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FINITE STRAIN GENERALIZATION OF SMALL-STRAIN CONSTITUTIVE RELATIONS FOR ANY FINITE STRAIN TENSOR AND ADDITIVE VOLUMETRIC-DEVIATORIC SPLIT

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Abstract—The paper deals with finite strain generalization of small-strain constitutive equations for isotropic materials for which the strain is split into a volumetric part and a deviatoric part (the latter characterizing the isochoric strain, i.e. a strain at constant volume). The volumetric-deviatoric split has so far been handled by a multiplicative decomposition of the transformation tensor; but the existing sophisticated complex constitutive models for small strains of cohesive pressure-sensitive dilatant materials, such as concrete and geomaterials, involve an additive decomposition and would be difficult to convert a multiplicative decomposition. It is shown that an additive decomposition of any finite strain tensor, and of the Green–Lagrange strain tensor in particular, is possible, provided that the higher-order terms of the deviatoric strain tensor are allowed to depend on the volumetric strain. This dependence is negligible for concrete and geomaterials because the volumetric strains are normally small, whether or not the deviatoric strains are large. Furthermore, the related question of the choice of the finite-strain measure to be used for the finite-strain generalization is analysed. A transformation of the Green–Lagrange finite strain tensor whose parameters approximately reflect the degrees of freedom equivalent to replacing the small strain tensor by any other possible finite strain measure is proposed. Finally a method by which the stress tensor that is work-conjugate to any finite strain tensor can be converted to the Green–Lagrange strain tensor is presented. Copyright © 1996 Elsevier Science Ltd.

1. INTRODUCTION

Driven by applications in metal forming, the plastic constitutive relations for large strains, requiring the finite strain theory, have been studied extensively and important results have been achieved (Fox, 1968; Green and Naghdi, 1971; Kröner, 1960; Lee, 1969; Lee and Lui, 1967; Lubliner 1986; 1990; Mandel, 1973; 1976; Mandel *et al.*, 1977; McMeeking and Rice, 1975; Naghdi and Trapp, 1975; Pinski *et al.*, 1983; Gurtin and Spear, 1983; Atluri, 1984; Rubinstein and Atluri, 1983; Simo, 1985; 1988; Simo and Ortiz 1985; Rice, 1970; with further references). The existing results, however, concern mostly metal plasticity. They are not applicable to cohesive frictional dilatant pressure-sensitive materials, e.g. concretes, rocks, soils. Yet, finite-strain constitutive relations for such materials are needed for the analysis of impact, blast or earthquake.

Normally, it is hardly possible to identify a finite-strain constitutive relation directly by fitting large-deformation test data. Rather, the feasible strategy is to generalize existing small-strain constitutive relations to finite strain by introducing only a few additional material parameters. The reason is that much is known about small-strain constitutive relations for such materials, while at the same time a uniform finite strain field is next to impossible to achieve in most types of large deformation tests. Therefore, the additional parameters of finite-strain generalization of the small-strain constitutive relation need to be calibrated by large deformation tests of structures, accepting nonuniformity of the strain field.

In the existing isotropic small-strain constitutive relations for materials such as concretes and geomaterials, the stresses and strains are normally separated into additive volumetric and deviatoric parts. It would be convenient to preserve the additivity of volumetric and deviatoric parts even for finite strains, but the existing formulation is a

multiplicative split of Green's deformation tensor. This paper, based on a recent conference presentation (Bažant, 1995), will examine whether an additive split may be possible. It will also address the question of the additional degrees of freedom which ought to be exhibited by the finite-strain generalization in order to reflect the arbitrariness in the choice of the finite strain measure, and study the related problem of converting the stress tensor work-conjugate to any finite strain tensor into the Green–Lagrange strain tensor.

2. VOLUMETRIC-DEVIATORIC SPLIT OF FINITE STRAIN

2.1. *Additivity for Green–Lagrange finite strain tensor*

Let us begin with an elementary, first-principles approach to the first problem. We consider the Green–Lagrange finite strain tensor,

$$\varepsilon_{ij} = \tfrac{1}{2}(F_{ki}F_{kj} - \delta_{ij}) = \tfrac{1}{2}(u_{i,j} + u_{j,i} + u_{k,i}u_{k,j}), \tag{1}$$

where the numerical subscripts refer to Cartesian coordinates X_i of material points in their initial locations (Lagrangean coordinates), $u_i = x_i - X_i$ = components of displacements of the material points, x_i = coordinates of material points in the final deformed state, $u_{i,j}$ = gradient u_i, $F_{i,j} = \partial x_i/\partial X_j = u_{i,j} + \delta_{ij}$ = components of the transformation tensor F, and δ_{ij} = Kronecker delta. The derivatives, denoted by a subscript preceded by a comma, are the derivatives with respect to X_i, e.g. $u_{i,j} = \partial u_i/\partial X_j$. For pressure-sensitive dilatant materials, ε_{ij} needs to be decomposed into volumetric and deviatoric finite strain tensors $\varepsilon_{V_{ij}}$ and $\varepsilon_{D_{ij}}$.

For small strains, the relative volume change is given by the trace ε_{kk}, but not for finite strains. The trace can, in fact, be greatly in error. This can be made conspicuous by the following example. Consider that $F_{11} = 0.50$, $F_{22} = 1.25$, $F_{33} = 1.55$, $F_{21} = -0.55$, with all other $F_{ij} = 0$. Calculations give $\varepsilon_{11} = -0.224$, $\varepsilon_{22} = 0.281$, $\varepsilon_{33} = 0.701$, $\varepsilon_{12} = \varepsilon_{21} = -0.344$, with all other $\varepsilon_{ij} = 0$. The relative volume change $(V_0 + \Delta V)/V_0 = \det \boldsymbol{F} - 1 = -0.0312$, while $\varepsilon_{kk} = 0.759$.

