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Chapter 3

Microplane Model for Strain-controlled Inelastic Behaviour

Z. P. Bažant

3.1 INTRODUCTION

Various heterogeneous brittle aggregate materials such as concretes, rocks, or sea ice, are inelastic but cannot be described as plastic, except at extremely high hydrostatic pressures. A characteristic property of such materials is that they exhibit strain-softening, i.e., a decline of stress at increasing strain, which results from progressive development of fracture. Since these materials can undergo strain-softening within a relatively large zone, a non-linear triaxial constitutive relation is needed for its description. There are, however, some important differences from the classical modelling of inelastic behaviour, i.e., from the theory of plasticity.

First, rather than determining the inelastic phenomena in terms of stresses, as in plasticity, one must determine them in terms of strains. This is because in terms of stresses the description is not unique, as two strains correspond to the same stress, in the case of strain-softening, while still only one stress corresponds to a given strain. Second, the normal inelastic strains are, in contrast to plasticity, important, in fact dominant. They describe the cumulative effect of microcracking. Third, the inelastic phenomena are highly oriented and happen almost independently on planes of various orientation within the material as a function of normal strains across the planes.

In the present work, it is proposed to describe this behaviour independently on planes of various orientations in the material, called microplanes, and then in a certain way superimpose the inelastic effects from all the planes. This type of approach has a long history. First proposed in 1938 by Taylor [1], the idea was exploited by Batdorf and Budianski in their slip theory of plasticity [2]. A number of subsequent investigators adopted this approach for plasticity of polycrystalline metals [2 to 6]. Zienkiewicz and Pande [7] and Pande *et al.* [8–9] developed an approach of this type in their multilaminate models for rocks and soils.

In the aforementioned models, the stress on each plane within the material is assumed to correspond to the same macroscopic stress and the inelastic stresses are superimposed. As mentioned, however, for certain materials the inelastic behaviour is predominantly strain-controlled, and it is then more appropriate to assume that strains, not stresses, correspond on the planes of all orientations to the same macroscopic strain. In this case it is necessary to superimpose in some way the inelastic stresses (relaxations) from the planes of all orientations. This approach was adopted for concrete and geomaterials in ref. [11] and was summarized in ref. [12]. In these works, the inelastic shear stresses on planes of all orientations within the material were neglected. However, although their role is no doubt secondary, in case of concrete and geomaterials at high hydrostatic pressures, they certainly have some effect. The purpose of this work is to generalize ref. [11] to include the effect of inelastic shear stresses.

3.2 BASIC HYPOTHESES

The macroscopic stress tensor will be denoted as σ_{ij}, and the macroscopic strain tensor as ε_{ij}. With regard to the interaction between the macro- and micro-levels, the following three hypotheses may be introduced.

Hypothesis I. The tensor of macroscopic stress, σ_{ij}, is a sum of a purely elastic macrostress σ^a_{ij} that is unaffected by inelastic processes on planes of various orientation, and an inelastic macrostress τ_{ij} which reflects the stress relaxations from microplanes of various orientations, i.e.,

$$\sigma_{ij} = \sigma^a_{ij} + \tau_{ij} \tag{3.1}$$

(latin lower case subscripts refer to cartesian coordinates $x_i, i = 1, 2, 3$).

Hypothesis II. The normal microstrain ε_N and the shear microstrain ε_T on each microplane of any orientation is the resolved component of the macroscopic strain tensor ε_{ij}.

Hypothesis III. There exist an independent stress–strain relation for each microplane of any orientation.

Hypothesis II is opposite to that made in the slip theory of plasticity, in which the stresses rather than strains on the planes of all orientations are assumed to be the resolved components of the macroscopic stress. One can offer three reasons for this. First, if the material state were characterized by stress rather than strain, the description would not be unique since, in the case of strain-softening, there are two strains corresponding to a given stress. Second, the relationship between the micro- and macro-levels would not be stable in the case of strain softening, which has been confirmed numerically. Third, the use of resolved strains, rather than stresses, appears to reflect the microstructure of a brittle aggregate material more realistically. In contrast to polycrystalline metals, brittle aggregate materials consist of hard inclusions embedded in a relatively soft matrix. The microstresses are far from uniform, having sharp extremes at the locations where the aggregate

pieces are nearest. The deformation of the thin layer of matrix between two aggregate pieces, which is the chief source of inelastic behaviour, seems to be determined mainly by the relative displacements of the centroids of the aggregate pieces, which roughly correspond to the macroscopic strain. The microplanes may be imagined to represent the thin layers of matrix and the bond interfaces between the adjacent aggregate pieces (Figure 3.1(a)), since microcracking is chiefly concentrated there.

According to Hypothesis I (equation (3.1)), the virtual work of stresses per unit volume may be written as $\delta W = \varepsilon_{ij}\delta\sigma_{ij} = \varepsilon^a_{ij}\delta\sigma^a_{ij} + \varepsilon^m_{ij}\delta\sigma^m_{ij}$, in which ε^a_{ij} and ε^m_{ij} represent the strains associated with the additional elastic stress and the stress resulting from the microplanes. At the same time, $\delta W = \varepsilon_{ij}\delta\sigma^a_{ij} + \varepsilon_{ij}\delta\sigma^m_{ij}$. Since both expressions must hold for any $\delta\sigma^a_{ij}$ and any $\delta\sigma^m_{ij}$, we must have $\varepsilon^a_{ij} = \varepsilon^m_{ij} = \varepsilon_{ij}$.

