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Nonlocal Model Based on Crack Interactions: A Localization Study

A micromechanically based enrichment of the nonlocal operator by a term taking into account the directional dependence of crack interactions (Bažant, 1992) can be expected to improve the performance of the nonlocal model. The aim of this paper is to examine this new model in the context of a simple localization problem reducible to a one-dimensional description. Strain localization in an infinite layer under plain stress is studied using both the old and the new nonlocal formulations. The importance of renormalization of the averaging function in the proximity of a boundary is demonstrated and the differences between the localization sensitivity of the old and new model are pointed out.

1 Introduction

As is now generally accepted, finite element analysis of distributed strain-softening damage, including its final localization into sharp fracture, requires the use of some type of localization limiter. A wide class of localization limiters is represented by the nonlocal continuum concept, which was introduced into continuum mechanics by Eringen (1965, 1966), Kröner (1967) and others, and was proposed as a localization limiter by Bažant, Belytschko and Chang (1984). An effective nonlocal damage model was developed by Pijaudier-Cabot and Bažant (1987) and Bažant and Lin (1988). It bears some resemblance to the crack band model (Bažant and Oh, 1983) and to the mesh-dependent softening modulus of Pietruszczak and Mróz (1981). A differential form of the nonlocal concept (Bažánt, 1984) was exploited in various gradient-dependent models (Schreyer and Chen, 1986; Lasry and Belytschko, 1988; de Borst and Mühlhaus, 1991). A more refined limiter of this type is the *micropolar continuum* (Cosserat and Cosserat, 1909), which was extended to strain-softening problems by Mühlhaus and Vardoulakis (1987). A computational model for the elastoplastic Cosserat continuum was formulated by de Borst and Sluys (1991). *Viscoplastic regularization* (Needleman, 1987) limits localization by adding rate-dependent terms to the constitutive equations.

2 New Approach to Nonlocal Averaging

One of the most powerful and computationally effective localization limiters is the concept of nonlocal averaging, first used in strain-softening analysis by Bažant (1984) and Bažant, et al. (1984). The nonlocal version (Bažant and Ožbolt, 1990) of the microplane model (Bažant and Prat, 1988) proved to be particularly efficient for the computer analysis of structures made of quasibrittle materials such as concrete. However, it also became clear that the classical nonlocal concept based on an isotropic weight function has its limitations and does not allow formulating a model universally applicable to the same material under different loading conditions. More specifically, it turned out that the values of the characteristic length required to fit experimental data for very different test geometries are significantly different and therefore cannot be regarded as a true material parameter. Moreover, the physical meaning of nonlocal averaging was not clear and theoretically supported, and so the nonlocal concept appeared as an artifice dictated merely by the need to regularize the governing differential equations.

To overcome these difficulties, a micromechanically based derivation of the nonlocal operator has recently been presented (Bažant, 1992). This led to certain modifications of the original approach. Both the original and the new approaches use the incremental form of the constitutive law

$$\Delta\sigma = \mathbf{C}_u : \Delta\epsilon - \Delta\bar{\mathbf{S}} \tag{1}$$

where $\Delta\sigma$, $\Delta\epsilon$ are the increments of the stress and strain tensor and $\mathbf{C}_u$ is the fourth rank stiffness tensor for unloading. The nonlocal inelastic stress increment $\Delta\bar{\mathbf{S}}$ is computed by applying a certain nonlocal operator on the local inelastic stress increment

$$\Delta\mathbf{S} = (\mathbf{C}_u - \mathbf{C}_t):\Delta\epsilon \tag{2}$$

The tangential stiffness tensor $\mathbf{C}_t$ corresponds to the given local constitutive law.

In the original formulation, the nonlocal operator represented weighted averaging over a certain neighborhood:

$$\Delta\bar{\mathbf{S}}(\mathbf{x}) = \int_V \Phi(\mathbf{x},\xi)\Delta\mathbf{S}(\xi)d\xi \tag{3}$$

The scalar weight function $\Phi(\mathbf{x},\xi)$ depends only on the distance $r = \|\mathbf{x} - \xi\|$ between the "source point" ξ and the "effect point" $\mathbf{x}$, and on the characteristic length l of the nonlocal continuum. The form of $\Phi(\mathbf{x},\xi)$ used in this study is a bell-shaped function with a compact support $\Phi(\mathbf{x},\xi) = \Phi_0[1-(r/l)^2]^2$ where Φ_0 is a normalizing factor such that $\int_{-\infty}^{\infty} \Phi(\mathbf{x},\xi)d\xi = 1$.

