Title no. 81-29

Bayesian Statistical Prediction of Concrete Creep and Shrinkage


by Zdeněk P. Bažant and Jenn-Chuan Chern

In present design practice, the statistical approach is used for strength but not for deformations, including creep and shrinkage. However, predicting concrete creep properties from design strength and concrete composition involves a large uncertainty, much larger than that of strength. It is shown that by carrying out some short-time creep measurements, even rather limited ones, the uncertainty can be drastically reduced, and extrapolation of short-time measurements can be made much more reliable. This is accomplished by developing a Bayesian approach to creep prediction. Prior information consists of the coefficient of variation of deviations from the creep law for concrete in general, as determined in a recent statistical analysis of the numerous creep data that exist in literature. This information is combined, according to Bayes' theorem, with the probability of a given concrete's creep values to yield the posterior probability distribution of the creep values for any load duration and age at loading.

Only a linear creep case is considered, and a normal distribution of errors is assumed for the given concrete as well as for the prior information. To demonstrate and verify the method developed, various creep data reported in literature are considered. Predictions made on the basis of only a part of the test data are compared with the rest of the data, and very good agreement is found. The effects of various amounts of measured data, and of various degrees of uncertainty in the prior information, are also illustrated. The present approach is recommended for concrete structures for which the creep deflections, creep-induced cracking, or creep buckling are of special concern, e.g., nuclear reactor vessels and containments, certain very large bridges, shells, or building frames.

Keywords: Bayes theorem; concretes; confidence limits; **creep properties**; errors; probability theory; regression analysis; **shrinkage; statistical analysis**; structural analysis; structural design.

Creep and shrinkage appear to be the most uncertain phenomena with which a designer of concrete structures must cope. The statistical variability of creep and shrinkage is much larger than that of concrete strength, yet so far statistical methods have been well developed only for the latter. This is partly because the problem is more difficult and partly because the consequences of a substantial error in predicting creep and shrinkage are generally less disastrous than they are for strength. Except for creep buckling, errors in creep prediction and shrinkage do not cause structural collapse but merely put the structure out of service due to excessive deflections or excessive cracking (which causes reinforcement corrosion). Nevertheless, for reasons of economy, it is very important to improve the prediction of the effect of creep and shrinkage in structures and, in particular, design structures for certain extreme rather than average creep predictions.

Probabilistic analysis of concrete creep and shrinkage has recently been rendered possible by extracting extensive statistical information from literature (see Reference 12, in which data for 80 different concretes, consisting of over 800 experimental curves and over 10,000 data points, have been analyzed statistically and organized in a computerized data bank). It appears that if no measurements for a given concrete to be used are made, the uncertainty in predicting its creep and shrinkage on the basis of the chosen concrete mix parameters and chosen design strength is enormous. Even with the most sophisticated and comprehensive prediction model,[12,13] prediction errors (confidence limits) exceeded with a 10 percent probability are about ± 31 percent of the mean prediction.[13] For the 1971 ACI Committee 209 Model,[1] which is much simpler, this increases to about ± 63 percent, and for the 1978 CEB-FIP Model Code[16] to about ± 76 percent. This is clearly an unsatisfactory state of affairs.

It has been demonstrated, however, that drastic improvement is possible if some experimental data, even very limited short-time data, are obtained for the particular concrete to be used.[12,13] For a creep-sensitive structure, such as a nuclear reactor vessel, large shell, or large bridge, the designer usually has at his disposal some limited short-time test data for his concrete. Considered by themselves, extrapolation of these data to long times would also be very uncertain (for an exam-

Received Apr. 28, 1983, and reviewed under Institute publication policies. Pertinent discussion will be published in the May-June 1985 ACI JOURNAL if received by Feb. 1, 1985.

Zdeněk P. Bažant, FACI, is a professor and director, Center for Concrete and Geomaterials, Northwestern University. Dr. Bažant is a registered structural engineer, serves as consultant to Argonne National Laboratory and several other firms, and is on editorial boards of five journals. He serves as chairman of RILEM Committee TC69 on creep, ASCE-EMD Committee on Properties of Materials, and IA-SMiRT Division H. His works on concrete and geomaterials, inelastic behavior, fracture, and stability have been recognized by a RILEM medal, ASCE Huber Prize and T.Y. Lin Award, IR-100 Award, Guggenheim Fellowship, Ford Foundation Fellowship, and election as Fellow of American Academy of Mechanics.

Jenn-Chuan Chern is a scientist at Argonne National Laboratory. In June 1984 he received his PhD in civil engineering from Northwestern University. Prior to coming to the U.S., he worked for two years as an engineering lieutenant in the Chinese Marine Corps and for one year as a structural engineer on the design and construction of petrochemical plants. His research interests include concrete creep, moisture effects, cracking, and high temperature effects on concretes, on which he co-wrote a state-of-the-art report for the Electric Power Research Institute. He has also developed a large computer code now in use in some government laboratories.

ple from a nuclear containment design, see Reference 8). A great improvement is, however, possible if the statistical information for the given concrete is combined with prior statistical information for all similar concretes, such as that presented in Reference 12 or 13. This is the subject of Bayesian statistical analysis.[4,14,22,25,27,29-31,36,37,40] Application of this concept to the present problem is the objective of this work.

From the physical viewpoint, the causes of randomness in concrete creep and shrinkage are basically threefold: (1) randomness due to uncontrollable variations in material properties; (2) randomness due to variations in the environment (weather); and (3) randomness of the creep increments due to the statistical nature of the creep mechanism itself.[18] The first of these three causes of uncertainty is by far the worst and may be largely eliminated by carrying out a few limited measurements and applying the Bayesian analysis that follows. The second and third causes then remain, and the worse one of these is randomness of environment. A separate study, based on spectral analysis of stochastic processes, has recently been devoted to this aspect. By environmental control in the laboratory, or due to mass concrete conditions, the second cause is largely eliminated, and if the properties of the given concrete are determined well by measurements, randomness of the creep mechanism then remains as the principal remaining cause of uncertainty. For the sake of simplicity, the study does not attempt to distinguish between the aforementioned different physical causes, but we should at least be aware that the purpose of Bayesian analysis is to eliminate the first of the three causes. Measurement errors from the creep predictions are also not eliminated.

The problem at hand exhibits various mathematical similarities with some other Bayesian problems.[4,14,25,27,30,31,35-38] This is true of Tang's[35] Bayesian analysis of future settlements of an oil platform for which some short-time settlements have been observed.

In this study, we aim only at determining the creep and shrinkage properties of a given concrete. Use of these properties in structural analysis is another problem that cannot be included in this study.

The error of measurement should be eliminated from test data used in the present analysis, since this error is not felt by the structures. If many readings at closely spaced time intervals are taken, it may be assumed that this error is approximately eliminated by hand-smoothing of the measured curves.

Linearization of creep law and its error

As a reasonable approximation for many practical situations, the creep law of concrete may be considered to be linear, i.e., obeying the principle of superposition. Creep is then fully characterized by the compliance function $J(t,t')$ (also called the creep function), which represents the strain (creep plus elastic strain) at age t caused by a unit constant uniaxial stress acting since concrete age t'.[6] For the purpose of statistical analysis, it is helpful to express the compliance function in linearized form

$$J(t, t') = \theta_1 \xi + \theta_2 \tag{1}$$

in which ξ is a certain reduced time and θ_1, θ_2 are material parameters. Various creep prediction formulas can be brought to the form of Eq. (1). We choose here the double power law[6,7,11-13] because it was shown to agree better with test data than other well-known formulas. For this law

$$\theta_1 = \frac{\phi_1}{E_0}, \quad \theta_2 = \frac{1}{E_0},$$
$$\xi = \xi(t, t') = (t'^{-m} + \alpha)(t - t')^n \tag{2}$$

in which E_0, ϕ_1, m, n, and α are material constants. Their typical values are $m = 0.3$, $n = 1/8$, $\alpha = 0.05$, and $\phi_1 = 2$ to 6; however, the scatter is considerable.[11] Empirical formulas for estimating these par[illegible]s from concrete strength and composition exist.[12,[illegible]] [illegible]able ξ may be called the reduced time; θ_1 is a creep [illegible] parameter, θ_2 is the instantaneous deformation (for infinitely rapid loading), and E_0 is called the asymptotic modulus, typically $E_0 \simeq E_c$ where E_c is the conventional elastic (static) modulus, which corresponds to a load of about two hour duration and involves much short-time creep. Note that the earlier use of E_c instead of E_0 in the power law for concrete creep does not permit an adequate representation of longtime creep because it results in a much too high value of n.[6]

All the material parameters in the creep law, i.e., E_0, ϕ_1, m, n, and α are, strictly speaking, random variables. However, to make our problem tractable, only those parameters that appear linearly in $J(t,t')$ can be considered random. According to the linearization in Eq. (1), these are the parameters θ_1 and θ_2, whose variations represent a vertical shift of the creep curves and a change in their overall slope. Although, in principle, it would be more realistic to apply factor analysis to reduce the number of random material parameters, we cannot do so since independent statistical data for these parameters are unavailable. Statistical information exists only for the compliance values, which leads us to the following statistical model

$$J(t, t') = \theta_1\xi + \theta_2 + e \tag{3}$$

where e is the error. We will consider that e has a normal distribution with mean 0 and standard deviation σ.

