Softening Instability: Part II—Localization Into Ellipsoidal Regions

Zdeněk P. Bažant
Professor,
Department of Civil Engineering,
Northwestern University,
Evanston, IL 60208
Mem. ASME

Extending the preceding study of exact solutions for finite-size strain-softening regions in layers and infinite space, exact solution of localization instability is obtained for the localization of strain into an ellipsoidal region in an infinite solid. The solution exploits Eshelby's theorem for eigenstrains in elliptical inclusions in an infinite elastic solid. The special cases of localization of strain into a spherical region in three dimensions and into a circular region in two dimensions are further solved for finite solids—spheres in 3D and circles in 2D. The solutions show that even if the body is infinite the localization into finite regions of such shapes cannot take place at the start of strain-softening (a state corresponding to the peak of the stress-strain diagram) but at a finite strain-softening slope. If the size of the body relative to the size of the softening region is decreased and the boundary is restrained, homogeneous strain-softening remains stable into a larger strain. The results also can be used as checks for finite element programs for strain-softening. The present solutions determine only stability of equilibration states but not bifurcations of the equilibrium path.

Introduction

The strain-localization solutions in the preceding paper (Bažant, 1987) deal with unidirectional localization of strain into an infinite planar band. If the body is finite, localization into such a band does not represent an exact solution because the boundary conditions cannot be satisfied. In this paper, we will seek exact solutions for multidirectional localization due to strain-softening in finite regions. In particular, we will study localization into ellipsoidal regions, including the special cases of a spherical region in three dimensions and a circular region in two dimensions. All definitions and notations from the preceding paper (Bažant, 1987, Part I) are retained.

Softening Ellipsoidal Region in Infinite Solid

This type of strain-localization instability can be solved by application of Eshelby's (1957) theorem for ellipsoidal inclusions with uniform eigenstrain. Consider an ellipsoidal hole, Fig. 1(*a*), in a homogeneous infinite medium that is elastic and is characterized by elastic moduli matrix $\mathbf{D}_u$. We imagine fitting and glueing into this hole an ellipsoidal plug of the same material, Fig. 1(*a*), which must first be deformed by uniform strain ϵ^0 (the eigenstrain) in order to fit into the hole perfectly (note that a uniform strain changes an ellipsoid into another ellipsoid). Then the strain in the plug is unfrozen, which causes the plug to deform with the surrounding medium to attain a new equilibrium state. The famous discovery of Eshelby (1957) was that if the plug is ellipsoidal and the elastic medium is homogeneous and infinite, the strain increment ϵ^e in the plug which occurs during this deformation is *uniform* and is expressed as

$$\epsilon^e_{ij} = S_{ijkm}\epsilon^0_{km}. \tag{1}$$

S_{ijkm} are components of a fourth-rank tensor which depend only on the ratios a_1/a_3 and a_2/a_3 of the principal axes of the ellipsoid and, and for the special case of isotropic materials, on Poisson ratio ν_u; see, e.g., Mura (1982) and Christensen (1979). Due to symmetry of ϵ^0_{ij} and ϵ^e_{ij}, $S_{ijkm} = S_{jikm} = S_{ijmk}$, but in general $S_{ijkm} \neq S_{kmij}$. Coefficients S_{ijkm} are, in general, expressed by elliptic integrals; see also Mura (1982). Extension of Eshelby's theorem to generally anisotropic materials was later accomplished at Northwestern University by Kinoshita and Mura (1971) and Lin and Mura (1973).

It will be convenient to rewrite equation (1) in a matrix form

$$\epsilon^e = \mathbf{Q}_u\epsilon^0 \tag{2}$$

or

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$$
\begin{Bmatrix} \epsilon_{11}^e \\ \epsilon_{22}^e \\ \epsilon_{33}^e \\ 2\epsilon_{12}^e \\ 2\epsilon_{23}^e \\ 2\epsilon_{31}^e \end{Bmatrix} = \begin{bmatrix} S_{1111} & S_{1122} & S_{1133} & S_{1112} & S_{1123} & S_{1131} \\ S_{2211} & S_{2222} & S_{2233} & S_{2212} & S_{2223} & S_{2231} \\ S_{3311} & S_{3322} & S_{3333} & S_{3312} & S_{3323} & S_{3331} \\ 2S_{1211} & 2S_{1222} & 2S_{1233} & 2S_{1212} & 2S_{1223} & 2S_{1231} \\ 2S_{2311} & 2S_{2322} & 2S_{2333} & 2S_{2312} & 2S_{2323} & 2S_{2331} \\ 2S_{3111} & 2S_{3122} & 2S_{3133} & 2S_{3112} & 2S_{3123} & 2S_{3131} \end{bmatrix} \begin{Bmatrix} \epsilon_{11}^0 \\ \epsilon_{22}^0 \\ \epsilon_{33}^0 \\ 2\epsilon_{12}^0 \\ 2\epsilon_{23}^0 \\ 2\epsilon_{31}^0 \end{Bmatrix} \tag{3}
$$

in which $\epsilon = (\epsilon_{11}, \epsilon_{22}, \epsilon_{33}, 2\epsilon_{12}, 2\epsilon_{23}, 2\epsilon_{31})^T$ and superscript T denotes the transpose of a matrix.

For isotropic materials, the only nonzero elements of matrix $\mathbf{Q}$ are those between the dashed lines marked in equation (3), which is the same as for the stiffness matrix. The factors 2 in matrix $\mathbf{Q}_u$ in equation (3) are due to the fact that the column matrix of strains is (6×1) rather than (9×1) and, therefore, must involve shear angles $2\epsilon_{12}$, $2\epsilon_{23}$ and $2\epsilon_{31}$ rather than tensorial shear strain components ϵ_{12},ϵ_{23} and ϵ_{31}, or else $\sigma^T \delta\epsilon$ where $\sigma^T = (\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}, \sigma_{23}, \sigma_{31})$ would not be a correct work expression. (The work expression $\sigma_{ij}\delta\epsilon_{ij}$, as well as the sum implied in equation (1) for each fixed i,j, has 9 terms in the sum, not 6.) For example, writing out the terms of equation (1) we have

$$
\begin{aligned} \epsilon_{11}^e &= \ldots + (S_{1112}\epsilon_{12}^0 + S_{1121}\epsilon_{21}^0) + \ldots \\ &= \ldots + S_{1112}(2\epsilon_{12}^0) + \ldots \end{aligned} \tag{4}
$$

while the factors 2 arise as follows

$$
\begin{aligned} 2\epsilon_{12}^e = 2[\ldots &+ S_{1233}\epsilon_{33}^0 + (S_{1112}\epsilon_{12}^0 + S_{1121}\epsilon_{21}^0) \\ &+ (S_{1223}\epsilon_{23}^0 + S_{1232}\epsilon_{32}^0) + \ldots] \\ = \ldots & (2S_{1233})\epsilon_{33}^0 + (2S_{1212})2\epsilon_{12}^0 \\ &+ (2S_{1223})2\epsilon_{23}^0 + \ldots \end{aligned} \tag{5}
$$