Based on analysis of the equations describing the interaction among microcracks in an elastic medium, the following gen-

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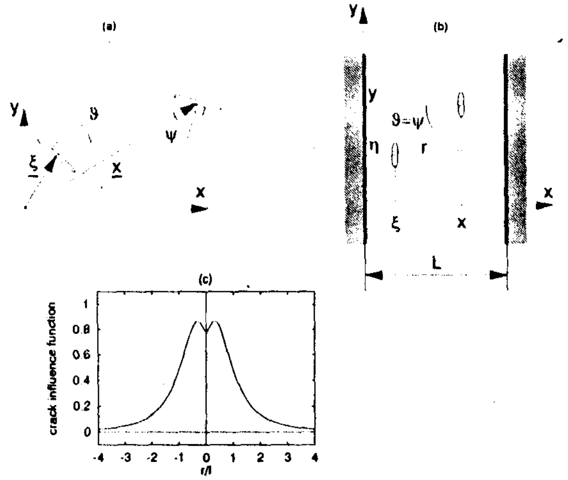

Fig. 1 (a) Orientation angle θ and ψ, (b) geometry of the infinite layer, (c) one-dimensional crack influence crack influence function $\Lambda(r)$

eralization of the nonlocal concept has recently been derived (Bažant, 1992):

$$\Delta\bar{S}^{(1)}(\mathbf{x}) = \int_V \Phi(\mathbf{x},\xi)\Delta S^{(1)}(\xi)d\xi + \int_V \Lambda(\mathbf{x},\xi)\Delta\bar{S}^{(1)}(\xi)d\xi \quad (4)$$

where $\Delta S^{(1)}$ is the increment of the maximum principal inelastic stress and $\Lambda(\mathbf{x},\xi)$ is the so-called crack influence function. The value of this function depends not only on the locations of the source point and the effect point, but also on the orientation of the principal directions at these points. The following form of the crack influence function for two-dimensional problems has been derived on the basis of a simplified analysis of crack interactions (Bažant, 1992):

$$\Lambda(\mathbf{x},\xi) = -\frac{k(r)}{2l^2}\left[\cos 2\theta + \cos 2\psi + \cos 2(\theta+\psi)\right] \quad (5)$$

where

$$k(r) = \left(\frac{\kappa lr}{r^2+l^2}\right)^2 \quad (6)$$

The angles θ and ψ characterize the orientations of two interacting cracks as shown in Fig. 1(a), and κ is a nondimensional parameter roughly equal to the ratio of the average crack size and the characteristic length.

3 Simplified One-Dimensional Problem

Before the new nonlocal formulation is implemented into finite element codes, its performance must be tested in basic situations. The most basic one is one-dimensional localization of damage into a straight band, taking place inside an infinite layer of thickness L (Fig. 1(b)). To make use of the simple expression for the crack influence function $\Lambda(\mathbf{x},\xi)$ in two dimensions (in contrast to the much more complicated form for a 3-D continuum given also in Bažant, 1992), this brief conference paper will consider a plane stress situation—the dimension of the layer in the z-direction is assumed to be so small that the corresponding normal stress σ_z is negligible. On the other hand, the dimension of the layer in the y-direction is very large and the corresponding normal strain ϵ_y is negligible. The layer is loaded by enforcing a uniform displacement in the x-direction at one of the fixed boundaries, which causes an increase of strain ϵ_x and a change of stress σ_x. To simplify the notation, the subscripts at σ_x and ϵ_x as well as at the corresponding stiffness coefficients will be dropped. A more detailed analysis with various ramifications and extensions will be presented in a separate journal article (Bažant and Jirásek, 1994, in press).