It is helpful to recall the elementary derivation of the Green–Lagrange finite strain tensor ε_{ij}. Consider the initial line segment $\mathrm{d}X_i$ transforming into $\mathrm{d}x_i$. The finite strain tensor ε_{ij} is defined by setting $\mathrm{d}x_k\mathrm{d}x_k - \mathrm{d}X_k\mathrm{d}X_k = 2\varepsilon_{ij}\mathrm{d}X_i\mathrm{d}X_j$. Substituting $x_i = X_i + u_i$, $\mathrm{d}x_k = x_{k,i}\mathrm{d}X_i$ and adopting the notation $x_{k,i} = \partial x_k/\partial X_i$, one gets

$$2\varepsilon_{ij}\mathrm{d}X_i\mathrm{d}X_j = x_{k,i}\mathrm{d}X_i x_{k,j}\mathrm{d}X_j - \mathrm{d}X_k\mathrm{d}X_k = [(X_k + u_k)_{,i}(X_k + u_k)_{,j} - \delta_{ij}]\mathrm{d}X_i\mathrm{d}X_j, \tag{2}$$

in which one may substitute $X_{k,i} = \partial X_k/\partial X_i = \delta_{ki}$. Since this relation must hold for any $\mathrm{d}X_i$, we have $2\varepsilon_{ij} = (\delta_{ki} + u_{k,i})(\delta_{kj} + u_{k,j}) - \delta_{ij} = u_{k,i} + u_{k,j} + u_{k,i}u_{k,j}$, which yields eqn (1).

Let us now proceed similarly, imagining that a small material element is deformed in two steps (Fig. 2a). In the first step, the element is subjected to pure volumetric (isotropic) expansion (i.e. same expansion in all directions), without any rotation. During this expansion, the point of initial coordinates X_i moves to a point of intermediate coordinates $\xi_i = X_i + u'_i$, and line segment $\mathrm{d}X_i$ transforms to line segment $\mathrm{d}\xi_i$ (Figs 1 and 2b). In the

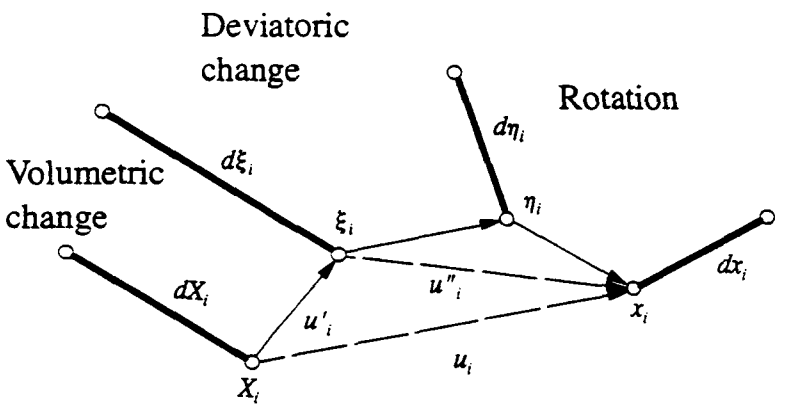

Fig. 1. Coordinates and displacements.

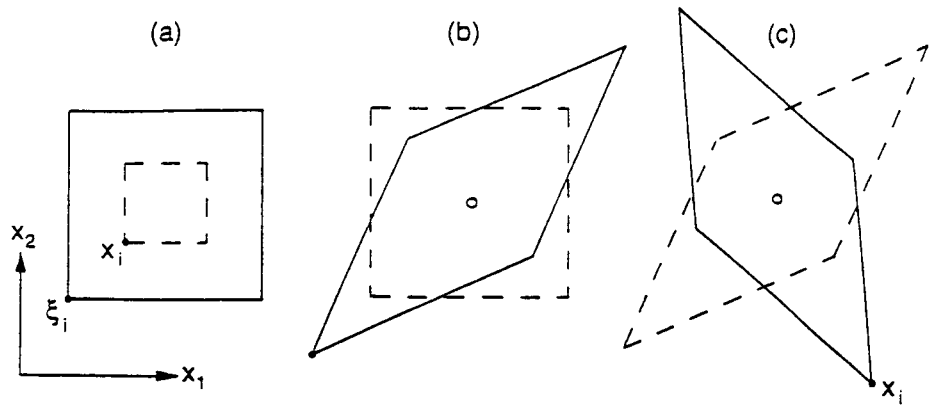

Fig. 2. (a) Volume expansion of elementary cube of material, (b) subsequent deviatoric deformation, and (c) subsequent rotation.

second step, the material element is transformed by deformation at no change of volume (Fig. 2b) and then is subjected to a rigid-body rotation (Fig. 2c), in which the volume change is also zero. In this transformation, the point at coordinates ξ_i moves to $x_i = X_i + u_i$, and segment $d\xi_i$ transforms to dx_i (Fig. 1). Let ε_0 be the engineering strain (or Biot strain) giving the relative volume change, that is, $d\xi_i = (1+\varepsilon_0)dX_i$. Then $(V_0+\Delta V)/V_0 = (1+\varepsilon_0)^3 = \det F_{ij} = J$, where J = Jacobian of the transformation from X_i to x_i, V_0 = initial volume of small material element, and ΔV = volume increment. So we have,

$$\varepsilon_0 = (\det F_{ij})^{1/3} - 1 \quad (F_{ij} = \delta_{ij} + u_{i,j}). \tag{3}$$

Let us now denote $u_i'' = x_i - \xi_i =$ displacements during the second transformation with zero volume change. Noting that $\partial\xi_k/\partial\xi_i = \delta_{ki}$, we may write

$$dx_k = d(\xi_k + u_k'') = \frac{\partial(\xi_k + u_k'')}{\partial\xi_i} d\xi_i = \left(\delta_{ki} + \frac{\partial u_k''}{\partial\xi_i}\right) d\xi_i. \tag{4}$$

