Scaling of Sea Ice Fracture—Part I: Vertical Penetration

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Based on the premise that large-scale failure of sea ice is governed by fracture mechanics, recently validated by Dempsey's in situ tests of fracture specimens of a record-breaking size, this two-part study applies fracture mechanics and asymptotic approach to obtain approximate explicit formulas for the size effect in two fundamental problems. In the present Part I, the load capacity of a floating ice plate subjected to vertical load is determined, and in Part II, which follows, the horizontal force exerted by an ice plate moving against a fixed structure is analyzed in a similar manner. The resulting formulas for vertical loading agree with previous sophisticated numerical fracture simulations as well with the limited field tests of vertical penetration that exist. The results contrast with the classical predictions of material strength or plasticity theories, which in general exhibit no size effect on the nominal strength of the structure. [DOI: 10.1115/1.1429932]

1 Introduction

Predictions of load capacity and failure of floating sea ice require good understanding of the scaling properties and size effect. Because small-scale laboratory tests of sea ice show hardly any notch sensitivity and do not exhibit fracture mechanics behavior, many studies from early to recent times have treated sea ice failure according to either plasticity or elasticity theory with a strength limit ([1–8]). Both theories exhibit no size effect. When size effects were observed in tests, they were generally attributed to randomness of material strength (e.g., [9]), captured by Weibull [10] theory stemming from the qualitative idea of Mariotte [11] and mathematically justified by extreme value statistics ([12]), see reviews in, e.g., [13–15]. However, the statistical explanation of size effect is, for the present problem, dubious because the maximum load is not reached at the initiation of fracture but only after large stable crack growth (in detail, see, e.g., [14,15]). In that case a nonlocal generalization of Weibull theory is required ([16,17]). The nonlocal probabilistic analysis shows that the statistical size effect becomes significant only of for very large structures failing at fracture initiation. Otherwise the energetic (deterministic) size effect dominates.

Many studies document the brittleness of ice (e.g., [18,19]). Various recent experiments ([20–22]) especially the remarkable in situ tests of Dempsey's team made with record-size specimens ([23–26]), indicate that on a scale exceeding about 0.5-m sea ice does follow cohesive (quasi-brittle) fracture mechanics, with a strong size effect, and on scales larger than about 10 m is very well described by linear elastic fracture mechanics (LEFM). The need for fracture mechanics approach and the presence of size effect is also suggested by the fact that the experimental load-deflection diagrams (e.g., [8]) exhibit no yield plateau but a gradual softening, i.e., a decrease of load with increasing deflection after the peak load has been reached. Analysis of acoustic observations, too, suggests a size effect ([27]).

The analysis of failure and especially the size effect must, therefore, be based on fracture mechanics. Many investigators have been applying to sea ice fracture problems the linear elastic fracture mechanics (LEFM) in which the fracture process zone at the crack tip is assumed to be infinitely small. However, as inspired from the field fracture tests of size effect by Dempsey et al. [23,24], the length of the fracture process zone of sea ice is of the order of several meters for horizontal propagation, while for vertical propagation it is roughly 25 cm. Therefore, the cohesive crack model or some of its approximations must be used. Two basic types of cohesive crack model need to be distinguished: (a) the *brittle-ductile* model, in which the stress-displacement relation has a long horizontal yield plateau, terminating by a sharp drop at a certain critical opening displacement, and (b) the *quasi-brittle* model, in which the cohesive crack-bridging stress gradually decreases according to a fixed law as a function of the opening displacement. The former was developed long ago for metals, and the latter more recently for concrete ([15]). It is the latter type which appears more appropriate for sea ice.

In view of the quasi-brittle behavior, the deterministic (energetic) size effects of quasi-brittle fracture ([14,15,28–32]) must get manifested, and must be expected to be strong, in all the problems in which large cracks grow stably prior to reaching the maximum load ([33,34]). This includes two fundamental problems to be addressed in Parts I and II of this study: (1) the vertical load capacity of floating ice plate (penetration fracture), and (2) the maximum horizontal force exerted on a fixed structure by a moving ice plate.

The vertical penetration problem has been analyzed by fracture mechanics at various levels of sophistication in several recent works. Bažant and Li [35,36] assumed that full-through bending cracks propagate radially from the loaded area, but this assumption now appears inapplicable except perhaps for very thin plates in which the horizontal forces due to dome effect nearly vanish. Dempsey with co-workers [37], in an elegant analytical solution of the problem, assumed that the radial cracks at maximum load emanating from the loaded area reach through only a part of the ice thickness. To make an analytical solution feasible, they made various simplifying assumptions, the main one being a uniform crack depth.

The aforementioned simplifications were avoided in a numerical simulation of penetration fracture in [38,39], which confirmed that indeed the cracks reach only through a part of the thickness and propagate at the maximum load stage mainly vertically, although the crack depth is not uniform. This numerical simulation indicated that for larger ice thicknesses there is a strong size effect, approaching the size effect of geometrically similar failures governed by LEFM, for which the nominal strength is proportional to $(\text{ice thickness})^{-1/2}$. This conclusion represents a sharp contrast with the classical solutions based on plasticity or elasticity with a strength limit. Such solutions inevitably imply the absence of any size effect.