According to Hypothesis II, the components of the strain vector $\boldsymbol{\varepsilon}^n$ on any microplane are

$$\varepsilon^n_j = n_i \varepsilon_{ji} \tag{3.2}$$

in which n_i are the cosines of the unit normal to the microplane. The normal microstrain, i.e., the normal component of strain vector $\boldsymbol{\varepsilon}^n$, may be denoted as ε_N, and the components of the vector of the shear component $\boldsymbol{\varepsilon}_T$ may be denoted as

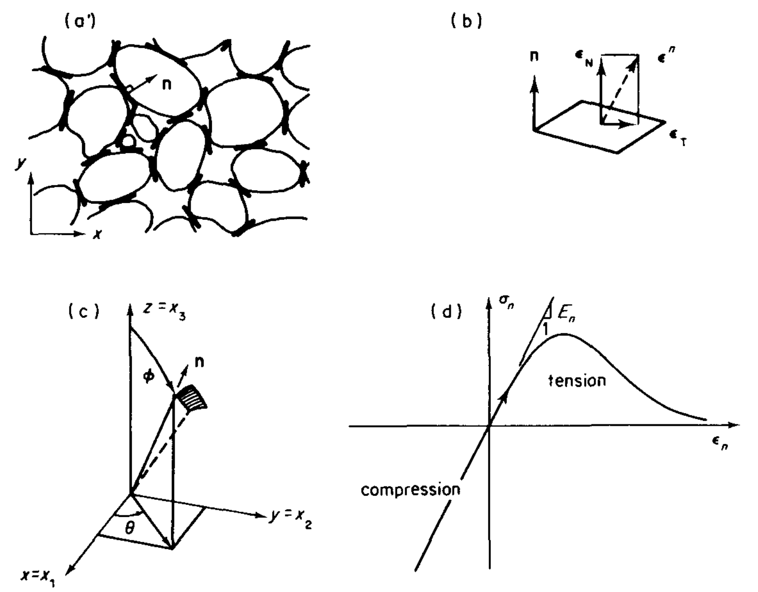

Figure 3.1 (a) Example of idealized microstructure, (b)–(c) explanation of notations, (d) stress–strain relation on a microplane ($\sigma_n = \tau_N, \varepsilon_n = \varepsilon_N$)

ε_{T_i}. With regard to the elastic parts of the shear components of stresses and strains on the microplane, we will now introduce an additional assumption, namely that the vectors of these shear components are parallel. This precludes anisotropic behaviour within each microplane, although overall anisotropy remains possible by considering different properties on various microplanes. According to Hypothesis III we may now write

$$\dot{\tau}_N = C\dot{\varepsilon}_N - \dot{\tau}''_N \qquad \dot{\tau}_{T_i} = B\dot{\varepsilon}_{T_i} - \dot{\tau}''_{T_i} \tag{3.3}$$

in which C and B are the elastic constants for the normal and shear response on the microplane, and τ''_N and τ''_{T_i} are the inelastic stress relaxations in the normal and tangential directions on the microplane. Superimposed dots denote time rates. For the magnitudes of the shear components, equation (3.3) implies $\dot{\tau} = B\dot{\varepsilon}_T - \dot{\tau}''_T$.

Further we need to specify the inelastic stress relaxations. For this purpose, we assume the existence of inelastic potentials f_β and loading surfaces $g_\beta (\beta = 1, \ldots, n)$ for each microplane. They must be defined in terms of strains rather than stresses, i.e.,

$$f_\beta(\varepsilon_N, \varepsilon_T) = 0 \qquad (\beta = 1, \ldots, n) \tag{3.4}$$

$$g_\beta(\varepsilon_N, \varepsilon_T) = 0 \tag{3.5}$$

The rates of inelastic stress relaxations may be assumed to be given by the normality rule

$$\dot{\tau}''_{ij} = \sum_{\beta=1}^{n} \frac{\partial f_\beta}{\partial \varepsilon_{ij}} \dot{\mu}_\beta \tag{3.6}$$

$$\dot{\mu}_\beta = h_\beta \frac{\partial g_\beta}{\partial \varepsilon_{km}} \dot{\varepsilon}_{km} H(\dot{\mu}_\beta) \tag{3.7}$$

in which h_β are material softening parameters depending on the current state of the material and possibly also its history, and H is Heaviside step function. Similarly to Drucker's stability postulate in plasticity, equations (3.6)–(3.7) can be easily derived from a more plausible hypothesis (postulate) for a strain cycle, called Il'yushin's postulate, as previously used for macroscopic inelastic theories based on loading surfaces in the strain space [13 to 20]. The Heaviside function in equation (3.7) distinguishes between loading and unloading.