The stress in the ellipsoidal plug, σ^e (which is uniform), may be expressed according to Hooke's law as

$$
\sigma^e = \mathbf{D}_u(\epsilon^e - \epsilon^0). \tag{6}
$$

After substituting $\epsilon^0 = \mathbf{Q}_u^{-1}\epsilon^e$, according to equation (2), we get $\sigma^e = \mathbf{D}_u(\epsilon^e - \mathbf{Q}_u^{-1}\epsilon^e)$ or

$$
\sigma^e = \mathbf{D}_u(\mathbf{1} - \mathbf{Q}_u^{-1})\epsilon^e \tag{7}
$$

where $\mathbf{1}$ is a unit 6×6 matrix. The surface tractions that the ellipsoidal plug exerts upon the surrounding infinite medium, Fig. 1(*a*), are $p_i^e = \sigma_{ij}^e n_j$, in which n_j denotes the components of a unit normal $\mathbf{n}$ of the ellipsoidal surface (pointed from the ellipsoid outward).

Consider now infinitesimal variations $\delta\mathbf{u}$, $\delta\epsilon$, $\delta\sigma$ from the initial equilibrium state of uniform strain ϵ^0 in an infinite homogeneous anisotropic solid (without any hole). The matrices of incremental elastic moduli corresponding to ϵ^0 are $\mathbf{D}_t$ for further loading and $\mathbf{D}_u$ for unloading, $\mathbf{D}_u$ being positive-definite. We imagine that the initial equilibrium state is disturbed by applying surface tractions δp_i over the surface of the ellipsoid with axes a_1, a_2, a_3, Fig. 1(*b*). We expect δp_i to produce loading inside the ellipsoid and unloading outside. We try to calculate the displacements δu_i produced by tractions δp_i at all loading points on the ellipsoid surface.

Let $\delta\epsilon_{ij}^e$, δu_i^e be the strain and displacement variations produced (by tractions δp_i) in the ellipsoid, and denote the net tractions acting on the softening ellipsoid as δp_i^t, and those on the rest of the infinite body, i.e., on the exterior of the ellipsoid, as δp_i^e. As for the distributions of δp_i^t and δp_i^e over the ellipsoid surface, we assume them to be such that $\delta p_i^t = \delta\sigma_{ij}^t n_j$ and $\delta p_i^e = \delta\sigma_{ij}^e n_j$ where $\delta\sigma_{ij}^t$ and $\delta\sigma_{ij}^e$ are arbitrary constants; $\delta\sigma_{ij}^t$ is the stress within the softening ellipsoidal region, which is uniform (and represents an equilibrium field), and $\delta\sigma_{ij}^e$ is a fictitious uniform stress in this region which would equilibrate δp_i^e. Stresses $\delta\sigma_{ij}^e$ in reality exist only on the outside of the ellipsoid surface.

Equilibrium requires that $\delta p_i = \delta p_i^t - \delta p_i^e$. The first-order work δW done by σ_{ij}^0 must vanish if the initial state is an equilibrium state. The second-order work done by δp_i may be calculated as

$$
\begin{aligned} \delta^2 W &= \int_S \frac{1}{2}\delta p_i \delta u_i^e dS \\ &= \frac{1}{2}\int_S \delta p_i^t \delta u_i^e dS - \frac{1}{2}\int_S \delta p_i^e \delta u_i^e dS \\ &= \frac{1}{2}\delta\sigma_{ij}^t \int_S n_j \delta u_i^e dS - \frac{1}{2}\delta\sigma_{ij}^e \int_S n_j \delta u_i^e dS \end{aligned} \tag{8}
$$

where S = surface of the softening ellipsoidal region. Note that $\delta\sigma_{ij}^e$ are not the actual stresses in the solid but merely serve the purpose of characterizing the surface tractions δp_i^e. Applying Gauss' integral theorem and exploiting the symmetry of tensors $\delta\sigma_{ij}^t$ and $\delta\sigma_{ij}^e$, we further obtain

$$
\begin{aligned} \delta^2 W &= \frac{1}{2}(\delta\sigma_{ij}^t - \delta\sigma_{ij}^e)\int_V \delta u_{i,j}^e dV \\ &= \int_V \frac{1}{2}(\delta\sigma_{ij}^t - \delta\sigma_{ij}^e)\frac{1}{2}(\delta u_{i,j}^e + \delta u_{j,i}^e) dV \\ &= \int_V \frac{1}{2}(\delta\sigma^t - \delta\sigma^e)^T \delta\epsilon^e dV \end{aligned} \tag{9}
$$

