Creep in Continuous Beam Built Span-by-Span

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Abstract: The long-term variation of bending moment distribution caused by creep in a continuous beam erected sequentially in span-length sections with overhangs is analyzed. A linear aging creep law is assumed. The problem involves changes of the structural system from statically determinate to indeterminate, a gradual increase in the number of redundant moments, and age differences between various cross sections. A system of Volterra integral equations for the history of support bending moments is derived. By considering infinitely many equal spans, which is good enough whenever there are more than a few spans, one can take advantage of a periodicity condition for the construction cycle; this reduces the problem to a single equation which is of a novel type in creep theory—an integral-difference equation involving time lags in the integrated unknown. The solution exhibits sudden jumps at times equal to multiples of the construction cycle. The jumps decay with time roughly in a geometric progression. Approximation of time integrals with finite sums yields a large system of simultaneous linear algebraic equations. These equations cannot be solved recurrently, step-by-step. By solving the large equation system with a computer, the effects of the duration of the construction cycle, of concrete age at assembly of span from segments, and of the overhang length are studied numerically.

Introduction

The stresses and deflections of modern concrete structures which are built sequentially from repetitively produced parts can be profoundly affected by the construction procedure. Neglect of its effect can lead to excessive cracking and deflections and, thus, endanger the serviceability. This paper focuses on one important problem of this sort, i.e., a multi-span continuous bridge beam of many spans which is erected one span after another—an efficient construction procedure which has recently become commonplace thoughout the world. An outstanding example is the Long Key Bridge, a prestressed concrete box girder in Florida which consists of 101 spans (15,20).

In contrast to metallic structures, the analysis of sequential erection must take into account creep. The creep problem is burdened by two complicating aspects: (1) The structural system changes from statically determinate to indeterminate, which occurs not once but repeatedly as the number of redundant bending moments grows with time; and (2) the concretes of individual sections and spans are of different ages and are loaded at different times. Creep problems which involve both these ingredients seem not to have been analyzed so far, and we will develop a general method for this purpose.

The analysis which follows is detailed but, admittedly, too complicated for use in regular design. Our aim here is principally to gain an understanding of the problem. The stress redistribution or excessive creep deflections which we calculate do not usually endanger the safety of the structure, i.e., they do not reduce the collapse load. Together with the cracking they cause, they may, however, severely shorten the life of the structure.

An exact solution of the problem based on the usual assumptions of linearity of creep and superposition leads to a system of integral equations in time. When there are many spans, the number of equations becomes a formidable obstacle. We propose here, as a novel idea, to avoid it by taking advantage of the periodicity due to sequential construction. If there are many spans (from one dilation joint to another), all spans except the terminal ones can be approximately treated as spans in a continuous beam of infinitely many spans. The histories of the support bending moments must then be the same, one lagging behind the other by the duration of the construction cycle. This periodicity condition reduces the number of integral equations to one. At the same time, however, it makes the type of equation more difficult mathematically, leading to an integral-difference equation rather than just an integral equation. Although no general theory of such equations seems to exist in mathematics, we will be able to solve this equation by the brute force approach, i.e., reduction to a large system of algebraic equations solved by a computer.

Numerous problems of stress redistributions due to a change in the structural system or to differences in concrete age have been solved since 1940 (see the surveys in Refs. 4, 7, 13, 14, and 21). The method of linear creep analysis of aging concrete structures of nonuniform age is, in principle, well known, and computer programs which can handle various problems of this type exist, e.g., NONSAP-C programs by Anderson, et al. (1,2) and CREEP80 by Bažant, et al. (12), programs by Huet (16), by Schade and Haas (24), by Kang and Scordelis (17,18), by VanZyl and Scordelis (26,27) by Marshall and Gamble (19), and others. The use of these general programs for the particular problem at hand is, however, unwieldy since it involves a far larger number of unknowns than the present approach and requires an extensive, tedious input. One contribution of the present work is a systematic mathematical characterization of the variables defining the construction sequence, changes in constraints and age differences, and this contribution can be applied also with the general solutions mentioned previously. These solutions, however, cannot handle the aforementioned periodicity condition since it destroys the recursive nature of the usual step-by-step algorithms of integration in time. Development of a method which can handle the periodicity condition is intended to be the main contribution of this study.