Equilibrium in the x-direction requires σ_x to be constant, but ϵ_x can in general vary as a function of x. The basic Eqs. (2), (1) and (4) can be rewritten as

$$\Delta S(x) = [C_u(x) - C_t(x)]\Delta\epsilon(x) \quad (7)$$

$$\Delta\sigma = C_u(x)\Delta\epsilon(x) - \Delta\bar{S}(x) \quad (8)$$

$$\Delta\bar{S}(x) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Phi(x,y,\xi,\eta)\Delta S(\xi)d\xi d\eta + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Lambda(x,y,\xi,\eta)\Delta\bar{S}(\xi)d\xi d\eta = \int_{-\infty}^{\infty}\hat{\Phi}(x,\xi)\Delta S(\xi)d\xi + \int_{-\infty}^{\infty}\hat{\Lambda}(x,\xi)\Delta\bar{S}(\xi)d\xi \quad (9)$$

where

$$\hat{\Phi}(x,\xi) = \int_{-\infty}^{\infty}\Phi(x,y,\xi,\eta)d\eta,\ \hat{\Lambda}(x,\xi) = \int_{-\infty}^{\infty}\Lambda(x,y,\xi,\eta)d\eta \quad (10)$$

In the simple situation considered, the directions of the maximum principal inelastic stress at all points are aligned (Fig. 1(b)), which implies that $\theta = \psi$. With the notation $\zeta = (x-\xi)/l$, the one-dimensional crack influence function can be expressed as

$$\hat{\Lambda}(x,\xi) = \frac{\pi\kappa^2}{l}\left[\frac{4\zeta^6+6\zeta^4+1.5\zeta^2+0.25}{(1+\zeta^2)^{3/2}} - 4|\zeta|^3\right] \quad (11)$$

A surprising fact is that the resulting function is positive for all values of its arguments (Fig. 1(c)), which contradicts the intuitively expected property $\int_{-\infty}^{\infty}\hat{\Lambda}(x,\xi)d\xi = 0$. This is a consequence of the lack of absolute integrability of the original two-dimensional crack influence function. The one-dimensional crack influence function can be shown to be integrable.

4 Bifurcation Analysis of Post-Peak Behavior

4.1 Formulation of the Problem. Let us first use a local stress-strain law linear up to the peak, with a constant slope C_0, and then softening with an initial slope C_s. Up to the peak, the tangential stiffness C_t and the unloading stiffness C_u are identical and equal to C_0, and thus the local inelastic stress increments given by (7) vanish at all points of the layer. The basic Eq. (9) has a trivial solution $\Delta\bar{S}(x) = 0$ and (8) then implies $\Delta\epsilon(x) = \Delta\sigma/C_0 = \text{const.}$, which means that the strains remain uniform up to the peak. After reaching the peak stress, the equilibrium path can bifurcate into several branches. The actual branch followed by the physical system usually corresponds to a localized solution, for which a part of the layer experiences further strain increase accompanied by softening while the rest unloads in an elastic way. The unloading modulus C_u is still equal to C_0 at all points of the layer, but the tangential modulus C_t remains equal to C_0 only in the unloading part (denoted by U) and jumps to C_s in the softening region (denoted by S). Equations (7) and (8) can be substituted into (9), and, introducing the unknown normalized strain increment $e(x) = -C_0\Delta\epsilon(x)/\Delta\sigma$ and a constant $\mu = 1 - C_s/C_0$, a single integral equation ensues:

$$\mu\int_S\hat{\Phi}(x,\xi)e(\xi)d\xi + \int_0^L\hat{\Lambda}(x,\xi)e(\xi)d\xi - e(x) = 1-\lambda(x) \quad (12)$$

The function $\lambda(x)$ on the right-hand side is defined by $\lambda(x) = \int_0^L\hat{\Lambda}(x,\xi)d\xi$. In addition to Eq. (12), an acceptable solution of the problem must satisfy the loading-unloading criterion

$$e(x) \geq 0 \text{ if } x\in S,\quad e(x)\leq 0 \text{ if } x\in U \quad (13)$$

4.2 Analysis of Nonlocal Model of Isotropic Averaging Type. Equation (12) can be discretized and solved numeri-

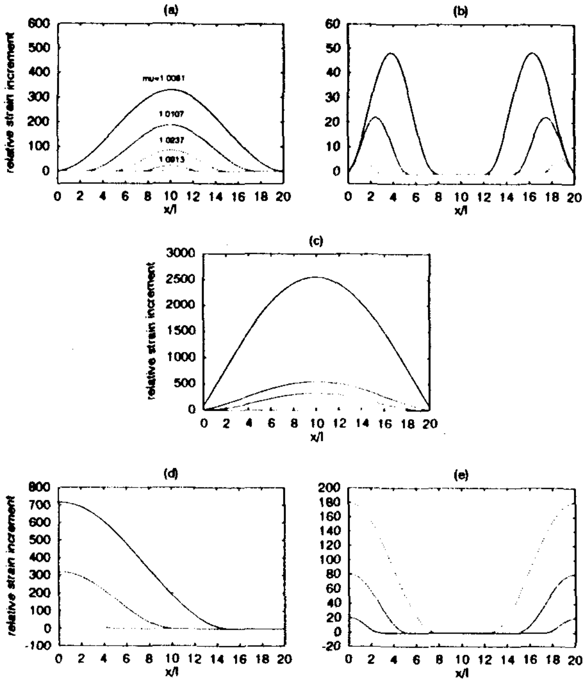

Fig. 2 Admissible solutions (old models): (*a*) U-S-U, (*b*) S-U-S, (*c*) S, (*d*) S-U with renormalization, (*e*) S-U-S with renormalization