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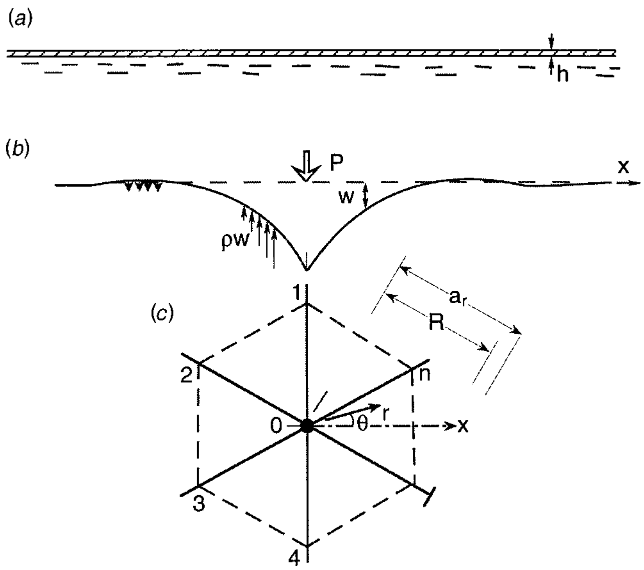

Fig. 1 Floating ice plate, its deflection under concentrated load and crack pattern

Analysis of another ice fracture problem, namely the large-scale thermal bending fracture of floating ice ([40,41]), also indicated a strong size effect, obeying, however, a different law. In this case, the critical temperature difference is not proportional to $(\text{ice thickness})^{-1/2}$, as in LEFM, but to $(\text{ice thickness})^{-3/8}$. The reason is that, at large scale, the cracks must propagate horizontally as bending cracks, rather than vertically across the thickness. A size effect following still another law was recently demonstrated for the fracture of ice subject to a line load ([42]).

As typical for all quasi-brittle materials, the size effect is very difficult to analyze for the normal sizes of interest, but becomes much simpler asymptotically for very large sizes as well as very small sizes ([14,15,43]). The philosophy of *asymptotic matching* ([44]) can then be employed to ''interpolate'' between the opposite asymptotic size effects. This furnishes an approximate solution for the size effect in the difficult intermediate range. This approach, pioneered and widely used in fluid mechanics (e.g., [45–47]), has been successfully employed in many studies of concrete and a more recently in studies of fiber composites and rock ([14,15]).

Static behavior until failure will be assumed in all of the present analysis. Situations in which the ice might acquire significant kinetic energy during a temporarily unstable fracture fracture propagation will not be considered. The creep of ice will not be explicitly considered and the elastic modulus of ice will be assumed to represent the effective modulus that approximately incorporates the effect of creep for the prevalent loading rate.

The purpose of the present two-part study, based on a recent workshop article ([43]), is to employ the asymptotic matching approach to deduce simple approximate formulas for the nominal strength of the ice plate as a function of the size as well as geometry. Such an approach helps intuitive understanding, clarifies the failure mechanism, facilitates optimization of engineering design, elucidates the role of energy release as the main source of size effect, and readily reveals how the material and geometry parameters control the size effect. Part I will deal with the vertical load, and Part II which follows with the horizontal load.

2 Problem Formulation

An ice plate floating on water behaves exactly as a plate on Winkler elastic foundation (Fig. 1(*a*,*b*)), with a foundation modulus equal to the specific weight of water, ρ. Failure under a vertical load is known to involve formation of radial bending cracks in a star pattern (shown in a plan view in Fig. 1(*c*) for the case of six cracks). As transpired from a simplified analytical study of Dempsey et al. [37] and from a detailed numerical simulation ([38,39]), these radial cracks do not reach through the full ice thickness before the maximum load is reached. Rather, they penetrate at maximum load to an average depth of about $0.8h$ and maximum depth $0.85h$ where h is the ice thickness (Fig. 2*a*). The maximum load is reached when polygonal (circumferential) cracks, needed to complete a failure mechanism, begin to form (dashed lines in Fig. 1(*c*)).

The nominal strength, which is a parameter of the maximum vertical load P, is defined for the vertical penetration problem as

$$\sigma_N = P/h^2. \tag{1}$$

In plasticity or any theory in which the material failure criterion is defined in terms of stresses and strains, the nominal strength (of a nonrandom material) is size independent for geometrically similar structures. The size effect in fracture and damage mechanics arises from the fact that the criterion of material failure (crack growth) is expressed in terms of energy (or stress-displacement relation).

Sea ice, unlike glacier ice, is not sufficiently confined to behave plastically (this is for example confirmed by the absence of yield plateau from the measured load-deflection diagrams seen, e.g., in [8]. Sea ice is a brittle material, and so the failure must be analyzed by fracture mechanics (e.g., [20–22,35,36,38–41,48]). The analysis must be based on the rate of energy dissipation at the crack front and the rate of energy release from the ice-water system. The energy release is associated with unloading, during which the ice deforms elastically, with a certain Young's modulus E (which depends on temperature and other factors).