Now, substituting $d\xi_i = (1+\varepsilon_0)\,dX_i$, we may further write:

$$dx_k\,dx_k = \left(\delta_{ki} + \frac{\partial u_k''}{\partial\xi_i}\right) d\xi_i \left(\delta_{kj} + \frac{\partial u_k''}{\partial\xi_j}\right) d\xi_j = (\delta_{ij} + 2d_{ij})(1+\varepsilon_0)^2\,dX_i\,dX_j \tag{5}$$

in which

$$d_{ij} = \frac{1}{2}\left(\frac{\partial u_i''}{\partial\xi_j} + \frac{\partial u_j''}{\partial\xi_i} + \frac{\partial u_k''}{\partial\xi_i}\frac{\partial u_k''}{\partial\xi_j}\right) \tag{6}$$

represents the Green–Lagrange finite strain tensor for the deviatoric transformation taken alone. The total change of length of the line segment may now be expressed as follows:

$$\begin{aligned} 2\varepsilon_{ij}\,dX_i\,dX_j &= dx_k\,dx_k - dX_k\,dX_k \\ &= [(1+\varepsilon_0)^2(\delta_{ij}+2d_{ij}) - \delta_{ij}]\,dX_i\,dX_j \\ &= \{[(1+\varepsilon_0)^2 - 1]\delta_{ij} + 2(1+\varepsilon_0)^2 d_{ij}\}\,dX_i\,dX_j \\ &= 2(\varepsilon_{V_{ij}} + \varepsilon_{D_{ij}})\,dX_i\,dX_j \end{aligned} \tag{7}$$

in which we introduced the notations:

$$\varepsilon_{\mathrm{D}_{ij}} = (1+\varepsilon_0)^2\, d_{ij} \tag{8}$$

$$\varepsilon_{\mathrm{V}_{ij}} = \varepsilon_{\mathrm{V}}\delta_{ij}, \quad \varepsilon_{\mathrm{V}} = \varepsilon_0 + \tfrac{1}{2}\varepsilon_0^2\,. \tag{9}$$

Comparing the first and last expressions in eqn (7), we conclude that

$$\varepsilon_{ij} = \varepsilon_{\mathrm{V}_{ij}} + \varepsilon_{\mathrm{D}_{ij}}. \tag{10}$$

Here $\varepsilon_{\mathrm{D}_{ij}}$ = additive finite strain tensor for deviatoric deformation; $\delta_{ij}\varepsilon_{\mathrm{V}} = \varepsilon_{\mathrm{V}_{ij}}$ = Green–Lagrange volumetric finite strain tensor, which is the same as the Green–Lagrange finite strain tensor for the initial volumetric transformation taken alone. As we see from eqn (10), the volumetric and deviatoric strain tensors, as defined here, are additive.

It should be noted that $\varepsilon_{\mathrm{D}_{ij}}$ is not a deviatoric tensor, i.e. its trace $\varepsilon_{\mathrm{D}_{kk}}$ in general does not vanish. For this reason, $\varepsilon_{\mathrm{D}_{ij}}$ would better be called isochoric, which means "at constant volume". However the term "deviatoric", which we use solely to refer to a deviation from volumetric deformation, has already become too established to change it.

The basic requirements for $\varepsilon_{\mathrm{V}_{ij}}$ and $\varepsilon_{\mathrm{D}_{ij}}$ to be volumetric and deviatoric tensors is that $\varepsilon_{\mathrm{V}_{ij}}$ must vanish for purely deviatoric deformation ($\varepsilon_0 = 0$) and $\varepsilon_{\mathrm{D}_{ij}}$ must vanish for purely volumetric deformation ($u_i'' = 0$). These requirements are obviously satisfied.

The fact that the additive deviatoric strains $\varepsilon_{\mathrm{D}_{ij}}$ are not independent of volume change calls for discussion. In this regard, three observations should be made:

(1) With the exception of highly porous materials, e.g. foams, ε_0 is normally very small and the strain is large only because of large deviatoric deformations. For example at hydrostatic pressure $\sigma_0 = -300{,}000$ psi (probably the highest tested), the volume strain of concrete of uniaxial compression strength $f_c' \approx 4000$ psi is only about $\varepsilon_0 = -3\%$ (Bažant *et al.*, 1986), in which case $(1+\varepsilon_0)^2 = 0.94 \approx 1$. Higher pressures can hardly ever occur in engineering applications. Typically, $|\sigma_0| < 10{,}000$ psi, and then $|\varepsilon_0| < 0.1\%$ and $(1+\varepsilon_0)^2$ differs from 1 by less than 0.2%. Therefore, the dependence of $\varepsilon_{\mathrm{D}_{ij}}$ on $(1+\varepsilon_0)^2$ does not appear a problem for such applications.

(2) In frictional materials, e.g. concretes, rocks or soils, the deviatoric stresses depend not only on the deviatoric strains, but also on the hydrostatic pressure σ_0, which in turn depends on ε_0. Thus, the dependence of $\varepsilon_{\mathrm{D}_{ij}}$ on ε_0 means that volumetric deformation alone modifies the value of deviatoric strain. This may cause some modeling difficulties, although it is probably possible to compensate for this effect at least approximately by modifying the effect of volumetric strain in the constitutive equation for deviatoric behavior.

(3) For highly compressible materials, e.g. foams, an acceptable form of additivity can be achieved only by using the logarithmic (Hencky) finite strain tensor, which will be discussed later.

2.2. *Additivity for a general class of finite strain tensors*

Let us now consider the question of additive decomposition for other types of finite strain tensors. First, let us briefly recall how the multiplicative decomposition of finite strain into volumetric and deviatoric parts (Flory, 1961; Sidoroff, 1974; Ogden, 1984; Simo, 1988; Lubliner, 1990) is obtained: Let η_i be the coordinates of material points after the deviatoric deformation and before the rigid body rotation $\boldsymbol{R}$. Then

$$\mathrm{d}x_i = \frac{\partial x_i}{\partial \eta_j}\frac{\partial \eta_j}{\partial X_l}\mathrm{d}X_l = R_{ij}U_{jl}\,\mathrm{d}X_l \tag{11}$$

$$\mathrm{d}x_i = \frac{\partial x_i}{\partial \eta_j}\frac{\partial \eta_j}{\partial \zeta_k}\frac{\partial \zeta_k}{\partial X_l}\mathrm{d}X_l = R_{ij}F_{\mathrm{D}_{jk}}U_{\mathrm{V}_{kl}}\,\mathrm{d}X_l. \tag{12}$$