Although the creep compliance depends separately on the current time t and the age at loading t', these two variables are grouped under one independent variable ξ. This is one essential idea of the present analysis, which permits us use of prior information on creep at any t and t'.

Bayesian estimation

We consider J as a random variable for which the governing law [Eq. (1) and (2)] is known with a high degree of certainty, but the material parameters θ_1, θ_2 are rather uncertain. Assume that the probability density distribution $f'(\theta_1, \theta_2)$ of the possible values of θ_1 and θ_2 is known from prior testing of various concretes. This distribution is called prior. The probability that θ_i $(i = 1, 2)$ lies within the intervals $(\theta_i^*, \theta_i^* + d\theta_i)$ may be denoted as $P(\theta_i^* < \theta_i < \theta_i^* + d\theta_i)$ and equals $f'(\theta_1, \theta_2)d\theta_1 d\theta_2$.

Now suppose that, for a given concrete of interest, certain compliance values J_j at reduced times ξ_j $(j = 1, 2, \ldots, N)$ have been measured. We want to exploit this information to obtain an improved (updated) probability density distribution $f''(\theta_1, \theta_2)$ of parameters θ_1, θ_2, given that compliances $J_1(\xi_1), \ldots, J_N(\xi_N)$ have been measured. This distribution is called posterior. The conditional probability that θ_i lies within the intervals $(\theta_i^*, \theta_i^* + d\theta_i)$, given that $J_j(\xi_j)$ has been measured, may be denoted as $P(\theta_i^* < \theta_i < \theta_i^* + d\theta_i | J_1, \ldots, J_N)$ and equals $f''(\theta_1, \theta_2)d\theta_1 d\theta_2$. According to Bayes' theorem[4,5,14,15,22,27,30,39,40]

$$P(\theta_i^* < \theta_i < \theta_i^* + d\theta_i | J_i, \ldots, J_N) = kP(J_1, \ldots, J_N|\theta_i)\, P(\theta_i^* < \theta_i < \theta_i^* + d\theta_i) \tag{4}$$

where k is a normalizing constant assuring that the sum (integral) of all probabilities $P(\theta_i^* < \theta_i < \theta_i^* + d\theta_i|J_1, \ldots, J_N)$ is unity, and $P(J_1, \ldots, J_N|\theta_i)$ is the probability of measuring values $J_j(\xi_j)$ under the condition that the parameter values are θ_i. Introducing the foregoing expressions for $P(\ldots)$ in terms of $f'(\theta_1, \theta_2)$ and $f''(\theta_1, \theta_2)$ and dividing by $d\theta_1 d\theta_2$, we obtain from Eq. (4)

$$f''(\theta_1, \theta_2) = kL(\theta_1, \theta_2; J_j)\, f'(\theta_1, \theta_2) \tag{5}$$

in which $L(\theta_1, \theta_2; J_j) = P(J_1, \ldots, J_N|\theta_i)$. This function is called the sample likelihood function. The normalizing constant k is determined from the condition

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f''(\theta_1, \theta_2)\, d\theta_1\, d\theta_2 = 1 \tag{6}$$

Supposing, for the sake of simplification, that the observed values $J_1, J_2, \ldots, J_N$ are statistically independent, we may express the sample likelihood function as

$$L(\theta_1, \theta_2; J_j) = \prod_{j=1}^{N} f_j(J_j|\theta_1, \theta_2) \tag{7}$$

Here, $f_j(J_j|\theta_1,\theta_2)$ is the probability density distribution of one random variable J_j, given that parameter values are θ_1 and θ_2, i.e., $f_j(J_j|\theta_1,\theta_2)d\theta_1 d\theta_2 dJ_j$ is the probability that the J_j-value lies within the interval $(J_j, J_j + dJ_j)$ under the condition that the parameter values lie within the intervals $(\theta_i, \theta_i + d\theta_i)$. For one particular concrete, parameters θ_1, θ_2 are fixed, and so $f_j(J_j|\theta_1,\theta_2)$ describes the scatter of J_j-values in one and the same concrete.

According to our statistical model [Eq. (3)], the compliance J_j for certain parameter values θ_1, θ_2 and a certain fixed reduced time ξ_j is a normal random variable with the mean $(\theta_1\xi_j + \theta_2)$ and standard deviation σ, i.e.

$$f_j(J_j|\theta_1, \theta_2) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left[-\frac{1}{2}\left(\frac{J_j - \theta_1\xi_j - \theta_2}{\sigma}\right)^2\right] \tag{8}$$

Standard deviation σ is to be evaluated for fixed θ_1 and θ_2, i.e., fixed material properties. Thus, σ characterizes the scatter of J for the given concrete for which measurements J_j were taken. If these measurements do not suffice to determine σ for this concrete, a typical value of σ for any similar concrete may be used for which plentiful data exist.

Assuming σ to be independent of reduced time ξ and substituting Eq. (8) into Eq. (7) and Eq. (7) into Eq. (5), we obtain the result

$$f''(\theta_1, \theta_2) = a_0 \exp\left[-\frac{1}{2}\sum_{j=1}^{N}\left(\frac{J_j - \theta_1\xi_j - \theta_2}{\sigma}\right)^2\right] f'(\theta_1, \theta_2), \qquad a_0 = k(\sqrt{2\pi}\,\sigma)^{-N} \tag{9}$$

The normalizing constant k follows from Eq. (6)

$$k = \left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} L(\theta_1, \theta_2; J_j)\, f'(\theta_1, \theta_2)\, d\theta_1\, d\theta_2\right]^{-1} = (\sqrt{2\pi}\,\sigma)^N \left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left[-\frac{1}{2}\sum_{j=1}^{N}\left(\frac{J_j - \theta_1\xi_j - \theta_2}{\sigma}\right)^2\right] f'(\theta_1, \theta_2)\, d\theta_1\, d\theta_2\right\}^{-1} \tag{10}$$

The posterior (updated) expectation of parameter θ_i, given that the observed compliance values were J_j, may be obtained as

$$\bar{\theta}_i = E(\theta_i|J_j) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \theta_i f''(\theta_1, \theta_2)\, d\theta_1\, d\theta_2 \tag{11}$$

where E denotes the expectation. The posterior (updated) probability that the compliance $J(\xi)$ at reduced time ξ will be less than some given value $\hat{J}$ may be calculated as

$$P\,[J(\xi) < \hat{J}] \;=\; \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P\,[J(\xi) < \hat{J} | \theta_1, \theta_2]\; f''(\theta_1, \theta_2)\, d\theta_1\, d\theta_2$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Phi\,[y\,(\theta_1, \theta_2)] = f''(\theta_1, \theta_2)\, d\theta_1\, d\theta_2 \qquad (12)$$

where Φ represents the cumulative normal distribution function

$$\Phi\,(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-x^2/2}\, dx \qquad (13)$$

and

$$y\,(\theta_1, \theta_2) = \frac{\hat{J} - \theta_1\xi - \theta_2}{\sigma} \qquad (14)$$

It is convenient to use a prior distribution that yields a posterior distribution of the same type; this is called a conjugate prior.[4] For a normal distribution, the conjugate one is again a normal distribution.[4] Therefore, for $f'(\theta_1,\theta_2)$ the normal distribution is chosen. Because two parameters are used, a bivariate prior distribution should be properly considered

$$f'\,(\theta_1, \theta_2) = \frac{1}{2\pi\,\sigma_1\sigma_2}\, \exp\left\{ -\frac{1}{2}\left[\left(\frac{\theta_1 - m_1}{\sigma_1}\right)^2 + \left(\frac{\theta_2 - m_2}{\sigma_2}\right)^2\right]\right\} \qquad (15)$$

where m_1, m_2 are the means and σ_1, σ_2 the standard deviations of θ_1, θ_2. Unfortunately, however, no statistical data have yet been published for the parameters θ_1 and θ_2.