Aside from the recently developed accurate computer solutions that permit the use of a rather general, or a completely general, compliance function for creep, there exist in the literature various approximate solutions (25) involving either an approximate solution method, or an approximate form of the compliance function. The error of these solutions with respect to the accurate computer solutions (based on same initial assumptions) was not determined, however, for our problem. In the course of the work on the present paper it has been found that for the cases where the structural system changes periodically, these approxi-

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mate solutions (e.g., the effective modulus method and the age-adjusted effective modulus method, or Trost's method) appear to yield unacceptably large errors compared to an accurate solution for the same initial assumptions of analysis, such as the solution presented here.

Mathematical Formulation of Problem

The code formulations on standard recommendations for creep analysis of concrete structures (22,23) are presently all based on the assumption that the creep strain depends linearly on the applied stress and the principle of superposition is valid with regard to changes of stress in time. This assumption agrees very well with the test data (3,4,14) when: (1) Stress magnitudes are within the service stress range; (2) no large sudden strain decrease occurs; (3) the moisture content does not change significantly; and (4) no significant cracking occurs. In practical situations such as ours, these assumptions, except the first one, are not satisfied too well, but due to unavailability of practical nonlinear solution methods these assumptions are still used as a crude approximation. This seems acceptable for the mean behavior of the cross section as a whole, yielding realistic values of the changes in deflections and bending moments, but the stress distributions obtained from such analysis should be considered strictly as nominal, not representing the actual stress.

Having adopted the principle of superposition, the creep properties are fully defined by the compliance function, $J(t,t')$ (also called the creep function), which represents the normal strain in concrete at age t caused by a unit sustained uniaxial stress acting since age t'. It is convenient to include both the elastic and creep strains in this function, and so $1/J(t',t') = E/(t')$ = Young's elastic modulus at age t'. Since the subsequent analysis is valid for any form of $J(t,t')$, we do not need at this point to specify any particular formula for $J(t,t')$; we will need it only at the end, for the purpose of numerical evaluation (Eq. 26). The fact that our analysis is applicable to any $J(t,t')$ allows one to introduce creep properties exactly as measured in tests, which avoids the error due to approximating the test data by some of the crude simplified formulas embodied in various existing codes or standard recommendations.

The strain history produced by a constant stress, $d\sigma(t')$, applied at age t' is, due to linearity, $\epsilon(t) = J(t,t')d\sigma(t')$. Then, considering a general uniaxial stress history, $\sigma(t')$, as a sum of infinitesimal stress increments, $d\sigma(t')$, and summing the strains produced by all these increments, we obtain the following well-known general uniaxial stress-strain relation (3,4):

$$\epsilon(t) = \int_0^t J(t,t')d\sigma(t') \quad (1)$$

Mathematically, this integral represents the Stieltjes integral which is valid even for a discontinuous variation of $\sigma(t')$. When $\sigma(t')$ is continuous and differentiable, we may obtain the usual (Riemann) integral by substituting $d\sigma(t') = [d\sigma(t')/dt']dt'$. When the stress history consists of sudden finite jumps, $\Delta\sigma_r$, at times t_r ($r = 1,2, \ldots N$), then the Stieltjes integral (Eq. 1) yields $\epsilon(t) = \Sigma_r J(t,t_r)\Delta\sigma_r$. We omit in Eq. 1 the shrinkage and