Here, F_{ij}, R_{ij}, U_{ij}, $U_{\mathrm{V}_{ij}}$ and $F_{\mathrm{D}_{ij}}$ are the components of the total transformation tensor $\boldsymbol{F}$,

rotation tensor $\boldsymbol{R}$, right-stretch tensor $\boldsymbol{U} = \sqrt{\boldsymbol{F}^T\boldsymbol{F}}$, and volumetric and deviatoric transformation tensors $\boldsymbol{U}_V$ and $\boldsymbol{F}_D$, respectively. By comparison of eqns (11) and (12),

$$\boldsymbol{U} = \boldsymbol{F}_D\boldsymbol{U}_V \quad \text{for} \quad U_{ji} = F_{D_{jk}}U_{V_{kl}} \tag{13}$$

(Flory, 1961; Sidoroff, 1974; Simo, 1988; Lubliner, 1990). Furthermore, $\boldsymbol{F} = \boldsymbol{R}\boldsymbol{U} = \boldsymbol{R}\boldsymbol{F}_D\boldsymbol{U}_V$. Because $\boldsymbol{U}_V = J^{1/3}\boldsymbol{I}$, we have, in view of eqn (13), $\boldsymbol{F}_D = J^{-1/3}\boldsymbol{U}$. So, tensor $\boldsymbol{F}_D$ is symmetric. Tensor $\boldsymbol{U}_V$ is of course symmetric too.

Consider a very general class of finite strain tensors, called the Doyle–Ericksen (1956) tensors (also, e.g. Bažant and Cedolin, 1991, Sec. 11.1):

$$\boldsymbol{\varepsilon}^{(m)} = m^{-1}(\boldsymbol{U}^m - \boldsymbol{I}) \quad \text{for} \quad m \neq 0, \quad \boldsymbol{\varepsilon}^{(m)} = \ln \boldsymbol{U} \quad \text{for} \quad m = 0 \tag{14}$$

where m is any real number; $\boldsymbol{\varepsilon}^{(2)}$ = Green–Lagrange tensor in eqn (1), $\boldsymbol{\varepsilon}^{(1)}$ = Biot finite strain tensor, and $\boldsymbol{\varepsilon}^{(0)}$ = logarithmic finite strain tensor. The following transformations are now possible for $m \neq 0$:

$$\begin{aligned}\boldsymbol{\varepsilon}^{(m)} &= m^{-1}[(\boldsymbol{F}^T\boldsymbol{F})^{m/2} - \boldsymbol{I}] = m^{-1}[(\boldsymbol{F}_D^T\boldsymbol{F}_D J^{2/3})^{m/2} - \boldsymbol{I}] \\ &= m^{-1}[(\boldsymbol{F}^T\boldsymbol{F}_D)^{m/2} J^{m/3} - \boldsymbol{I}] = \boldsymbol{\varepsilon}_D^{(m)} + \boldsymbol{\varepsilon}_V^{(m)}\end{aligned} \tag{15}$$

in which

$$\boldsymbol{\varepsilon}_V^{(m)} = m^{-1}[(\boldsymbol{U}_V)^{m/2} - \boldsymbol{I}] = m^{-1}(J^{m/3} - 1)\boldsymbol{I} \tag{16}$$

$$\boldsymbol{\varepsilon}_D^{(m)} = m^{-1}[(\boldsymbol{F}_D^T\boldsymbol{F}_D)^{m/2} - \boldsymbol{I}] = m^{-1}J^{m/3}(\boldsymbol{U}_D^{(m)} - \boldsymbol{I}) \tag{17}$$

and $\boldsymbol{U}_D = (\boldsymbol{F}_D^T\boldsymbol{F}_D)^{1/2} = \boldsymbol{F}_D$ = right stretch tensor for the deviatoric transformation alone. Similarly, for $m = 0$:

$$\boldsymbol{\varepsilon}^{(0)} = \ln\sqrt{\boldsymbol{F}^T\boldsymbol{F}} = \ln\sqrt{\boldsymbol{F}_D^T\boldsymbol{F}_D J^{2/3}} = \boldsymbol{\varepsilon}_D^{(0)} + \boldsymbol{\varepsilon}_V^{(0)} \tag{18}$$

where

$$\boldsymbol{\varepsilon}_V^{(0)} = \ln\sqrt{\boldsymbol{U}_V^T\boldsymbol{U}_V} = (\ln J)\boldsymbol{I}, \quad \boldsymbol{\varepsilon}_D^{(0)} = \ln\sqrt{\boldsymbol{F}_D^T\boldsymbol{F}_D} = \ln \boldsymbol{U}_D. \tag{19}$$

Note that for $m = 2$, eqns (16) and (17) reduce to eqns (8) and (9). Further, note that tensor $\boldsymbol{\varepsilon}_V^{(m)}$ vanishes when the volume change is zero (or $J = 1$) and that tensor $\boldsymbol{\varepsilon}_D^{(m)}$ vanishes when the deformation is a pure isotropic expansion ($\boldsymbol{F}_D = \boldsymbol{I}$) and that they are both independent of rigid body rotation. This confirms that they indeed represent the volumetric and deviatoric deformations. The tensor $\boldsymbol{\varepsilon}_V^{(m)}$ is volumetric because it vanishes for purely deviatoric deformations ($\boldsymbol{F}_D = \boldsymbol{F}$, $\boldsymbol{U}_V = \boldsymbol{I}$), and tensor $\boldsymbol{\varepsilon}_{D_{ij}}^{(m)}$ is deviatoric because it vanishes for purely volumetric deformations ($\boldsymbol{F}_D = \boldsymbol{I}$).

