounded medium; $\int_V \alpha(\mathbf{x} - \xi)\mathrm{d}V(\xi)$ is a constant if the unrestricted averaging domain does not tend to protrude outside the boundaries. The weight function α $(\mathbf{x} - \xi)$ is often taken as a bell-shaped function, the simplest expression of which is

$$\alpha(\mathbf{x}, \xi) = \begin{cases} (1 - |\xi - \mathbf{x}|^2/R^2)^2 & \text{if } 0 \leqslant |\xi - \mathbf{x}| \leqslant R \\ 0 & \text{if } R \leqslant |\xi - \mathbf{x}| \end{cases} \tag{3}$$

where R, called the interaction radius, is proportional to the material characteristic length ℓ, $R = \rho_0 \ell$, $|x - \xi|^2 = (x_i - \xi_i)^2$ and $\alpha(\mathbf{x}, \mathbf{x}) = 1$. The coefficient ρ_0 is determined so that the volume under function α is equal to the volume of the uniform distribution over radius R. Another convenient bell-shaped weight function is the Gauss' error function

$$\alpha(\mathbf{x}, \xi) = \exp\left(-\pi \frac{|\xi - \mathbf{x}|^2}{\ell^2}\right) \tag{4}$$

which has unbounded support (its interaction radius is $R = \infty$). The bell-shaped function (3) is often used in the numerical applications since it has a limited support (its interaction radius is $R = \rho_0 \ell$). On the other hand, Gauss' error function is often convenient for analytical solutions.

Originally Vermeer and Brinkgreve (1994), and later Planas et al. (1996), and Strömberg and Ristinmaa (1996) (see also Bažant and Planas, 1998), introduced a refinement of the standard nonlocal formulation in the form of an over-nonlocal formulation in which the local and the nonlocal variables are linearly combined as follows:

$$\hat{f}(\mathbf{x}) = m\bar{f}(\mathbf{x}) + (1-m)f(\mathbf{x}) \tag{5}$$

here $\hat{f}(\mathbf{x})$ is the over-nonlocal average of the variable $f(\mathbf{x})$, $\bar{f}(\mathbf{x})$ is the nonlocal variable obtained from Eq. (1), and m is an empirical coefficient (over-nonlocal parameter). Strömberg and Ristinmaa (1996) called it the 'mixed local and nonlocal model'. Planas et al. (1996) called this formulation a 'nonlocal model of the second kind'. The previous studies of this formulation, concerned with different constitutive laws, confirmed that spurious localization is avoided if $m > 1$. For a uniaxial stress field, Planas et al. (1996) rigorously proved that the localization zone is finite if and only if $m > 1$. It was also proven (Bažant and Planas, 1998) that, for uniaxial stress, the formulation with $m > 1$ is equivalent, in terms of strain rate, to the nonlocal damage model of Pijaudier-Cabot and Bažant (1987). The refinement of Eq. (5) can also be obtained by rewriting the normalized nonlocal weight function, Eq. (2), in the following way:

$$\alpha_m^*(\mathbf{x},\boldsymbol{\xi}) = (1-m)\delta(\boldsymbol{\xi}-\mathbf{x}) + m\frac{\alpha(\boldsymbol{\xi}-\mathbf{x})}{\int_V \alpha(\boldsymbol{\xi}-\mathbf{x})\mathrm{d}V(\boldsymbol{\xi})} \tag{6}$$

where δ denotes the Dirac delta function; m is here called the *over-nonlocal* parameter, because $m > 1$ assigns to the nonlocal averaging integral in the foregoing equation excess weight compared to the standard nonlocal formulation, in which $m = 1$. Thus, the over-nonlocal operator can be defined as

$$\hat{f}(\mathbf{x}) = \int_V \alpha_m^*(\mathbf{x},\xi)f(\xi)\mathrm{d}V(\xi) \tag{7}$$

in which the subscript of α indicates that averaging operator depends on m.

3. Localization analysis of nonlocal integral-type model with strain-dependent yield limit

The localization analysis of a strain-softening nonlocal plasticity model, in which the yield limit depends on the total strains, will now be considered in the dynamic context. The capability of a model of this kind to maintain the well-posedness of the problem may be investigated by examining the propagation speed of stress acceleration waves. We must ensure that the propagation speed would not become imaginary. The wave propagation in local and nonlocal strain-softening materials was solved by Bažant and Chang (1984) and Bažant and Belytschko (1985) (see also Bažant and Cedolin, 1991, Sec. 13.1). Here we will follow the approach of, among others, Lasry and Belytschko (1988), Pijaudier-Cabot and Benallal (1993), Comi and Rizzi (2000), and Borino et al. (2003). Consider a one-dimensional continuum model for the localization zone defined as

$$\sigma(\hat{\epsilon}) = E_0\hat{\epsilon}\mathrm{e}^{-\hat{\epsilon}/\epsilon_1} \tag{8}$$

where E_0 is (undamaged) Young's modulus, $\hat{\epsilon}$ is the over-nonlocal strain and ϵ_1 is a material parameter representing the strain at the stress peak. Outside the localization zone the material is unloading elastically, i.e., $\dot{\sigma}(\epsilon) = E_0\dot{\epsilon}$, where ϵ is the local strain and the superior dot denotes the time rate. In the following, only the post-peak behavior is considered, i.e., the strain $\epsilon(x)$ is assumed to be larger than the strain at the peak stress ϵ_1. The integral-type over-nonlocal model, in which the over-nonlocal strain is given by Eq. (7), may be written for a one-dimensional bar as

$$\hat{\epsilon}(x) = \int_L \alpha_m^*(\xi - x)\epsilon(\xi)\mathrm{d}\xi \tag{9}$$

where, for an unbounded solid and for the Gaussian weight function, we have $\int_{-\infty}^{+\infty} \alpha(z)\mathrm{d}z = \ell$. Let us consider the one-dimensional equation of motion linearized around an initial strained homogeneous state $\hat{\epsilon}(x) = \epsilon(x) = \epsilon_0 = \text{const.}$;

$$\dot{\sigma}_{,x} = \rho \dot{u}_{,tt} \tag{10}$$

where $\dot{u}$ is the time rate of the incremental displacement u perturbing the initial state, and ρ is the mass density of the bar. The relation between the rate of deformation and the time derivative of the incremental displacement rate is

$$\dot{\epsilon} = \dot{u}_{,x} \tag{11}$$

We assume that all the material is in a softening state (this is a classical assumption in the analysis of localization). An elastic solid with the corresponding incremental (tangential) stiffness tensor is called the ''linear comparison solid'' (Hill, 1958). Its non-trivial equilibrium solution (different from the initial homogeneous state) yields bifurcation points in statics. Starting from a homogeneous state, and assuming the solid to be so large that the boundary effect is negligible, we consider a harmonic wave of frequency ω and wave number k propagating through a very long bar with a velocity that is the real part of

$$\dot{u}(x,t) = \dot{u}_0 \mathrm{e}^{\mathrm{i}(kx - \omega t)} \tag{12}$$

where i = imaginary unit. The time derivatives of Eqs. (9) and (8) are

$$\dot{\hat{\epsilon}}(x) = \int_L \alpha_m^*(\xi - x)\dot{u}_{,x}(\xi)\mathrm{d}\xi \tag{13}$$

$$\dot{\sigma}(\hat{\epsilon}) = E_0\,\mathrm{e}^{-\epsilon_0/\epsilon_1}\left(1 - \frac{\epsilon_0}{\epsilon_1}\right)\dot{\hat{\epsilon}} \tag{14}$$

where $\alpha_m(\xi - x) = (1 - m)\delta(\xi - x) + m\alpha\,(\xi - x)/l$ and $\delta(\xi - x)$ is the Dirac delta function. The derivative of Eq. (14) with respect to the spatial coordinate x is

$$\dot{\sigma}(\hat{\epsilon})_{,x} = E_0\,\mathrm{e}^{-\epsilon_0/\epsilon_1}\left(1 - \frac{\epsilon_0}{\epsilon_1}\right)\dot{\hat{\epsilon}}_{,x} \tag{15}$$

Substitution of the derivative of Eq. (13) with respect to x into Eq. (15), together with the linearized equation of motion (Eq. (10)), yields

$$E_0\mathrm{e}^{-\epsilon_0/\epsilon_1}\left(1 - \frac{\epsilon_0}{\epsilon_1}\right)\frac{\partial}{\partial x}\left\{(1 - m)\dot{u}_{,x} + \frac{m}{\ell}\int_L \alpha(\xi - x)\dot{u}_{,x}(\xi)\mathrm{d}\xi\right\} = \rho\dot{u}_{,tt} \tag{16}$$

Substitution of Eq. (12) and its partial derivatives with respect to x and t into Eq. (16) provides

$$\left\{E_0\mathrm{e}^{-\epsilon_0/\epsilon_1}\left(1 - \frac{\epsilon_0}{\epsilon_1}\right)k^2\left[1 - m + \frac{m}{\ell}A(k)\right] - \rho\omega^2\right\}\dot{u}_0\,\mathrm{e}^{\mathrm{i}(kx - \omega t)} = 0 \tag{17}$$

where

$$A(k) = \int_L \alpha(z)\mathrm{e}^{\mathrm{i}kz}\mathrm{d}z \tag{18}$$

For an infinite bar ($L \to \infty$), $A(k)$ is the Fourier transform of the weight function $\alpha(z)$ (this property was exploited in Bažant and Chang (1984) for comparing various weight functions; see also Bažant and Cedolin, 1991, Eqs. 13.10.10–13.10.14). Eq. (17) must be verified at every point x and for every instant t. This condition yields the dependence of the phase velocity, c_p, on the wave number k. The angular frequency for this kind of nonlocal integral model is

$$\omega = c_e k \left[e^{-\epsilon_0/\epsilon_1} \left(1 - \frac{\epsilon_0}{\epsilon_1}\right) \left(1 + \frac{m}{\ell} A(k) - m\right) \right]^{1/2} \tag{19}$$

where $c_e = \sqrt{E_0/\rho}$ is the elastic propagation velocity in the bar. Therefore, the phase velocity, $c_p = \omega/k$, and the group velocity, $c = \omega_{,k}$, are given by

$$c_p = c_e \left[e^{-\epsilon_0/\epsilon_1} \left(1 - \frac{\epsilon_0}{\epsilon_1}\right) \left(1 + \frac{m}{\ell} A(k) - m\right) \right]^{1/2} \tag{20}$$

$$c = \frac{\omega}{k} + c_e k \frac{m}{\ell} \sqrt{e^{-\epsilon_0/\epsilon_1} \left(1 - \frac{\epsilon_0}{\epsilon_1}\right)} A(k)_{,k} \left(1 + \frac{m}{\ell} A(k) - m\right)^{-1/2} \tag{21}$$

Because the phase velocity is function of the wave number k (Eq. (20)), the waves are dispersive. Let us consider the Gauss weight function (Eq. (4)) $\alpha(z) = \exp(-\pi \ z^2/\ell^2)$, for which $\alpha(0) = 1$ and $\int_{-\infty}^{+\infty} \alpha(z) dz = \ell$. Because $\int_{-\infty}^{+\infty} \exp(-z^2) dz = \sqrt{\pi}$, the Fourier transform (Eq. (18)) is $A(k) = \ell e^{-k^2\ell^2/4\pi}$. Substituting this value of $A(k)$ into Eqs. (19) and (21), we get the following expressions for the angular frequency and group velocity:

$$\omega = c_e k \left\{ e^{-\epsilon_0/\epsilon_1} \left(1 - \frac{\epsilon_0}{\epsilon_1}\right) \left[1 + m\left(e^{-\frac{k^2\ell^2}{4\pi}} - 1\right)\right] \right\}^{1/2} \tag{22}$$

$$c = \frac{\omega}{k} + c_e k m \sqrt{e^{-\epsilon_0/\epsilon_1} \left(1 - \frac{\epsilon_0}{\epsilon_1}\right)} \left(-\frac{k\ell}{2\pi}\right) e^{-\frac{k^2\ell^2}{4\pi}} \left[1 + m\left(e^{-\frac{k^2\ell^2}{4\pi}} - 1\right)\right]^{-1/2} \tag{23}$$

Note that, for $m = 0$ or for $\ell \to 0$ (local strain-softening model), the angular frequency, phase velocity and group velocity are not functions of the wave number k. This means that the waves are non-dissipative: the model is not able to change the shape of an arbitrary loading wave into a stationary wave reproducing the localization band.