By projecting the microplane strain vector (equation (3.2)) on to the direction $\mathbf{n}$ of the normal, we obtain the magnitude of the normal strain component on the microplane and its vector:

$$\varepsilon_N = n_j \varepsilon_j^n = n_j n_k \varepsilon_{jk}, \qquad (\varepsilon_N)_i = n_i n_j n_k \varepsilon_{jk} \tag{3.8}$$

The magnitude of the strain vector on the microplane is $|\boldsymbol{\varepsilon}^n| = (\varepsilon_j^n \varepsilon_j^n)^{1/2} = (n_i \varepsilon_{ji} n_k \varepsilon_{jk})^{1/2}$. The vector of the tangential (shear) strain component is (see

Figure 3.1(c)) $\boldsymbol{\varepsilon}_T = \boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_N$, and its magnitude is $\varepsilon_T = [|\boldsymbol{\varepsilon}^n|^2 - (\varepsilon_N)^2]^{1/2}$ or $\varepsilon_T = |\boldsymbol{\varepsilon}_T| = (\varepsilon_{T_j}\varepsilon_{T_j})^{1/2}$. Thus, we obtain the following expressions for the vector and the magnitude of the shear strain component on the microplane:

$$\varepsilon_{T_i} = (n_k \delta_{ij} - n_i n_j n_k)\varepsilon_{jk} \tag{3.9}$$

$$\varepsilon_T = [n_i \varepsilon_{ji} n_k (\varepsilon_{jk} - n_j n_m \varepsilon_{km})]^{1/2} \tag{3.10}$$

According to equation (3.6), the normal and tangential components of the stress relaxation rate on the microplane can be expressed as

$$\dot{\tau}''_N = \sum_{\beta=1}^{n} \frac{\partial f_\beta}{\partial \varepsilon_N}\dot{\mu}_\beta, \qquad \dot{\tau}''_T = \sum_{\beta=1}^{n} \frac{\partial f_\beta}{\partial \varepsilon_T}\dot{\mu}_\beta, \qquad \dot{\tau}''_{T_i} = \sum_{\beta=1}^{n} \frac{\partial f_\beta}{\partial \varepsilon_{T_i}}\dot{\mu}_\beta \tag{3.11}$$

The derivatives of the inelastic potential in these equations may be calculated as

$$\frac{\partial f_\beta}{\partial \varepsilon_{T_i}} = \frac{\partial f_\beta}{\partial \varepsilon_T}\frac{\partial \varepsilon_T}{\partial \varepsilon_{T_i}} = \frac{\partial f_\beta}{\partial \varepsilon_T}\frac{\partial}{\partial \varepsilon_{T_i}}(\varepsilon_{T_j}\varepsilon_{T_j})^{1/2} = \frac{\partial f_\beta}{\partial \varepsilon_T}\frac{\varepsilon_{T_i}}{\varepsilon_T} \tag{3.12}$$

We see that the vector normal to the potential surface f_β is parallel to the vector of the tangential component of strain on the microplane, i.e.,

$$\dot{\boldsymbol{\varepsilon}}''_T \parallel \boldsymbol{\varepsilon}_T, \qquad \dot{\boldsymbol{\varepsilon}}_T \parallel \boldsymbol{\varepsilon}_T \tag{3.13}$$

The derivatives of the inelastic potentials and loading functions appearing in equations (3.6)–(3.7) may be calculated as

$$\frac{\partial f_\beta}{\partial \varepsilon_{ij}} = p_{ij}\frac{\partial f_\beta}{\partial \varepsilon_N} + q'_{ij}\frac{\partial f_\beta}{\partial \varepsilon_T}, \qquad \frac{\partial g_\beta}{\partial \varepsilon_{ij}} = p_{ij}\frac{\partial g_\beta}{\partial \varepsilon_N} + q'_{ij}\frac{\partial g_\beta}{\partial \varepsilon_T} \tag{3.14}$$

in which we introduce the notation

$$p_{ij} = \frac{\partial \varepsilon_N}{\partial \varepsilon_{ij}}, \qquad q'_{ij} = \frac{\partial \varepsilon_T}{\partial \varepsilon_{ij}} \tag{3.15}$$

From equation (3.8) we can further calculate

$$p_{ij} = \frac{\partial}{\partial \varepsilon_{ij}}(n_p n_q \varepsilon_{pq}) = \delta_{ip}\delta_{jq} n_p n_q = n_i n_j \tag{3.16}$$

while from equation (3.10) we obtain

$$q'_{ij} = \frac{1}{2\varepsilon_T}\frac{\partial}{\partial \varepsilon_{ij}}(n_p \varepsilon_{qp} n_r \varepsilon_{qr} - n_p n_q \varepsilon_{pq} n_r n_s \varepsilon_{rs})$$

which reduces to

$$q'_{ij} = \frac{1}{\varepsilon_T} n_j n_r (\varepsilon_{ir} - n_i n_s \varepsilon_{rs}) \tag{3.17}$$