Equations (15) and (18) show that the volumetric-deviatoric split can be formulated as additive for any choice of finite strain tensor. However, same as before, $\boldsymbol{\varepsilon}_{D_{ij}}^{(m)}$ in general depends on J, i.e. on the volume change. In this regard, the same comments apply as stated before. However, the case of logarithmic (or Hencky) strain ($m = 0$, eqns 18 and 19) is an exception. The logarithm strain provides the only additive volumetric-deviatoric decomposition in which the deviatoric finite strain tensor is independent of volume change. Unfortunately, this advantageous property is offset by greater complexity in converting the stress tensor that is work-conjugate to logarithmic strain to the 2PK and Cauchy stress tensors.

3. CONSTITUTIVE ADJUSTMENT EQUIVALENT TO CHANGING FINITE STRAIN MEASURE

If practical calculations are limited to the Green–Lagrange strain tensor, it is necessary to consider in the finite strain constitutive equation the degrees of freedom that are equivalent to an arbitrary change in the finite strain measure. Such degrees of freedom have been ignored in the past practice.

When the finite strain tensor used as the input of a constitutive subroutine is the Green–Lagrange strain, the output is the work-conjugate stress tensor called the second Piola–Kirchhoff (2PK) stress tensor. The Cauchy stress tensor S_{ij}, which is also called the true stress tensor and represents the actual forces acting on a small unit cube cut out from the deformed material, is easily calculated as $S_{ij} = \sigma_{km}F_{ik}F_{jm}/J$ (see, e.g. eqns 11.2.9 and 11.2.13 in Bažant and Cedolin, 1991). If a finite element program calculates some other finite strain tensor, such as Biot's or the logarithmic one, this strain tensor may first be converted to the Green–Lagrange finite strain tensor so that the output stress tensor would be 2PK.

A change in the choice of finite strain tensor is manifested by a change of the corresponding objective stress rate (Bažant and Cedolin, 1991, eqn 11.3.19); but this, in turn, has been shown to be equivalent to a change in the tangential stiffness moduli of the material (Bažant, 1971; Bažant and Cedolin, 1991, eqn 11.4.4). Therefore, a change in the choice of finite strain tensor must be equivalent to a certain change in the functions describing the constitutive properties which introduces additional degrees of freedom in the fitting of test data. The question is how to formulate these degrees of freedom in an approximate but effective manner.

Any finite strain tensor $\tilde{\boldsymbol{\varepsilon}}$, of components $\tilde{\varepsilon}_{ij}$, may be expressed as

$$\tilde{\boldsymbol{\varepsilon}} = \boldsymbol{G}(\boldsymbol{\varepsilon}) \tag{20}$$

where $\boldsymbol{G}$ is a tensor-valued one-to-one nonsingular mapping of Green–Lagrange strain tensor $\boldsymbol{\varepsilon}$ having components ε_{ij}. Most generally, this function may be defined in terms of the spectral decomposition:

$$\tilde{\varepsilon}_{jk} = \sum_{i=1}^{3} g(\varepsilon_{(i)}) n_j^{(i)} n_k^{(i)} \tag{21}$$

where $n_j^{(i)}$ (with $i = 1, 2, 3$) are the direction cosines of the principal axes of tensor $\boldsymbol{\varepsilon}$ (which are the same for any other tensor $\tilde{\boldsymbol{\varepsilon}}$, i.e. all the finite strain tensors are coaxial); $\varepsilon_{(i)}$ = principal Green–Lagrange strains; and $g(\varepsilon)$ is any real continuous monotonically increasing function such that $g(0) = 0$ and $g'(0) = 1$, defined for $\varepsilon \in (-0.5, \infty)$ [note that $\varepsilon = -0.5$ corresponds to $u_{1,1} = -1$ and thus represents the case in which a finite line segment is compressed into a point; values $\varepsilon \leqslant -0.5$ are physically impossible]. Equation (21) is analogous to that given by Ogden (1984) and Rice (1993) in terms of the principal stretches $\lambda_{(i)} = \sqrt{1+2\varepsilon_i}$. The Doyle–Ericksen finite strain tensors (14) are obtained by setting

$$\begin{aligned} g(\varepsilon) &= \frac{1}{m}[(1+2\varepsilon)^{m/2} - 1] \quad \text{for} \quad m \neq 0 \\ &= \tfrac{1}{2}\ln(1+2\varepsilon) \quad \text{for} \quad m = 0 \end{aligned} \tag{22}$$

where m can be any real constant.

As established by Hill (1966), the stress tensor $\tilde{\boldsymbol{\sigma}}$ that is conjugate to $\tilde{\boldsymbol{\varepsilon}}$ must be defined energetically by $\tilde{\boldsymbol{\sigma}} : \delta\tilde{\boldsymbol{\varepsilon}} = \boldsymbol{\sigma} : \delta\boldsymbol{\varepsilon}$. This variational equation requires that the work of the conjugate stresses on any variations of the corresponding finite strains must be the same for any chosen finite strain tensor. Rearranging this variational equation as $[\tilde{\boldsymbol{\sigma}} : (\partial\tilde{\boldsymbol{\varepsilon}}/\partial\boldsymbol{\varepsilon}) - \boldsymbol{\sigma}] : \delta\boldsymbol{\varepsilon} = 0$, we note that it is satisfied if and only if $[\ldots] = 0$ or

$$\boldsymbol{\sigma} = \tilde{\boldsymbol{\sigma}} : \frac{\hat{c}\boldsymbol{G}(\boldsymbol{\varepsilon})}{\hat{c}\boldsymbol{\varepsilon}} \quad \text{or} \quad \sigma_{ij} = \tilde{\sigma}_{kl} \frac{\hat{c}G_{kl}(\boldsymbol{\varepsilon})}{\hat{c}\varepsilon_{ij}} \tag{23}$$

in which $\tilde{\sigma}_{kl}$, G_{kl} = components of tensors $\tilde{\boldsymbol{\sigma}}$, $\boldsymbol{G}$; and the colon refers to a doubly contracted tensorial product. Alternatively, rearranging the variational equation as $[\boldsymbol{\sigma} : (\hat{c}\boldsymbol{\varepsilon}/\hat{c}\tilde{\boldsymbol{\varepsilon}}) - \tilde{\boldsymbol{\sigma}}] : \delta\boldsymbol{\varepsilon} = 0$, we further note that it is satisfied if and only if