The phase velocity, $c_p = \omega/k$, and the angular frequency are real if the wave number is such that the term under the square root in the following equation is non-negative

$$c_p = c_e \left\{ e^{-\epsilon_0/\epsilon_1} \left(1 - \frac{\epsilon_0}{\epsilon_1}\right) \left[1 + m\left(e^{-\frac{k^2\ell^2}{4\pi}} - 1\right)\right] \right\}^{1/2} \tag{24}$$

For $m \leqslant 1$, the term under the square root in Eq. (24) is negative. This means that the phase velocity remains real and positive only if $m > 1$ (precisely, for finite k, $m > C$, where C is a constant larger than 1). For $m \leqslant 1$, we have a situation in which the stability in the sense of Lyapunov is lost: a small disturbance $\dot{u}$ results in an unbounded response (more precisely, given any positive bound, no small enough positive disturbance exists such that the response would remain within the bound). Thus, in the case of the standard nonlocality, $m = 1$, regularization is not achieved.

Let us find the critical wave number, k_{cr}, that causes the phase velocity and the angular frequency to vanish ($c_p = \omega = 0$) (in other words, causes a static bifurcation);

$$k_{\mathrm{cr}} = \frac{2\sqrt{\pi}}{\ell}\sqrt{\ln\left(\frac{m}{m-1}\right)} \tag{25}$$

The corresponding critical wavelength is

$$\lambda_{\mathrm{cr}} = 2\pi/k_{\mathrm{cr}} = \ell\sqrt{\pi}\left[\ln\left(\frac{m}{m-1}\right)\right]^{-1/2} \tag{26}$$

As we can see from Eq. (25), the critical wave number has a real value only if $m > 1$. Eqs. (25) and (26) are plotted in Fig. 3 for $\ell = 25$ mm. Only the waves with wavelength $\lambda \leqslant \lambda_{\mathrm{cr}}$ do propagate. If $k < k_{\mathrm{cr}}$ or $\lambda > \lambda_{\mathrm{cr}}$ we obtain a situation where a small disturbance $\dot{u}$ leads to unbounded response and therefore is not a stable state in the sense of Lyapunov. The waves of wavelengths larger than λ_{cr} cannot propagate because they fit within the strain-softening region. Due to the material dissipative behavior, the high frequencies are damped, and this leads to a stationary harmonic localization wave of wavelength λ_{cr}. Thus the critical wavelength is related to the internal localization length. This internal length, which is an increasing function of the parameter m, is defined only for $m > 1$. For $0 \leqslant m \leqslant 1$, the internal length, λ_{cr} given by Eq. (26) has an imaginary value. Therefore, the integral nonlocal model, in which the stress in the softening region depends on the nonlocal strain, can yield a complete regularization only if $m > 1$.

As a numerical example, we assume, for the nonlocal constitutive law in Eq. (8), the following material data: $E = 30{,}000$ N/mm^2, $\rho = 2.5 \times 10^{-8}$ Ns^2/mm^4, $\hat{\epsilon} = 0.001$, $\epsilon_1 = 0.0001$, $\ell = 25$ mm and $m = 2$, which yield the linear elastic wave speed $c_{\mathrm{e}} = 1095$ m/s. For these values of the parameters, Fig. 1 shows the plots of the angular frequency as a function of the wave number k, and Fig. 2 the group velocity c ($c = \omega_{,k}$) and the phase velocity c_{p} as a function of the wave number.

The over-nonlocal constitutive law was then implemented in a finite element code. A one-dimensional bar of length 250 mm and cross-section area of 625 mm^2 was discretized by 100 bar elements with two nodes each. A weak element is inserted in the middle of the bar. Fig. 4 shows the strain distributions along

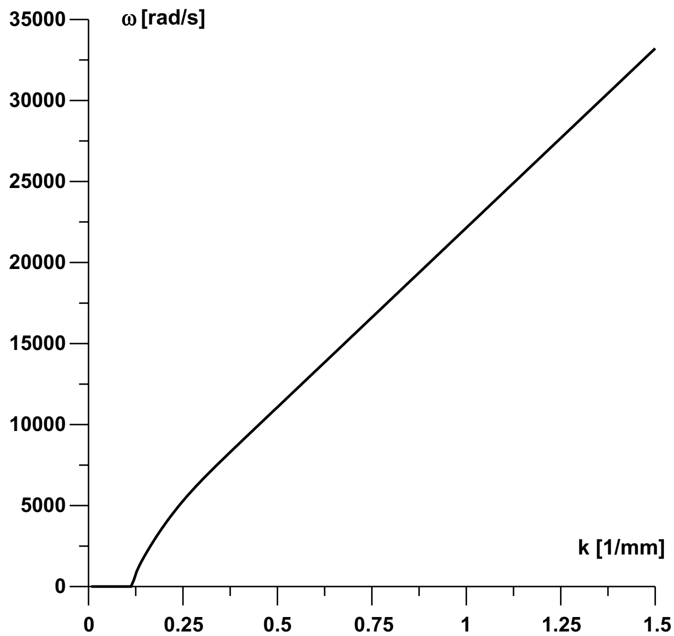

Fig. 1. Dispersion relation: angular frequency ω versus wave number k.

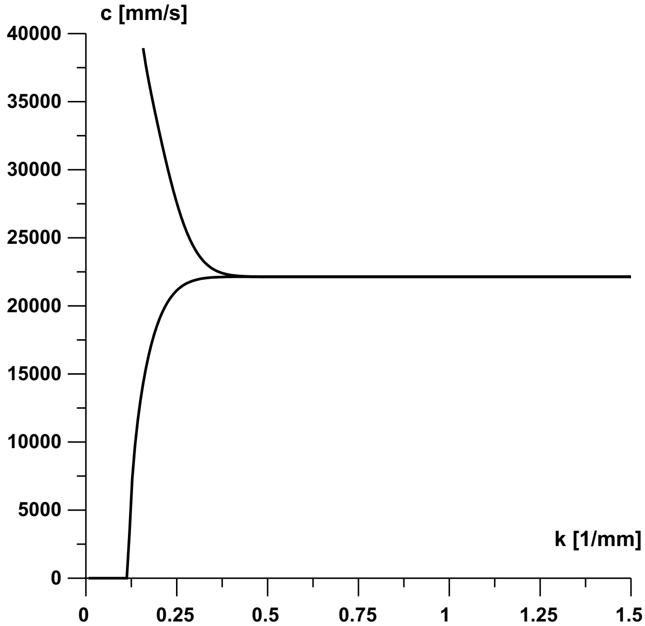

Fig. 2. Group velocity and phase velocity as functions of wave number k.

the bar for different values of the over-nonlocal parameter m. The numerical results confirm what was observed from our analysis of the propagation of a harmonic wave: the strain in the localization band (Figs. 4d–f) has a realistic shape only for the over-nonlocal formulation with $m > 1$. The effect of the over-nonlocal parameter m on the localization band is further illustrated in Fig. 4: an increase of m lengthens the localization band.

The same behavior has been obtained analytically in Eq. (26) for the critical wavelength, which is plotted in Fig. 3b. Note that Eq. (26) gives an approximate value of the localization band: $\lambda_{cr}(m = 1.5) = 42.3$ mm, $\lambda_{cr}(m = 2) = 53.22$ mm, and $\lambda_{cr}(m = 2.5) = 62$ mm. We can compare these values with the width of localization band obtained from the numerical simulations in Fig. 4. For $m = 1.5$, 2 and 2.5, respectively, the strain localizes in a band of about 48 mm, 61.5 mm, and 75 mm.

The paper of Bažant and Di Luzio (2004) (and also the thesis of Di Luzio, 2002; based on preceding collaboration with Bažant on that paper) applied the over-nonlocal formulation for the regularization of the microplane model M4, which can be considered a complex 3D constitutive law where the yield limits soften as a function of the total strain. Their previous conclusions agree with the results obtained for the simple nonlocal one-dimensional constitutive law given by Eq. (8).

4. Localization analysis of nonlocal plasticity

Consider now a simple plasticity model in the one-dimensional case. Local plasticity with linear hardening is described by the equations

$$\sigma = E(\epsilon - \epsilon_p) \tag{27}$$

$$f(\sigma, \sigma_Y) = |\sigma| - \sigma_Y \quad \text{and} \quad \sigma_Y = \sigma_0 + H\kappa \tag{28}$$

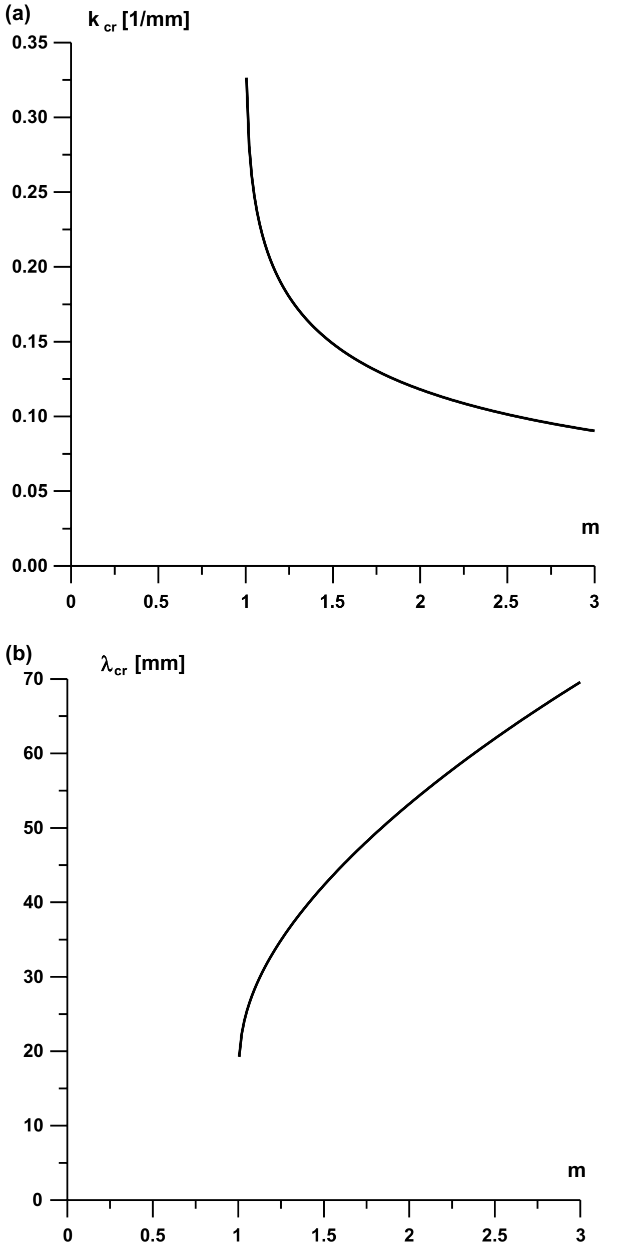

Fig. 3. Critical wave number (a) and critical wavelength (b) as functions of the over-nonlocal parameter m.