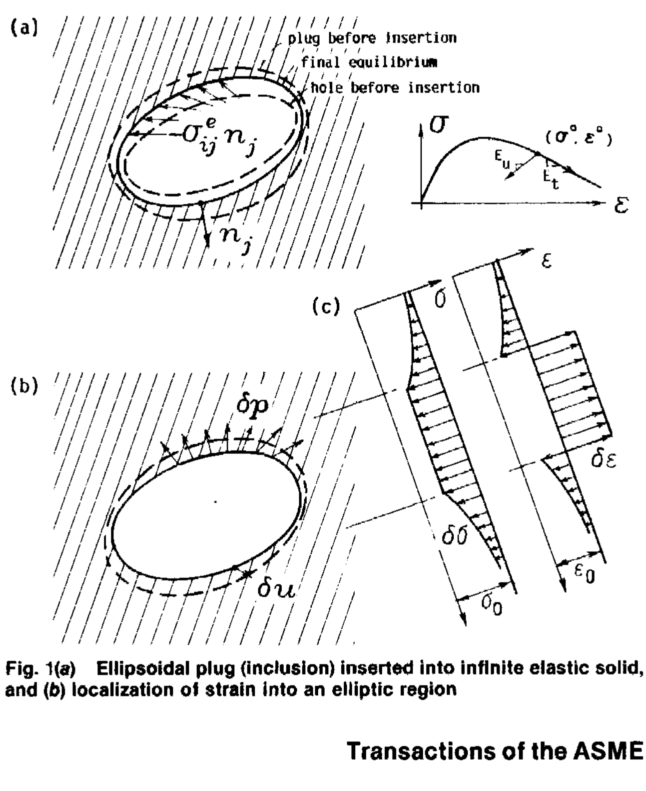

Fig. 1(*a*) Ellipsoidal plug (inclusion) inserted into infinite elastic solid, and (*b*) localization of strain into an elliptic region

where V = volume of the softening ellipsoidal region, and subscripts preceded by a comma denote partial derivatives. We changed here to matrix notation and also recognized that $\frac{1}{2}(\delta u^e_{i,j} + \delta u^e_{j,i}) = \delta\epsilon^e_{ij}$. Now we may substitute

$$\delta\sigma^t = \mathbf{D}_t \delta\epsilon^e \tag{10}$$

or $\delta\sigma^{t^T} = \delta\epsilon^{e^T}\mathbf{D}_t$ (assuming $\mathbf{D}^T_t = \mathbf{D}_t$). According to equation (7), we also have, as a key step

$$\delta\sigma^e = \mathbf{D}_u(1 - \mathbf{Q}_u^{-1})\delta\epsilon^e. \tag{11}$$

In contrast to our previous consideration of the elastic ellipsoidal plug made of the same elastic material, equations (2)-(7), the sole meaning of $\delta\sigma^e$ now is to characterize the tractions δp^e_i acting on the ellipsoidal surface of the infinite medium lying outside the ellipsoid. Noting that the integrand in equation (9) is constant, we thus obtain

$$\delta^2 W = \frac{1}{2}\delta\epsilon^{e^T}\mathbf{Z}\delta\epsilon^e V = \frac{1}{2}Z_{ijkm}\delta\epsilon^e_{ij}\delta\epsilon^e_{km}V \tag{12}$$

in which $\mathbf{Z}$ denotes the following 6×6 matrix

$$\mathbf{Z} = \mathbf{D}_t + \mathbf{D}_u(\mathbf{Q}_u^{-1} - 1). \tag{13}$$

Equation (12) defines a quadratic form. If the initial uniform strain ϵ^0 is such that the associated $\mathbf{D}_t$ and $\mathbf{D}_u$ give $\delta^2 W > 0$ for all possible $\delta\epsilon^e_{ij}$, then no localization in an ellipsoidal region can begin from the initial state of uniform strain ϵ^0 spontaneously, i.e., without applying loads δp. If, however, $\delta^2 W$ is negative for some $\delta\epsilon^e_{ij}$, the localization leads to a release of energy which is first manifested as a kinetic energy and is ultimately dissipated as heat. Such a localization obviously increases entropy of the system, and so it will occur, as required by the second law of thermodynamics. Therefore, the necessary condition of stability of a uniform strain field in an infinite solid is that matrix $\mathbf{Z}$ given by equation (13) must be *positive-definite*.

The expressions for Eshelby's coefficients S_{ijkm} from which matrix $\mathbf{Q}_u$ is formed (see, e.g., Mura, 1982) depend on the ratios a_1/a_3, a_2/a_3 of the axes of the ellipsoidal localization region. They also depend on the ratios of the unloading moduli D^u_{ijkm}. If, e.g., the unloading behavior is assumed to be isotropic, they depend on the unloading Poisson ratio ν_u. Matrix $\mathbf{D}_u$ is determined by ν_u and unloading Young's modulus E_u. If, just for the sake of illustration, the loading behavior is assumed to be also isotropic, matrix $\mathbf{D}_t$ is determined by ν_t and E_t (Poisson's ratio and Young's modulus for loading). E_t, ν_t, E_u, ν_u, in turn, depend on the strain ϵ^0_{ij} at the start of localization. Since a division of $\mathbf{Z}$ by E_u does not affect positive-definiteness, only the ratio $\mathrm{E}_t/\mathrm{E}_u$ matters. Thus, $\mathbf{Z}$ is a function of the form

$$\mathbf{Z} = \mathrm{E}_u\hat{\mathbf{Z}}\left(\frac{a_1}{a_3}, \frac{a_2}{a_3}, \nu_u, \nu_t, \frac{\mathrm{E}_t}{\mathrm{E}_u}\right)$$

$$= \mathrm{E}_u\bar{\mathbf{Z}}\left(\frac{a_1}{a_3}, \frac{a_2}{a_3}, \epsilon^0_{ij}\right) \tag{14}$$

where $\hat{\mathbf{Z}}$ and $\bar{\mathbf{Z}}$ are nondimensional matrix functions.

Note that matrix $\mathbf{Z}$, which decides the localization instability, is independent of the size of the ellipsoidal localization region. This is the same conclusion as already made for a planar localization band in an infinite solid. No doubt, the size of the localization ellipsoid would matter for finite-size solids, same as it does for localization bands in layers.

The previously obtained solution for a planar localization band in an infinite solid must be a special case of the present solution for an ellipsoid. Localization in line cracks also must be obtained as a special case for $a_3 \to \infty$; however, the present solution is not realistic for this case since energy dissipation due to strain-softening is finite per unit volume and, therefore, vanishes for a crack (the volume of which is zero). For this case, it would be necessary to include the fracture energy (surface energy) in the energy criterion of stability, same as in fracture mechanics. In this study, though, we take the view that, due to material heterogeneity, it makes no sense to apply a continuum analysis to localization regions whose width is less than a certain length h proportional to the maximum size of material inhomogeneities.