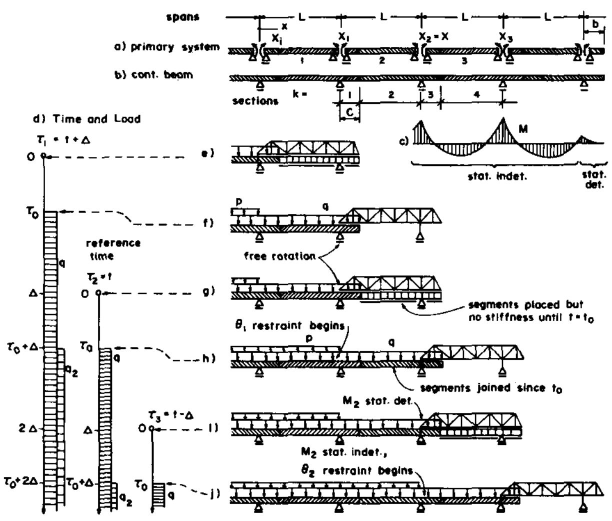

FIG. 1.—Sequential Construction Scheme, Time and Age, and Redundant Moment

thermal expansion terms since we do not plan to study their effects here.

Consider now a continuous beam shown in Fig. 1(*b*), in which we introduce as the statically indeterminate forces the bending moments on top of the supports, X_i ($i = \ldots 1, 2, 3 \ldots$) (Fig. 1(*a*)). According to the principle of virtual work, the deformation (rotation) in the sense of X_i on the primary system (Fig. 1(*a*)) may be calculated as

$$\delta_{X_i}(t) = \int_{(x)} \bar{M}_i(x)\kappa(x,t)dx \quad (2)$$

in which x = length coordinate; $\bar{M}_i(x)$ = bending moment on the primary system due to $X_i = 1$; κ = curvature of the beam; and t = reference time. The differences in concrete age may be characterized by Δ_x = given age of concrete at location x when $t = 0$. Then $t = -\Delta_x$ is the reference time when the concrete at location x is cast, and $t + \Delta_x$ = the local concrete age when the reference time is t. According to the creep law (Eq. 1):

$$\kappa(x,t) = \frac{1}{I(x)} \int_{-\Delta_x}^{t} J_x(t + \Delta_x, t' + \Delta_x)dM(x,t') \quad (3)$$

in which $I(x)$ = centroidal moment of inertia of the cross section at location x; and $M(x,t')$ = bending moment in the continuous beam at reference time t' and location x, and $dM(x,t') = [dM(x,t')/dt']dt'$ if the variation of M is continuous. We append subscript x to J to indicate possible variation of creep properties, not just age, along the beam. However, we will not consider this possibility in our numerical computations.

Since Eq. 3 is fundamental for the subsequent analysis it may be helpful to briefly review how it may be derived. According to the principle of virtual work, $\kappa(x,t) = \int_{(x)} \int_{(z)} \bar{\sigma}\epsilon(t)$ in which z = cross-sectional depth, $b = b(z)$ = its width at level z, $\bar{\sigma} = z/I$ = normal stress due to a unit bending moment, I = centroidal moment of inertia of the cross section.

Substituting here $\epsilon(t)$ according to Eq. 1 in which $d\sigma(t') = dM(x,t')z/I$, and noting that $\int_{(z)} z^2 bdz = I$, one then obtains Eq. 3.

Although the case of arbitrary age variation along the beam, including a continuous one, would not be much more difficult, we now assume that the beam may be subdivided in sections $k = 1, 2, 3, \ldots$ for each of which the concrete age is uniform, i.e., $\Delta_x = \Delta_k = \text{const}$. Eqs. 2–3 then provide

$$\delta_{X_i}(t) = \sum_k \int_{(k)} \frac{\bar{M}_i(x)}{I(x)} \int_{t'=-\Delta_k}^{t'=t} J(t + \Delta_k, t' + \Delta_k) dM(x,t') dx \quad (4)$$