The evolution law of the plastic strain is written as

$$\dot{\epsilon}_{\mathrm{p}} = \dot{\kappa}\frac{\partial f}{\partial \sigma} = \dot{\kappa}\,\mathrm{sgn}(\sigma) \tag{29}$$

$$\dot{\kappa} \geqslant 0 \quad f(\sigma, \sigma_{\mathrm{Y}}) \leqslant 0 \quad \dot{\kappa} f(\sigma, \sigma_{\mathrm{Y}}) = 0 \tag{30}$$

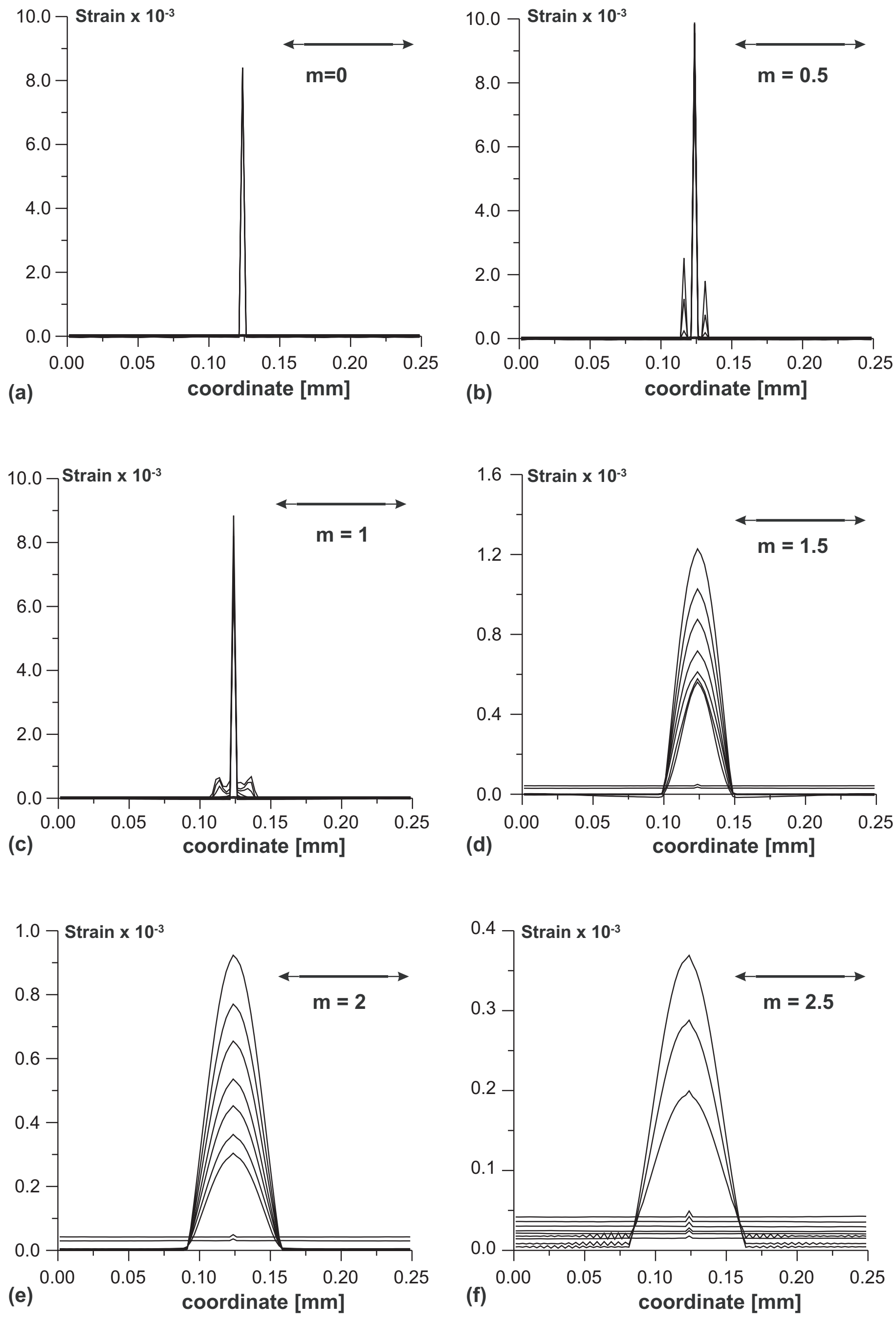

Fig. 4. Strain distribution along the bar for different values of the over-nonlocal parameter m: (a) $m = 0$; (b) $m = 0.5$; (c) $m = 1$; (d) $m = 1.5$; (e) $m = 2$; (f) $m = 2.5$.

where σ is the stress (considered to be positive, i.e. tensile), ϵ is the strain, E_0 is Young's modulus, ϵ_p is the plastic strain, σ_Y is the current yield stress, H is the plastic modulus (for strain-softening considered as negative), σ_0 is the initial yield limit, and κ is the plastic softening parameter, taken to coincide with the cumulative plastic strain, i.e., $\dot{\kappa} = |\dot{\epsilon}_p|$.

Consider the one-dimensional equation of motion, linearized about an initial state of homogeneous strain, Eq. (10); $\epsilon(x) = \epsilon_0 = \text{constant}$, and $\hat{\kappa}(x) = \kappa(x) = \kappa_0 = \text{constant}$. The initial equilibrium state is assumed to be a homogeneous strain-softening state. The harmonic wave solutions of frequency ω and wave number k are the following form

$$\dot{u}(x,t) = \dot{u}_0 e^{i(kx-\omega t)}, \qquad \dot{\kappa}(x,t) = \dot{\kappa}_0 e^{i(kx-\omega t)} \tag{31}$$

The over-nonlocal average of the plastic variable and its time derivative are

$$\hat{\kappa}(x) = \int_L \alpha_m^*(\xi - x)\kappa(\xi)d\xi \tag{32}$$

$$\dot{\hat{\kappa}}(x) = \int_L \alpha_m^*(\xi - x)\dot{\kappa}(\xi)d\xi \tag{33}$$

where $\alpha_m^*(\xi - x) = (1-m)\delta(\xi - x) + m\alpha^*(\xi - x)$ and $\delta(\xi - x)$ is the Dirac delta function.

(a) Let us consider the first nonlocal formulation of softening plasticity proposed by Bažant and Lin (1988), who replaced the plastic strain in the stress–strain law (27) by its nonlocal average

$$\sigma = E(\epsilon - \hat{\epsilon}_p) \quad \text{and} \quad f(\sigma, \sigma_Y) = E(\epsilon - \hat{\epsilon}_p) - (\sigma_0 + H\kappa) \tag{34}$$

Remembering that $\dot{\epsilon} = \dot{u}_{,x}$, differentiating Eq. (34) with respect to time and spatial coordinate x, and using the linearized equation of motion (Eq. (10)), we obtain

$$E(\dot{u}_{,xx} - \dot{\hat{\kappa}}_{,x}) = \rho\dot{u}_{,tt} \tag{35}$$

Starting from an initial uniform plastic strain state, the loading–unloading condition (Eq. (28)) requires that

$$\dot{f} = E\dot{u}_{,x} - E\dot{\hat{\kappa}} - H\dot{\kappa} = 0 \tag{36}$$

Substituting the harmonic wave solutions in Eq. (31) and their partial derivatives with respect to x and t into Eqs. (35) and (36), we obtain the following system of two equations

$$\left\{(Ek^2 - \rho\omega^2)\dot{u}_0 + Eik\left[1 - m + \frac{m}{\ell}A(k)\right]\dot{\kappa}_0\right\}e^{i(kx-\omega t)} = 0 \tag{37}$$

$$\left\{Eik\dot{u}_0 - \left[E\left(1 - m + \frac{m}{\ell}A(k)\right) + H\right]\dot{\kappa}_0\right\}e^{i(kx-\omega t)} = 0 \tag{38}$$

For $L \to \infty$, $A(k)$ is the Fourier transform of the weighting function $\alpha(z)$, Eq. (18) (Bažant and Chang, 1984). Eqs. (37) and (38) must be verified at every point x and for every instant t. Since this is a homogeneous linear system of two equations for $\dot{u}_0$ and $\dot{\kappa}_0$, the solution differs from the trivial one only if the determinant of the coefficient matrix is equal to zero, i.e.,

$$-EHk^2 + \rho\omega^2\left[H + E\left(1 - m + \frac{m}{\ell}A(k)\right)\right] = 0 \tag{39}$$

This equation implies that no wave number k makes the phase velocity to vanish, i.e., $c_p = \omega/k = 0$. Consequently the harmonic waves are non-dissipative and the constitutive model is unable to reproduce the localization band due to the softening. Therefore, for each value of the parameter m the plasticity model with the average of the plastic strain is not able to restore the well-posedness of the problem when the softening occurs. This plasticity model was already analyzed by Bažant and Jirásek (2002), who concluded that this nonlocal formulation exhibit locking at a later stage of softening process.

(b) Consider now a simple over-nonlocal integral-type plasticity model, in which the plastic variable is replaced by its over-nonlocal counterpart in the current yield stress. The constitutive relations are given by

$$\sigma = E(\epsilon - \epsilon_{\mathrm{p}}) \quad \text{and} \quad \sigma_{\mathrm{Y}} = \sigma_0 + H\hat{\kappa} \tag{40}$$

Jirásek and Rolsoven (2003) called Eq. (40) the basic nonlocal plasticity model. Differentiating Eq. (27) with respect to time and spatial coordinate x and using the linearized equation of motion (Eq. (10)), we obtain

$$E(\dot{u}_{,xx} - \dot{\kappa}_{,x}) = \rho\dot{u}_{,tt} \tag{41}$$

Starting from an initial uniform plastic strain state, the loading–unloading condition (Eq. (28)) requires that

$$\dot{f} = E\dot{u}_{,x} - E\dot{\kappa} - H\dot{\hat{\kappa}} = 0 \tag{42}$$

Substituting the harmonic wave solutions of Eq. (31) and their partial derivatives with respect to x and t into Eqs. (41) and (42), we obtain the following system of two equations

$$\left\{(Ek^2 - \rho\omega^2)\dot{u}_0 + E\mathrm{i}k\dot{\kappa}_0\right\}\mathrm{e}^{\mathrm{i}(kx-\omega t)} = 0 \tag{43}$$

$$\left\{E\mathrm{i}k\dot{u}_0 - \left[E + H\left(1 - m + \frac{m}{\ell}A(k)\right)\right]\dot{\kappa}_0\right\}\mathrm{e}^{\mathrm{i}(kx-\omega t)} = 0 \tag{44}$$

The previous expressions, Eqs. (43) and (44), defines a homogeneous linear system of two equations for $\dot{u}_0$ and $\dot{\kappa}_0$, and its solution differs from the trivial one only if the determinant of the coefficient matrix is equal to zero, i.e.,

$$-\left[E + H\left(1 - m + \frac{m}{\ell}A(k)\right)\right]\left(Ek^2 - \rho\omega^2\right) + E^2k^2 = 0 \tag{45}$$

For statics ($\omega \to 0$), Eq. (45) becomes

$$\left[1 - m + \frac{m}{\ell}A(k)\right]EHk^2 = 0 \tag{46}$$

The harmonic wave perturbations about the initial state, Eq. (31), do not remain homogeneous if Eq. (46) is satisfied. Thus we look for bifurcation points. Let us consider the Gauss weight function, Eq. (4), $\alpha(z) = \exp(-\pi z^2/\ell^2)$, such that $\alpha(0) = 1$ and $\int_{-\infty}^{+\infty}\alpha(z)\mathrm{d}z = \ell$. Noting that $\int_{-\infty}^{+\infty}\exp(-z^2)\mathrm{d}z = \sqrt{\pi}$, Eq. (18) gives the Fourier transform $A(k) = \ell\mathrm{e}^{-k^2\ell^2/4\pi}$. Substituting this value of $A(k)$ into Eq. (46), we search for the value of $k = k_{\mathrm{cr}}$ for which Eq. (46) can be satisfied for $k \neq 0$;

$$k_{\mathrm{cr}} = \frac{2\sqrt{\pi}}{\ell}\sqrt{\ln\left(\frac{m}{m-1}\right)} \tag{47}$$

The corresponding critical wavelength is

$$\lambda_{\mathrm{cr}} = 2\pi/k_{\mathrm{cr}} = \ell\sqrt{\pi}\left[\ln\left(\frac{m}{m-1}\right)\right]^{-1/2} \tag{48}$$

Note that for $m \to 0$, or for $\ell \to 0$ (the local strain-softening model), the critical wavelength λ_{cr} tends to zero, i.e., the strain localizes into a point. In the case of standard nonlocality, $m = 1$, the first term in Eqs. (45) and (46) is always non-zero for each value of k. This implies that the waves are non-dissipative: the shape of an arbitrary loading wave cannot change to a stationary wave representing the localization band. For $m \leqslant 1$, the critical wavelength (48), which measures the localization band, is not defined and, consequently, the strain localizes into a point, like in a local formulation. On the other hand, for $m > 1$

there exist some values of k, given by Eq. (47), which satisfy Eqs. (45) and (46) implying a finite dimension for the localization zone given by Eq. (48). As we can see from Eqs. (47) and (48), we obtain the same results of Section 3: Eqs. (25) and (26) plotted in Fig. 3 with $\ell = 25$ mm for a softening model with strain-dependent yield limit. The previous localization analysis reveals that the basic nonlocal plasticity model, in which the plastic softening parameter is replaced by its nonlocal counterpart, provides a finite localization band if and only if an over-nonlocal averaging is used.