To circumvent this difficulty, we may consider the univariate distribution of the values of $J(\xi)$ and assume it to be normal

$$f\,[J(\xi)] = \frac{1}{\sqrt{2\pi}\,\sigma_J(\xi)} \exp\left[-\frac{1}{2}\left(\frac{\bar{J}(\xi) - J(\xi)}{\sigma_J(\xi)}\right)^2\right] \qquad (16)$$

Here, $\bar{J}(\xi)$ = mean of the (prior) observations of J at reduced time ξ, and $\sigma_J(\xi)$ = their standard deviation, which may be calculated as $\sigma_J(\xi) = \omega_J \bar{J}(\xi)$ where ω_J = coefficient of variation of the prior data, which is considered, for the sake of simplicity, to be independent of reduced time ξ. Extensive data on the values of ω_J are given in Table III of Reference 12 and in Table 2 of Reference 13.

Now, we come to an important step; if we substitute $J(\xi) = \theta_1\xi + \theta_2$, this distribution becomes a function of θ_1 and θ_2. Furthermore, noting that $\Delta J = \xi\,\Delta\theta_1 + \Delta\theta_2$, we see that the probability (or frequency) of error $\Delta J = \xi\,\Delta\theta_1$ at $\Delta\theta_2 = 0$ is the same as the probability (or frequency) of error $\Delta\theta_1$, and the probability of error $\Delta J = \Delta\theta_2$ at $\Delta\theta_1 = 0$ is the same as the probability of error $\Delta\theta_2$. Therefore, on substitution of $J(\xi) = \theta_1\xi + \theta_2$, Eq. (16) may be regarded as an approximate probability density distribution of θ_1 and θ_2, i.e.

$$f'\,(\theta_1, \theta_2) = \frac{1}{\sqrt{2\pi}\,\sigma_J(\xi)} \exp\left[-\frac{1}{2}\left(\frac{\bar{J}(\xi) - \theta_1\xi - \theta_2}{\sigma_J(\xi)}\right)^2\right] \qquad (17)$$

Replacement of the bivariate distribution [Eq. (15)] with a univariate distribution [Eq. (17)] is a crucial step in the present analysis, allowing the use of existing statistics of creep data. A similar step was used by Tang et al.[37] in their analysis of settlement of oil platforms.

Numerical integration

Integrals of the type of Eq. (12) must be evaluated numerically. Caution is required, since the integral extends over an infinite domain. From the practical viewpoint, this is usually the most sensitive task in Bayesian analysis, and various studies have been devoted to it.[17,26] None of them, however, seems to be directly applicable to the present integrand.

Fig. 1(a) shows the shape of the exponents of the function $f''(\theta_1,\theta_2)$ for the case when $f'(\theta_1,\theta_2) = 1$ (dif-

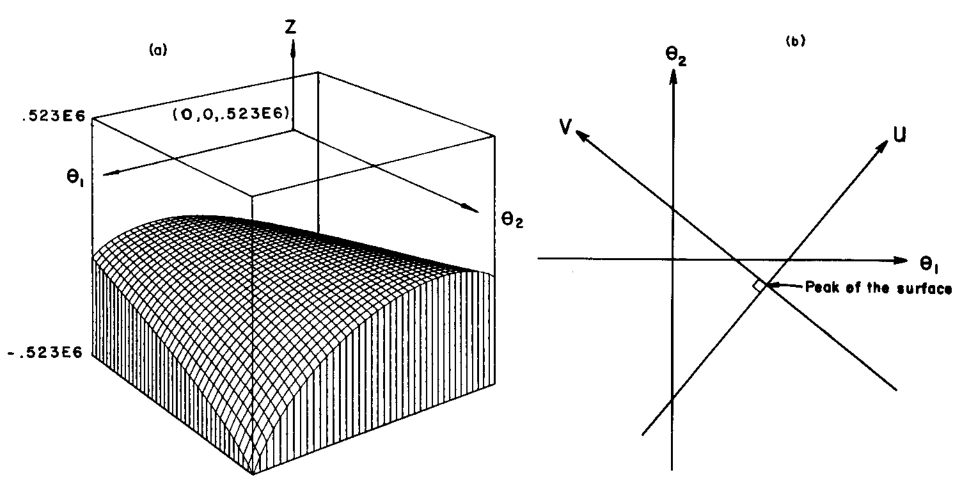

Fig. 1—(a) Exponent of function $f''(\theta_1, \theta_2)$; (b) transformation of coordinate system

fuse prior); the integration was done with 40 integration points for both θ_1 and θ_2, and the statistical parameters of the test data by Rostasy[33] were considered. The surface decays rapidly in one direction and extremely slowly in another. For this reason, it is necessary to introduce new independent variables u and v, which convert the bivariate normal distribution to a standard form.

Function $f''(\theta_1,\theta_2)$ that needs to be integrated in the integrand of Eq. (12) may be written, according to Eq. (9), in the form

$$f''(\theta_1, \theta_2) = a_0 \exp\left[-(A\theta_2^2 + B\theta_1^2 + 2D\theta_1\theta_2 + 2E\theta_2 + 2F\theta_1 + G)\right] f'(\theta_1, \theta_2) \quad (18)$$

with coefficients

$$\begin{aligned}
A &= \frac{N}{2\sigma^2} + \frac{1}{2\sigma_f^2(\xi)}, \quad B = \frac{1}{2\sigma^2}\sum_{j=1}^{N} \xi_j^2 + \frac{\xi^2}{2\sigma_f(\xi)^2}, \\
D &= \frac{1}{2\sigma^2}\sum_{j=1}^{N} \xi_j + \frac{\xi}{2\sigma_f(\xi)^2}, \\
E &= -\frac{1}{2\sigma^2}\sum_{j=1}^{N} J_j - \frac{\bar{J}(\xi)}{2\sigma_f(\xi)^2}, \\
F &= -\frac{1}{2\sigma^2}\sum_{j=1}^{N} \xi_i J_i - \frac{\xi\bar{J}(\xi)}{2\sigma_f(\xi)^2}, \\
G &= \frac{1}{2\sigma^2}\sum_{j=1}^{N} J_j^2 + \frac{\bar{J}(\xi)^2}{2\sigma_f(\xi)^2}
\end{aligned} \quad (19)$$

where N = number of observed creep data. In the special case of a diffuse prior [$f'(\theta_1, \theta_2)$ = const.], the second terms in these expressions vanish because $\sigma_f(\xi) \rightarrow \infty$. We seek to transform Eq. (18) to the form

$$f''(\theta_1, \theta_2) = a_0 e^{-u^2} e^{-v^2} f'(\theta_1, \theta_2) \quad (20)$$

where u, v are new variables given by a certain linear transformation [Fig. 1(b)]

$$u = a\theta_2 + b\theta_1 + c, \quad v = (a\theta_2 - b\theta_1 + d)\sqrt{e} \quad (21)$$

which transforms the quadratic polynomial in Eq. (18) to its principal coordinates. Comparison of the coefficients in Eq. (19) and in Eq. (20) and (21) yields the relations

$$a^2 + eb^2 = A, \quad b^2 + ea^2 = B, \quad (22a, b)$$

$$ab(1 - e) = D, \quad ac - ebd = E, \quad (22c, d)$$

$$bc + ead = F, \quad c^2 + ed^2 = G \quad (22e, f)$$

Further, one needs to solve from these equations a, b, c, d, e under the conditions that a, b, c, d, e are real and $a > 0$, $b > 0$, and $e > 0$. After some tedious algebraic manipulations, one can show that such a solution exists, and one solution (with $e < 1$) is

$$\begin{aligned}
a &= \left(\frac{A - eB}{1 - e^2}\right)^{1/2}, \quad b = \left(\frac{B - eA}{1 - e^2}\right)^{1/2}, \\
c &= \frac{aE + bF}{a^2 + b^2}, \quad d = \frac{aF - bE}{e(a^2 + b^2)}, \\
e &= \frac{A + B - [(A - B)^2 + 4D^2]^{1/2}}{A + B + [(A - B)^2 + 4D^2]^{1/2}}
\end{aligned} \quad (23)$$

which can be verified by substitution into Eq. (22a-f). Another solution exists (with $e > 1$), but it is equivalent since it merely corresponds to interchanging u and v. The inverse of the linear transformation in Eq. (21) is

$$\begin{aligned}
\theta_1 &= [a(e^{-1/2}v - d) + b(u - c)](a^2 + b^2)^{-1} \\
\theta_2 &= [a(u - c) - b(e^{-1/2}v - d)](a^2 + b^2)^{-1}
\end{aligned} \quad (24)$$

and the Jacobian of the transformation in Eq. (21) is

$$\begin{aligned}
|J| &= \begin{vmatrix} \partial u/\partial\theta_1, & \partial u/\partial\theta_2 \\ \partial v/\partial\theta_1, & \partial v/\partial\theta_2 \end{vmatrix} = \begin{vmatrix} a, & b \\ -b\sqrt{e}, & a\sqrt{e} \end{vmatrix} \\
&= (a^2 + b^2)\sqrt{e}
\end{aligned} \quad (25)$$

In the integrals in Eq. (11) and (12), we may now substitute

$$f''(\theta_1, \theta_2)\, d\theta_1\, d\theta_2 = \frac{1}{\pi} e^{-u^2 - v^2}\, du dv \quad (26)$$

in which Eq. (24) must be substituted for θ_1 and θ_2. The integration over u and v is to be carried out from $-\infty$ to ∞.