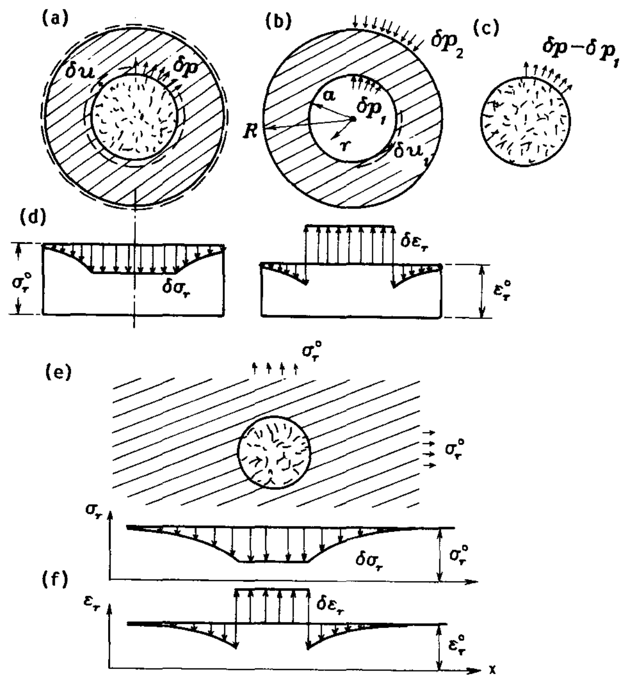

Fig. 2 Localization of strain into spherical and circular regions

Spherical Softening Region in a Sphere or Infinite Solid

Localization in a spherical region is a special case of the preceding solution for ellipsoidal regions. However, the solution may now be easily obtained even when the solid is finite. Consider a spherical hole of radius a inside a sphere of radius R, (Fig. 2(*a*)). We assume polar symmetry of the deformation field and restrict our attention to materials that are isotropic for unloading. As shown by Lamé (1852) (see, e.g., Timoshenko and Goodier, 1970, p. 395), the elastic solution for the radial displacements and the radial normal stresses at a point of radial coordinate r is

$$u = Ar + Dr^{-2}, \quad \sigma_r = \mathrm{E}_u(\bar{A} - 2\bar{D}r^{-3}) \tag{15}$$

where $\bar{A} = A/(1-2\nu_u)$, $\bar{D} = D/(1+\nu_u)$; E_u, ν_u = Young's modulus and Poisson's ratio of the sphere, and A, D = arbitrary constants to be found from the boundary conditions.

We now consider a solid sphere of radius R which is initially under uniform hydrostatic stress σ^0 and strain ϵ^0 ($\sigma^0 = \sigma^0_{kk}/3$, $\epsilon^0 = \epsilon^0_{kk}/3$), and seek the conditions for which the initial strain may localize in an unstable manner into a spherical region of radius a. Such localization may be produced by applying on the solid sphere at $r = a$ radial outward tractions δp (i.e. pressure) uniformly distributed over the spherical surface of radius a, Fig. 2(*a*). To determine the work of δp, we need to calculate the radial outward displacement δu_i at $r = a$. We will distinguish several types of boundary conditions on the outer surface $r = R$.

(*a*) Outer Surface Kept Under Constant Load. As the boundary condition during localization, we assume that the initial radial pressure p^0_2 applied at outer surface $r = R$ is held constant, i.e., $\delta p_2 = 0$. For $\delta\sigma_r = -\delta p_1$ at $r = a$ and $\delta\sigma_r = -\delta p_2 = 0$ at $r = R$, equation (11) may be solved to yield $\bar{A} = a^3\delta p_1/\mathrm{E}(R^3 - a^3)$, $\bar{D} = \bar{A}R^3/2$, and from equation (15)

$$\delta u_1 = \frac{\delta p_1}{C_u}, \quad \frac{1}{C_u} = \frac{a}{\mathrm{E}_u (R^3 - a^3)} \left[(1 - 2\nu_u) a^3 + \frac{1 + \nu_u}{2} R^3 \right]. \quad (16)$$

The inner spherical softening region of radius a, Fig. 2(*c*), is assumed to remain in a state of uniform hydrostatic stress and strain, and its strain-softening properties to be isotropic, characterized by E_t and ν_t. Thus the strains for $r<a$ are $\delta\epsilon = \delta u_1/a$, the stresses are $\delta\sigma = 3K_t\, \delta u_1/a = C_t \delta u_1$ with $C_t = 3K_t/a$; K_t = bulk modulus for further loading (softening); $3K_t = \mathrm{E}_t/(1-2\nu_t)$, with $\mathrm{E}_t<0$, $K_t<0$, for softening. The surface tractions acting on the softening region, Fig. 2(*c*), are equal to $C_t\delta u_1$ where $C_t = 3K_t/a$. Hence, by equilibrium, the total distributed traction at surface $r=a$ must be $\delta_p = C_u \delta u_1 + C_t \delta u_1$ and the work done by δp is $\Delta W = 4\pi a^2 (\tfrac{1}{2} \delta p \delta u_1) = 2\pi a^2 (C_u + C_t) \delta u_1^2$. Thus the necessary condition of stability of the initial uniform strain ϵ^0 in the solid sphere is $C_u + C_t > 0$, which yields the necessary condition for stability

$$-\frac{\mathrm{E}_t}{\mathrm{E}_u} < \frac{2(1-2\nu_t)(R^3 - a^3)}{2(1-2\nu_u)a^3 + (1+\nu_u)R^3}. \quad (17)$$

Changing $<$ to $>$, we obtain the sufficient condition for strain-localization instability.