$$\tilde{\boldsymbol{\sigma}} = \boldsymbol{\sigma} : \frac{\hat{c}\boldsymbol{H}(\tilde{\boldsymbol{\varepsilon}})}{\hat{c}\tilde{\boldsymbol{\varepsilon}}} \quad \text{or} \quad \tilde{\sigma}_{ij} = \sigma_{kl} \frac{\hat{c}H_{kl}(\tilde{\boldsymbol{\varepsilon}})}{\hat{c}\tilde{\varepsilon}_{ij}} \tag{24}$$

where $\boldsymbol{H}(\tilde{\boldsymbol{\varepsilon}})$ is the tensor-valued mapping inverse to $\boldsymbol{G}(\boldsymbol{\varepsilon})$, such that $\boldsymbol{\varepsilon} = \boldsymbol{H}(\tilde{\boldsymbol{\varepsilon}})$.

Let us now restrict consideration to nonlinear constitutive laws for which, within a certain range, the stress can be considered a function of strain (this is for certain reasons appropriate within a limited range even for finite strain plasticity; Simo and Ortiz, 1985). We may express the constitutive law for any chosen finite strain measure in the form $\tilde{\boldsymbol{\sigma}} = \tilde{\boldsymbol{f}}(\tilde{\boldsymbol{\varepsilon}})$. In this constitutive relation, the stress tensor $\tilde{\boldsymbol{\sigma}}$ that is conjugate to $\tilde{\boldsymbol{\varepsilon}}$ ought to be used, and $\tilde{\boldsymbol{f}}$ is a tensor-valued function that defines the constitutive law. According to eqns (20) and (23),

$$\boldsymbol{\sigma} = \tilde{\boldsymbol{f}}[\boldsymbol{G}(\boldsymbol{\varepsilon})] : \frac{\hat{c}\boldsymbol{G}(\boldsymbol{\varepsilon})}{\hat{c}\boldsymbol{\varepsilon}} = \boldsymbol{f}(\boldsymbol{\varepsilon}). \tag{25}$$

Conversely, if the constitutive law is defined for the Green–Lagrange strain measure in the form $\boldsymbol{\sigma} = \boldsymbol{f}(\boldsymbol{\varepsilon})$, it follows from eqn (24) that

$$\tilde{\boldsymbol{\sigma}} = \boldsymbol{f}[\boldsymbol{H}(\tilde{\boldsymbol{\varepsilon}})] : \frac{\hat{c}\boldsymbol{H}(\tilde{\boldsymbol{\varepsilon}})}{\hat{c}\boldsymbol{\varepsilon}} = \tilde{\boldsymbol{f}}(\tilde{\boldsymbol{\varepsilon}}). \tag{26}$$

Calculation of the 6×6 matrix of the components of fourth-rank tensor $\hat{c}\boldsymbol{G}(\boldsymbol{\varepsilon})/\hat{c}\boldsymbol{\varepsilon}$ in eqn (25) is not easy. Since general powers of tensors are involved, the spectral representation of tensors needs to be introduced, and a system of six linear equations, resulting from 12 variational conditions with six restrictions, must be set up and solved numerically.

Equation (25) demonstrates that any constitutive law given by tensor-valued function $\tilde{\boldsymbol{f}}$ that relates the conjugate stress tensor $\tilde{\sigma}_{ij}$ to the strain tensor $\tilde{\varepsilon}_{ij}$ must be equivalent to the constitutive law given by a certain other tensor-valued function $\boldsymbol{f}$ that relates the 2PK stress tensor σ_{ij} to the Green–Lagrange strain tensor ε_{ij}. Conversely, eqn (26), demonstrates again that any constitutive law given by tensor-valued function $\boldsymbol{f}$ that relates the 2PK stress tensor σ_{ij} to the Green–Lagrange strain tensor ε_{ij} must be equivalent to the constitutive law given by tensor-valued function $\tilde{\boldsymbol{f}}$ that relates the conjugate stress tensor $\tilde{\sigma}_{ij}$ to $\tilde{\varepsilon}_{ij}$. So, it is not only inconvenient, but in principle unnecessary to consider any finite strain measure other than the Green–Lagrange strain measure.

The foregoing conclusion complements the previous conclusion that, in incremental elastic (hypoelastic) constitutive laws, all objective stress rates associated with different strain measures are equivalent. Changing from one objective stress rate to another is tantamount to changing the tensor of tangential elastic moduli as a linear function of the initial stress (Bažant, 1971; Bažant and Cedolin, 1991, Section 11.4). The objective stress rate associated with the Green–Lagrange strain tensor is Truesdell's rate, and no other objective stress rate needs to be used.

4. GENERALIZATION OF SMALL-STRAIN CONSTITUTIVE RELATION TO FINITE STRAIN

Consider now that the small-strain constitutive law $\boldsymbol{s} = \phi(\boldsymbol{e})$ is given with $\boldsymbol{e}$, $\boldsymbol{s}$ = small (linearized) strain tensor and stress tensor. To generalize this constitutive law to finite strain, a widespread practice has been to simply replace the small strain tensor $\boldsymbol{e}$ by the

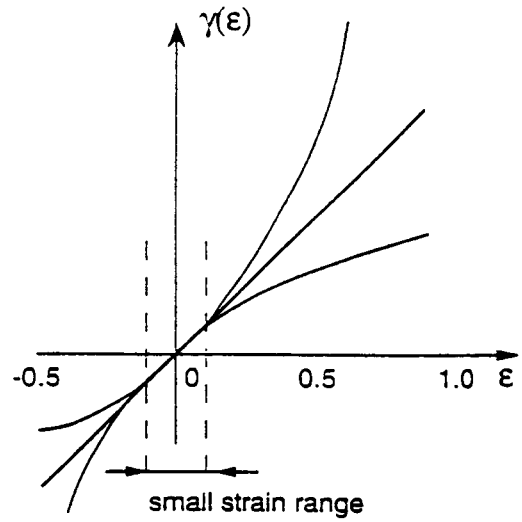

Fig. 3. Finite strain transformation functions.