The behavior of the ice plate may be described by the plate bending theory. Dimensional analysis, or transformation of the partial differential equation of a plate on Winkler foundation to dimensionless coordinates, shows that the behavior of the plate is fully characterized by the characteristic length

$$L = (D/\rho)^{1/4} \tag{2}$$

where $D = Eh^3/12(1-\nu^2)$ = cylindrical stiffness of the ice plate; ν = Poisson ratio of ice.

3 Energy Release and Equilibrium of Fractured Ice Plate

Superposing the expressions for the stress intensity factor K_I of the part-through radial bending crack of depth a (Fig. 3*b*,*d*) produced by bending moment M and normal force N (per unit length), one has

$$K_I = \frac{\sqrt{\pi a}}{h}\left[\frac{6M}{h}F_M(\alpha) + NF_N(\alpha)\right] \tag{3}$$

where

$$F_M(\alpha) = \sqrt{\frac{2}{\pi\alpha}\tan\frac{\pi\alpha}{2}}\left(\cos\frac{\pi\alpha}{2}\right)^{-1} \times\left[0.923 + 0.199\left(1-\sin\frac{\pi\alpha}{2}\right)^4\right] \tag{4}$$

$$F_N(\alpha) = \sqrt{\frac{2}{\pi\alpha}\tan\frac{\pi\alpha}{2}}\left(\cos\frac{\pi\alpha}{2}\right)^{-1} \times\left[0.752 + 2.02\alpha + 0.37\left(1-\sin\frac{\pi\alpha}{2}\right)^3\right] \tag{5}$$

([15,49,50]) with an error less than 0.5 percent over the entire range $\alpha \in (0,1)$. According to Irwin's relation, the energy release rate is

$$\mathcal{G} = \frac{K_I^2}{E'} = \frac{N^2}{E'h}g(\alpha) \tag{6}$$

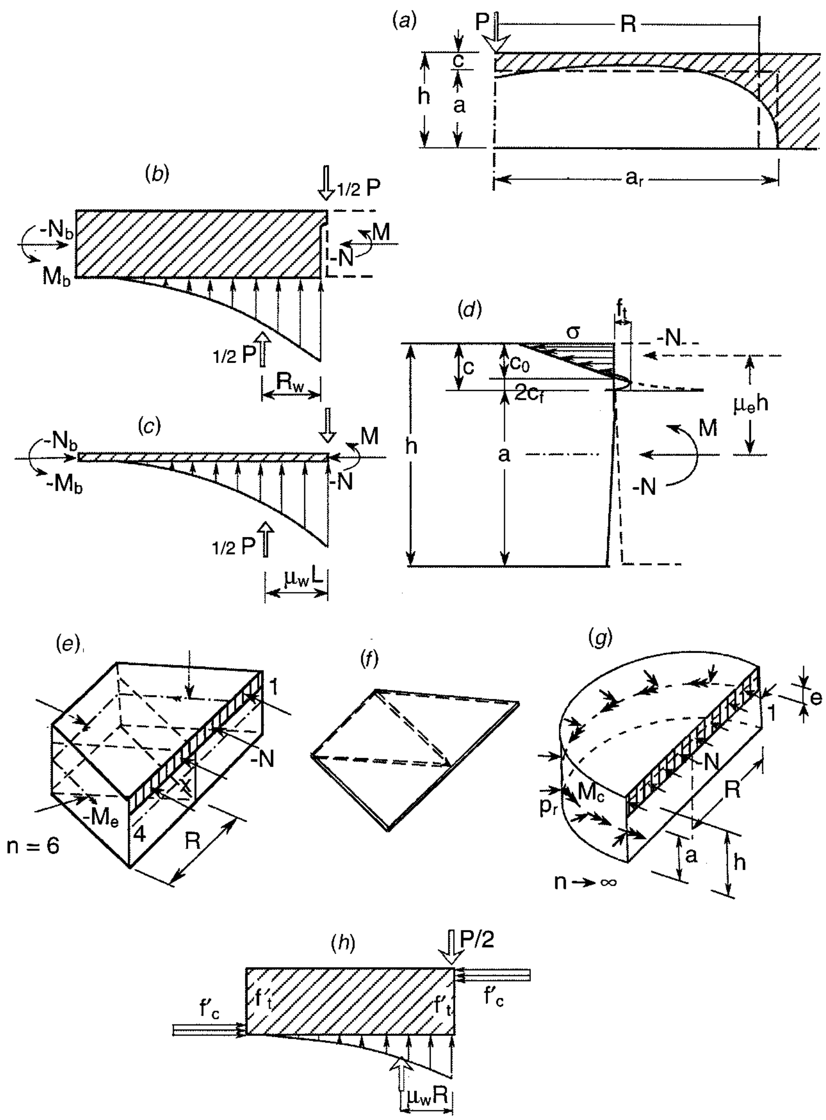

Fig. 2 Analysis of vertical penetration fracture: (*a*) crack profile and (*b–h*) forces acting on element 123401 in Fig. 1

where $E' = E/(1-\nu^2)$ and g is a dimensionless function,

$$g(\alpha) = \pi\alpha\left[\frac{6e}{h}F_M(\alpha) + F_N(\alpha)\right]^2 \quad (\alpha = a/h). \tag{7}$$

$e = -M/N =$ eccentricity of the normal force resultant in the cross section (positive when N is above the midplane).