The integrals may be efficiently evaluated using the Hermite-Gaussian formula which has, for the variable, the form $\int_{-\infty}^{\infty} f(x)e^{-x^2}dx = \Sigma_n w_n f(x_n)$, where w_n is certain known weights.[34] In the case of two variables [Eq. (26)], this integration formula yields for the integral in Eq. (12) the approximation

$$P[J(\xi) < \hat{J}] = \frac{1}{\pi}\sum_{k=1}^{K}\sum_{m=1}^{M} w_k w_m\, \Phi(y_{km}) \quad (27)$$

in which K and M are the chosen numbers of integration points for coordinates u and v, and $\Phi(y_{km})$ is given by Eq. (13)

$$y_{km} = \frac{1}{\sigma}(\hat{J} - \theta_{1_{km}}\xi - \theta_{2_{km}}) \quad (28)$$

and $\theta_{1_{km}}$, $\theta_{2_{km}}$ are evaluated from Eq. (24) for $u = u_k$ and $v = v_m$.

Eq. (27) is the final result to be used in computer calculations.

Examples

To demonstrate the theory, consider the test data by McDonald.[28] From the fitting of these data, one obtains double-power law parameters m = 0.305, n = 0.147, and $\alpha = 0.059$,[12] and based on the scatter of these data (fixed θ_1, θ_2), it can be roughly estimated that $\sigma \simeq 0.01 \times 10^{-6}$/psi. The prior information may be determined from the BP model[12] using the following parameters: cylinders 15.2 × 40.6 cm, sealed, at 23 C, 28 day cylindrical strength 6300 psi, water-cement-sand-gravel ratio 0.425 : 1 : 2.03 : 2.62, cement type II, and age at loading $t' = 90$ days. Furthermore, the value of coefficient of variation for all concretes (prior information) may be considered as $\omega_J = 0.24$, according to Reference 12 (provided the BP model is used).

To get an idea of the usefulness of the Bayesian approach to creep prediction, the prediction for varying amounts of data for the given concrete (i.e., for progressively better likelihood functions) is given in Fig. 2. First Fig. 2 shows the predicted 90 percent probability band and the mean for the concrete of McDonald if the prediction was based strictly on the prior information, i.e., if the data of McDonald were not known to us. Furthermore, Fig. 2 also shows the Bayesian prediction of the mean creep and the 90 percent probability band when only the first four data points of McDonald are known, and also when the first eight points are known.

It should be noted that the 90 percent probability band progressively narrows with the increasing amount of data for the concrete of McDonald and that the median prediction comes closer to the remaining later data not used in the prediction. The most significant improvement of prediction is brought about by considering the first data point [Fig. 3(a)].

Also note that the width of the 90 percent probability band (scatter band) increases with increasing time. This effect is particularly pronounced when assuming constant ω_J; if σ_J were assumed constant, the width of the probability band would not increase so markedly with time.

From Fig. 3(b) and 3(c) the effect of prior information (i.e., of the statistics from Reference 12) can be seen. Fig. 3(b) shows three curves: Curve A is the median prediction based on prior information only; Curve B is the prediction updated on the basis of first four data points and based on full prior information as shown in Reference 12; and Curve C is the prediction based on the same first four data points if this prior information is considered to be highly uncertain, i.e., the information on coefficients of variation from Reference 12 is ignored and only the mean prediction formulas from the reference are used. Fig. 3(c) shows the comparison for the 90 percent probability band. For this particular data the prediction based on the diffuse prior ($\sigma_J \to \infty$) happens to be quite good. However, this is not so for all data, e.g., for those of Rostasy et al., as Fig. 4(b) demonstrates.

From these comparisons, it can be seen that, for concrete creep, the benefit of the prior information is smaller than the benefit of obtaining at least some

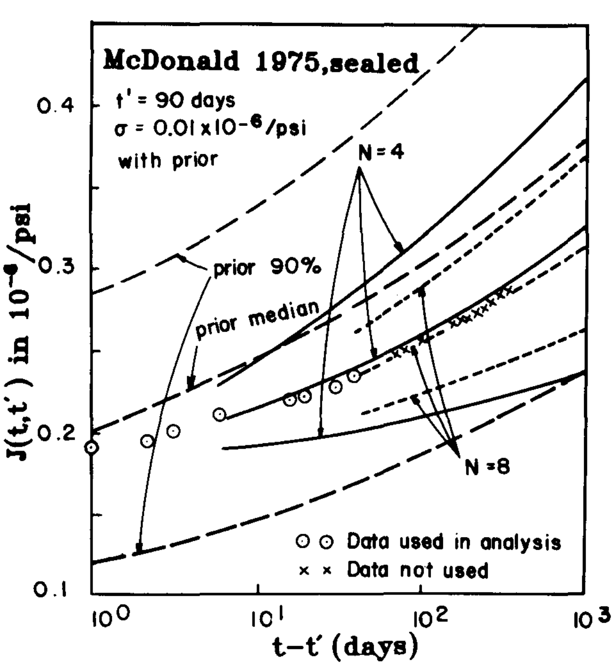

Fig. 2—Prior, posterior median and probability band ω_{90} for McDonald's test data

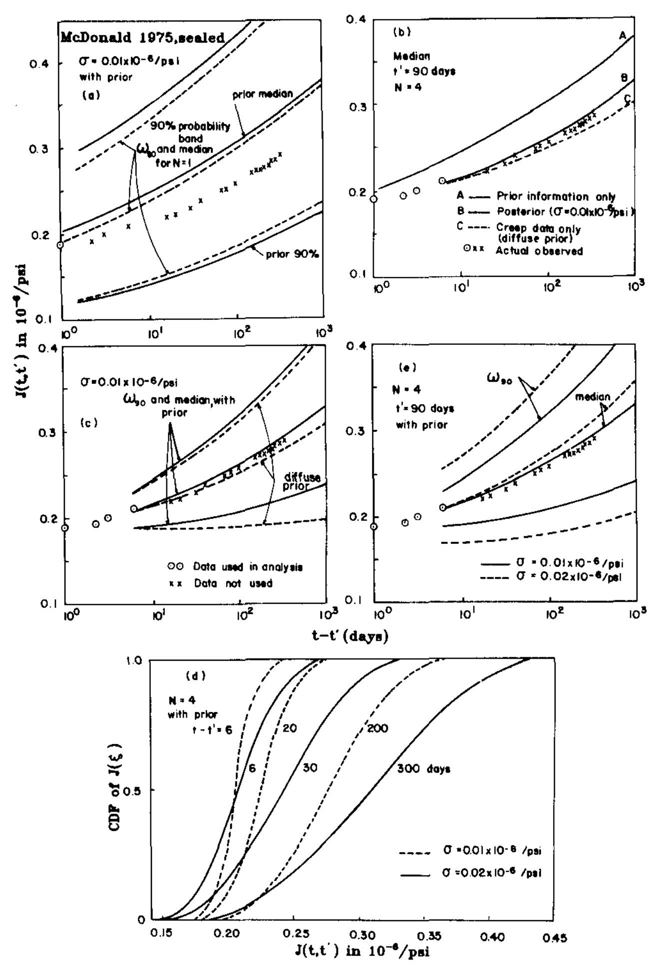

Fig. 3—Various types of posterior predictions for McDonald's data: (a) Prediction based on one data point; (b) median prediction based on a different number of data points; (c) prediction for actual and diffuse prior; (d) cumulative density function J(ξ); and (e) predictions based on different values of standard deviation