Green–Lagrange finite strain tensor ε, or some other finite strain tensor. However, such a practice is incorrect. Although it would be inconvenient to use strain measures other than Green–Lagrange, the degrees of freedom that are equivalent to the use of other possible finite strain measures must be introduced in the correct generalization.

So one must consider, at least approximately, all possible transformations of the small-strain constitutive law that are equivalent to passing from function $f(\varepsilon)$ to any other function $\tilde{f}(\tilde{\varepsilon})$ given by eqns (25) and (26). Because such transformations can arbitrarily change all the terms of the Taylor series expansion of the constitutive law higher than the linear terms, the finite-strain generalization of the small-strain constitutive law may consist of a replacement of the small strain tensor $\boldsymbol{e}$ in this law by some strain transformation function $\gamma(\boldsymbol{\varepsilon})$ [Fig. 3] of the following properties: (1) it must be isotropic; (2) the linear term of its Taylor series expansion is $\boldsymbol{\varepsilon}$, with the constant term vanishing; and (3) the transformation is one-to-one (nonsingular). This means that the generalization of the small-strain constitutive law to finite strain may be considered in the form

$$f(\boldsymbol{\varepsilon}) = \phi[\gamma(\boldsymbol{\varepsilon})]. \tag{27}$$

It must be emphasized that γ is not an alternative strain measure. Rather, transformation γ accounts for both the transformation of the strain tensor and the inverse transformation of the conjugate stress tensor, in a combined manner.

In view of the Cayley–Hamilton theorem, the most general possible transformation may be written as

$$\gamma_{ij}(\boldsymbol{\varepsilon}) = \varepsilon_{ik} p(\delta_{kj} + q d_{kj} + r d_{kl} d_{lj}) \tag{28}$$

where p, q, r are smooth functions of the invariants of $\boldsymbol{\varepsilon}$. In practice, however, the available test data for large deformations of structures will hardly suffice to calibrate these functions. As the simplest approximation which should suffice in practice, one may assume that $r = 0$, $p = (1 + \varepsilon_0)^n$ and n, q are two constants to be calibrated by large deformation experiments. Note that $n = -2$ cancels the dependence of $\varepsilon_{D_{ij}}$ on ε_0, which confirms that this dependence is not objectionable.

A rather simple formulation which is equivalent to function $\gamma(\boldsymbol{\varepsilon})$ is possible for the microplane model, to which the present formulation has been applied (Bažant *et al.*, 1995).

5. CONVERSION OF GENERAL WORK-CONJUGATE STRESSES TO 2PK

To convert the general work-conjugate stress tensor to the 2PK, we introduce the spectral representation (21) into the variational equation $\sigma_{kl}\delta\varepsilon_{kl} = \tilde{\sigma}_{kl}\delta\tilde{\varepsilon}_{kl}$. In taking the variations, we must take into account the rotation of the principal axes of strain. The result is:

$$\sigma_{kl}\left[\sum_{i=1}^{3}\delta\varepsilon_{(i)}n_k^{(i)}n_l^{(i)} + \delta\Omega_{kj}\varepsilon_{jl} - \varepsilon_{kj}\delta\Omega_{jl}\right] = \tilde{\sigma}_{kl}\left[\sum_{i=1}^{3} g'(\varepsilon_{(i)})\delta\varepsilon_{(i)}n_k^{(i)}n_l^{(i)} + \delta\Omega_{kj}\varepsilon_{jl} - \varepsilon_{kj}\delta\Omega_{jl}\right] \tag{29}$$

in which $\delta\Omega_{ij}$ are the components of the antisymmetric tensor called the spin of Lagrangean principal axes (e.g. Ogden, 1984, p.128). Its components are:

$$\text{for } i<j: \quad \delta\Omega_{ij} = -\delta\Omega_{ji} = n_k^{(i)}\delta n_k^{(j)}; \quad \text{for} \quad i=j: \quad \delta\Omega_{ij} = 0. \tag{30}$$

Equation (29) can be rearranged to the form: $(\ldots)\delta\varepsilon_{(1)} + (\ldots)\delta\varepsilon_{(2)} + (\ldots)\delta\varepsilon_{(3)} + (\ldots)\delta\Omega_{(12)} + (\ldots)\delta\Omega_{(13)} + (\ldots)\delta\Omega_{(23)} = 0$. The six variations $\delta\varepsilon_{(1)}$, $\delta\varepsilon_{(2)}$, $\delta\varepsilon_{(3)}$, $\delta\Omega_{12}$, $\delta\Omega_{13}$ and $\delta\Omega_{23}$ are independent and can be chosen arbitrarily. It follows that the six (unwritten) expressions in parentheses must vanish. This leads to the following system of six equations:

$$\sigma_{kl}n_k^{(i)}n_l^{(i)} = \tilde{\sigma}_{kl}g'(\varepsilon_{(i)})n_k^{(i)}n_l^{(i)} \quad \text{for} \quad i = 1,2,3 \tag{31}$$

$$\sigma_{lk}\varepsilon_{jl} - \sigma_{jl}\varepsilon_{lk} = \tilde{\sigma}_{lk}\tilde{\varepsilon}_{jl} - \tilde{\sigma}_{jl}\tilde{\varepsilon}_{lk} \qquad \text{for} \quad kj = 12, 23, 31. \tag{32}$$