In a dispersion analysis of the plasticity model, we substitute the over-nonlocal average in the stress–strain relation and in the current yield stress ($\sigma = E(\epsilon - \hat{\epsilon}_{\rm p})$ and $\sigma_{\rm Y} = \sigma_0 + H\hat{\kappa}$). This leads to the same results as obtained in Eqs. (47) and (48).

(c) The reduction of the resulting yield stress in ductile materials is mainly due to the growth and coalescence of voids. The void growth leads to a reduction of the effective area which results into an overall softening (decrease of the nominal yield stress). Because the growth of defects is a damage process, this phenomenon is referred to as ductile damage. In this section, a plasticity model motivated by the concept of ductile damage, proposed by Geers, Engelen and coworkers (Geers et al., 2001; Engelen et al., 2003), will be considered in an over-nonlocal integral-type version. Recently, an extension to large deformations of the ductile damage model has been presented (Geers et al., 2003; Geers, 2004). This feature is of practical relevance for the application of such model to, e.g., metallic materials. This model is characterized by the softening–hardening law written in the multiplicative format, in which the yield function in the one-dimensional case is given by

$$f(\sigma, \kappa, \hat{\kappa}) = |\sigma| - [1 - \omega_{\rm p}(\hat{\kappa})](\sigma_0 + h\kappa) \tag{49}$$

where the plastic modulus h is a positive constant. In the yield function (49), the current yield stress is defined by two terms: the term $\sigma_0 + h\kappa$ which represent the linear plastic hardening of the bulk material and the term $1 - \omega_{\rm p}$ which reduces the yield stress for the growth of damage in the material. An important feature is that while the hardening is governed by the local cumulative plastic strain, the softening is driven by the nonlocal cumulative plastic strain. The essential component of the ductile damage model is the hardening–softening law written in the multiplicative format (Eq. (49)), which differs from the hardening–softening law of Eq. (28) written in an additive format. A linear dependence of $\omega_{\rm p}$ on $\hat{\kappa}$ is given by a piecewise linear law

$$\omega_{\rm p}(\hat{\kappa}) = \begin{cases} 0 & \text{if } \hat{\kappa} \leqslant \kappa_i \\ \dfrac{\hat{\kappa} - \kappa_i}{\kappa_f - \kappa_i} & \text{if } \kappa_i \leqslant \hat{\kappa} \leqslant \kappa_f \\ 1 & \text{if } \hat{\kappa} \geqslant \kappa_f \end{cases} \tag{50}$$

where κ_i is a material parameter which characterizes the damage threshold and κ_f is a material parameter which specifies the plastic deformation at failure (i.e., at zero yield stress). For this kind of over-nonlocal ductile damage model, Eqs. (41) and (42) are written as

$$E(\dot{u}_{,xx} - \dot{\kappa}_{,x}) = \rho \dot{u}_{,tt} \tag{51}$$

$$\dot{f} = E\dot{u}_{,x} - E\dot{\kappa} - (1 - \omega_{\rm p})h\dot{\kappa} + (\sigma_0 + h\kappa_0)\dot{\omega}_{\rm p} = 0 \tag{52}$$

Localization takes place only in the softening range, in which, for $\kappa_i \leqslant \hat{\kappa} \leqslant \kappa_f$, we have

$$\omega_{\rm p} = \frac{\kappa_0 - \kappa_i}{\kappa_f - \kappa_i} \quad \text{and} \quad \dot{\omega}_{\rm p} = \frac{\dot{\hat{\kappa}}}{\kappa_f - \kappa_i} \tag{53}$$

Using Eqs. (51)–(53) and starting from a uniform initial state ($\epsilon(x) = \epsilon_0 = $ const and $\hat{\kappa}(x) = \kappa(x) = \kappa_0 = $ const.), the system of two equations that characterizes the localization problem is now given by

$$\left[(Ek^2 - \rho\omega^2)\dot{u}_0 + Eik\dot{\kappa}_0\right]\mathrm{e}^{\mathrm{i}(kx - \omega t)} = 0 \tag{54}$$

$$\left\{Eik\dot{u}_0 + \left[-E - \frac{\kappa_f - \kappa_0}{\kappa_f - \kappa_i}h + \frac{\sigma_0 - h\kappa_0}{\kappa_f - \kappa_i}\left(1 - m + \frac{m}{\ell}A(k)\right)\right]\dot{\kappa}_0\right\}e^{i(kx-\omega t)} = 0 \tag{55}$$

Setting the determinant of the coefficient matrix equal to zero, we get

$$\left[-\frac{\kappa_f - \kappa_0}{\kappa_f - \kappa_i}h + \frac{\sigma_0 - h\kappa_0}{\kappa_f - \kappa_i}\left(1 - m + \frac{m}{\ell}A(k)\right)\right]\left[Ek^2 - \rho\omega^2\right] + E\rho\omega^2 = 0 \tag{56}$$

In statics ($\omega \to 0$), Eq. (56) becomes

$$\left[-\frac{\kappa_f - \kappa_0}{\kappa_f - \kappa_i}h + \frac{\sigma_0 + h\kappa_0}{\kappa_f - \kappa_i}\left(1 - m + \frac{m}{\ell}A(k)\right)\right]Ek^2 = 0 \tag{57}$$

As done in the foregoing for other plasticity models, first we substitute the value of $A(k) = \ell e^{-k^2\ell^2/4\pi}$ into Eq. (57), and then we search for the value of the critical wave number ($k \neq 0$) for which Eq. (57) can be satisfied

$$k_{cr} = \frac{2\sqrt{\pi}}{\ell}\sqrt{\ln\left(\frac{m}{h(\kappa_f - \kappa_0)/(\sigma_0 + h\kappa_0) + m - 1}\right)} \tag{58}$$

The corresponding critical wavelength is

$$\lambda_{cr} = 2\pi/k_{cr} = \ell\sqrt{\pi}\left[\ln\left(\frac{m}{h(\kappa_f - \kappa_0)/(\sigma_0 + h\kappa_0) + m - 1}\right)\right]^{-1/2} \tag{59}$$

Note that for $m \to 0$, or for $\ell \to 0$ (local strain-softening model), the critical wavelength λ_{cr} tends to zero, i.e., the strain localizes in a point. For each value of the parameter m, in particular for $m = 1$ (standard nonlocality), the critical wavelength, which measures the localization band, is well defined and, consequently, the strain localizes into a band with a finite dimension which is plotted in Fig. 5 for different values of the initial softening parameter κ_0 (assuming $\kappa_i = 0.0001$, $\kappa_f = 0.01$, $h = 10^2$ MPa, $\sigma_0 = 2$ MPa and $\ell = 25$ mm). Another important feature of this nonlocal plasticity model is that the critical wavelength in Eq. (59) is a decreasing function of the softening parameter κ_0. Thus, the width of the localization band tends to reduce to zero as the softening parameter increases ($\lambda_{cr} \to 0$ for $\kappa_0 \to \kappa_f$ and $m = 1$), i.e., there is convergence to a discrete crack. Note that this important feature happens only for the standard nonlocality ($m = 1$). We may conclude that the ductile damage model proposed by Geers, Engelen and coworkers in the integral-type nonlocal version provides a full regularization of the softening problem even for the standard nonlocal formulation with $m = 1$. Thus, only the basic nonlocal plasticity model with $m > 1$ and the nonlocal (with $m = 1$) ductile damage model of Geers, Engelen and coworkers seem to be able to describe correctly the localization problem due to the softening. Our conclusions confirm in a simple way the earlier results of Jirásek and Rolsoven (2003) and Rolshoven (2003) obtained by using a different approach for the one-dimensional analysis.

Consider now the class of thermodynamically based nonlocal plasticity models with internal variables (Nilson, 1997; Svedberg and Runesson, 1998; Borino et al., 1999). Upon specifying a suitable expression for the free energy density and for the energy dissipation density, one can construct a mechanical constitutive theory. The nonlocal generalization of plasticity models of this class follows from the assumption that the free energy density depends not only on the local value of the state variables but also on their nonlocal average. Models of this class are characterized by the so-called dual averaging, in which the dual averaging operator differs from the original one only in the order of arguments of the weight function α (cf. Bažant and Jirásek, 2002; Jirásek and Rolsoven, 2003). Therefore the models of this class (Nilson, 1997; Svedberg and Runesson, 1998; Borino et al., 1999) can be easily brought again within the realm of plasticity models considered in the foregoing (see for details Rolshoven, 2003).

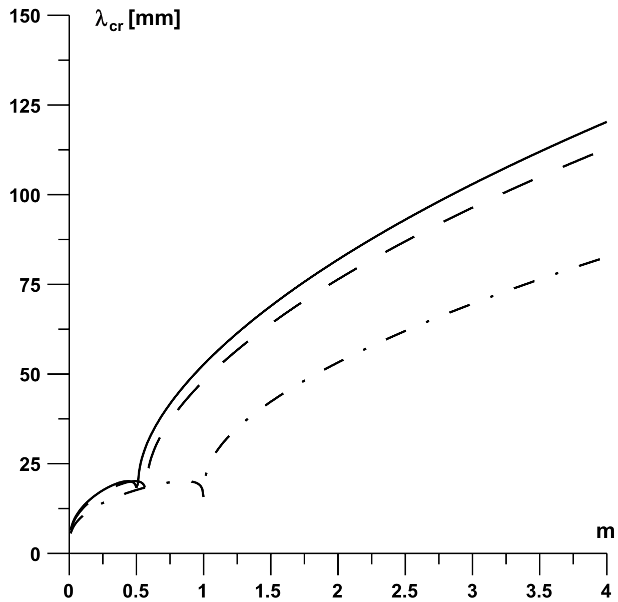

Fig. 5. Critical wavelength as function of the over-nonlocal parameter m for different value of the initial softening parameter: $\kappa_0 = 0.0001$ for the solid line, $\kappa_0 = 0.001$ for the dashed line and $\kappa_0 = 0.00999$ for the dashed-dotted line.