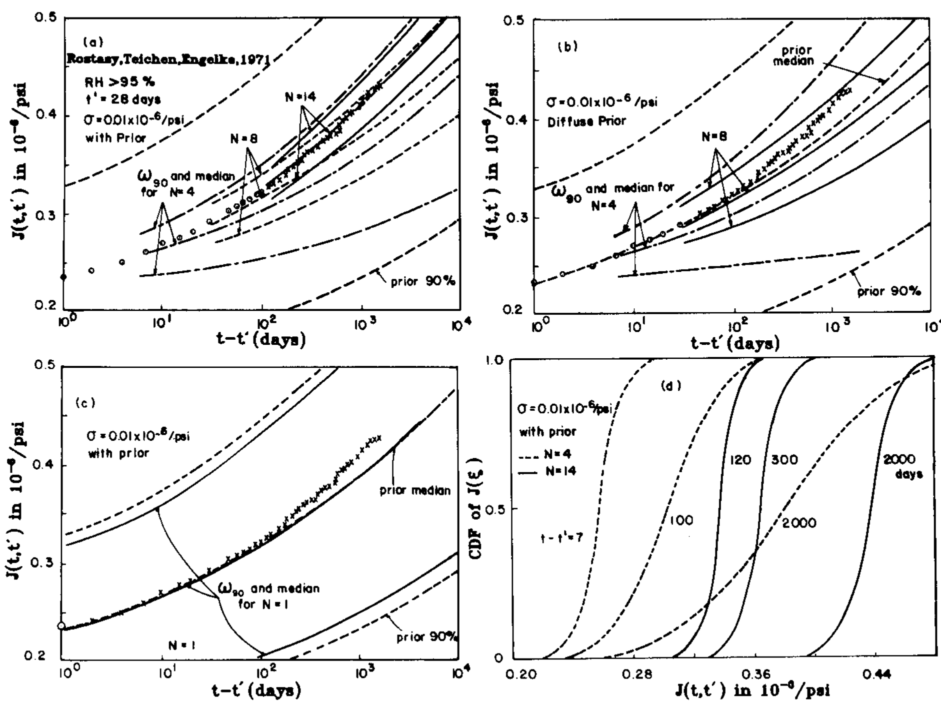

Fig. 4—Various types of posterior prediction for Rostasy et al.'s data: (a) Prediction based on prior and different numbers of data points; (b) same as (a) but for diffuse prior; (c) prediction based on first data point only; and (d) cumulative density function J(ξ)

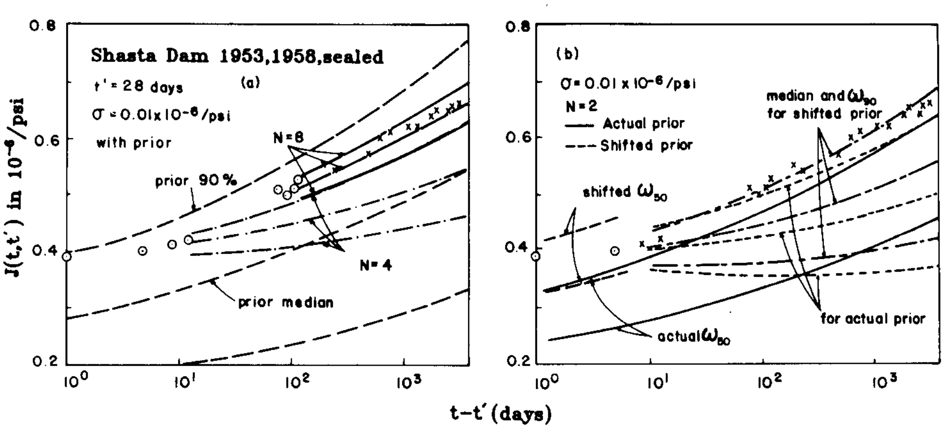

Fig. 5—Various types of posterior predictions for Shasta Dam concrete: (a) Predictions based on actual prior and various numbers of data points; and (b) effect of shifting the prior upwards

short-time creep data for the given concrete. (A similar conclusion was reached empirically in Reference 13.) This last conclusion is further reinforced by the prediction in Fig. 3(a), which is based on the full prior but only the first data point of McDonald and on the knowledge that σ is roughly 0.01×10^{-6}/psi for these data. The mean prediction agrees surprisingly well with the rest of the data of McDonald. (This case is called a "dominant prior" and a "soft Bayesian prediction."[27])

From Fig. 3(d) an idea of the shape of the cumulative probability density function Φ (y) can be drawn for the $J(\xi)$ values at various $t - t'$, obtained for McDonald's data, according to Eq. (12) or (27).

To show that similar results are obtained for other creep data from literature, predictions have been calculated for the test data of Rostasy et al.[33] and for Shasta Dam data[23,24]; see Fig. 4(a) through (d) and 5(a) and (b). There are some differences, however. For the data of Rostasy et al., the measured points happen to lie closer to the median of the prior, much more so than for other test series. Nevertheless, use of the prior still does improve the prediction based on a reduced number of data points (N = 4, N = 8, and N = 14). For Shasta Dam concrete, the opposite is the case; the measured data lie far off the prior median prediction, and this is why the posterior curves are so different. Especially note that the first two data points combined with the prior give a very poor extrapolation to longer times, much poorer than that for the concrete of Rostasy et al. and of McDonald.

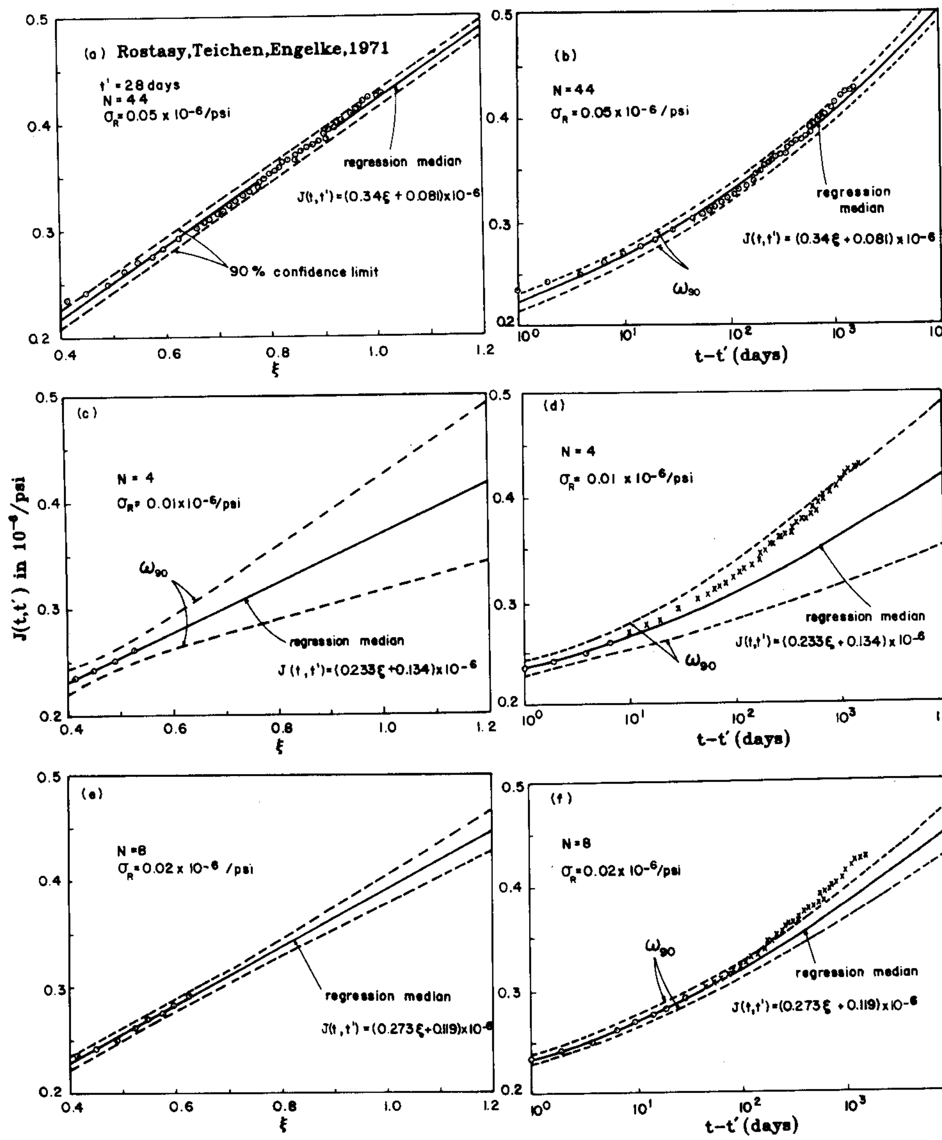

Fig. 6—Regression analysis of Rostasy et al.'s data for various numbers of points. (a), (c), and (e) in linear plots and (b), (d), and (f) same as in log (t - t') plot

The prior information greatly modifies the prediction when the measured data lie outside the 50 percent probability band ω_{50} of the prior. An example of this is artificially constructed in Fig. 5(b), in which the actual prior median was deliberately shifted upward, keeping the data as measured. We see that such a shift has a great effect on the median and makes the posterior probability band wider.