The last tensor is non-symmetric. Later we will need its symmetric part, which reads

$$q_{ij} = \frac{1}{2\varepsilon_{\mathrm{T}}}[n_k(n_j\varepsilon_{ik} + n_i\varepsilon_{jk}) - 2a_{ijkm}\varepsilon_{km}] \tag{3.18}$$

in which we introduce the notation

$$a_{ijkm} = n_i n_j n_k n_m \tag{3.19}$$

We need to establish now the equilibrium relation between the microstresses on the microplanes of all orientations and the macroscopic stress tensor. We may use for this purpose the principle of virtual work, which requires that the virtual work of macroscopic stress rates on any macroscopic strain variations within a small unit sphere of unit radius be equal to the virtual work done on all the microplanes tangential to the unit sphere. This condition may be written as follows

$$\begin{aligned}\delta\dot{W} &= \frac{4\pi}{3}\dot{\tau}_{ij}\delta\varepsilon_{ij} = 2\int_S(\dot{\tau}_{\mathrm{N}}\delta\varepsilon_{\mathrm{N}} + \dot{\tau}_{\mathrm{T}}\delta\varepsilon_{\mathrm{T}})f(\mathbf{n})\,\mathrm{d}S \\ &= 2\int_S(C\dot{\varepsilon}_{\mathrm{N}}\delta\varepsilon_{\mathrm{N}} + B\dot{\varepsilon}_{\mathrm{T}_i}\delta\varepsilon_{\mathrm{T}_i} - \dot{\tau}''_{ij}\delta\varepsilon_{ij})f(\mathbf{n})\,\mathrm{d}S\end{aligned} \tag{3.20}$$

where S is the surface of a unit hemisphere, and $f(\mathbf{n})$ describes the frequency of microplanes as a function of orientation $\mathbf{n}$.

Substituting here from equations (3.8), (3.9), and (3.6), we may obtain the relation

$$\dot{\tau}_{ij}\delta\varepsilon_{ij} = \frac{3}{2\pi}\int_S\left(Cn_kn_m\dot{\varepsilon}_{km}n_in_j + Bb'_{ijkm}\dot{\varepsilon}_{km} - \sum_{\beta=1}^{n}\frac{\partial f_\beta}{\partial\varepsilon_{ij}}\dot{\mu}_\beta\right)\delta\varepsilon_{ij}f(\mathbf{n})\,\mathrm{d}S \tag{3.21}$$

in which

$$\begin{aligned}b'_{ijkm} &= (n_m\delta_{rk} - n_rn_kn_m)(n_j\delta_{ri} - n_rn_in_j) \\ &= n_jn_m\delta_{ik} - n_jn_kn_mn_i - n_mn_in_jn_k + n_in_jn_kn_m(n_rn_r) \\ &= \delta_{ik}n_jn_m - a_{ijkm}\end{aligned} \tag{3.22}$$

This fourth order tensor is symmetric when ij is interchanged with km but non-symmetric when i is interchanged with j or k is interchanged with small m. The tensor may be written as a sum of a symmetric part and an antisymmetric part,

$$b'_{ijkm} = b_{ijkm} + \tilde{b}_{ijkm} \tag{3.23}$$

in which the symmetric part is

$$b_{ijkm} = \tfrac{1}{4}(\delta_{ik}n_jn_m + \delta_{jk}n_in_m + \delta_{im}n_jn_k + \delta_{jm}n_in_k) - a_{ijkm} \tag{3.24}$$

For the antisymmetric part it is true that $\tilde{b}_{ijkm}\delta\varepsilon_{ij}\dot{\varepsilon}_{km} = 0$ for any $\delta\varepsilon_{ij}$. Therefore,

$$b'_{ijkm}\delta\varepsilon_{ij}\dot{\varepsilon}_{km} = b_{ijkm}\delta\varepsilon_{ij}\dot{\varepsilon}_{km} \tag{3.25}$$

Furthermore, using equations (3.14) and (3.7), we may express

$$\begin{aligned}\sum_{\beta=1}^{n}\frac{\partial f_\beta}{\partial\varepsilon_{ij}}\dot{\mu}_\beta\delta\varepsilon_{ij} = \sum_{\beta=1}^{n}h_\beta&\left[p_{ij}\frac{\partial f_\beta}{\partial\varepsilon_{\mathrm{N}}} + (q_{ij} + \tilde{q}_{ij})\frac{\partial f_\beta}{\partial\varepsilon_{\mathrm{T}}}\right] \\ &\times\delta\varepsilon_{ij}\left[p_{km}\frac{\partial g_\beta}{\partial\varepsilon_{\mathrm{N}}} + (q_{km} + \tilde{q}_{km})\frac{\partial g_\beta}{\partial\varepsilon_{\mathrm{T}}}\right]\dot{\varepsilon}_{km}H(\dot{\mu}_\beta)\end{aligned} \tag{3.26}$$

in which $\tilde{q}_{ij}$ is the antisymmetric part of q'_{ij}, i.e., $\tilde{q}_{ij} = q'_{ij} - q_{ij}$. Noting that the antisymmetric parts give zero products with symmetric tensors, i.e., $\tilde{q}_{ij}\delta\varepsilon_{ij} = 0$, $\tilde{q}_{km}\delta\varepsilon_{km} = 0$, we find that