(*b*) **Outer Surface Kept Fixed.** In this case we assume that during localization $\delta u = 0$ at $r=R$, and $\delta\sigma_r = -\delta p_1$ at $r=a$. From equation (15), we may then solve $D = -AR^3$ and

$$A = -\frac{\delta p_1}{\mathrm{E}_u} \left(\frac{1}{1-2\nu_u} + \frac{2R^3}{(1+\nu_u)a^3} \right)^{-1} \quad (18)$$

which yields

$$\delta p_1 = C_u \delta u_1, \quad C_u = \frac{\mathrm{E}_u a^2}{R^3 - a^3} \left(\frac{1}{1-2\nu_u} + \frac{2R^3}{(1+\nu_u)a^3} \right). \quad (19)$$

The work done on the solid sphere by tractions $\delta p = C_u \delta u_1 + C_t \delta u_1$ applied at surface $r=a$ is $\Delta W = 2\pi a^2 (C_u + C_t)\, \delta u_1^2$ where $C_t = \mathrm{E}_t/(1-2\nu_t)a$ as before. Thus the necessary stability condition is $C_u + C_t > 0$, which can now be reduced to the condition

$$-\frac{\mathrm{E}_t}{\mathrm{E}_u} < \frac{1-2\nu_t}{R^3 - a^3} \left(\frac{a^3}{1-2\nu_u} + \frac{2R^3}{1+\nu_u} \right). \quad (20)$$

Assuming that $|\mathrm{E}_t|$ increases continuously after the peak of the stress-strain diagram as ϵ_0 is increased, instability develops at the value $a = a_{cr}$ which minimizes $|\mathrm{E}_t|$ under the restriction $h/2 \le a \le R$ where h is the given minimum admissible size of the strain-softening region, representing a material property. For the case of prescribed pressure at the boundary $r=R$, we find from equation (17) that $a_{cr} = R$, which corresponds to $\mathrm{E}_t = 0$. So the sphere becomes unstable right at the start of strain-softening, i.e., no strain-softening can be observed when the boundary is not fixed. For the case of a fixed (restrained) boundary at $r=R$, equation (20), one can verify that Min $|\mathrm{E}_t|$ is finite and occurs at a_{cr} = Min $a = h/2$ (provided that $\nu_u \ge 0$).

For $R/a \to \infty$, equation (20) yields the stability condition for the case of infinite solid fixed at infinity, Fig. 2(*e, f*)

$$-\frac{\mathrm{E}_t}{\mathrm{E}_u} < \frac{2(1-2\nu_t)}{1+\nu_u} \quad (21)$$

It is interesting that for $R/a \to \infty$ equation (17) yields the same condition, but this limit case is of questionable significance since we found that $a_{cr} = R$ when the boundary is not fixed. Note that the stability condition in equation (21) does not depend on the radius a of the softening region; yet, unlike the softening in a layer in infinite solid, solved before, instability does not begin at the peak of stress-strain diagram (where $\mathrm{E}_t = 0$) but begins only at a certain finite negative slope of the stress-strain diagram. This slope can in fact be rather steep ($-\mathrm{E}_t = 2\mathrm{E}_u$ for $\nu_t = \nu_u = 0$).

Circular Softening Region in a Planar Disk or Infinite Plate

Working in two dimensions, consider a circular hole of radius a inside a homogeneous isotropic circular disk of radius R Fig. 2(*a,b*). We assume a plane-stress state and, then, according to Lamé's solution, the radial displacement u and the radial normal stresses are (see, e.g., Flügge, 1962, p. 37–13; or Timoshenko-Goodier, 1970, p. 70)

$$u = \frac{1-\nu_u}{\mathrm{E}_u} \frac{a^2 p_1 - R^2 p_2}{R^2 - a^2} r + \frac{1+\nu_u}{\mathrm{E}_u} \frac{R^2 a^2 (p_1 - p_2)}{(R^2 - a^2) r} \quad (22)$$

$$\sigma_r = \frac{R^2 a^2 (p_2 - p_1)}{(R^2 - a^2) r^2} + \frac{a^2 p_1 - R^2 p_2}{R^2 - a^2},$$

$$\epsilon_2 = \frac{2\nu_u (p_2 R^2 - p_1 a^2)}{\mathrm{E}_u (R^2 - a^2)} \quad (23)$$

where p_1 and p_2 are the pressures applied along the hole perimeter and along the outer perimeter of the disk, respectively, and ϵ_2 is the transverse strain in the plate, which is independent of r. Depending on the boundary conditions, we distinguish three cases:

(*a*) **Outer Boundary Kept Under Constant Load.** As the boundary condition during the strain localization instability, *we now assume that the initial radial pressure p_2 applied at the* outer boundary $r=R$ of the disk is held constant, i.e., $\delta p_2 = 0$. Equation (22) then yields for $\delta u_1 = \delta u$ at $r = a$, Fig. 2(*b*), the relation

$$\delta u_1 = \frac{\delta p_1}{C_u}, \quad \frac{1}{C_u} = \frac{a}{\mathrm{E}_u} \left(\nu_u + \frac{R^2 + a^2}{R^2 - a^2} \right). \quad (24)$$

The inner circular softening region of radius a, Fig. 2(*c*), is assumed to remain in a uniform state of stress and strain. Thus the strains for $r<a$ are $\delta\epsilon_r = \delta u_1/a$. Assuming the plate to be thin compared to radius a, we may assume the strain-softening region to be also in a plane-stress state, and then $\delta\epsilon_r = \delta\sigma_r (1-\nu_t)/\mathrm{E}_t$. Hence,

$$\delta\sigma_r = C_t \delta u_1, \quad C_t = \frac{1}{a} \frac{\mathrm{E}_t}{1-\nu_t}. \quad (25)$$

Now, we consider uniformly distributed outward tractions δp to be applied along the circle $r=a$ on the solid disk (without the hole). By equilibrium, $\delta p = C_u \delta u_1 + C_t \delta u_1$ and the work done by δp is $\Delta W = 2\pi a (\tfrac{1}{2}\, \delta p \delta u_1) = \pi a (C_u + C_t) \delta u_1^2$. Thus the necessary condition of stability of the initial state of uniform strain is $C_u + C_t > 0$. According to equations (24) and (25), the stability condition for plane stress becomes

$$-\frac{\mathrm{E}_t}{\mathrm{E}_u} < \frac{1-\nu_t}{\nu_u + \dfrac{R^2 + a^2}{R^2 - a^2}} \quad \text{(thin plate).} \quad (26)$$