The history of M must now be described taking into account the change of the structural system from a statically determinate one to a statically indeterminate one, according to the particular construction sequence used. We assume that the bridge is erected with the help of a traveler truss supported directly on the piers. As is now becoming popular for box girder bridges (15,20), the beam is assembled of short precast segments which are placed next to each other and temporarily supported by the truss (Fig. 1(*e,g,i*)). The precast segments are joined by prestressing tendons when their age is τ_0. At the same instant, the traveler truss is moved ahead to the next span and the newly installed segments (sections $k = 2, 3$ in Fig. 1(*b*)) start carrying the load. The newly built overhand section, $k = 3$, starts acting as a cantilever, which is a statically determinate situation, and the previously built overhang section, $k = 1$, together with the newly built section, $k = 2$, of the span, begins to act as a statically indeterminate beam span. Thus, the reference times at which sections $k = 1, 2, 3, 4$, defined in Fig. 1(*b*), begin carrying load are $t = \tau_0 - \Delta_k$ in which

$$\Delta_1 = \Delta; \quad \Delta_2 = 0; \quad \Delta_3 = 0; \quad \Delta_4 = -\Delta \quad (5)$$

and Δ = duration of the cycle of construction. The times at which these sections become statically indeterminate are $t = \tau_0 - \Delta'_k$ in which

$$\Delta'_1 = 0; \quad \Delta'_2 = 0; \quad \Delta'_3 = -\Delta; \quad \Delta'_4 = -\Delta \quad (6)$$

Sections $k = 2, 4$ are statically indeterminate from the moment they receive the first load. Section $k = 1$ acts as a statically determinate cantilever from $t' = \tau_0 - \Delta$ to $t' = \tau_0$, and section $k = 3$ from $t' = \tau_0$ to $t' = \tau_0 + \Delta$ (i.e., each of them for a period of duration Δ). In general, the statically determinate bending moments may be written as

$$M(x,t') = \Sigma M_j^C(x) q_j(t') \quad (\text{for} \quad \tau_0 - \Delta_k \le t' < \tau_0 - \Delta'_k) \quad (7)$$

in which q_j = load parameters; and $M_j^C(x)$ = the bending moments in the overhang corresponding to $q_j = 1$ (for uniform load, $M_j^C(x) = C_j^2/2$ in which C_j = distance from the end of overhang). We should also note that during the statically determinate period, $X_i(t)$ is known and may be written as

$$\text{for} \quad t_1^i - \Delta \le t' < t_1^i: \quad X_i(t) = q_j(t) M_j^C(x) \quad (8)$$

in which $t_1^i = \tau_0 + (i - 1)\Delta$ = time when X_i becomes statically indeterminate and M_j^C = support moment from the overhang, and $M_j^C = -C^2/2$ for uniform load $q_j = 1$.

For $t' \ge \tau_0 - \Delta'_k$, all sections are statically indeterminate and their bending moments may be expressed as

$$M(x,t') = \sum_j \bar{M}_j(x) X_j(t') + \sum_m M_m^L(x) q_m(t') \quad (\text{for} \quad t' \ge \tau_0 - \Delta'_k) \quad (9)$$

in which $\bar{M}_j(x)$ = bending moments in the primary system. For $X_j = 1$, $q_m(t')$ = load parameters at reference time t' ($m = 1, 2, \ldots$), and $M_m^L(x)$ = bending moments in the primary system for $q_m = 1$ (Fig. 1(*a*)).

Equations 8–9 may be combined into one expression for the bending moment history of all sections $k = 1, 2, 3, 4$ for all times:

$$M(x,t') = \{H[t' - (\tau_0 - \Delta_k)] - H[t' - (\tau_0 - \Delta'_k)]\} \sum_j M_j^C(x) q_j(t')$$

$$+ H[t' - (\tau_0 - \Delta'_k)] \left[\sum_j \bar{M}_j(x) X_j(t') + \sum_m M_m^L(x) q_m(t') \right] \quad (10)$$

in which $H[t]$ = Heaviside step function. Note that the term $\{ \ldots \}$ is nonzero (one) only for a period of duration either Δ (for $k = 1, 3$) or 0 (for $k = 2, 4$).