5. Localization analysis of nonlocal damage model

Following the approach of Pijaudier-Cabot and Benallal (1993), let us now analyze the localization of an over-nonlocal version of the scalar continuous damage model proposed by Pijaudier-Cabot and Bažant (1987) and studied by Bažant and Pijaudier-Cabot (1988). The constitutive relation in one-dimension and the loading function are

$$\sigma = (1-d)E\epsilon \quad f(\widehat{Y}, d) = \int_0^{\widehat{Y}} F(z)\mathrm{d}z - d \tag{60}$$

where d is the damage variable, function F defines the damage evolution, and $\widehat{Y}$ is the over-nonlocal average of the energy release rate;

$$\widehat{Y}(x) = \int_V \alpha^*_m(\xi - x)Y(\xi)\mathrm{d}\xi \quad Y(x) = \frac{1}{2}E\epsilon(x)^2 \tag{61}$$

In the case of associated damage, the evolution of the damage variable is controlled by the conditions:

$$\dot{d} \geqslant 0 \quad f \leqslant 0 \quad \dot{d}f = 0 \tag{62}$$

Using the same approach as in Section 3, (Eqs. (10)–(17)), the loading consistency condition for damage (Eq. (60)) requires

$$\dot{f}(\widehat{Y}, d) = F(\widehat{Y})\dot{\widehat{Y}} - \dot{d} = 0 \tag{63}$$

Assuming an initial state of uniform damage and uniform strain field, i.e., $\hat{\epsilon}(x) = \epsilon(x) = \epsilon_0 = \text{const.}$ and $d(x) = d_0 = \text{const.}$, we differentiate Eq. (60) with respect to time and obtain the rate constitutive relation:

$$\dot{\sigma} = (1-d_0)E\dot{\epsilon} - \dot{d}E\epsilon_0 \tag{64}$$

Obtaining the rate of the damage variable from Eq. (63) and substituting it into Eq. (64), we have

$$\dot{\sigma} = (1 - d_0)E\dot{\epsilon} - E\epsilon_0 F(\widehat{Y})\dot{\widehat{Y}} = (1 - d_0)E\dot{\epsilon} - E^2\epsilon_0^2 F(\widehat{Y}) \int_L \alpha_m^*(\xi - x)\dot{\epsilon}(\xi)\mathrm{d}\xi \tag{65}$$

This equation applies to all the volume. Differentiating it with respect to x and using Eq. (10), we obtain the following equation of motion

$$(1 - d_0)E\dot{u}_{,xx} - E^2\epsilon_0^2 F(\widehat{Y})\frac{\partial}{\partial x}\int_L \alpha_m^*(\xi - x)\dot{\epsilon}(\xi)\mathrm{d}\xi = \rho\dot{u}_{,tt} \tag{66}$$

Consider now a harmonic wave perturbation of frequency ω and wave number k propagating through an infinite bar (Eq. (12)). Substituting Eq. (12) into Eq. (66), we obtain:

$$\left\{(1 - d_0)Ek^2 - E^2\epsilon_0^2 F(\widehat{Y})k^2\left(1 - m + \frac{m}{\ell}A(k)\right) - \rho\omega^2\right\}\dot{u}_0 \mathrm{e}^{\mathrm{i}(kx-\omega t)} = 0 \tag{67}$$

A non-zero (non-trivial) solution exists only if the braced term vanishes, which yields

$$\omega^2 = k^2\frac{E}{\rho}\left[1 - d_0 - E\epsilon_0^2 F(\widehat{Y})\left(1 - m + \frac{m}{\ell}A(k)\right)\right] \tag{68}$$

Consider the same constitutive law and the same parameters as Pijaudier-Cabot and Benallal (1993);

$$F(\widehat{Y}) = \frac{b_1 + 2b_2(\widehat{Y} - Y^0)}{\left[1 + b_1(\widehat{Y} - Y^0) + b_2(\widehat{Y} - Y^0)^2\right]^2} \tag{69}$$

where $b_1 = 605\ \mathrm{MPa}^{-1}$, $b_2 = 54{,}200\ \mathrm{MPa}^{-2}$, $Y^0 = 60\times 10^{-6}$ MPa, $E = 32{,}000$ MPa and $\nu = 0.2$. Because we assumed an infinite body and a uniform initial state, we have $\widehat{Y} = Y = \frac{1}{2}E\epsilon_0^2$ and $d_0 = \int_0^{\widehat{Y}} F(z)\mathrm{d}z$. Using $A(k) = \ell\mathrm{e}^{-k^2\ell^2/4\pi}$, the critical wave number k_{cr} that makes the phase velocity and the angular frequency vanish ($c_{\mathrm{p}} = \omega = 0$) is

$$k_{\mathrm{cr}} = \frac{2\sqrt{\pi}}{\ell}\left[\ln\left(\frac{mE\epsilon_0^2 F(\widehat{Y})}{1 - d_0 + (m-1)E\epsilon_0^2 F(\widehat{Y})}\right)\right]^{1/2} \tag{70}$$

and the corresponding critical wavelength is

$$\lambda_{\mathrm{cr}} = 2\pi/k_{\mathrm{cr}} = \ell\sqrt{\pi}\left[\ln\left(\frac{mE\epsilon_0^2 F(\widehat{Y})}{1 - d_0 + (m-1)E\epsilon_0^2 F(\widehat{Y})}\right)\right]^{-1/2} \tag{71}$$

Fig. 6 shows the plot of the critical wave number k_{cr} and the critical wavelength λ_{cr} for two different values of the initial uniform strain ϵ_0. Because, in the case of standard nonlocality ($m = 1$), the critical wavelength is well defined, the nonlocal damage model does not necessitate an over-nonlocal formulation. The standard nonlocality suffices to correct the problems of local softening for a damage model with the average of the damage energy release rate. Moreover, the critical wavelength of Eq. (71) goes to zero upon complete failure which means that the localization band converges to a discrete crack, as also shown by Peerlings et al., 1996a,b.

The same dispersion analysis, obtaining the same result, can be carried out when the loading function (Eq. (60)) is made to depend on the over-nonlocal average of strain:

$$f(Y(\hat{\epsilon}), d) = \int_0^{Y(\hat{\epsilon})} F(z)\mathrm{d}z - d \tag{72}$$

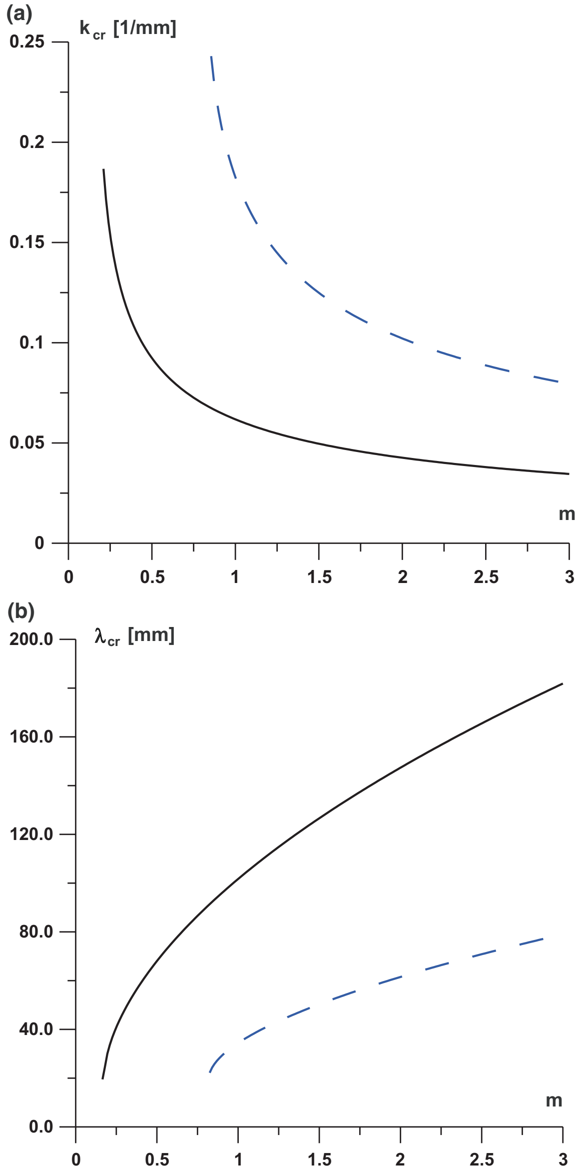

Fig. 6. Critical wave number (a) and critical wavelength (b) as functions of the over-nonlocal parameter m for $\epsilon = 0.0003$ (solid line) and for $\epsilon = 0.0008$ (dashed line).

6. Gradient-type model derived from over-nonlocal integral-type model

Instead of introducing strong nonlocality through spatial interaction, one can impose weak nonlocality by introducing first or higher-order gradients into the constitutive relations. The gradient-type models can be considered as differential approximation of the integral-type nonlocal models. This correspondence helps to explain the role of the over-nonlocal parameter. Let the local variable f be expanded around x into the Taylor series:

$$f(z) = f(x) + f_{,x}(z-x) + \frac{1}{2} f_{,xx}(z-x)^2 + o[(z-x)^2] \tag{73}$$

Averaging the variable $f(z)$ through Eq. (7) and neglecting the terms of third and higher order in terms of the distance $|z-x|$, we obtain:

$$\hat{f}(x) \simeq f(x) \int_L \alpha_m^*(y-x)\mathrm{d}y + f_{,x} \int_L (y-x)\alpha_m^*(y-x)\mathrm{d}y + \frac{1}{2} f_{,xx} \int_L (y-x)^2 \alpha_m^*(y-x)\mathrm{d}y \tag{74}$$

The weight function $\alpha(x-y)$ is normalized such that $\alpha(0) = 1$ and $\int_L \alpha(z)\mathrm{d}z = \ell$. Moreover, in an infinite specimen (or at points sufficiently far from the boundary of a finite specimen), the function $\alpha(x-y)$ depends only on the distance $|z-x|$. In Eq. (74), due to symmetry the term $\int_L (y-x)\alpha(x-y)\mathrm{d}y$ vanishes and the term $\int_L (y-x)^2 \alpha(x-y)\mathrm{d}y$ is equal to ℓ^2. Consequently, the expression in Eq. (74) may be written as

$$\hat{f}(x) = f(x) + m\frac{\ell^2}{2} f(x)_{,xx} = f(x) + \frac{\ell_0^2}{2} f(x)_{,xx} \tag{75}$$

Therefore, the gradient-type models display a similar behavior as the nonlocal models, even though examples in literature show considerable differences in the solutions.

In the following, localization analysis will be carried out for the constitutive laws which are the same as considered in the foregoing, except for being regularized by explicit and implicit gradient enhancement. The explicit gradient models are those directly obtained by simply introducing in the constitutive law certain gradient terms. They are weakly nonlocal, because only the infinitely close neighborhood of a point directly affects the response. The implicit gradient models are those in which the constitutive law is made to depend on a nonlocal field that is obtained as the solution of a separate differential equation (Helmholtz equation). The implicit models are strongly nonlocal, just like the integral-type models, because a non-vanishing neighborhood of a point directly affects the solution. Many studies have addressed the gradient enhancement of plasticity model (e.g., de Borst et al., 1995) or the damage model (e.g., Peerlings et al., 1996a,b; Askes et al., 2000; Peerlings et al., 2001; Askes and Sluys, 2002; Peerlings et al., 2002).