$$\sum_{\beta=1}^{n}\frac{\partial f_\beta}{\partial\varepsilon_{ij}}\dot{\mu}_\beta = R_{ijkm}\dot{\varepsilon}_{km} \tag{3.27}$$

in which

$$R_{ijkm} = \sum_{\beta=1}^{n}h_\beta\left(p_{ij}\frac{\partial f_\beta}{\partial\varepsilon_{\mathrm{N}}} + q_{ij}\frac{\partial f_\beta}{\partial\varepsilon_{\mathrm{T}}}\right)\left(p_{km}\frac{\partial g_\beta}{\partial\varepsilon_{\mathrm{T}}} + q_{km}\frac{\partial g_\beta}{\partial\varepsilon_{\mathrm{T}}}\right)H(\dot{\mu}_\beta) \tag{3.28}$$

Substituting equations (3.27) and (3.25) into the variational virtual work relation in equation (3.21), and noting that this relation must hold for any variation $\delta\varepsilon_{ij}$, we find that

$$\dot{\tau}_{ij} = D^{\mathrm{e}}_{ijkm}\dot{\varepsilon}_{km} - \dot{\tau}''_{ij} \tag{3.29}$$

or

$$\dot{\tau}_{ij} = D^{\mathrm{m}}_{ijkm}\dot{\varepsilon}_{km} \tag{3.30}$$

in which

$$\begin{aligned}D^{\mathrm{m}}_{ijkm} &= \frac{3}{2\pi}\int_S(a_{ijkm}C + b_{ijkm}B - R_{ijkm})f(\mathbf{n})\,\mathrm{d}S, \\ D^{\mathrm{e}}_{ijkm} &= \frac{3}{2\pi}\int_S(a_{ijkm}C + b_{ijkm}B)f(\mathbf{n})\,\mathrm{d}S\end{aligned} \tag{3.31}$$

$$\dot{\tau}''_{ij} = \frac{3}{2\pi}\int_S R_{ijkm}f(\mathbf{n})\,\mathrm{d}S\dot{\varepsilon}_{km} \tag{3.32}$$

Here D^{m}_{ijkm} is the tensor of tangential moduli corresponding to the microplanes, D^{e}_{ijkm} is the elastic part of this tensor, and $\dot{\sigma}''_{ij}$ is a tensor of the rate of inelastic stress relaxation. The integrals in equations (3.31) and (3.32) extend over the surface S of a unit hemisphere.

Consider now the special case of isotropic materials, for which $f(\mathbf{n}) = 1$. For this case the elastic stiffness matrix D^{e}_{ijkm} must be equivalent to an isotropic material stiffness matrix characterized by some shear modulus G^{m} and Poisson ratio ν^{m}. Their values may be easily calculated. To this end, consider a uniaxial strain rate $\dot{\varepsilon}_{33} = 1$ while all other components of $\dot{\varepsilon}_{ij} = 0$. We may now substitute $n_1 = \sin\phi\cos\theta$, $n_2 = \sin\phi\sin\theta$, $n_3 = \cos\phi$ and equations (3.29) and (3.31) for

$\dot{\tau}''_{ij} = 0$ then give

$$\dot{\tau}_{33} = \frac{3}{2\pi}\left[C\int_0^{\pi}\int_0^{\pi}\cos^4\phi\sin\phi\,d\phi\,d\theta + B\int_0^{\pi}\int_0^{\pi}\cos^2\phi\sin^3\phi\,d\phi\,d\theta\right]\dot{\varepsilon}_{33}$$
$$= \tfrac{1}{5}(3C + 2B) \tag{2.33}$$

$$\dot{\tau}_{11} = \frac{3}{2\pi}(C - B)\int_0^{\pi}\int_0^{\pi}\sin^3\phi\cos^2\phi\cos^2\theta\,d\theta\,d\phi\,\dot{\varepsilon}_{33} = \tfrac{1}{5}(C - B) \tag{2.34}$$

According to Hooke's law, $\dot{\tau}_{11}/\dot{\tau}_{33} = \nu^m/(1-\nu^m)$ for uniaxial strain, and so $\nu^m = \dot{\tau}_{11}/(\dot{\tau}_{11} + \dot{\tau}_{33})$. Substituting from equations (3.33) and (3.34), we thus get

$$\nu^m = \frac{C - B}{4C + B} \tag{2.35}$$

Furthermore, according to Hooke's law we have, for uniaxial strain, $\nu^m = \dot{\tau}_{11}/(\dot{\tau}_{11} + \dot{\tau}_{33})$, from which we may solve

$$G^m = \frac{1}{10}\frac{(1-2\nu^m)}{(1-\nu^m)}(3C + 2B) \tag{3.36}$$

From equations (3.35) and (3.36) we may solve the constants C and B from desired values of G^m and ν^m. Equation (3.35) yields the following values of Poisson ratio:

B/C	ν^m
$=0$	$=0.25$
14/59	0.18
1	0
∞	-1

It is interesting to observe that Poisson ratios greater than 0.25 cannot be obtained. The range appears suitable for geomaterials.