To relate M and N to vertical load P, let us consider element 12341 of the plate (Figs. 1(*c*) and 2(*e,f,g*)), limited by a pair of opposite radial cracks and the initiating polygonal cracks. The depth to the polygonal cracks at maximum load is zero, as they just initiate, and since the cracks must form at the location of the maximum radial bending moment, the vertical shear force on the planes of these cracks is zero. The distance R of the polygonal cracks from the vertical load P may be expected to be proportional to the characteristic length L since this is the only length constant in the differential equation governing the problem, and so we may set $R = \mu_R L$ where dimensionless μ_R is assumed to be a constant.

In each narrow radial sector, the resultant of the water pressure due to deflection w (Fig. 2(*b,c*)) is located at a certain distance r_w from load P. Since r_w can be solved from the differential equation for w, and since the solution depends only on one parameter, the characteristic length L, r_w must be proportional to L. Integration over the area of a semi-circle of radius r_w yields the resultant of water pressure acting on the whole element 12341. Again, the distance of this resultant, whose magnitude it $P/2$, from load P must be proportional to L, i.e., may be written as

$$R_w = \mu_m L \tag{8}$$

where μ_w is a constant that can be solved from the differential equation of plate deflections. Of course, μ_w is a constant only as long as the behavior is elastic, which is exactly true only if the crack depth a is constant. Although the crack is growing, we will assume that its rate of growth is small enough so that μ_w would be approximately constant.

For the sake of simplicity, we assume the normal force N and bending moment M on the planes of the radial cracks and the polygonal cracks to be uniform. The condition of equilibrium of horizontal forces acting on element 12341 in the direction normal to the radial cracks is then simple; it requires the normal forces on the planes of the polygonal cracks to be equal to the normal force N acting in the radial crack planes. The axial vectors of the moments M_c acting on the polygonal sides are shown in Fig. 2(*e,g*) by double arrows. Summing the projections of these axial vectors from all the polygonal sides of the element, one finds that their moment resultant with axis in the direction 14 is $2RM_c$, regardless of the number n of radial cracks. So, upon setting $R=\mu_R L$, the condition of equilibrium of the radial cracks with the moments about axis 14 (Fig. 2(b,c,e,g)) located at midthickness of the cross section may be written as

$$2(\mu_R L)M+2(\mu_R L)M_c-\frac{1}{2}P(\mu_w L)=0. \tag{9}$$

Furthermore, we must take into account condition (6) of vertical propagation of the radial bending cracks, which may be written as $\mathcal{G}=G_f$ where G_f is the fracture energy of ice. Thus, the critical value of normal force (compressive, with eccentricity e) may be written as

$$N=-\sqrt{\frac{E'G_f\ h}{g(\alpha)}}. \tag{10}$$

Depending on the energy release rate $g(\alpha_0)$ of the actual crack of length $a_0=\alpha_0 D$ (excluding the cohesive zone), there are two kinds of deterministic size effect: (a) the size effect due to energy release of a large crack, characterized by a large value of $g(\alpha_0)$, and (b) the size effect at crack initiation $(\alpha_0=0)$, characterized by $g(\alpha_0)=0$. They are governed by different laws ([14,15,30,32,51]), and both must be expected to occur in ice penetration.

4 Size Effect on Flexural Strength at Initiation of Polygonal Cracks

Consider first the initiation of the polygonal cracks. Since $\alpha_0=0$ and $g(\alpha_0)=0$, the initiation criterion is that the normal stress σ reaches the tensile strength f_t' of the ice. However, the crack can begin to propagate only after a boundary layer of distributed microcracking, representing the fracture process zone, forms at the top surface of ice ([14,15,30,51,52]). The half-depth of this layer, denoted as D_b, is a material constant (which should be roughly equal to the fracture process zone length c_f introduced later). Note that the boundary layer D_b has been shown to explain the experimentally observed size effect on the modulus of rupture in the bending tests of concrete ([15,52]).

Although the crack initiation can be handled by the energy release function, it is simpler to consider the stress redistribution in the cross section caused by softening in the boundary layer ([52]). The easiest way to obtain a nominal strength formula that is correct up to the first two terms of the expansion in terms of powers of $1/h$ is to write the condition that the elastically calculated normal stress σ_e should be equal to the tensile strength of ice, f_t', at the middle of the boundary layer of thickness $2D_b$, rather than at ice surface. So the crack initiation criterion is $\sigma_e+N/h=f_t'$ where, according to the bending stress formula, $\sigma_e=M_c(h/2-D_b/2)/(h^3/12)$. This yields the crack initiation criterion:

$$\frac{6M_c}{h^2}q(h)+\frac{N}{h}=f_t' \tag{11}$$

where $q(h)=1-D_b/h$. This form of the criterion, however, becomes meaningless when $h<2D_b$, i.e., when the ice is thinner than the cracking layer thickness. It can be correct only when h is sufficiently larger than $2D_b$, i.e., asymptotically for $h/D_b\to\infty$. So it is desirable to modify function $q(h)$ so as to obtain a formula approximately applicable through the entire size range. This can be achieved by considering a range of σ_N formulas that have the same first two terms of the large-size asymptotic expansion in $1/h$ as (11), and then choosing that which gives the correct value of the small-size nominal strength. Such a kind of approach is known as asymptotic matching.