7. Localization analysis of an explicit gradient-type model with strain-dependent yield limit

Consider now a constitutive law in which the yield limit is a function of the total strain (Eq. (8)). The softening now depends on the strain and its second spatial derivative. Thus, Eq. (13) is replaced by

$$\dot{\hat{\epsilon}} = \dot{\epsilon} + \frac{\ell^2}{2} \dot{\epsilon}_{,xx} \tag{76}$$

Introducing the derivative of the previous equation with respect to x, together with the linearized equation of motion (Eq. (10)), into Eq. (15), one gets

$$E_0 \mathrm{e}^{-\epsilon_0/\epsilon_1} \left(1 - \frac{\epsilon_0}{\epsilon_1}\right) \left(\dot{u}_{,xx} + \frac{\ell}{2} \dot{u}_{,xxxx}\right) = \rho \dot{u}_{,tt} \tag{77}$$

Substituting Eq. (12) and its partial derivatives with respect to x and t into Eq. (77), one obtains

$$\left\{ E_0 \mathrm{e}^{-\epsilon_0/\epsilon_1} \left(1 - \frac{\epsilon_0}{\epsilon_1}\right) k^2 \left(-1 + \frac{\ell}{2} k^2\right) - \rho\omega^2 \right\} \dot{u}_0 \, \mathrm{e}^{\mathrm{i}(kx-\omega t)} = 0 \tag{78}$$

Because the expression in the braces must vanish, the angular frequency for this kind of gradient model is

$$\omega = c_{\mathrm{e}} k \left[\mathrm{e}^{-\epsilon_0/\epsilon_1} \left(1 - \frac{\epsilon_0}{\epsilon_1} \right) \left(\frac{\ell}{2} k^2 - 1 \right) \right]^{1/2} \tag{79}$$

where c_{e} is the elastic propagation velocity in the bar. Therefore, the phase velocity, $c_{\mathrm{p}} = \omega/k$, and the group velocity, $\mathrm{c} = \omega_{,k}$, are given by

$$c_{\mathrm{p}} = c_{\mathrm{e}} \left[\mathrm{e}^{-\epsilon_0/\epsilon_1} \left(1 - \frac{\epsilon_0}{\epsilon_1} \right) \left(\frac{\ell}{2} k^2 - 1 \right) \right]^{1/2} \tag{80}$$

$$c = \frac{\omega}{k} + c_{\mathrm{e}} k^2 \frac{\ell}{2} \sqrt{\frac{\mathrm{e}^{-\epsilon_0/\epsilon_1} (1 - \frac{\epsilon_0}{\epsilon_1})}{\frac{\ell}{2} k^2 - 1}} \tag{81}$$

Because the phase velocity is a function of the wave number k (Eq. (80)), the waves are dispersive. The phase velocity, $c_{\mathrm{p}} = \omega/k$, and the angular frequency are real if the wave number is such that the term under the square root in Eq. (80) is non-negative. From this, one finds the critical wave number, k_{cr}, which makes the phase velocity and the angular frequency vanish ($c_{\mathrm{p}} = \omega = 0$):

$$k_{\mathrm{cr}} = \frac{\sqrt{2}}{\ell} \tag{82}$$

The corresponding critical wavelength is

$$\lambda_{\mathrm{cr}} = 2\pi/k_{\mathrm{cr}} = \sqrt{2}\pi\ell \tag{83}$$

Only the harmonic waves with wavelength $\lambda \leqslant \lambda_{\mathrm{cr}}$ can propagate. If $k < k_{\mathrm{cr}}$ or $\lambda > \lambda_{\mathrm{cr}}$, we have a situation in which an arbitrarily small disturbance $\dot{u}$ leads to a finite change in the response and, therefore, the initial state is not stable (in the sense of Lyapunov). So, the explicit gradient model, in which the stress in the softening region depends on the strain and its second derivative, is capable of regularizing the softening problem with a localization into a band of finite width, given by Eq. (83).

8. Localization analysis of explicit gradient-type plasticity model

This section will examine the same plasticity models as in Section 4 except that the nonlocal variable will now be replaced by its approximation in Eq. (75). Thus, Eq. (33) is replaced by

$$\dot{\hat{\kappa}} = \dot{\kappa} + \frac{\ell^2}{2} \dot{\kappa}_{,xx} \tag{84}$$

(a) First we consider the plasticity model in which the plastic strain in the stress–strain law is replaced by the sum of the variable and its second derivative.

Assuming that all the material is initially in a softening homogeneous state, one gets from Eqs. (35) and (36), together with Eq. (84), the following equation system

$$E \left(\dot{u}_{,xx} - \dot{\kappa}_{,x} - \frac{\ell^2}{2} \dot{\kappa}_{,xxx} \right) = \rho \dot{u}_{,tt} \tag{85}$$

$$\dot{f} = E \dot{u}_{,x} - E \left(\dot{\kappa} + \frac{\ell^2}{2} \dot{\kappa}_{,xx} \right) - H \dot{\kappa} = 0 \tag{86}$$

Consider harmonic waves of frequency ω and wave number k, propagating along the bar with the velocity and cumulative plastic strain field given by Eq. (31). Substituting Eq. (31) and its partial derivatives with respect to x and t into Eqs. (85) and (92), we obtain

$$\left[(Ek^2-\rho\omega^2)\dot{u}_0+E\mathrm{i}k\left(1-\frac{\ell^2}{2}k^2\right)\dot{\kappa}_0\right]\mathrm{e}^{\mathrm{i}(kx-\omega t)}=0 \tag{87}$$

$$\left\{E\mathrm{i}k\dot{u}_0-\left[H+E\left(1-\frac{\ell^2}{2}k^2\right)\right]\dot{\kappa}_0\right\}\mathrm{e}^{\mathrm{i}(kx-\omega t)}=0 \tag{88}$$

Eqs. (87) and (88) must be verified at every point x and for every instant t. Because this is a homogeneous system of two linear equations, a non-trivial solution exists if and only if the determinant of the coefficient matrix vanishes;

$$(Ek^2-\rho\omega^2)\left[\left(1-\frac{\ell^2}{2}k^2\right)E+H\right]-E^2k^2\left(1-\frac{\ell^2}{2}k^2\right)=0 \tag{89}$$

In statics ($\omega=0$), we have

$$HEk^2=0 \tag{90}$$

The explicit gradient enhancement of the plasticity model (Eq. (34)) gives harmonic waves that are non-dissipative. Thus the constitutive model is unable to reproduce the localization band due to the softening.

(b) Consider now the so-called basic nonlocal plasticity model (Section 4), in which the yield stress, Eq. (28) depends on the cumulative plastic strain and its second derivative, Eq. (84). Eqs. (41) and (42), together with Eq. (84), give the following equation system (cf. Bažant and Jirásek, 2002) and by Jirásek and Rolsoven, 2003):

$$E(\dot{u}_{,xx}-\dot{\kappa}_{,x})=\rho\dot{u}_{,tt} \tag{91}$$

$$\dot{f}=E\dot{u}_{,x}-E\dot{\kappa}-H\left(\dot{\kappa}+\frac{\ell^2}{2}\dot{\kappa}_{,xx}\right)=0 \tag{92}$$

Let us analyze harmonic waves of frequency ω and wave number k, in which the velocity and cumulative plastic strain field are given by Eq. (31). Substituting Eq. (31) and its partial derivatives with respect to x and t into Eqs. (91) and (92), we obtain

$$\left[(Ek^2-\rho\omega^2)\dot{u}_0+E\mathrm{i}k\dot{\kappa}_0\right]\mathrm{e}^{\mathrm{i}(kx-\omega t)}=0 \tag{93}$$

$$\left\{E\mathrm{i}k\dot{u}_0-\left[E+H\left(1-\frac{\ell^2}{2}k^2\right)\right]\dot{\kappa}_0\right\}\mathrm{e}^{\mathrm{i}(kx-\omega t)}=0 \tag{94}$$

Eqs. (93) and (94) must be satisfied at every point x and every instant t. Because this is a homogeneous system of two linear equations, a non-trivial solution exists if and only if the determinant of the coefficient matrix is equal to zero

$$(Ek^2-\rho\omega^2)\left[\left(1-\frac{\ell^2}{2}k^2\right)H+E\right]-E^2k^2=0 \tag{95}$$

In statics ($\omega=0$), we have

$$\left(1-\frac{\ell^2}{2}k^2\right)HEk^2=0 \tag{96}$$

The perturbation does not remain homogeneous about the initial state if Eq. (96) is satisfied. So we look for bifurcation points. The non-zero value of $k = k_{cr}$ that satisfies Eq. (96) is

$$k_{cr} = \frac{\sqrt{2}}{\ell} \tag{97}$$

The corresponding critical wavelength is

$$\lambda_{cr} = \sqrt{2}\pi\ell \tag{98}$$

For $\ell \to 0$ (i.e., for a local strain-softening model), the critical wavelength λ_{cr} tends to zero, which means that the strain localizes into a point. For $\ell > 0$, the critical wavelength has always a finite real value. Therefore, we conclude that the explicit gradient-type model is able to regularize the problem of softening plasticity.

(c) Consider now the ductile damage model (Geers et al., 2001; Engelen et al., 2003) in which the yield stress, Eq. (49), now depends on the cumulative plastic strain and its second derivative as expressed by Eq. (84). For the linear softening in Eq. (50), the equation system that governs the localization problem of the ductile damage model with implicit gradient enhancement is given by

$$E(\dot{u}_{,xx} - \dot{\kappa}_{,x}) = \rho\dot{u}_{,tt} \tag{99}$$

$$\dot{f} = E\dot{u}_{,x} - E\dot{\kappa} - \frac{\kappa_f - \kappa_0}{\kappa_f - \kappa_i}h\dot{\kappa} + \frac{\sigma_0 + h\kappa_0}{\kappa_f - \kappa_i}\left(\dot{\kappa} + \frac{\ell^2}{2}\dot{\kappa}_{,xx}\right) = 0 \tag{100}$$

If the harmonic waves of frequency ω and wave number k, propagating along the bar, Eq. (31), are substituted in Eqs. (99) and (100), one obtains:

$$\left[\left(Ek^2 - \rho\omega^2\right)\dot{u}_0 + E\mathrm{i}k\dot{\kappa}_0\right]\mathrm{e}^{\mathrm{i}(kx-\omega t)} = 0 \tag{101}$$

$$\left\{E\mathrm{i}k\dot{u}_0 + \left[-E - \frac{\kappa_f - \kappa_0}{\kappa_f - \kappa_i}h + \frac{\sigma_0 + h\kappa_0}{\kappa_f - \kappa_i}\left(1 - \frac{\ell^2}{2}k^2\right)\right]\dot{\kappa}_0\right\}\mathrm{e}^{\mathrm{i}(kx-\omega t)} \tag{102}$$

The homogeneous system of two linear equations (Eqs. (101) and (102)) has a non-trivial solution if and only if the determinant of the coefficient matrix vanishes;

$$\left(Ek^2 - \rho\omega^2\right)\left[-E - \frac{\kappa_f - \kappa_0}{\kappa_f - \kappa_i}h + \frac{\sigma_0 + h\kappa_0}{\kappa_f - \kappa_i}\left(1 - \frac{\ell^2}{2}k^2\right)\right] + E^2k^2 = 0 \tag{103}$$

In statics ($\omega = 0$), we have

$$\left[-\frac{\kappa_f - \kappa_0}{\kappa_f - \kappa_i}h + \frac{\sigma_0 + h\kappa_0}{\kappa_f - \kappa_i}\left(1 - \frac{\ell^2}{2}k^2\right)\right]Ek^2 = 0 \tag{104}$$

The non-zero value of k that satisfies Eq. (104) is

$$k_{cr} = \frac{\sqrt{2}}{\ell}\left(1 - \frac{\kappa_f - \kappa_0}{\sigma_0 + h\kappa_0}h\right)^{1/2} \tag{105}$$

The corresponding critical wavelength is

$$\lambda_{cr} = \sqrt{2}\pi\ell\left(1 - \frac{\kappa_f - \kappa_0}{\sigma_0 + h\kappa_0}h\right)^{-1/2} \tag{106}$$

Since, for $\ell > 0$, the critical wavelength always has a finite real value, the explicit gradient enhancement of the ductile damage model is able to regularize the problem of softening plasticity. In Fig. 7 the critical

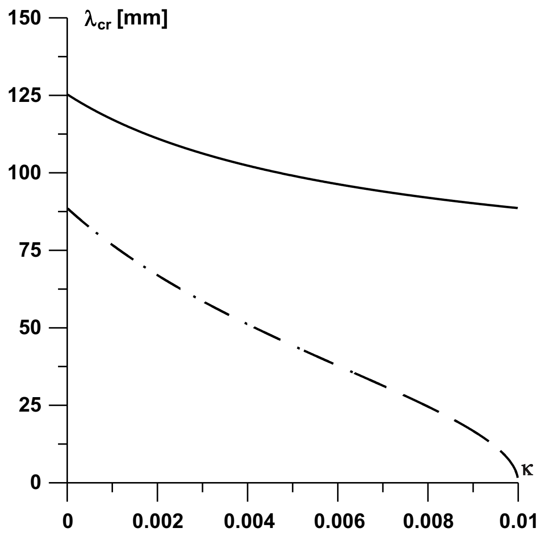

Fig. 7. Critical wavelength versus initial softening parameter for ductile damage model: solid line for the explicit gradient enhancement, dashed line for the implicit gradient enhancement and dashed-dotted line for the nonlocal integral-type formulation.

wavelength versus the initial softening parameter κ_0 is plotted as the solid line. Note that the explicit gradient enhancement of the ductile damage model provides a critical wavelength which does not tend to zero at complete failure ($\kappa_0 \to \kappa_f$).