In some situations, however, an adjustment of the Poisson ratio provided by the system of microplanes may be needed. For example, one might desire for some material an overall Poisson ratio $\nu > 0.25$, or one might simply need for the best fit of test data a different Poisson ratio for the microplane system than for the material as a whole. Such an adjustment of Poisson ratio is made possible by equation (3.1) (Hypotehsis I). Let the additional elastic stresses σ^a_{ij} be given by an isotropic stress–strain relation to ε_{ij}, characterized by shear modulus G^a and Poisson ratio ν^a. Then, for uniaxial strain $\dot{\varepsilon}_{33} = 1$ $(\dot{\varepsilon}_{11} = \dot{\varepsilon}_{22} = 0)$ we have $\dot{\sigma}_{33} = 2G(1-\nu)/(1-2\nu)$. Summing the stress from the microplane system and the additional elastic stress, we also have $\dot{\sigma}_{33} = 2G^a(1-\nu^a)/(1-2\nu^a) + 2G^m(1-\nu^m)/(1-2\nu^m)$. From these equations we may solve

$$G^a = \frac{1-2\nu^a}{1-\nu^a}\left(\frac{G(1-\nu)}{1-2\nu} - \frac{G^m(1-\nu^m)}{1-2\nu^m}\right) \tag{3.37}$$

This relation permits us to choose the elastic constants of the material as a whole, as well as of the microplane system, and also choose the Poisson ratio for the additional elastic stress.

In general, the total tangential elastic moduli are

$$D_{ijkm} = D^m_{ijkm} + D^a_{ijkm} \tag{3.38}$$

For isotropic materials, D^a_{ijkm} represent the elastic moduli tensor of an isotropic material;

$$D^a_{ijkm} = G^a(\delta_{ik}\delta_{jm} + \delta_{jk}\delta_{im}) + \frac{2\nu^a}{1-2\nu^a}G^a\delta_{ij}\delta_{km} \tag{3.39}$$

in which δ_{ij} = Kronecker delta = 1 if $i = j$, and 0 if $i \neq j$.

3.3 CASE OF ZERO SHEAR RELAXATIONS ON MICROPLANES

For tensile strain-softening of concrete, it seems that one may neglect the shear stress relaxations and consider only the normal stress relaxations on the microplanes, which correspond to the formation of microcracks in the direction of the microplanes. In this case $B = \tau_{T_i} = \tau''_{T_i} = 0$. One may consider here for each microplane only one loading surface $(\beta = 1)$, $f_1 = g_1 = \varepsilon_N =$ const. In this case we get

$$\dot{\tau}_N = [C - h_1 H(\dot{\mu}_1)]\dot{\varepsilon}_N \tag{3.40}$$

The relationship between the normal stress and the normal strain on the microplane may be conveniently described by the formula

$$\tau_N = F(\varepsilon_N) = C\varepsilon_N \exp(-k\varepsilon_N^n) \qquad (n \simeq 2) \tag{3.41}$$

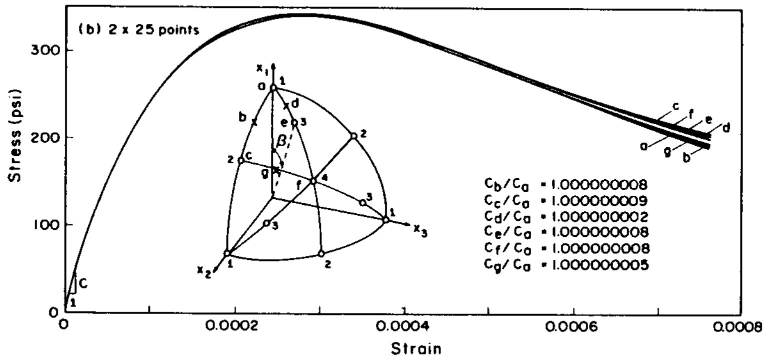

Figure 3.2 Distribution of integration points for 2 × 25-point formula defined in Table 3.1, and response curves for uniaxial stress applied at directions a, b,.. ,f.

in which C, k, and n are material constants. For large ε_N the value of τ_N becomes essentially zero ($n = 2$ is a suitable exponent). Comparing this with equation (3.40), we have

$$h_1 = C - F'(\varepsilon_N) = C - C(1 - kn\varepsilon_N^n)\exp(-k\varepsilon_N^n) \tag{3.42}$$

The present special case has been considered in refs [11 and 12], and good fits of various tensile strain-softening test data have been demonstrated. Since, in absence of shear relaxations, the microplane model yields Poisson ratio 0.25, the additional elastic deformation was used to correct this value to 0.18, typical of concrete. There was a slight difference from the present formulation in that the superimposed additional elastic value was strain rather than stress. The present formulation is, however, more efficient. One of the comparisons with the test data of Evans and Marathe [21], made in ref. [11], is reproduced in Figure 3.2, and the associated stress–strain curve considered for the microplanes is shown in Figure 3.1(d).