When $h=2D_b$, i.e., when the distributed cracking zone encompasses essentially the whole depth of plate, the moment at failure can be approximately determined as the plastic bending moment M_p. If f_c'/f_t', with f_c' = compression strength of ice, is about 1, then the plastic stress distribution is symmetric bi-rectangular and $M_p/M_e=1.5$, where M_e = elastically calculated bending moment for which $\sigma=f_c'/f_t'$ at ice surface. If f_c'/f_t' were very large, then the stress distribution would be a single rectangle balanced by a concentrated compression force at ice surface, and in that case M_p/M_e would be equal to 3. The real value must lie in between, but probably closer to 1.5. We will safely assume that $M_p/M_e=1.5$. So we should seek a formula for $q(h)$ that gives this ratio for $h=D_b$ and has a large-size asymptotic expansion of the form $1-D_b(1/h)+(\cdot)(1/h)^2+\ldots$. There are many such formulas but the simplest one is

$$q(h)=\frac{1+D_b/h}{1+2D_b/h}. \tag{12}$$

This is verified by the asymptotic expansion:

$$\frac{1+D_b/h}{1+2D_b/h}=\left(1+\frac{D_b}{h}\right)\left(1-\frac{2D_b}{h}+\frac{4D_b^2}{h^2}-\ldots\right)$$
$$=1-\frac{D_b}{h}+\frac{(\cdot)}{h^2}+\frac{(\cdot)}{h^3}+\ldots. \tag{13}$$

5 Size Effect on Nominal Vertical Penetration Strength

Aside from the stress redistribution at initiation of polygonal cracks ([52]), there is another deterministic source of size effect—the energy release due to vertical propagation of the radial bending cracks ([28]). The bending moment

$$M=-Ne=-N\mu_e h \tag{14}$$

may be substituted into (9); here the normal force N is defined to be positive when tensile, although the actual value of N is negative (compression); and $\mu_e=e/h$ = dimensionless parameter whose value at maximum load may be assumed to be approximately constant. This assumption is indicated by the numerical simulations in [38,39], from which it further transpires that $\mu_e\approx 0.45$, as a consequence of the fact that the average crack depth a at maximum load is about $0.8h$ (in any case, $\mu_e<0.5$, and so a possible error in μ_e cannot have a large effect). The value 0.45 approximately corresponds to the correct number of cracks in the star pattern; if there were more cracks, the depth would be smaller, if fewer, larger.

After substituting (14), we may express M_c from (9) and substitute it into (11). Then, taking into account (10), we obtain after rearrangements the equation:

$$\sigma_N=\frac{2\mu_R}{3\mu_w}\left[\left(6\mu_e+\frac{1}{q(h)}\right)\sqrt{\frac{E'G_f}{hg(\alpha)}}+\frac{f_t'}{q(h)}\right] \tag{15}$$

where $q(h)$ is given by (12).

Now we need to decide how the values of α at maximum load should vary with ice thickness h. To this end, note that ice is a quasibrittle material. This is evidenced by the fact that at small laboratory scale it is notch-insensitive and exhibits no size effect while at large scale it behaves according to LEFM ([20,24]). Therefore, at the tip of the vertically propagating radial crack, there must exist a finite fracture process zone (FPZ) of a certain characteristic depth $2c_f$ which is a material property. This zone was modeled in the numerical simulations of Bažant and Kim [38,39] as a yielding zone. The tip of the equivalent LEFM crack lies approximately in the middle of the FPZ, i.e., at a distance c_f from the actual crack tip ([15]), whose location is denoted as a_0.

If the locations of the center of the FPZ in structures of different sizes were geometrically similar, i.e., if α at maximum load were the same for all h, then the size effect would be the same as in LEFM. Experience with testing of quasi-brittle materials ([15]), as well as with cohesive crack and nonlocal damage simulations, shows the locations of the center of FPZ are usually not geometrically similar. Rather, similar locations are those of the actual crack tip. Thus the value of $\alpha_0 = a_0/h$ may be expected to be approximately constant when ice plates of different thicknesses h are compared. Denoting $g'(\alpha_0) = dg(\alpha_0)/d\alpha_0$, one may introduce the approximation

$$g(\alpha) \approx g(\alpha_0) + g'(\alpha_0)(c_f/D). \tag{16}$$

Substituting this into (15) and rearranging, one gets for the size effect the formula

$$\sigma_N = \frac{4\mu_R}{\mu_w}\left(\mu_e + \frac{1}{6q(h)}\right)\sqrt{\frac{E'G_f}{hg(\alpha_0) + c_f g'(\alpha_0)}} + \frac{\mu_R}{3\mu_w}\frac{f_t'}{q(h)}. \tag{17}$$