9. Localization analysis of implicit gradient-type plasticity models

Another important class of gradient-type models consists of the implicit gradient-type models, in which Eq. (84) is rearranged as follows

$$\dot{\hat{\kappa}} - \frac{\ell^2}{2}\dot{\hat{\kappa}}_{,xx} = \dot{\kappa} \tag{107}$$

which is the Helmholtz equation.

(a) First we consider the plasticity model in which the plastic strain in the stress–strain law is now replaced by the nonlocal variable obtained by the solution of Eq. (107). Two differentiations of Eq. (36) with respect to x, together with Eq. (107), give

$$\dot{\hat{\kappa}} = \dot{\kappa} + \frac{\ell^2}{2E}(E\dot{u}_{,xxx} - H\dot{\kappa}_{,xx}) \tag{108}$$

Rearranging Eqs. (35), (36) and (108), we obtain the following system of equations

$$E\left(\dot{u}_{,xx} - \frac{\ell^2}{2}\dot{u}_{,xxxx}\right) - E\dot{\kappa}_{,x} + H\frac{\ell^2}{2}\dot{\kappa}_{,xxx} = \rho\dot{u}_{,tt} \tag{109}$$

$$\dot{f} = E\left(\dot{u}_{,x} - \frac{\ell^2}{2}\dot{u}_{,xxx}\right) - (H+E)\dot{\kappa} + H\frac{\ell^2}{2}\dot{\kappa}_{,xx} = 0 \tag{110}$$

Substituting the harmonic waves of frequency ω and wave number k from Eq. (31) and its partial derivatives with respect to x and t into Eqs. (109) and (110), one gets

$$\left[\left(Ek^2-\rho\omega^2+E\frac{\ell^2}{2}k^4\right)\dot{u}_0+\mathrm{i}k\left(E+H\frac{\ell^2}{2}k^2\right)\dot{\kappa}_0\right]\mathrm{e}^{\mathrm{i}(kx-\omega t)}=0 \tag{111}$$

$$\left[\mathrm{i}Ek\left(1+\frac{\ell^2}{2}k^2\right)\dot{u}_0-(E+H+H\frac{\ell^2}{2}k^2)\dot{\kappa}_0\right]\mathrm{e}^{\mathrm{i}(kx-\omega t)}=0 \tag{112}$$

The last two equations must be verified at every point x and every instant t. Since this is a system of two homogeneous linear equations, a non-trivial solution exists if and only if the determinant of the coefficient matrix vanishes, i.e.,

$$\left(Ek^2+E\frac{\ell^2}{2}k^4-\rho\omega^2\right)\left(E+H+H\frac{\ell^2}{2}k^2\right)-Ek^2\left(E+H\frac{\ell^2}{2}k^2\right)\left(1+\frac{\ell^2}{2}k^2\right)=0 \tag{113}$$

For statics ($\omega=0$), Eq. (113) becomes

$$Ek^2H\left(1+\frac{\ell^2}{2}k^2\right)=0 \tag{114}$$

This equation implies that there exists no non-zero critical wave number. This means that the waves are non-dissipative. So the plasticity model with implicit gradient-type enhancement of the plastic strain is not able to capture the localization of the harmonic wave in a band. This is the same behavior as observed for the nonlocal (Eq. (39)) and the explicit gradient-type (Eq. (90)) formulation of this plasticity model.

(b) Consider now the basic nonlocal plasticity model (Section 4) with an implicit gradient-type enhancement. Two differentiations of Eq. (42) with respect to x, together with Eq. (107), give

$$\dot{\hat{\kappa}}=\dot{\kappa}+\frac{E\ell^2}{2H}(\dot{u}_{,xxx}-\dot{\kappa}_{,xx}) \tag{115}$$

Rearranging Eqs. (41), (42) and (115), we obtain the following system of equations

$$E(\dot{u}_{,xx}-\dot{\kappa}_{,x})=\rho\dot{u}_{,tt} \tag{116}$$

$$\dot{f}=E\dot{u}_{,x}-E\dot{\kappa}-H(\dot{\kappa})+\frac{E\ell^2}{2H}(\dot{u}_{,xxx}-\dot{\kappa}_{,xx})=0 \tag{117}$$

Consider now harmonic wave solutions of frequency ω and wave number k, with velocity and cumulative plastic strain field given by Eq. (31). Substituting Eq. (31) and its partial derivatives with respect to x and t into Eqs. (116) and (117), we get

$$\left[\left(Ek^2-\rho\omega^2\right)\dot{u}_0+\mathrm{i}Ek\dot{\kappa}_0\right]\mathrm{e}^{\mathrm{i}(kx-\omega t)}=0 \tag{118}$$

$$\left[\mathrm{i}Ek\left(1+\frac{\ell^2}{2}k^2\right)\dot{u}_0-\left(E+H+E\frac{\ell^2}{2}k^2\right)\dot{\kappa}_0\right]\mathrm{e}^{\mathrm{i}(kx-\omega t)}=0 \tag{119}$$

The last two equations must be verified at every point x and every instant t. Since this is a system of two homogeneous linear equations, a non-trivial solution exists if and only if the determinant of the coefficient matrix vanishes, i.e.,

$$\left(Ek^2-\rho\omega^2\right)\left(E+H+E\frac{\ell^2}{2}k^2\right)-E^2k^2\left(1+\frac{\ell^2}{2}k^2\right)=0 \tag{120}$$

For $\omega=0$ we have

$$EHk^2=0 \tag{121}$$

This equation implies that the phase velocity $c_p = \omega/k$ is not a function of the wave number. So, the waves are non-dissipative. Therefore, the model is unable to change the shape of an arbitrary loading wave into a stationary wave reproducing the localization band. We must conclude that the basic nonlocal plasticity model with implicit gradient-type enhancement is unable to restore the well-posedness of the problem when softening occurs. This means that the implicit gradient formulation of the basic nonlocal plasticity model exhibits the same behavior as the nonlocal basic plasticity model with $m = 1$ (Section 4). This agrees with the conclusions of Peerlings et al. (2001) for the damage model.

(c) Consider now the ductile damage model (Geers et al., 2001; Engelen et al., 2003), in which the yield stress, Eq. (49), now depends on the cumulative plastic strain defined by Eq. (107). Two differentiations of Eq. (42) with respect to x, together with Eq. (107), furnish

$$\dot{\hat{\kappa}} = \dot{\kappa} + \frac{\ell^2}{2}\left(-E\dot{u}_{,xxx} + E\dot{\kappa}_{,xx} + \frac{\kappa_f - \kappa_0}{\kappa_f - \kappa_i} h\dot{\kappa}_{,xx}\right)\frac{\kappa_f - \kappa_i}{\sigma_0 + h\kappa_0} \tag{122}$$

Using the linear softening given by Eq. (50), the equation system that governs the localization problem now has the form:

$$E(\dot{u}_{,xx} - \dot{\kappa}_{,x}) = \rho\dot{u}_{,tt} \tag{123}$$

$$\dot{f} = E\dot{u}_{,x} - E\dot{\kappa} - \frac{\kappa_f - \kappa_0}{\kappa_f - \kappa_i} h\dot{\kappa} + \frac{\sigma_0 + h\kappa_0}{\kappa_f - \kappa_i}\dot{\kappa} + \frac{\ell^2}{2}\left(E\dot{\kappa}_{,xx} - E\dot{u}_{,xxx} + \frac{\kappa_f - \kappa_0}{\kappa_f - \kappa_i} h\dot{\kappa}_{,xx}\right) = 0 \tag{124}$$

If harmonic waves of frequency ω and wave number k, Eq. (31), are substituted in Eqs. (123) and (124), one gets

$$\left[(Ek^2 - \rho\omega^2)\dot{u}_0 + E\mathrm{i}k\dot{\kappa}_0\right]\mathrm{e}^{\mathrm{i}(kx-\omega t)} = 0 \tag{125}$$

$$\left\{E\mathrm{i}k\left(1 + \frac{\ell^2}{2}k^2\right)\dot{u}_0 + \left[-E\left(1 + \frac{\ell^2}{2}k^2\right) - \frac{\kappa_f - \kappa_0}{\kappa_f - \kappa_i} h\left(1 + \frac{\ell^2}{2}k^2\right) + \frac{\sigma_0 + h\kappa_0}{\kappa_f - \kappa_i}\right]\dot{\kappa}_0\right\}\mathrm{e}^{\mathrm{i}(kx-\omega t)} \tag{126}$$

The homogeneous system of two linear equations (Eqs. (125) and (126)) has a non-trivial solution if and only if the determinant of the coefficient matrix vanishes;

$$(Ek^2 - \rho\omega^2)\left[-E\left(1 + \frac{\ell^2}{2}k^2\right) - \frac{\kappa_f - \kappa_0}{\kappa_f - \kappa_i} h\left(1 + \frac{\ell^2}{2}k^2\right) + \frac{\sigma_0 + h\kappa_0}{\kappa_f - \kappa_i}\right] + E^2k^2\left(1 + \frac{\ell^2}{2}k^2\right) = 0 \tag{127}$$

In statics ($\omega = 0$), we have

$$\left[-\frac{\kappa_f - \kappa_0}{\kappa_f - \kappa_i} h\left(1 + \frac{\ell^2}{2}k^2\right) + \frac{\sigma_0 + h\kappa_0}{\kappa_f - \kappa_i}\right]Ek^2 = 0 \tag{128}$$

The non-zero value of $k = k_{\mathrm{cr}}$ that satisfies Eq. (128) is

$$k_{\mathrm{cr}} = \frac{\sqrt{2}}{\ell}\left(\frac{\sigma_0 + h\kappa_0}{(\kappa_f - \kappa_0)h} - 1\right)^{1/2} \tag{129}$$

The corresponding critical wavelength is

$$\lambda_{\mathrm{cr}} = \sqrt{2}\pi\ell\left(\frac{\sigma_0 + h\kappa_0}{(\kappa_f - \kappa_0)h} - 1\right)^{-1/2} \tag{130}$$

Since $\ell > 0$, the critical wavelength, given by Eq. (130), always has a finite real value. This implies that the implicit gradient enhancement of the ductile damage model achieves regularization of the problem due to softening, as previously shown in a different way by Rolshoven (2003). In Fig. 7, the critical wavelength is plotted as a function of the initial softening parameter κ_0. The plot shows how the nonlocal integral-type generalization of the ductile damage model exhibits the same localization properties as the implicit gradient enhancement (dashed-dotted line and dashed line in Fig. 7). Moreover, the two critical wavelength in Eqs. (59) and (130), which differ from that obtained for the explicit gradient enhancement, Eq. (106), ensure convergence to zero upon complete failure ($\kappa_0 \rightarrow \kappa_f$).