It is interesting to contrast the Bayesian analysis with simple statistical regression based on only measured data and prior knowledge of the formulas for the mean values; see Fig. 6(a), (c), and (e) showing the linear regressions made in the ξ scale as well as corresponding plots in $\log(t - t')$ scale in Fig. 6(b), (d), and (f). If only a few data points, e.g., four points in Fig. 6(c) are used, the probability band rapidly widens with time, while for a long data series [Fig. 6(a)] it remains narrow. In the latter case, the probability band might be narrower than that obtained from Bayesian analysis with all the prior information, but the Bayesian approach is more realistic.

It should be noted that, while extrapolating data of statistical regression, one is predicting statistical properties of the future creep of the particular specimen measured. On the other hand, in our Bayesian analysis, we are predicting statistical properties of creep of all the specimens that could possibly be made from the given concrete. It is the latter case which is of interest for design, and the statistical variability for this case is obviously larger.

Determining standard deviation σ for the likelihood function is important. σ characterizes the scatter of J-values at some fixed time when a great number of identical creep tests on a given concrete (fixed θ_1,θ_2) is performed under the same conditions. Such data have recently been presented by Cornelissen, Reinhardt et al., and Alou and Wittmann.[3,19,20,32] One typical example from the Reinhardt et al.'s results is shown in Fig. 7, from which one can see that for their concrete roughly the value $\sigma = 0.02 \times 10^{-6}$/psi would be appropriate for characterizing the likelihood function for the sum of creep and shrinkage strains. Strictly speaking, σ de-

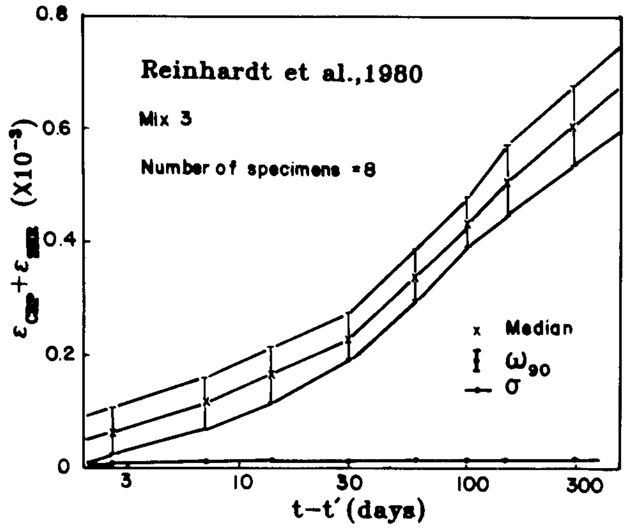

Fig. 7—Reinhardt et al.'s statistical data for identical specimens

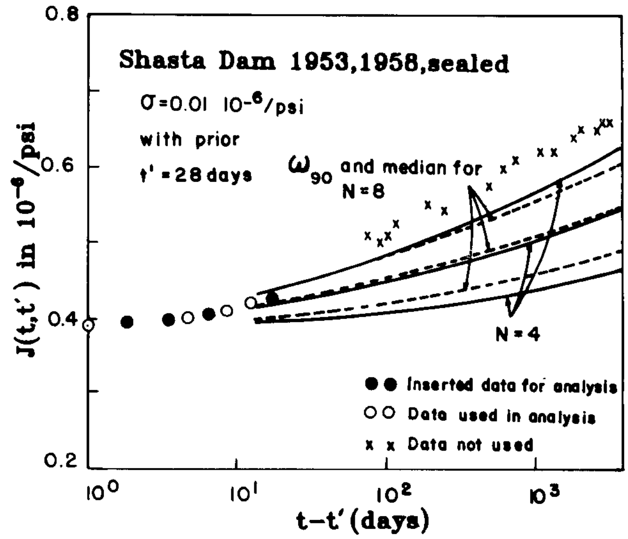

Fig. 8—Effect of spacing of data points on the predictions

pends on time, as can be seen from Fig. 7, although it was neglected in the foregoing analysis. To avoid the need for many tests for the given concrete, it may also be assumed that σ is the same as observed before on similar concretes. Note that the σ-value for the likelihood function is different from the value of standard deviation that results from a regression analysis of one creep test (with many J-values for different times but the same specimen, as shown in Fig. 6).

The effect of various choices of σ (for the likelihood function) is shown in Fig. 3(e). A smaller σ, if justified, gives a distinctly better median prediction and a narrower probability band.

As can be seen from these examples, the value of standard deviation σ for the concrete under consideration has a great influence on the width of the scatter band for creep extrapolation. At the same time, the direct information on σ is usually scant, since only few measurements are normally carried out for the concrete at hand. In such situations, the value of σ has to be based on an analysis of the data for various similar concretes. For these predictions, the standard deviations for the data of Rostasy et al. and Shasta Dam data were estimated as $\omega = 0.01 \times 10^{-6}$/psi. For the prior predictions according to Reference 12, the following information was needed: Shasta Dam data — Cylinders 15.2 × 66 cm, 21 C, sealed, 28 day cylinder strength 3230 psi, water-cement-sand-gravel ratio 0.58 : 1 : 2.5 : 7.1, cement type IV, $m = 0.376$, $n = 0.127$, $\alpha = 0.043$, and $t' = 28$ days. Rostasy et al. data — Cylinders 20 cm diameter and 140 cm length, environmental relative humidity $\geqslant$ 95 percent, temperature 20 C, 28 day cube strength 6500 psi, water-cement-sand-gravel ratio 0.41 : 1 : 2.43 : 3.15 (by weight), and age at loading $t' = 28$ days.

One notable simplification in our statistical model is that the values $J_1, J_2, \ldots, J_N$, as well as similar data for the prior, are implied to be statistically independent, but in reality they are not completely so. If, for example, the creep value is high, compared to the mean, at $\xi = 1000$, it will be high at $\xi = 1001$ and will be quite close to the value at $\xi = 1000$. This is so because, in the physical mechanism of creep, the randomness arises through creep increments rather than their accumulated values.[18] The correlation of adjacent J-values becomes weaker with increasing time intervals between these values, and for relatively sparse data our assumption of statistical independence of $J_1, J_2, \ldots, J_N$, implied in Eq. (7), is probably quite good. Obviously, one should avoid using very dense data. Nevertheless, the effect of data density does not seem to be critical, especially for the mean predictions. What happens when additional data points are inserted between each two adjacent points of Rostasy's data has been checked. The resulting mean prediction remained almost the same (see Fig. 8). On the other hand, the effect of these inserted points on the 90 percent probability band was stronger (Fig. 8).

The foregoing problems with the lack of independence of adjacent data values can only be completely avoided if creep is considered as a stochastic process in time.[18] However, Bayesian analysis in such a context would be difficult. The same problem is encountered in the analysis of continuous data records in general, and various simple methods of accounting for the correlation of adjacent data values in a time series have been devised.[21]

A related question is: How should the experimentalist properly choose the times of reading the strain in a creep test? Optimally, the readings should be uniformly distributed when plotted in the ξ-scale. This spacing does not correspond to a uniform spacing in either t and t' or log t and log t'. However, a uniform spacing in log $(t - t')$ and log t' is close to optimal. Crowding the readings in some segment of the ξ-scale is equivalent to assigning a larger weight to the corresponding measurements, which introduces subjective bias.

Prediction of shrinkage and drying creep

The present method of analysis can also be applied to predicting shrinkage of concrete. Instead of $J(t,t')$, the basic variable is then the shrinkage strain $Y = \epsilon_{sh}$, which depends on current time t and age t_0 at the start of drying, $\epsilon_{sh} = \epsilon_{sh}(t,t_0)$. According to References 12 and 13, the shrinkage law may be written as $\epsilon_{sh} = \epsilon_{sh\infty}k_h(1 + \tau_{sh}/\bar{t})^{-1/2}$ where $\bar{t} = t - t_0$, $\epsilon_{sh\infty}$ = constant for a given concrete, k_h = function of humidity, and based on diffusion theory, $\tau_{sh} = \tau_1 D^2$ where τ_1 = function of t_0 and D = effective thickness of concrete member. Including the error, we may write this law in the linearized form

$$Y = \theta_1\xi + \theta_2 + e \tag{29}$$

in which $Y = 1/\epsilon^2_{sh}$, $\xi = 1/\bar{t}$, $\theta_1 = \tau_{sh}(k_h\,\epsilon_{sh_\infty})^{-2}$, $\theta_2 = (k_h\,\epsilon_{sh_\infty})^{-2}$, and e = error. Here, θ_1 and θ_2 may again be considered as random material parameters whose random scatter corresponds to an uncertainty in ϵ_{sh_∞} and τ_{sh}. Since the form of Eq. (29) is identical to Eq. (1), the present Bayesian analysis can be followed.