The results of numerical simulations in [39] were found to be quite well represented by the simple classical size effect law with large-size residual strength σ_r proposed in [53] which reads

$$\sigma_N = \sigma_0\left(1 + \frac{h}{h_0}\right)^{-1/2} + \sigma_r. \tag{18}$$

Formula (17) is now seen to reduce to this law when $q(h) \approx 1$, i.e., when D_b is negligible, in which case then

$$\sigma_0 = \frac{4\mu_R\mu_e}{\mu_w}\sqrt{\frac{E'G_f}{c_f g'(\alpha_0)}}, \quad h_0 = c_f\frac{g'(\alpha_0)}{g(\alpha_0)}, \quad \sigma_r = \frac{\mu_e}{3\mu_w}f_t'. \tag{19}$$

Furthermore, the numerical simulations in [39] indicated that $\sigma_r \approx 0$. This means that the contribution of the tensile strength f_t' governing the initiation of the polygonal cracks must be negligible, which in turn implies a negligible role for $q(h)$.

The terms in (17) containing D_b anyway decrease with increasing h much more rapidly than (18)—they decrease with increasing h as $1/h$, compared to $1/\sqrt{h}$. Consequently, they must become negligible for not too large h regardless of the value of D_b.

Same as (18), formula (17) plotted as $\log\sigma_N$ versus $\log h$ approaches for large h a downward inclined asymptote of slope $-1/2$ (Fig. 3(g)). This characterizes the large-size asymptote of the size effect law in (17).

How does the number n of the radial cracks enter the solution? It does not appear in the present solution for the maximum load. The reason is that the number of cracks is decided at the beginning of loading, long before the maximum load is attained.

It is interesting to contrast the size effect obtained here with that deduced for large-scale thermal bending fracture of floating ice, which was shown to be ([40])

$$\Delta T \propto h^{-3/8} \tag{20}$$

where ΔT is the temperature difference between the bottom and top of the ice plate, which is proportional to the maximum thermal stress before fracture. The large-size asymptotic size effect for fracture under vertical loads would have to follow also the $-3/8$ power law if the cracks at maximum load penetrated through the full thickness of ice and force N were negligible ([35,40,42,54]). But this turned out not to be the case ([37–39,55,56]).

6 Comments on Plasticity Approach

In contrast to the brute-force numerical simulations conducted before, the approximate analytical derivation of size effect is intuitively instructive. It clarifies the reasons why there must be a deterministic size effect in penetration of floating ice. The size effect could be absent only if the material behaved plastically.

If the sea ice were a plastic material, the stress distributions on element 12341 would be as shown in Fig. 2(h), where f_t' and f_c' denote the tensile and compressive yield strengths. Taking the moment equilibrium condition of this element, one can easily show that the nominal strength would in that case be expressed as

$$\sigma_N = \frac{4\mu_R}{\mu_w}(f_c'^{-1} + f_t'^{-1})^{-1} \tag{21}$$

which exhibits no size effect. Plasticity, however, requires that the material strength at all the points of the failure surface be mobilized at the same time, which is impossible for a quasi-brittle (softening) material such as sea ice.

7 Closing of Part I

The simplified asymptotic analysis of size effect in vertical penetration of the ice plate confirms the inevitability of a strong size effect for larger ice thicknesses, approaching the size effect of LEFM. This conclusion does not disagree with experiments and is supported by previous numerical studies summarized in the Appendix. Part II which follows will apply a similar approach to the problem of an ice plate moving against a fixed structure. It will be seen that size effects must again be expected, but their nature is rather different.

Appendix

Review of Previous Numerical Fracture Analysis of Size Effect. To supplement the analytical approach, it may be useful to review recent detailed numerical simulation of fracture of floating ice caused by a vertical load ([38,39]). The fracture pattern (for the case of six radial cracks) is shown in Fig. 3(a). The radial cracks at maximum load penetrate through only a part of ice thickness ([26,55]); Fig. 3(b,c). The radius of each crack is divided by nodes into vertical strips in each of which the vertical crack growth obeys Rice and Levy's [57] "nonlinear line-spring" model relating the normal force N and bending moment M in the cracked cross section to the relative displacement Δ and rotation θ (Fig. 3(b)).

The analysis is based on a simplified version of the cohesive crack model in which the vertical crack growth in each vertical strip is initiated according to a strength criterion. The cross section behavior is considered elastic-plastic until the yield envelope in the (N, M) plane is crossed by the point (N, M) corresponding to fracture mechanics. For ease of calculations, a nonassociated plastic flow rule corresponding to the vector $(d\Delta, d\theta)$ based on fracture mechanics is assumed.