10. Localization analysis of an explicit gradient-type damage model

The constitutive model considered in Section 5 is now regularized by an explicit gradient-type formulation. The rate of the nonlocal energy release rate is given by

$$\dot{\widehat{Y}}(x) = \dot{Y}(x) + \frac{\ell^2}{2}\dot{Y}(x)_{,xx} \tag{131}$$

Using the definition of the rate of the damage energy release rate and substituting it into Eq. (131), we have

$$\dot{\widehat{Y}}(x) = E\epsilon\left(\dot{\epsilon} + \frac{\ell^2}{2}\dot{\epsilon}_{,xx}\right) \tag{132}$$

Then, substituting the last expression for the nonlocal rate of the damage energy release rate into Eq. (65), we obtain the following equation of motion:

$$(1-d_0)E\dot{u}_{,xx} - E^2\epsilon_0^2 F(\widehat{Y})\left(\dot{u}_{,xx} + \frac{\ell^2}{2}\dot{u}_{,xxxx}\right) = \rho\dot{u}_{,tt} \tag{133}$$

Consider again the harmonic wave perturbation, Eq. (12), propagating through the bar. Substituting Eq. (12) and its partial derivatives into Eq. (133), we obtain

$$\left\{(1-d_0)Ek^2 - E^2\epsilon_0^2 F(\widehat{Y})k^2\left(1 - \frac{\ell^2}{2}k^2\right) - \rho\omega^2\right\}\dot{u}_0\,\mathrm{e}^{\mathrm{i}(kx-\omega t)} = 0 \tag{134}$$

This equation has a non-trivial solution if and only if the braced term in Eq. (134) vanishes. This yields

$$\omega^2 = k^2\frac{E}{\rho}\left[1 - d_0 - E\epsilon_0^2 F(\widehat{Y})\left(1 - \frac{\ell^2}{2}k^2\right)\right] \tag{135}$$

The critical wave number, k_{cr}, which makes the phase velocity and the angular frequency vanish ($c_{\mathrm{p}} = \omega = 0$), is

$$k_{\mathrm{cr}} = \frac{\sqrt{2}}{\ell}\sqrt{\frac{d_0 - 1 + E\epsilon_0^2 F(\widehat{Y})}{E\epsilon_0^2 F(\widehat{Y})}} \tag{136}$$

and the corresponding critical wavelength is

$$\lambda_{\mathrm{cr}} = 2\pi/k_{\mathrm{cr}} = \pi\sqrt{2}\ell\sqrt{\frac{E\epsilon_0^2 F(\widehat{Y})}{d_0 - 1 + E\epsilon_0^2 F(\widehat{Y})}} \tag{137}$$

The critical wavelength is plotted in Fig. 8 (dotted line). The behavior of the explicit gradient-type damage model differs from the nonlocal damage model in terms of the width of the localization band (see Fig. 8). For the complete loss of material integrity, $d = 1$, the explicit gradient-type model provides a finite width of the localization band equal to $\pi\sqrt{2}\ell$.

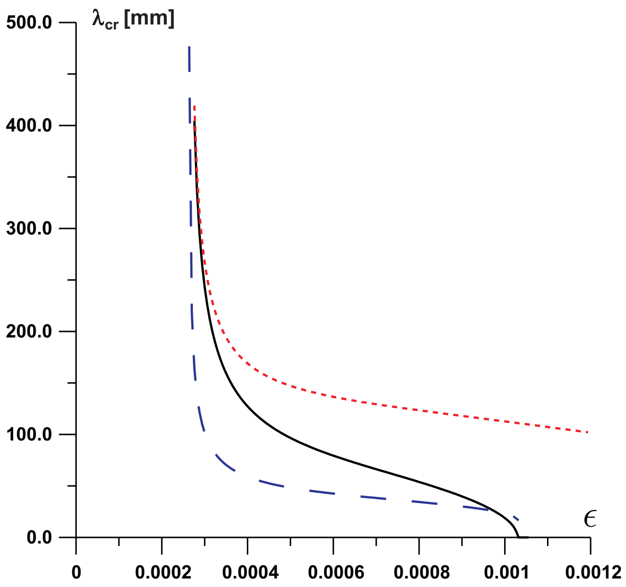

Fig. 8. Critical wavelength as function of the initial strain ϵ for an explicit gradient enhanced damage model (dotted line), for an implicit gradient enhanced model (solid line) and for a nonlocal damage model (dashed line).

11. Localization analysis of an implicit gradient-type damage model

As before, consider again an implicit gradient-type model, in which the rate of the nonlocal energy release rate is the solution of the following Helmholtz partial differential equation

$$\dot{\widehat{Y}}(x) - \frac{\ell^2}{2}\dot{\widehat{Y}}(x)_{,xx} = \dot{Y}(x) \tag{138}$$

Differentiating the condition $\dot{f} = 0$ (in Eq. (60)) twice with respect to x, we get

$$\dot{d}_{,xx} = F(\widehat{Y})\dot{\widehat{Y}}_{,xx} \tag{139}$$

Substituting $\dot{f} = 0$ and the last equation into Eq. (138), we have

$$\dot{d} - \frac{\ell^2}{2}\dot{d}_{,xx} = F(\widehat{Y})\dot{Y} \tag{140}$$

Then, substituting this equation into Eq. (64), we get

$$\dot{\sigma} = (1 - d_0)E\dot{\epsilon} - E\epsilon_0\dot{d} = (1 - d_0)E\dot{\epsilon} - E\epsilon_0\left(F(\widehat{Y})\dot{Y} + \frac{\ell^2}{2}\dot{d}_{,xx}\right) \tag{141}$$

and substituting this expression into the equation of motion (Eq. (10)), we obtain

$$(1 - d_0)E\dot{u}_{,xx} - E^2\epsilon_0^2F(\widehat{Y})\dot{u}_{,xx} - E\epsilon_0\frac{\ell^2}{2}\dot{d}_{,xxx} = \rho\dot{u}_{,tt} \tag{142}$$

Consider now the harmonic wave perturbation of the form:

$$\dot{u}(x,t) = \dot{u}_0\mathrm{e}^{\mathrm{i}(kx-\omega t)}, \quad \dot{d}(x,t) = \dot{d}_0\mathrm{e}^{\mathrm{i}(kx-\omega t)} \tag{143}$$

In the system of Eqs. (142) and (139), we now substitute the previous equations for the velocity field and the damage field, obtaining

$$\left\{\left[(1-d_0)Ek^2 - E^2\epsilon_0^2 F(\widehat{Y})k^2 - \rho\omega^2\right]\dot{u}_0 - E\epsilon_0\frac{\ell^2}{2}\mathrm{i}k^3\dot{d}_0\right\}\mathrm{e}^{\mathrm{i}(kx-\omega t)} = 0 \tag{144}$$

$$\left\{E\epsilon_0 F(\widehat{Y})\mathrm{i}k\dot{u}_0 - \left(1+\frac{\ell^2}{2}k^2\right)\dot{d}_0\right\}\mathrm{e}^{\mathrm{i}(kx-\omega t)} = 0 \tag{145}$$

A non-trivial solution exists if and only if the determinant of the foregoing system of equations vanishes. This yields

$$\omega^2 = k^2\frac{E}{\rho}\left(1 - d_0 - E\epsilon_0^2 F(\widehat{Y}) + \frac{E\epsilon_0^2 F(\widehat{Y})\ell^2 k^2}{2+\ell^2 k^2}\right) \tag{146}$$

The critical wave number, k_{cr}, that makes the phase velocity and the angular frequency vanish ($c_{\mathrm{p}} = \omega = 0$) is

$$k_{\mathrm{cr}} = \frac{\sqrt{2}}{\ell}\sqrt{\frac{d_0 - 1 + E\epsilon_0^2 F(\widehat{Y})}{1-d_0}} \tag{147}$$

and the corresponding critical wavelength is

$$\lambda_{\mathrm{cr}} = 2\pi/k_{\mathrm{cr}} = \pi\sqrt{2}\ell\sqrt{\frac{1-d_0}{d_0 - 1 + E\epsilon_0^2 F(\widehat{Y})}} \tag{148}$$

The critical wavelength is plotted in Fig. 8 (solid line). We observe that the behaviors of the implicit-type gradient model and the nonlocal damage model (dashed line) are similar; see Fig. 6. Both models predict a narrowing localization band, the width of which tends to zero for $d \to 1$. This is not surprising because the implicit model exhibits strong nonlocality. The same conclusion was reached by Peerlings et al. (2001) using a different damage model.

12. Conclusions

The purpose of the paper is to show a simple analytical derivation of the localization properties of various nonlocal softening models based on linearized wave propagation analysis.

1. For a softening constitutive law in which the yield limits are functions of the total strain, the internal length assumes a real positive value only if one adopts the nonlocal concept of Vermeer and Brinkgreve (1994), termed here 'over-nonlocal', in which a nonlocal variable obtained by spatial averaging with non-negative weights is multiplied by a factor $m > 1$, and the local variable multiplied by negative factor $1 - m$ is then superposed. The propagation wave velocity has a real value if and only if $m > 1$.
2. The localization analyses performed reveal that the effectiveness of the regularization achieved by integral-type nonlocality depends on the particular type of nonlocal plasticity model adopted. For the nonlocal plasticity model, in which the plastic strain is replaced by its nonlocal average in the stress–strain law, the harmonic waves are non-dissipative and the constitutive model is unable to reproduce the localization band due to the softening. The basic nonlocal plasticity model, in which the yield stress depends on the nonlocal cumulative plastic strain, provides correct regularization only if the over-nonlocal formulation is adopted. In this case, the critical wavelength, which characterizes the width of the

localization band, has a positive real value if and only if the over-nonlocal parameter m is greater than 1. The nonlocal integral-type ductile damage model, proposed by Geers et al. (2003) in the implicit gradient-type version, is able to give a correct localization behavior even for the standard nonlocality with $m = 1$ (i.e., the critical wavelength, has positive real value also for $m = 1$). Thus, among the plasticity models investigated, only the basic nonlocal plasticity model with $m > 1$ and the nonlocal ductile damage model of Geers et al. (2003) are able to describe the localization due to the softening, which confirms the conclusions already made by Jirásek and Rolsoven (2003) and by Rolshoven (2003).
3. When a nonlocal damage model is considered, the over-nonlocal approach is not needed (i.e., $m = 1$).
4. The localization analysis of the nonlocal integral-type models also shows that a narrowing of the critical wavelength is obtained only for the nonlocal generalizations of the ductile damage model and the damage model. This means that the width of the localization band tends to zero at complete material failure (i.e., formation of discrete crack).
5. The analysis also clarifies the role of the over-nonlocal parameter m when the nonlocal integral-type model is approximated by a gradient-type model. There is no difference between the nonlocal and over-nonlocal formulations in the gradient-type models. The localization analysis of the explicit gradient-type model shows that it is able to keep the problem well-posed for constitutive laws in which the yield limit is a function of the total strain. The dispersion analysis demonstrates that the basic nonlocal plasticity model, the damage model and the ductile damage model enhanced with an explicit gradient formulation provide a correct regularization of the softening problem.
6. The localization analysis of the implicit gradient-type model based on the Helmoltz equation reveals that this formulation is able to regularize the softening behavior of the damage model and the ductile damage model, but not the basic nonlocal plasticity model. This confirms in a simple way the early results of Askes et al. (2000), Peerlings et al. (2001, 2002), by showing that the implicit gradient-type model and the integral-type nonlocal model behave in the same way with respect to regularization of the softening problem.
7. The localization analysis of the gradient-type models shows that a narrowing of the localization band (i.e., critical wavelength converging to zero upon complete failure) is obtained only for the implicit gradient enhancement of the ductile damage model and the damage model. By contrast, the explicit gradient enhancement of all the considered softening models gives a critical wavelength that does not tend to zero at complete failure. These results confirm the earlier conclusion of Peerlings et al. (1996a,b) on the damage model.

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