3.4 NUMERICAL INTEGRATION ON THE SURFACE OF A SPHERE

In general situations, the integral in equation (3.31) over the surface of a unit hemisphere has to be evaluated numerically, approximating it by a finite sum:

$$D^m_{ijkm} = \sum_{\alpha=1}^{n} 6w_\alpha[(a_{ijkm}C + b_{ijkm}B - R_{ijkm})f(\mathbf{n})]_\alpha, \quad \sum_\alpha w_\alpha = \tfrac{1}{2} \tag{3.43}$$

in which α refers to the values evaluated at certain numerical integration points on the spherical surface (i.e., certain characteristic directions), and w_α are the weights associated with the integration points. In finite element programmes for incremental loading, the numerical integration needs to be carried out a great number of times. Therefore, a very efficient numerical integration formula is required. For the slip theory of plasticity, a similar integration was performed using a rectangular grid in the plane of spherical coordinates θ and ϕ. This approach is, however, computationally inefficient since the integration points are crowded around the poles, and since, in the $\theta - \phi$ plane, the singularity arising from the poles takes away the benefit from a use of a higher-order integration formula.

Optimally, the integration points should be distributed over the spherical surface as uniformly as possible. A perfectly uniform distribution is obtained when the microplanes normal to the α-directions are the faces of a regular polyhedron. However, a regular polyhedron with the greatest number of sides is the icosahedron, for which $N = 10$ ($2N$ is the number of faces), and this number appears insufficient (a formula for this case was presented by Albrecht and Collatz [22]). The need for greater accuracy is indicated when the response curves in a uniaxial tensile test with strain-softening are calculated for various

Table 3.1 Direction cosines and weights for 2 × 25 points with error of 10th order (after Bažant and Oh [11],

α	x_1^α	x_2^α	x_3^α	w^α
1	1	0	0	0.012 698 410 58
2	0	1	0	0.012 698 410 58
3	0	0	1	0.012 698 410 58
4	0.707 106 781 2	0.707 106 781 2	0	0.022 574 956 12
5	0.707 106 781 2	− 0.707 106 781 2	0	0.022 574 956 12
6	0.707 106 781 2	0	0.707 106 781 2	0.022 574 956 12
7	0.707 106 781 2	0	− 0.707 106 781 2	0.022 574 956 12
8	0	0.707 106 781 2	0.707 106 781 2	0.022 574 956 12
9	0	0.707 106 781 2	− 0.707 106 781 2	0.022 574 956 12
10	0.301 511 335 4	0.301 511 335 4	0.904 534 039 8	0.020 173 335 57
11	0.301 511 335 4	0.301 511 335 4	− 0.904 534 039 8	0.020 173 335 57
12	0.301 511 335 3	− 0.301 511 335 4	0.904 534 039 8	0.020 173 335 57
13	0.301 511 335 4	− 0.301 511 335 4	− 0.904 534 039 8	0.020 173 335 57
14	0.301 511 335 4	0.904 534 039 8	0.301 511 335 4	0.020 173 335 57
15	0.301 511 335 4	0.904 534 039 8	− 0.301 511 335 4	0.020 173 335 57
16	0.301 511 335 4	− 0.904 534 039 8	0.301 511 335 4	0.020 173 335 57
17	0.301 511 335 4	− 0.904 534 039 8	− 0.301 511 335 4	0.020 173 335 57
18	0.904 534 039 8	0.301 511 335 4	0.301 511 335 4	0.020 173 335 57
19	0.904 534 039 8	0.301 511 335 4	− 0.301 511 335 4	0.020 173 335 57
20	0.904 534 039 8	− 0.301 511 335 4	0.301 511 335 4	0.020 173 335 57
21	0.904 534 039 8	− 0.301 511 335 4	− 0.301 511 335 4	0.020 173 335 57
22	0.577 350 269 2	0.577 350 269 2	0.577 350 269 2	0.021 093 751 17
23	0.577 350 269 2	0.577 350 269 2	− 0.577 350 269 2	0.021 093 751 17
24	0.577 350 269 2	− 0.577 350 269 2	0.577 350 269 2	0.021 093 751 17
25	0.577 350 269 2	− 0.577 350 269 2	− 0.577 350 269 2	0.021 093 751 17

$\beta = 25.239401°$.

orientations of the uniaxial stress with regard to the α-directions. Ideally, the response curves for any orientation should be identical. However, large discrepancies are found for a ten-point formula.

Bažant and Oh [23] derived numerical integration formulas with more than 10 points, which give consistent results even in the strain-softening range. The most efficient formulas, with an almost uniform spacing of α-directions, are obtained by certain subdivisions of the faces of an icosahedron or a dodecahedron [23]. Such formulas do not exhibit orthogonal symmetries. Other formulas which do were also derived [23]. Taylor series expansions on a sphere were used and weights w_α were solved from the condition that the greatest possible number of terms of the expansion of the error would cancel out. The angular directions of certain integration points were further determined from the condition that the error term of the expansion be minimized. In this manner, formulas involving 16, 21, 25, 33, 37, and 61 points were established, with errors of 8th, 10th, and 12th order. Table 3.1 defines one of these numerical integration formulas, having 25 points for a hemisphere; this formula exhibits orthogonal symmetry [23]. The directions of the integration points are illustrated in Figure 3.2, and also shown are the stress–strain diagrams calculated for various directions of uniaxial tensile