cally. For the sake of simplicity, consider first the old nonlocal model, i.e., set $\kappa = 0$. Several admissible solutions for an assumed localization band in the middle of the layer (referred to as the unloading-softening-unloading, or U-S-U, pattern) are depicted in Fig. 2(*a*). More localized solutions required a higher value of μ, i.e., a steeper slope of the descending part of the local constitutive law. Solutions with a large softening zone are possible for smaller values of μ, however, μ must always be larger than 1, which means that the local law must exhibit softening.

Other sets of solutions can be constructed for other localization patterns. It turns out, however, that, for the old nonlocal model with $\kappa = 0$, the solutions are invariant with respect to a shift along the x-axis and therefore the S-U localization profiles have the same shape as the U-S-U profiles and are only shifted to the boundary. Similarly, the S-U-S profiles can be obtained by moving two identical U-S-U profiles to both boundaries (Fig. 2(*b*)). This seems to be a deficiency, because the presence of boundaries would no doubt affect the shape of the localization profiles and the corresponding values of μ. At the same time, the strain increment profiles evaluated under the assumption of loading only (the S type of localization pattern) are highly nonuniform, with strain concentration in the middle of the layer (Fig. 2(*c*)). This means that the boundaries repel strain localization.

The picture substantially changes if the averaging function $\Phi(x,\xi)$ is renormalized in the vicinity of the boundaries, i.e., the normalizing condition $\int_{-\infty}^{\infty} \Phi(x,\xi)d\xi = 1$ is replaced by $\int_V \Phi(x,\xi)d\xi = 1$ (V is the domain of the body), and the original averaging function in an infinite body $\Phi(x,\xi) = \Phi_0[1-(r/l)^2]^2$ is replaced by a normalized function

$$\Phi_n(x,\xi) = \frac{\Phi(x,\xi)}{\int_V \Phi(x,\xi)d\xi} \tag{14}$$

When the normalized averaging function is implemented, the model is able to exactly reproduce uniform strain increments and also the shape of the S-U and S-U-S localization patterns becomes more reasonable (Fig. 2(*d,e*)). This formulation is therefore adopted for the subsequent development.

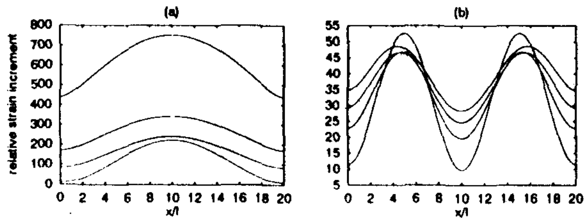

Fig. 3 Nonuniform solutions of the *S* type (new model)

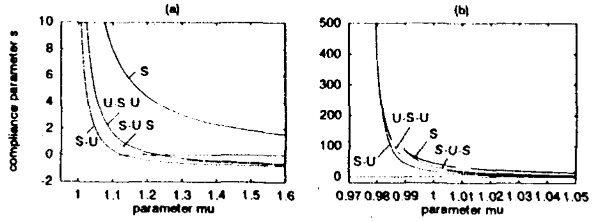

Fig. 4 Compliance parameter for different localization patterns (new model)

4.3 Analysis of New Nonisotropic Tensorial Nonlocal Model. The first striking difference between the old nonlocal model with $\kappa = 0$ and the new model with $\kappa > 0$ is that uniform strain increments are possible only if the local constitutive law has a linear elastic part. As soon as any nonlinearity occurs, strain increments become nonuniform. Figure 3 shows several solutions derived under the assumption of loading only for $\kappa = 0.1$ and μ ranging (*a*) from 0.979 to 0.985 and (*b*) from 1.002 to 1.008.