As another limiting case, we may consider a long cylinder of radius R (and length $>> R$), in which case the softening region is forced to have along the cylinder axis the same strain ϵ_z as the unloading region. However, the softening region is not in a plane-strain state either. Assuming that the planes normal to the cylinder axis remain plane, consider now that, *unlike before, the axial stresses σ_z are nonzero. We must im-*pose the equilibrium condition that the resultants of σ_z^t in the softening region and of σ_z^u in the unloading region cancel each other, i.e., $\pi(R^2 - a^2)\sigma_z^u = -\pi a^2 \sigma_z^t$. We leave it to a possible user to work out the solution in detail and we now restrict our attention to the case $a << R$, for which, according to equation (20) ($\delta p_2 = 0$), we have $\delta\epsilon_z = 2\nu_u \delta p_1 a^2/(a^2 - R^2)\mathrm{E}_u \simeq 0$ (for $r \ge a$), while also $\delta\sigma_z = 0$. Therefore, we may assume for the incremental deformation in the strain-softening region a state of plane strain. The solution may then be obtained simply by replacing E_t with $E_t' = E_t/(1-\nu_t^2)$, and ν_t with $\nu_t' = \nu_t/(1-\nu_t)$ (the unloading region remains in plane stress in this case). Equation (26) thus transforms to

$$-\frac{E_t}{E_u} < \frac{(1+\nu_t)(1-2\nu_t)}{\nu_u + \dfrac{R^2+a^2}{R^2-a^2}} \quad \text{(long cylinder, } a << R). \tag{27}$$

When a/R is not very small, the solution may be expected to lie between equations (26) and (27).

(*b*) **Outer Boundary Kept Fixed.** In this case, we have during localization $\delta u = 0$ at $r = R$ and $\delta\sigma_r = -\delta p_1$ at $r = a$. Taking the variations of equations (22) at $r = R$ and $r = a$, we get

$$\frac{1-\nu_u}{E_u}\,\frac{a^2\delta p_1 - R^2\delta p_2}{R^2-a^2}R + \frac{1+\nu_u}{E_u}\,\frac{R^2a^2(\delta p_1-\delta p_2)}{(R^2-a^2)R} = 0 \tag{28}$$

$$\delta u_1 = \frac{1-\nu_u}{E_u}\,\frac{a^2\delta p_1 - R^2\delta p_2}{R^2-a^2}a + \frac{1+\nu_u}{E_u}\,\frac{R^2a^2(\delta p_1-\delta p_2)}{(R^2-a^2)a}. \tag{29}$$

Eliminating δp_2 from these two equations, we get the relation $\delta p_1 = C_u \delta u_1$ with

$$\frac{1}{C_u} = \frac{1}{E_u(R^2-a^2)}\left\{a[(1-\nu_u)a^2 + (1+\nu_u)R^2] - \frac{4R^2a^3}{(1-\nu_u)R^2+(1+\nu_u)a^2}\right\}. \tag{30}$$

By the same reasoning as before, the necessary condition for the stability of the initial uniform strain ϵ^0 is $C_u + C_t > 0$ where C_t is again given by equation (25). This condition yields

$$-\frac{E_t}{E_u} < \frac{(1-\nu_t)(R^2-a^2)}{R^2+a^2+\nu_u(R^2-a^2) - \dfrac{4R^2a^2}{(1-\nu_u)R^2+(1+\nu_u)a^2}} \quad \text{(thin plate).} \tag{31}$$

For the case of a long cylinder of length $>> R$ and with $a << R$, we may obtain the solution again by replacing E_t, ν_t with E_t', ν_t'. This yields

$$-\frac{E_t}{E_u} < \frac{(1+\nu_t)(1-2\nu_t)(R^2-a^2)}{R^2+a^2+\nu_u(R^2-a^2) - \dfrac{4R^2a^2}{(1-\nu_u)R^2+(1+\nu_u)a^2}} \quad \text{(long cylinder, } a << R). \tag{32}$$

Instability develops at the value $a = a_{cr}$ which minimizes $|E_t|$ under the restriction that $h/2 \le a \le R$. For the case of prescribed load at outer boundary, we find from equation (26) or (27) that $a_{cr} = R$, which corresponds to $E_t = 0$. Thus the disk becomes unstable right at the start of strain-softening, i.e., no strain-softening can be observed. For the case of a fixed (restrained) boundary at $r = R$, we find that, for $a \to R$, lim $(-E_t/E_n) = \infty$ (to verify it one needs to substitute $a = R - \delta$ and consider $\delta \to 0$); consequently Min $|E_t|$ is finite, and it is found to occur at a_{cr} = Min $a = h/2$.

For $R/a \to \infty$, equation (31) or (32) yields the stability condition for the case of infinite plate fixed at infinity, Fig. 2(*e*, *f*)

$$-\frac{E_t}{E_u} < \frac{1-\nu_t}{1+\nu_u} \quad \text{for thin plate} \tag{33}$$

$$-\frac{E_t}{E_u} < \frac{(1+\nu_t)(1-2\nu_t)}{1+\nu_u} \quad \text{for massive solid.} \tag{34}$$

It is interesting that for $R/a \to \infty$ equations (26) and (27)

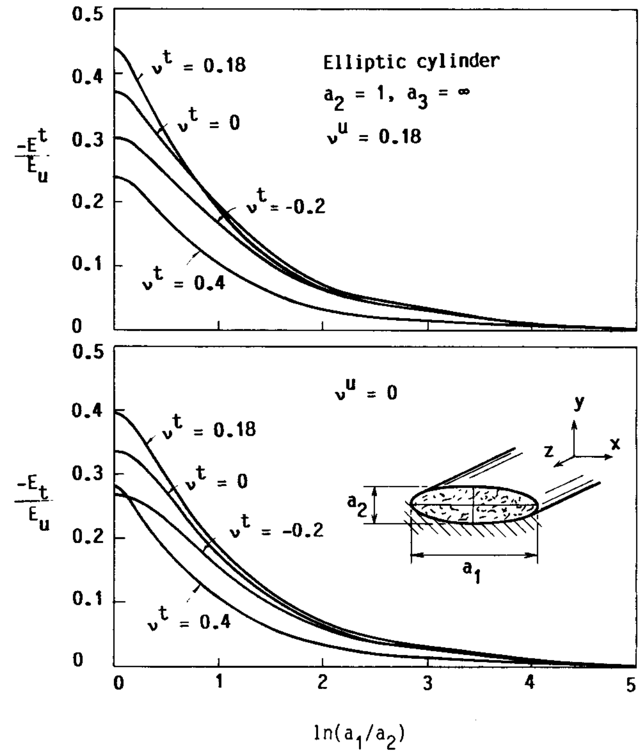

Fig. 3(*a*, *b*)