Although the double power law [Eq. (1) and (2)] may be applied, in an approximate sense, to drying concrete members,[11] it should be properly restricted to creep at constant water content, called basic creep. The additional creep due to variable moisture content, called drying creep, must be modeled differently because it depends on cross section thickness and has a different time and age dependence. The creep law describing both basic creep and drying creep was given in References 12 and 13. This law may still be written in the form of Eq. (1), provided that ξ is redefined as

$$\xi = (t_0'^{-m} + \alpha)(t - t')^n + \frac{\phi_d}{\phi_1}\frac{k_h'}{E_0}t'^{-m/2}\left(1 + \frac{3\tau_{sh}}{t - t'}\right)^{-0.35} \tag{30}$$

The coefficients and functions in the added drying creep term are given in Eq. (12) of Reference 13; ϕ_d is a function of temperature and of $t' - t_0$ where t_0 is the age at the start of drying, k_h' is a function of environmental humidity h, and τ_{sh} is the same as in Eq. (29) for shrinkage (as diffusion theory indicates). With the definition of reduced time ξ according to Eq. (29), all our preceding Bayesian analysis [Eq. (4)-(28)] remains applicable.

Instead of considering the ratio $\bar{\phi}_d/\phi_1$ in Eq. (30) to be fixed by the prediction model from Reference 12, $\theta_3 = \bar{\phi}_d/E_0$ may be introduced as a third random parameter characterizing the drying creep term separately from the basic creep term (double power law). Then Eq. (3) must be replaced by

$$J(t, t') = \theta_1\xi + \theta_3\eta + \theta_2 + e \tag{31}$$

where η is another reduced time for the drying creep term, and e is the error. The required generalization of the preceding analysis would be relatively straightforward, but the numerical integration of $f''(\theta_1,\theta_2,\theta_3)$ would be considerably more tedious. Due to the lack of meaningful separate statistics for the drying creep term, a generalization of this type would hardly make sense at present.

Ramifications and possible refinements

The best method for linearizing the creep law is an interesting problem. The linearization in Eq. (1) and (2) for creep without drying has one obvious disadvantage. The normal distribution admits errors of any magnitude, and a negative error in J of a large magnitude can make J negative, which is physically impossible. For the same reason, large negative errors are less likely than equally large positive errors in J, which is not reflected in a normal distribution for J. This situation could be remedied by introducing an asymmetric distribution for J, e.g., the log-normal distribution.

With the double power law, an asymmetric distribution of J appears naturally by introducing the following alternative linearization of the double power law

$$Y = \theta_1\xi + \theta_3\eta + \theta_2 + e \tag{32}$$

where

$$\begin{aligned} Y &= \log(J - 1/E_0), \quad & \xi &= \log(t - t'), \\ \eta &= \log(t'^{-m} + \alpha), \quad & \theta_1 &= n, \\ \theta_2 &= \log(\phi_1/E_0) \end{aligned} \tag{33}$$

Parameter θ_3 is added for reasons of generality, even though in the double power law $\theta_3 = 1$. Use of a normal distribution for Y would then correspond to a log-normal distribution for $(J - 1/E_0)$, preventing errors that would cause J to be less than $1/E_0$. Another possibly advantageous feature of Eq. (32) and (33) is that the (equally likely) errors would be larger the larger the J-value (longer times), as expected. The fact that the age at loading t' appears in Eq. (32) in an independent variable different from the variable characterizing load duration $t - t'$ is also an advantage. This approach, however, would have the disadvantage that the instantaneous deformation $1/E_0$ would be deterministic (zero error) and would have to be determined in advance.

Another questionable aspect of the present statistical model, which would be avoided by Eq. (32) and (33), is the fact that Eq. (1) or (3) cannot distinguish between t' and $(t - t')$, and consequently imply that the error is the same for all the combinations of $(t - t')$ and t' that yield the same value of $\xi = (t'^{-m} + \alpha)(t - t')^n$. Consider, e.g., that measurements on a given concrete are made only for the age at loading $t' = 28$ days and terminate at load duration $t - t' = 60$ days. Then, for $n = 1/8$, $m = 0.3$, and $\alpha = 0.05$, the corresponding ξ is 0.520. For $t' = 1000$ days, the same ξ-value is reached at $(t - t') = 5835$ days. At these times, the standard deviation of J is supposed to be the same according to the present model, while obviously it should be much larger than for $t' = 28$ days and $(t - t') = 60$ days if all measurements were confined to $t' = 28$

days. Thus, if the measurements for a given concrete are limited to a narrow range of t' and cover a broad range of $(t - t')$, extrapolation in t' has a larger error than the present model would predict. This problem may be overcome by using separate variables for $t - t'$, as in Eq. (32) and (33), but here again the difficulty is to obtain the prior statistical information for such an approach.

Various other creep prediction formulas may be brought to the linear two-variable form of Eq. (1) or Eq. (32). This includes Branson's formula used in the ACI Committee 209 recommendation[2] as well as the log-double power law[9] and the triple power law[10] — laws that represent an improvement of the double power law for basic creep. The present method of analysis is applicable for all these laws; the only change needed is to redefine ξ.

Further interesting questions arise with regard to the effect of reduced time ξ on the statistics. For example, in Eq. (8) for the likelihood function, standard variation σ is considered independent of ξ. However, it might be also reasonable to assume that $\sigma = \omega \bar{J}(\xi)$ where ω is a fixed coefficient of variation and $\bar{J}(\xi)$ is the mean of given data at ξ. This assumption would lead to larger errors at large $\bar{J}$ and smaller errors at small $\bar{J}$. A similar question arises for the effect of ξ on the statistics of the prior. In Eq. (17) it was assumed that $\sigma_J(\xi) = \omega_J \bar{J}_0(\xi)$ where ω_J is fixed. Alternatively, one might assume that σ_j is independent of ξ, in which case the coefficient of variation ω_J would decrease with increasing ξ. The present statistical data from creep testing do not give a clear answer to these questions.

It should also be kept in mind that the statistical approach based on creep formulas is, in itself, a simplification. The fundamental law governing creep is not an algebraic formula but a certain evolution law described by a differential or integral equation in time. Its proper stochastic generalization is a random process in time.[18] This approach would be particularly appropriate under general conditions of time variable stress (or temperature, pore humidity), and a combination of a random process with Bayesian analysis would be a better treatment of our problem.

Conclusions

1. For predicting creep (or shrinkage) in creep-sensitive structures, it is important to carry out some short-time measurements for the given concrete and then extrapolate them to very long times by combining the measured data with prior statistical information on creep of concrete in general. This may be accomplished using the Bayesian statistical approach.

2. The creep law needs to be linearized by introducing a certain reduced time combining the creep duration and age at loading.

3. The statistical variability of material parameters for the prior may be determined on the basis of the statistical variability of the compliance values, which was previously determined in a study of most test data from literature, involving over 800 measured curves.

4. The standard variation for the likelihood function characterizing the given concrete may be estimated, without a large set of measurements, on the basis of recent statistical creep observations by Wittmann, Alou, Cornelissen, and Reinhardt.

5. A certain transformation of variables permits determining the posterior (updated) probability density distribution of material parameters by integrating numerically with the help of Hermite-Gaussian formula.

6. A strong improvement in the mean longtime prediction can be achieved by the Bayesian approach even if only a few short-time measurements are made, provided that they do not greatly differ from the mean of the prior. Extending the measurements in time does not bring too much further improvement in the mean long time prediction, but it significantly further reduces the coefficient of variation of the longtime prediction.

7. When the measured short-time data lie outside the 50 percent probability band of the prior, Bayesian use of the prior greatly modifies the longtime extrapolation compared to that obtained by statistical regression of measured data alone.

ACKNOWLEDGMENT

Partial financial support under U. S. National Science Foundation Grant CEE-830148 to Northwestern University is gratefully acknowledged. Mary Hill is thanked for her excellent secretarial assistance.

REFERENCES

1. ACI Committee 209, "Prediction of Creep, Shrinkage, and Temperature Effects in Concrete Structures," *Designing for Effects of Creep, Shrinkage, Temperature in Concrete Structures*, SP-27, American Concrete Institute, Detroit, 1971, pp. 51-93.