Beside the solution with all points softening, several other admissible solutions may exist for the same value of the parameter μ, i.e., for the same post-peak slope of the local constitutive law. It can be proven (Bažant, 1988) that the branch that will actually be followed by the real system is the one with the steepest descent. Stability of different branches of the global response existing for a given local constitutive law can be evaluated by introducing a nondimensional compliance parameter

$$s = -\frac{C_0 \Delta\bar{\epsilon}}{\Delta\sigma} = -\frac{C_0}{L\Delta\sigma}\int_0^L \Delta\epsilon(x)dx = \frac{1}{L}\int_0^L e(x)dx \tag{15}$$

whose values are positive for post-peak softening and negative if snapback occurs; $s = 0$ corresponds to a vertical drop in the global load-displacement diagram indicating a loss of stability under displacement control.

It follows from the definition of the compliance parameter s that the actual branch is that which minimizes s. To study the effect of μ on the localization pattern and on the post-peak slope, the compliance parameter was evaluated for various types of solutions and plotted against μ. Figure 4 shows such a plot for $\kappa = 0.1$, $L = 20l$ and a renormalized averaging function Φ_n. It is clear that the pattern S-U dominates in all situations covered by this plot. This means that the strain tends to localize into a band at one boundary (the S-U pattern) rather than into a band in the middle (the U-S-U pattern) or into two symmetric bands at both boundaries (the S-U-S pattern). The decrease of s with increasing μ indicates that the post-peak slope of the load-displacement diagram is getting steeper as the sope of the local constitutive law becomes steeper. At $\mu = 1.14$, s becomes negative, which corresponds to the occurrence of a snapback. Beyond this limit, the test cannot be performed in a stable manner by controlling only the relative displacement of the boundaries.

The characteristic values $\bar{\mu}_i$ of the operator on the left-hand side of (12) mark important points where the number of ad-

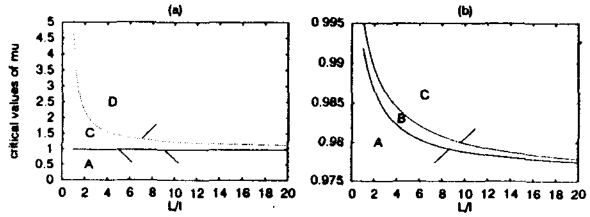

Fig. 5 Critical values of μ depending on the layer width (new model): (*a*) global picture, (*b*) magnified

missible solutions or the character of the solution change (Fig. 5). The first characteristic value $\tilde{\mu}_1$ corresponds to the transition from global hardening to global softening. Between $\tilde{\mu}_1$ and $\tilde{\mu}_2$, there is only one admissible solution for each value of μ. This solution is nonuniform but all the points are softening. At $\tilde{\mu}_2$, solutions of the S-U type start existing and as their compliance parameter s is smaller than that of the S type solutions, the actual response follows the localized branch. In contrast to the old nonlocal model with $\kappa = 0$, the S type solution ceases to be admissible at some value of μ and it changes into a U-S-U solution. However, the S type solution is "reborn" at the third characteristic value $\tilde{\mu}_3$ along with an S-U-S solution and at some higher value of μ it changes its character again. The compliance parameter is plotted against the parameter μ for the most important localization patterns in Fig. 4. The figure reveals that the actual solution is of the S-U type for all values of $\mu > \tilde{\mu}_1$. This was the case for the old nonlocal model, too, but an important difference can be noticed: The generalized nonlocal model allows strain localization even when $\mu < 1$, i.e., even when the local constitutive law exhibits hardening rather than softening. The hardening slope must, however, be sufficiently small. This is demonstrated in Fig. 5, showing the following four regions:

- A—global hardening,
- B—global softening without localization, strain increments remain uniform,
- C—localization into a softening band, stable during displacement control,
- D—snapback in the global load-displacement diagram immediately after peak.

5 Conclusions

1 The conventional nonlocal model with an isotropic averaging function without renormalization cannot capture strain localization at the boundaries. Localized strain profiles are invariant with respect to a shift and not affected by the proximity of the boundary. With renormalization, the strain increments localize into a band at one boundary if the post-peak slope of the local constitutive law exceeds a certain minimum value, which depends on the size of the layer.

2 The new nonlocal model, which contains an integral describing the effect of orientation-dependent crack interactions, leads to nonuniform strain profiles as soon as the local constitutive law deviates from linearity. The global load-displacement diagram can start softening even before the peak in the local constitutive law is reached. Similarly, the solution can bifurcate already in the (locally) hardening regime.

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