The following ice characteristics have been used in calculations: tensile strength $f_t' = 0.2$ MPa, fracture toughness $K_c = 0.1\ \text{MPa}\sqrt{\text{m}}$, Poisson ratio $\nu = 0.29$, and Young' modulus $E = 1.0$ GPa, with the corresponding values: fracture energy $G_f = K_c^2/E = 10\ \text{J/m}^2$, and Irwin's fracture characteristic length $l_0 = (K_c/f_t')^2 = 0.25$ m (this value happens to be about the same as for concrete).

Figure 3(e) displays, with a strongly exaggerated vertical scale, the calculated crack profiles at subsequent loading stages. Fig. 3(f) shows the numerically calculated plot of the radial crack length a versus the ice thickness h ("fracture length" means the radial length of open crack, and "plastic length" the radial length up to the tip of plastic zone). This plot reveals that, except for very thin ice, the radial crack length $a \approx c_h h$ where $c_h \approx 24$ for the typical ice properties assumed.

The data points in Fig. 3(g) show, in logarithmic scales, the numerically obtained size effect plot of the normalized nominal strength $\sigma_N = P/h^2$ versus the relative thickness of the ice (note that according to plasticity or elasticity with strength criterion, this plot would be a horizontal line). The initial horizontal portion, for which there is no size effect, corresponds to ice thinner than about 20 cm.

Since the model in [38,39] includes plasticity, it can reproduce the classical solutions with no size effect, depending on the input values of ice characteristics. The ice thickness at the onset of size effect depends on the ratio of ice thickness to the fracture characteristic length, h/l_0. For realistic ice thicknesses h ranging from 0.1 m to 6 m, the computer program would yield perfectly plastic

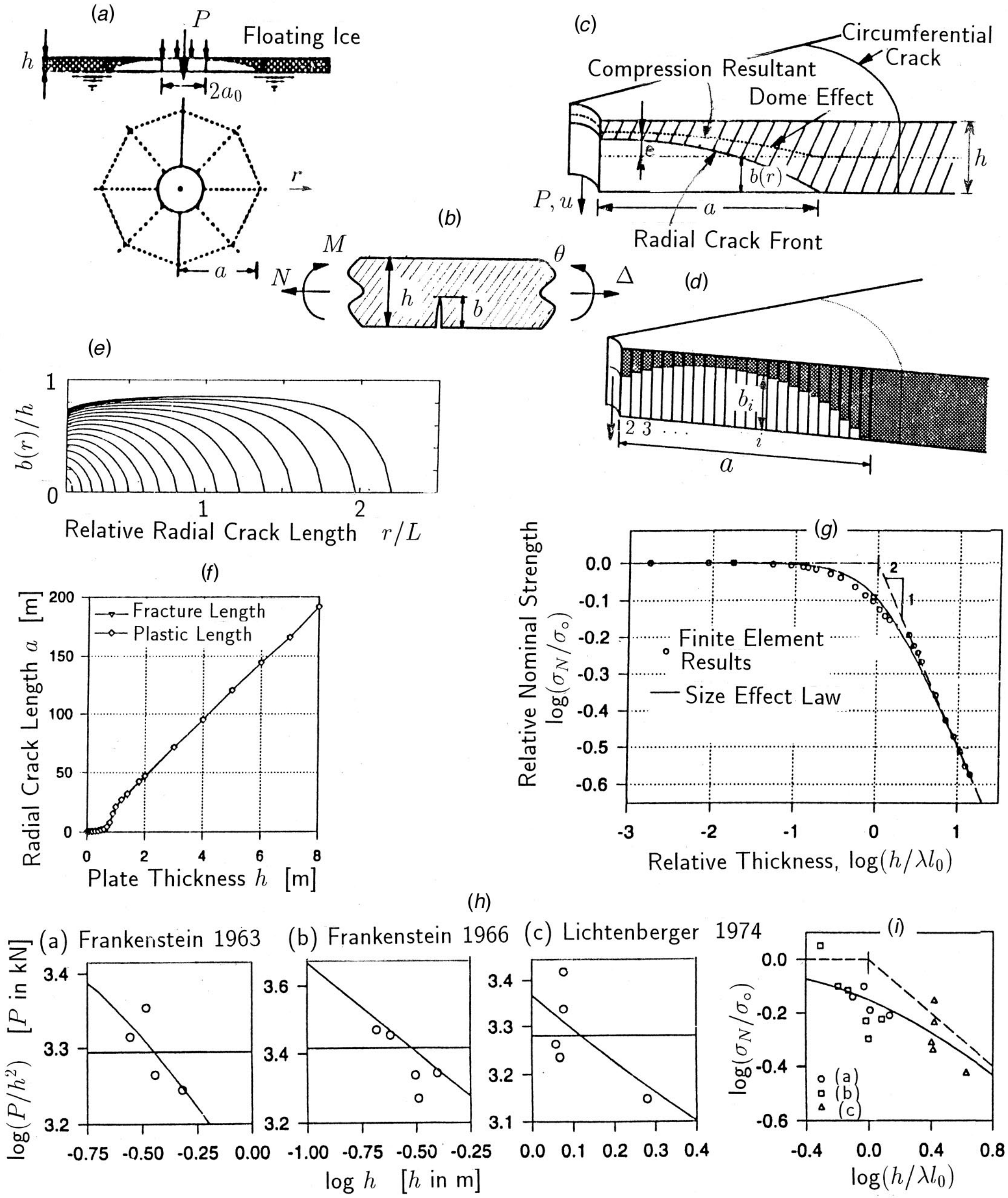

Fig. 3 Vertical penetration fracture problem analyzed by Bažant and Kim [38,39] main numerical results, and comparison with field tests of Frankenstein [59,60] and Lichtenberger [61]