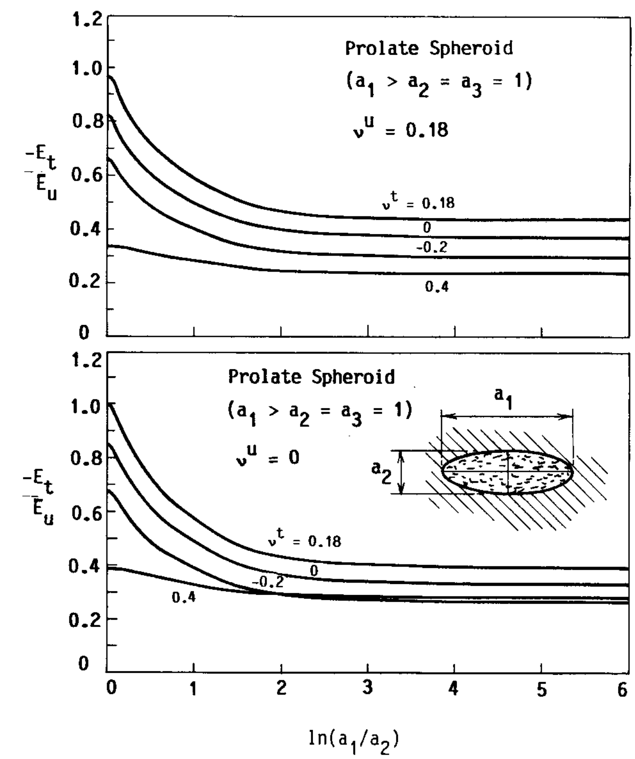

Fig. 3(*c*, *d*)

Fig. 3 Tangential modulus E_t at the limit of stability against localization into ellipsoidal region as a function of ratio a_1/a_2 of principal axes of ellipsoid

yield the same conditions, but these limits are of questionable significance since we found that $a_{cr} = R$ when the boundary is not fixed. Note that the stability conditions in equations (33) and (34) are independent of the size of the localization region, same as we found it for spherical localization regions and layers.

Softening Annulus or Shell

Solution on the basis of equations (22) and (23) or (25) is also possible for curved softening bands limited by two circles (an annulus) or by two spherical surfaces (a shell). In this case, the softening layer is not in a homogeneous state of strain and stress. The resulting formulas are more complicated. They represent a transition between the solution for a softening circle (or cylinder, sphere) and a softening band.

Numerical Examples and Discussion of Results

Figure 3 shows some numerical results for localization of strain into ellipsoidal domains in infinite space. The results were calculated for domains in the shape of infinitely long elliptic cylinder ($a_3 \to \infty$ and various a_1/a_2) as well as prolate spheroid ($a_1 > a_2 = a_3$, various ratios a_1/a_2). The material was assumed to be incrementally isotropic, with matrices $\mathbf{D}_t$ and $\mathbf{D}_u$ characterized by Young's moduli E_t and E_u, and Poisson ratios $\nu_u = 0.18$ with various values of ν_t. The assumption of incremental isotropy is here made for the sake of simplicity. In reality, the incremental moduli at strain-softening must be expected to be anisotropic, except when the initial state is a purely volumetric strain. A subsequent paper (Bažant and Lin, 1987) gives numerical results for incrementally anisotropic moduli corresponding to von Mises plasticity and nonassociated Drucker-Prager plasticity with a negative plastic modulus.

Matrix $\mathbf{Z}$, equation (13), was evaluated by computer on the basis of S_{ijkm} taken from Mura (1982, equations (11.22) and (11.29)). The smallest eigenvalue of matrix $\mathbf{Z}$ was calculated by a computer library subroutine. Iterative search by Newton method was made to find the value of E_t/E_u for which the smallest eigenvalue is zero and is about to become negative, which indicates loss of positive-definiteness.

The results are plotted in Figs. 3(*a–d*). For the infinite cylinder, Fig. 3(*a,b*), as a_1/a_2 increases, the localization instability occurs at smaller $|E_t/E_u|$. The case $a_1/a_2 \to \infty$ corresponds to an infinite planar band, and the results are identical to those obtained before for this case. In particular, $|E_t|$ tends to 0 as $a_1/a_2 \to \infty$; i.e., instability occurs right at the peak of the stress-strain diagram.

For the prolate spheroid, Fig. 3(*c, d*), the instability also occurs at decreasing $|E_t/E_u|$ as a_1/a_2 increases, but for $|E_t| \to \infty$, which corresponds to an infinite circular tube, a finite value of $|E_t|$, depending on Poisson's ratio, is still required for instability. This limiting case is equivalent to two-dimensional localization in a circular region, equation (34). On the other hand, the case $a_1/a_2 = 1$, Fig. 3(*c,d*), is equivalent to localization into a spherical region, equation (21).

Figures 4(*a–d*) shows the plots for various incremental Poisson's ratios of $|E_t/E_u|$ at incipient localization instability as a function of R/a for the following cases: (1) sphere, displacement fixed, equation (20); (2) localization in planar band in uniaxial extension (Equation (12) of Bažant, 1987, $R/a = L/h$); (3) localization in planar shear band (Bažant, 1987, equation (13), $R/a = L/h$).