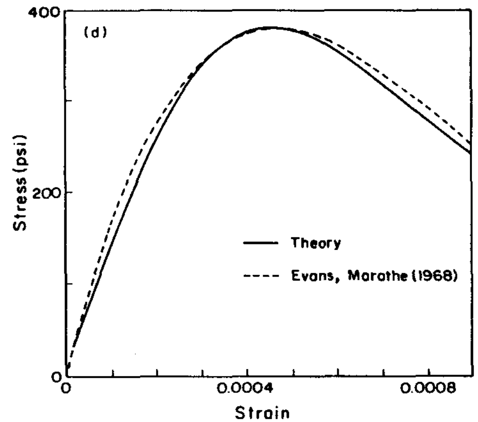

Figure 3.3 Comparison with test data of Evans and Marathe [21], after Bažant and Oh [11]

stress with regard to the integration points (directions a, b, c, d, ...); the spread of the response curves characterizes the range of numerical errors. For crude calculations, the lowest required number of integration points is 16 [23].

3.5 APPLICATION TO ANISOTROPIC CREEP OF CLAY

As another application, we may demonstrate an adaptation of the microplane model to describe creep of an anisotropically consolidated clay and to correlate the stress–strain relation to known information about the distribution of the frequency of platelets of various orientations within the clay. This problem has been studied, for example, in ref. [24], using a micromechanics model in which triangular cells of mutually sliding clay platelets are constrained to the same macroscopic strains ε_{ij}, same as here. It appeared, however, that this approach becomes quite complicated in the three-dimensional case, although it is not very difficult for two-dimensional analysis. In three dimensions, the present type of microplane model seems appropriate.

In treating clay, the stress tensor σ_{ij} must be interpreted as the effective stress tensor, i.e., $\sigma_{ij} = t_{ij} - \delta_{ij}p$, in which p = pore-water pressure and t_{ij} = total stress in the solid-water system. Let us consider only the case of deviatoric creep, for which the normal stiffness on the microplanes may be neglected, i.e. $C = \tau''_N = 0$. The microplanes of the present model may be interpreted, in the case of clay, as the planes of sliding in contact of adjacent clay platelets. As is well known, the sliding is governed by the rate-process theory, which yields the relation

$$\dot{\varepsilon}_T = k_1 \sinh(k_2 \tau_T) \tag{3.44}$$

in which $k_1 = 2A(kT/h)t^{-m}\exp(-Q/RT)$ and $k_2 = V_a/RT$. Here T is the absolute temperature, Q is the activation energy of creep, R is the universal gas constant, k is the Boltzmann constant, h is the Planck constant, V_a is the activation volume, and A, m are empirical constants. For the vectors of the tangential stress and strain components on the microplanes, equation (3.44) may be generalized, in the inverted form, as follows

$$\tau_{T_i} = \frac{1}{k_2}\frac{\dot{\varepsilon}_{T_i}}{\dot{\varepsilon}_T}\sinh^{-1}\left(\frac{\dot{\varepsilon}_T}{k_1}\right) \tag{3.45}$$

This equation now replaces equation (3.3) of the present model, and equations (3.6)–(3.11) become unnecessary. By following the same analysis as before, one obtains the macroscopic stress–strain relation, replacing equation (3.29) as follows

$$\sigma_{jk} = \eta_{jkrs}\dot{\varepsilon}_{rs}, \qquad \eta_{jkrs} = \frac{3}{2\pi}\int_S b_{jkrs}\frac{1}{k_2\dot{\varepsilon}_T}\sinh^{-1}\left(\frac{\dot{\varepsilon}_T}{k_1}\right) f(\mathbf{n})\,\mathrm{d}S \tag{3.46}$$

Here η_{jkrs} represents the fourth order tensor of current viscosities, and $f(\mathbf{n})$ represents the distribution function for the frequency of clay platelets of various orientations. For some clays, this distribution function has been measured experimentally, using X-ray scattering techique. Applicability of equation (3.43) to test results is presently being studied at Northwestern University by J. K. Kim.

Complete description of clays further requires superposing equations for the volume change. This may be best accomplished on the basis of the critical state theory, for example, in a manner recently described by Pande *et al.* [8–9].

3.6 CONCLUDING REMARKS

The microplane model allows great versatility in constitutive modelling. The present form of the model, in which the strains on microplanes of all orientations correspond to the same macroscopic strain, appears suitable for materials which exhibit progressive microcracking and tensile strain-softening. Constraining the microstructure to the same macroscopic strain is also important for numerical reasons, not merely for the purpose of stability and uniqueness of representation. The computational work required by a model of this kind is not as large as one might think. The work required is greatly reduced by the recent development of efficient numerical integration formulas for a spherical surface.

One advantage of the model is that the stress–strain relations are primarily defined on the microplane level, on which one does not need to heed the tensorial invariance requirements which are a source of great difficulty in constitutive modelling. Tensorial invariance is ensured subsequently, by combining the responses from microplanes of all orientations within the material.

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