Since these equations apparently are unavailable in the literature, it may be convenient for the readers to give them in matrix component form, in which they need to be programmed:

$$\begin{bmatrix} (n_1^{(1)})^2 & (n_2^{(1)})^2 & (n_3^{(1)})^2 & 2n_1^{(1)}n_2^{(1)} & 2n_2^{(1)}n_3^{(1)} & 2n_3^{(1)}n_1^{(1)} \\ (n_1^{(2)})^2 & (n_2^{(2)})^2 & (n_3^{(2)})^2 & 2n_1^{(2)}n_2^{(2)} & 2n_2^{(2)}n_3^{(2)} & 2n_3^{(2)}n_1^{(2)} \\ (n_1^{(3)})^2 & (n_2^{(3)})^2 & (n_3^{(3)})^2 & 2n_1^{(3)}n_2^{(3)} & 2n_2^{(3)}n_3^{(3)} & 2n_3^{(3)}n_1^{(3)} \\ \varepsilon_{12} & -\varepsilon_{21} & 0 & \varepsilon_{22}-\varepsilon_{11} & -\varepsilon_{31} & \varepsilon_{32} \\ 0 & \varepsilon_{23} & -\varepsilon_{32} & \varepsilon_{13} & \varepsilon_{33}-\varepsilon_{32} & -\varepsilon_{12} \\ \varepsilon_{13} & 0 & -\varepsilon_{31} & \varepsilon_{23} & -\varepsilon_{21} & \varepsilon_{33}-\varepsilon_{11} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{Bmatrix}$$

$$= \begin{Bmatrix} \tilde{\sigma}_{jk}g'(\varepsilon_{(1)})n_j^{(1)}n_k^{(1)} \\ \tilde{\sigma}_{jk}g'(\varepsilon_{(2)})n_j^{(2)}n_k^{(2)} \\ \tilde{\sigma}_{jk}g'(\varepsilon_{(3)})n_j^{(3)}n_k^{(3)} \\ \tilde{\sigma}_{1k}\tilde{\varepsilon}_{k2} - \tilde{\sigma}_{2k}\tilde{\varepsilon}_{k1} \\ \tilde{\sigma}_{2k}\tilde{\varepsilon}_{k3} - \tilde{\sigma}_{3k}\tilde{\varepsilon}_{k2} \\ \tilde{\sigma}_{3k}\tilde{\varepsilon}_{k1} - \tilde{\sigma}_{1k}\tilde{\varepsilon}_{k3} \end{Bmatrix}. \tag{33}$$

Because $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ are symmetric tensors, eqn (32) may simply be written as

$$\boldsymbol{\sigma\varepsilon} - \boldsymbol{\varepsilon\sigma} = \tilde{\boldsymbol{\sigma}}\tilde{\boldsymbol{\varepsilon}} - \tilde{\boldsymbol{\varepsilon}}\tilde{\boldsymbol{\sigma}} \tag{34}$$

or

$$\text{asym}\,(\boldsymbol{\sigma\varepsilon}) = \text{asym}\,(\tilde{\boldsymbol{\sigma}}\tilde{\boldsymbol{\varepsilon}}) \tag{35}$$

where "asym" denotes the antisymmetric part of the tensor.

The foregoing results may be summarized in the following simple theorem: σ is work-conjugate to ε if and only if (i) $\tilde{\sigma}\delta\tilde{\varepsilon} = \sigma\delta\varepsilon$ for nonrotating principal directions, and (ii) tensor asym $(\tilde{\sigma}\tilde{\varepsilon})$ is invariant with respect to the choice of finite strain tensor.

For the special case of Biot strain tensor $\tilde{\varepsilon} = U - I$ $(m = 1)$, it has been checked that the aforementioned theorem (or eqns 32 and 35) is satisfied by the well-known simple relation $\tilde{\sigma} = (U\sigma + \sigma U)/2$ (see, e.g. Bažant and Cedolin, 1991, eqn 11.2.21). Only in this case, the principal values and principal axes of the Green–Lagrange strain tensor need not be calculated. For any other finite strain tensor, they need to be calculated, which may cause a grave penalty in terms of computer time if the stress tensor needs to be converted for each integration point of each finite element in each loading step. For this reason, the logarithmic strain, which has the advantage that the additive deviatoric strain is independent of volume change, will prove difficult to use in practice (another reason is the difficulty of calculating its rate; see, e.g. Gurtin and Spear, 1983; Hill, 1970).

6. CONCLUSIONS

(1) In generalizing a small-strain constitutive equation with inelastic dilatancy and pressure sensitivity to finite strain, the following three conditions must be satisfied: (1) the volumetric strain tensor must express the relative volume change of the material exactly, and (2) it must vanish for purely deviatoric (isochoric) deformations, and (3) the deviatoric (isochoric) strain tensor must vanish for purely volumetric deformation.

(2) The additive volumetric-deviatoric finite strain decomposition is advantageous for generalizing small-strain constitutive relations with such decomposition of finite strain. In general, the volumetric and deviatoric finite strain tensors for the volumetric and deviatoric transformations are not additive, but they can be made additive (without violating the foregoing three conditions) if the finite strain tensor for the deviatoric transformation is scaled by a function of volumetric strain. The consequence is that, except for the logarithmic (Hencky) strain tensor, the deviatoric (isochoric) strain tensor changes if the volumetric strain does. This feature is acceptable for materials such as concrete or rock in which the volumetric strains cannot be very large even if the deviatoric (isochoric) strains are very large. Even if the volumetric strain is large, it may be possible to compensate for this feature by the constitutive relation. Additive decomposition with deviatoric (isochoric) strain independent of volume change can be achieved if and only if the logarithmic (Hencky) finite strain measure is used, but at the cost of unnecessary complication of analysis.

(3) Although no loss of generality is incurred if the finite strain generalization of a small-strain constitutive model is based on the Green–Lagrange finite strain measure (with the work-conjugate second Piola–Kirchhoff stress tensor), it is insufficient to simply replace the small (linearized) strain tensor by the Green–Lagrange finite strain tensor. Rather, the small strain tensor in the constitutive law must be replaced by a sufficiently general tensorial transformation of the Green–Lagrange finite strain whose parameters approximately reflect the degrees of freedom equivalent to choosing any different finite strain measure.

(4) The stress tensor $\tilde{\sigma}$ that is work-conjugate to any chosen finite strain tensor ε may be calculated from the condition that (i) the work expression $\tilde{\sigma}\delta\tilde{\varepsilon}$ for nonrotating principal directions, and (ii) tensor asym $(\tilde{\sigma}\tilde{\varepsilon})$ in general, are invariant with respect to the choice of finite strain tensor.

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