2. ACI Committee 209, "Prediction of Creep, Shrinkage, and Temperature Effects in Concrete Structures," (ACI 209R-82), American Concrete Institute, Detroit, 1982, 108 pp.

3. Alou, F., and Wittmann, F. H., "Étude Experimentale de la Variabilité du Retrait du Béton," *Preprints*, International Symposium on Fundamental Research on Creep and Shrinkage of Concrete, Lausanne, Sept. 1980, pp. 61-77.

4. Ang, A. H-S, and Tang, W. H., *Probability Concepts in Engineering Planning and Design*, V. 1, Basic Principles, John Wiley and Sons, New York, 1975, pp. 329-359.

5. Bayes, T., "An Essay toward Solving a Problem in the Doctrine of Chances," *Philosophical Transactions* (London), V. 53, 1763, pp. 376-398, and V. 54, 1764, pp. 298-310. Also, Deming, W. E., "Facsimile of Two Papers by Bayes," Department of Agriculture, Washington, D. C., 1940.

6. Bažant, Z. P., "Mathematical Models for Creep and Shrinkage of Concrete," *Creep and Shrinkage in Concrete Structures*, John Wiley and Sons, Chichester, 1982, pp. 163-258.

7. Bažant, Z. P., "Theory of Creep and Shrinkage in Concrete Structures: A Précis of Recent Developments," *Mechanics Today*, V. 2, 1975, pp. 1-93.

8. Bažant, Zdeněk,; Carreira, Domingo J.; and Walser, Adolf, "Creep and Shrinkage in Reactor Containment Shells," *Proceedings*, ASCE, V. 101, ST10, Oct. 1975, pp. 2118-2131.

9. Bažant, Z. P., and Chern, J. C., "Log-Double Power Law for Basic Creep of Concrete," *Report* No. 83-10/679, Center for Concrete and Geomaterials, Northwestern University, Evanston, 1983, 25 pp.; also ACI JOURNAL in press.

10. Bažant, Z. P., and Chern, J. C., "Triple-Power Law for Basic Creep of Concrete," *Report* No. 83-11/679t, Center for Concrete and Geomaterials, Northwestern University, Evanston, 1983, 27 pp.

11. Bažant, Z. P., and Osman, E., "Double Power Law for Basic Creep of Concrete," *Materials and Structures, Research and Testing* (RILEM, Paris), V. 9, No. 49, Jan-Feb. 1976, pp. 3-11.

12. Bažant, Z. P., and Panula, L., "Practical Prediction of Time-Dependent Deformations of Concrete," *Materials and Structures, Research and Testing* (RILEM, Paris), V. 11, No. 65, Sept.-Oct. 1978, pp. 307-328, V. 11, No. 66, Nov.-Dec. 1978, pp. 415-434, and V. 12, No. 69, Jan.-Feb. 1979, pp. 169-183.

13. Bažant, Zdeněk P., and Panula, Liisa, "Creep and Shrinkage Characterization for Analyzing Prestressed Concrete Structures," *Journal*, Prestressed Concrete Institute, V. 25, No. 3, May-June 1980, pp. 86-122.

14. Benjamin, Jack R., and Cornell, C. Allin, *Probability, Statistics and Decision for Civil Engineers*, McGraw-Hill Book Co., New York, 1970, pp. 524-641.

15. Box, G. E. P., and Tiao, G. C., *Bayesian Inference in Statistical Analysis*, Addison-Wesley Publishing Co., Reading, 1973, 588 pp.

16. *CEB-FIP Model Code for Concrete Structures*, 3rd Edition, Comité Euro-International du Béton/Fédération Internationale de la Précontrainte, Paris, 1978, 471 pp.

17. Chen, X., and Lind, N. C., "A New Method of Fast Probability Integration," *Paper* No. 171, Solid Mechanics Division, University of Waterloo, June 1982, 17 pp.

18. Cinlar, Erhan; Bažant, Zdeněk P.; and Osman, El Mamoun, "Stochastic Process for Extrapolating Concrete Creep," *Proceedings*, ASCE, V. 103, EM6, Dec. 1977, pp. 1069-1088.

19. Cornelissen, H., "Creep of Concrete—A Stochastic Quantity," PhD thesis, Technical University, Eindhoven, 1979. (in Dutch, with extended English summary)

20. Cornelissen, H., "Creep of Concrete—A Stochastic Quantity," *Preprints*, International Symposium on Fundamental Research on Creep and Shrinkage of Concretes, Lausanne, Sept. 1980, pp. 95-110.

21. Corotis, Ross B., "Statistical Analysis of Continuous Data Records," *Proceedings*, ASCE, V. 100, TE1, Feb. 1974, pp. 195-206.

22. DeGroot, M. H., *Optimal Statistical Decisions*, McGraw-Hill Book Co., New York, 1970, 489 pp.

23. Hanson, J. A., "A Ten-Year Study of Creep Properties of Concrete," *Concrete Laboratory Report* No. Sp-38, U. S. Bureau of Reclamation, Denver, July 1958.

24. Harboe, E. M., et al., "A Comparison of the Instantaneous and the Sustained Modulus of Elasticity of Concrete," *Concrete Laboratory Report* No. C-854, U. S. Bureau of Reclamation, Denver, Mar. 1958.

25. Kay, J. N., "A Bayesian Approach to Soils Engineering Problems," PhD dissertation, Northwestern University, Evanston, June 1971, 89 pp.

26. Lind, Niels C., "Optimal Reliability Analysis of Fast Convolution," *Proceedings*, ASCE, V. 105, EM3, June 1979, pp. 447-452.

27. MacFarland, W. J., "Bayes' Equation, Reliability, and Multiple Hypothesis Testing," *IEEE Transactions. Reliability*, V. R-21, No. 3, 1972, pp. 136-147.

28. McDonald, J. E., "Time-Dependent Deformation of Concrete Under Multiaxial Stress Conditions," *Technical Report* No. C-75-4, U. S. Army Engineer Waterways Experiment Station, Vicksburg, Oct. 1975.

29. Pratt, J. W.; Raiffa, H.; and Schlaifer, R., *Introduction to Statistical Decision Theory*, McGraw-Hill Book Co., New York, 1965, 985 pp.

30. Raiffa, H., and Schlaifer, R., *Applied Statistical Decision Theory*, Harvard University Press, Cambridge, 1961, 356 pp.

31. Rao, H. G., "Application of Bayesian Analysis for Wind Energy Site Evaluation," PhD dissertation, Northwestern University, Evanston, June 1981.

32. Reinhardt, H. W.; Pat, M. G. M., and Wittmann, F. H., "Variability of Creep and Shrinkage of Concrete," *Preprints*, International Symposium on Fundamental Research on Creep and Shrinkage of Concrete, Lausanne, Sept. 1980, pp. 79-93.

33. Rostasy, F. S.; Teichen, K.-Th.; and Engelke, H., "Beitrag zur Klärung der Zusammenhanges von Kriechen und Relaxation bei Normal-beton," *Heft* 139, Amtliche Forschungs-und Materialprüfungsanstalt für das Bauwesen, Otto-Graf-Institut, Universität Stuttgart, Strassenbau und Strassenverkechrstechnik, 1972.

34. Stroud, A. H., and Secrest, O., *Gaussian Quadrature Formulas*, Prentice-Hall, Inc., Englewood Cliffs, 1966, 374 pp.

35. Tang, Wilson H., "Probabilistic Evaluation of Penetration Resistance," *Proceedings*, ASCE, V. 105, GT10, Oct. 1979, pp. 1173-1191.

36. Tang, W. H., "Updating Reliability of Offshore Structures," *Probabilistic Methods in Structural Engineering*, American Society of Civil Engineers, New York, 1981, pp. 139-156.

37. Tang, W. H.; Michols, K.; and Kjekstad, O., "Probabilistic Stability Analysis of Gravity Platform," *Proceedings*, 3rd International Conference on Structural Safety and Reliability, Trondheim, June 1981, in *Structural Safety and Reliability*, T. Moan and M. Shinozuka, editors, Elsevier Scientific Publishing Company, New York, 1981, pp. 197-210.

38. Veneziano, D., and Faccioli, E., "Bayesian Design of Optimal Experiments for the Estimation of Soil Properties," *Proceedings*, 2nd International Conference on Application of Probability and Statistics to Soil and Structural Engineering, Aachen, Sept. 1975, pp. 191-214.

39. von Mises, R., *Probabilities, Statistics and Truth*, Dover Publications, New York, 1981, 244 pp.

40. Winkler, R. L., *Introduction to Bayesian Inference and Design*, Holt, Rinehart and Winston, New York, 1972, 563 pp.