response with no size effect if the fracture characteristic length l_0 were at least 100× larger, i.e., at least 25 m. This would, for instance, happen if either f_t' were at least 10× smaller ($f_t' \leqslant 0.01$ MPa or K_c at least 10× larger ($K_c \geqslant 10\ \mathrm{MPa}\sqrt{\mathrm{m}}$). The entire diagram in Fig. 3(*g*) would then be horizontal.

Larger values of l_0 are of course possible in view of statistical scatter, but nothing like 100× larger. For example, by fitting size effect data ([23,24]) from in situ tests at Resolute, one gets $K_c \approx 2.1\ \mathrm{MPa}\sqrt{\mathrm{m}}$, and with $f_t' \approx 2$ MPa one has the fracture characteristic length $l_0 = (K_c/f_t')^2 = 1$ m. But this larger value would not make much difference in the size effect plot in Fig. 3(*g*). The reason that these values were not used in the plot in Fig. 3(*g*) was that they correspond to long-distance horizontal propagation of fracture, rather than vertical growth of fracture.

The curve in Fig. 3(*g*) is the optimum fit of the numerically calculated data points by the generalized size effect law proposed in Bažant [58]. The final asymptote has slope −1/2, which means that the asymptotic size effect is $\sigma_N \propto h^{-1/2}$, the same as for LEFM with similar cracks, and not $h^{-3/8}$ as proposed by Slepyan [35,40,54]. The −3/8 power scaling would have to be true if the radial cracks at maximum load were full-through bending cracks. The −1/2 power scaling may be explained by the fact that during failure the bending cracks are not full-through and propagate mainly vertically, which is supported by the calculated crack profiles in Fig. 3(*e*).

By fitting of the data points in Fig. 3(*g*), spanning over four orders of magnitude of ice thickness *h*, the following prediction formula in the form of the generalized size effect law ([15,41]) has been calibrated (see the curve in Fig. 3(*g*)):

$$P_{\max} = \sigma_N h^2, \quad \sigma_N = B f_t' [1 + (h/\lambda_0 l_0)^r]^{-1/2r} \tag{22}$$

with $B = 1.214$, $\lambda_0 = 2.55$, $m = 1/2$, $r = 1.55$, and $l_0 = 0.25\,\mathrm{m}$ ($f_t' = 0.2\,\mathrm{MPa}$ in Fig. 1(g)).

Only very limited field test data exist. The data points in the size effect plots in Fig. 1(h) represent the results of the field tests by [59–61], and the curves show the optimum fits with the size effect formula verified by numerical calculations (note that if the size effect were absent, these plots of nominal strength would have to be horizontal). After optimizing the size effect law parameters by fitting the data in the three plots in Fig. 3(h), the data and the optimum fit are combined in the dimensionless plot in Fig. 3(i).

Interesting discussions of ([38,39]) were published by Dempsey [62] and Sodhi [63] and rebutted. One objection raised by Sodhi was the neglect of creep in Bažant and Kim's analysis. Intuition suggests that the influence of creep might be like that of plasticity, which tends to increase the process zone size, thereby making the response less brittle and the size effect weaker. But the opposite is true ([15]).

The influence of creep on scaling of brittle failures of concrete, which is doubtless quite similar from the mechanics viewpoint (albeit different in physical origin), was studied in depth at Northwestern University, along with the effect of the crack propagation velocity; see, e.g., [15,34,64] and especially [65,66]. The conclusion from these studies, backed by extensive fracture testing of concrete and rock at very different rates, is that creep in the material always makes the size effect due to cracks stronger (unless creep actually prevents crack initiation). In the logarithmic size effect plot of nominal strength versus structure size, it causes a shift to the right, toward the LEFM asymptote, which means that the size effect is intensified by creep. The slower the loading (or the longer its duration), the closer to LEFM is the size effect in a cracked structure.

The physical reason, clarified by numerical solutions of stress profiles with a rate-dependent cohesive crack model ([66]), is that the highest stresses in the fracture process zone get relaxed by creep, which tends to reduce the effective length of the fracture process zone. The shorter the process zone, the higher is the brittleness of response and the stronger is the size effect. This explains why experiments on notched concrete specimens consistently show the size effect to be more pronounced at a slower loading ([15]). A similar behavior might be expected for ice. It thus transpires that, in order to take the influence of creep on the size effect approximately into account, it suffices to reduce the value of fracture energy (or fracture toughness) and decrease the effective length of the fracture process zone.

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