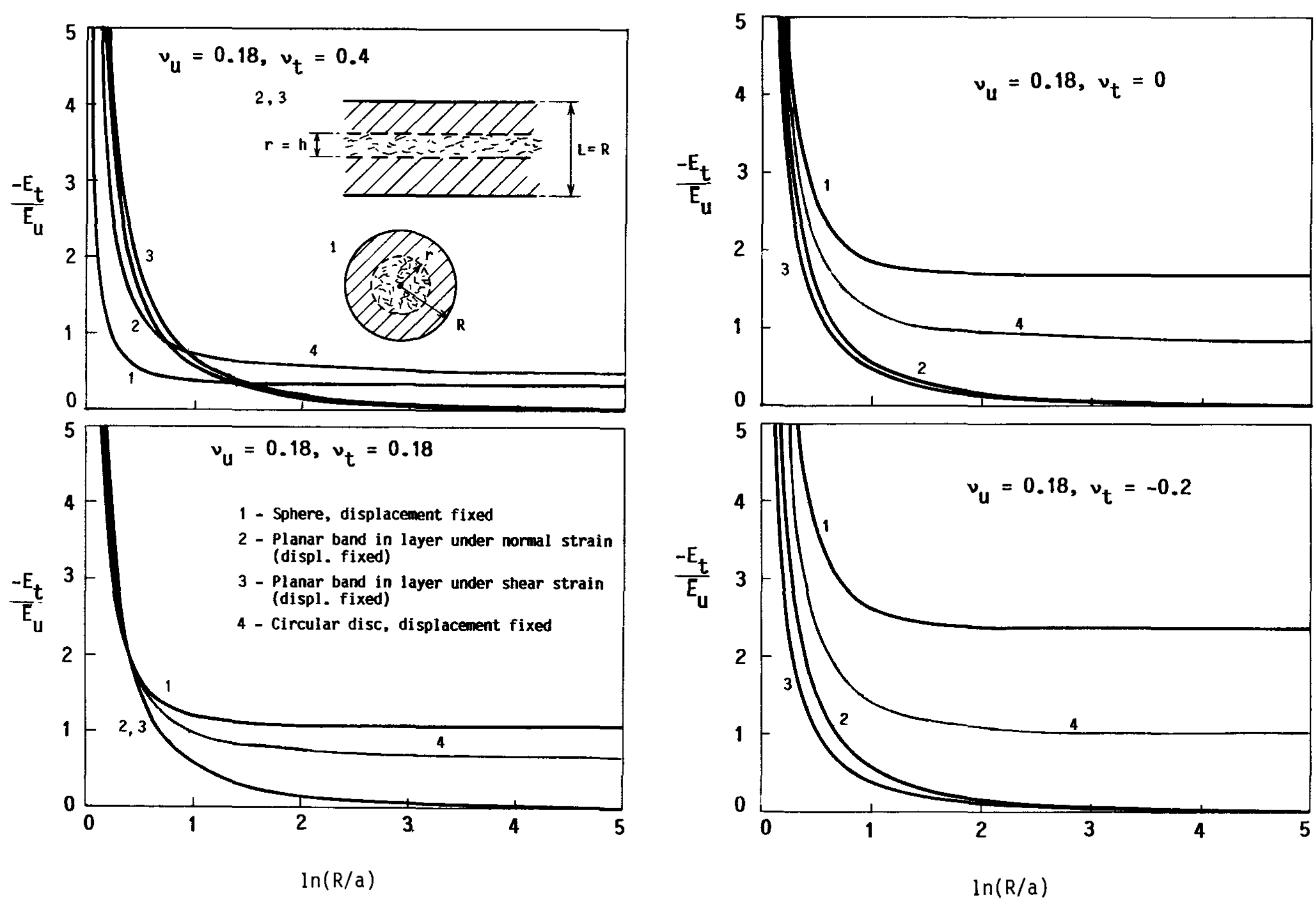

Fig. 4(*a, b*)

Fig. 4(*c, d*)

Fig. 4 Tangential modulus E_t at the limit of stability against localization into spherical and circular regions of radius *r* within a sphere or disk of radius *R*

We see from Fig. 4 that the value of E_t at instability depends strongly on the relative size $R(a)$ of the body as well as the Poisson's ratios. For infinite body size ($R/a \to \infty$), instability of planar bands occurs at $E_t = 0$, i.e., at the peak of stress-strain diagram. The same happens under load-controlled conditions for the cases of spherical or circular regions with $R/a \to 1$ (i.e., the smallest possible body size for which the body remains at homogeneous strain). For $R/a > 1$, the spherical or circular regions generally require a finite slope $|E_t|$ to produce localization instability, provided the boundary is under prescribed displacement during the localization.

The results show that a localization instability in the form of a planar band always develops at a smaller $|E_t|$, and thus at a smaller initial strain, than the localization instability in the form of an ellipsoidal softening region. That does not mean, however, that the planar band would always occur in practice. A planar localization band cannot accommodate the boundary conditions of a finite solid restrained on its boundary, and a localization region similar to an ellipsoid may then be expected to form. It is remarkable how slowly the slope $|E_t|$ at instability decreases as a function of a_1/a_2. The value of the aspect ratio that is required to reduce $|E_t|$ at instability from about 0.4 to about 0.04 of E_u is $a_1/a_2 = e^3 = 20$. This means that if a very long planar softening band cannot be accommodated within a given solid, the deformation at softening instability is considerably increased.

The present solutions represent upper bounds on $|E_t|$ at actual localization. When the stability condition for some of the previously considered softening regions is violated, instability with such a region is possible and must occur since it leads to an increase of entropy. However, it is possible that localization into some form of region that we could not solve would occur earlier, at a smaller initial strain. For this reason, the present stability conditions are only necessary rather than sufficient. However, their opposites (i.e., $<$ changed to $>$) represent sufficient conditions for instability.

In the preceding analysis of ellipsoidal softening regions, we solved only the case of infinite solids and were unable to examine the effect of the boundary conditions at infinity. Now, from the fact that spherical and circular softening regions are special cases of ellipsoidal ones, we must conclude that our solution for ellipsoidal region is applicable only if the ellipsoidal region is of finite size, which is guaranteed only if the infinite body is fixed at infinity rather than having prescribed loads at infinity. Otherwise the limit cases $a_{cr} = R$ for spherical or circular softening regions discussed after (equations (20) and (32)) would not be satisfied.

Conclusion

The solutions to multidirectional localization problems with ellipsoidal, spherical and circular localization regions indicate that, in general, a loss of positive-definiteness of matrix $\mathbf{D}_t$ of tangential moduli for loading does not necessarily produce instability. Rather, the stability criterion requires positive-definiteness of a certain weighted average of the incremental moduli matrices $\mathbf{D}_t$ and $\mathbf{D}_u$ for loading and unloading. The weights depend on the relative size of the body. Not only for finite but also for infinite bodies restrained at the boundary, unstable strain localization into finite-size ellipsoidal regions cannot take place for a certain range of nonpositive-definite tangential moduli matrices. By contrast, unstable strain localization into an infinitely long planar band of finite thickness occurs in an infinite space as soon as $\mathbf{D}_t$ loses positive-definiteness.

The present results can be used to check the correctness of finite-element programs for strain-softening.

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