Note that when the sections are cast in situ monolithically on a formwork supported by the traveler truss, our preceding equations would require modification to take into account the stiffness of monolithic concrete with the supporting truss before age τ_0, and creep due to M after age τ_0. We will not pursue this alternative, but the long time behavior should not be very different from our case of precast assembled segments.

Writing the differential of Eq. 10, substituting it into Eq. 4, and rearranging, we obtain

$$\delta_{X_i}(t) = \sum_k \left[\sum_j \theta_{ij}^k(t) + \phi_i^k(t) + \phi_i^{kC}(t) \right] \quad (11)$$

$$\text{in which} \quad \theta_{ij}^k(t) = f_{ij}^k \int_{t'=-\Delta_k}^{t'=t} J(t + \Delta_k, t' + \Delta_k) H[t' - (\tau_0 - \Delta'_k)] dX_j(t') \quad (12)$$

$$\phi_i^k(t) = \sum_m a_{im}^k \int_{t'=-\Delta_k}^{t'=t} J(t + \Delta_k, t' + \Delta_k) H[t' - (\tau_0 - \Delta'_k)] dq_m(t') \quad (13)$$

$$\phi_i^{kC}(t) = \sum_j a_{ij}^{kC} \int_{t'=-\Delta_k}^{t'=t} J(t + \Delta_k, t' + \Delta_k) \{H[t' - (\tau_0 - \Delta_k)]$$

$$- H[t' - (\tau_0 - \Delta'_k)]\} dq_j(t') \quad (14)$$

$$f_{ij}^k = \int_{(k)} \frac{\bar{M}_i(x)\bar{M}_j(x)}{I(x)} dx; \quad a_{im}^k = \int_{(k)} \frac{\bar{M}_i(x) M_m^L(x)}{I(x)} dx;$$

$$a_{ij}^{kC} = \int_{(k)} \frac{\bar{M}_i(x) M_j^C(x)}{I(x)} dx \quad (15)$$

Here, f_{ij}^k and a_{im}^k represent the flexibility coefficients of the primary sys-

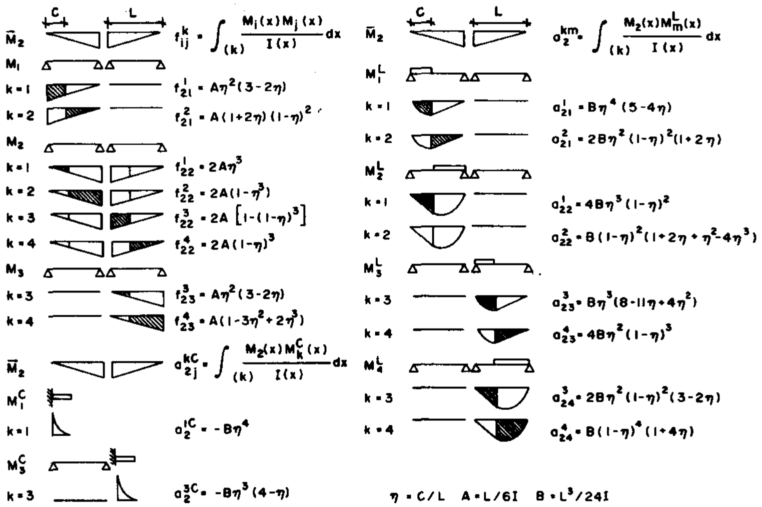

FIG. 2.—List of Flexibility Coefficients

tem; and a_{ij}^{kC} the flexibility coefficient due to the overhang (Fig. 2).

The compatibility conditions, which express the fact that, after reference time $t_1^i = \tau_0 + (i-1)\Delta$ (Fig. 1(*d*)), the change of deformation (rotation) of the primary system in the sense of X_i must be zero, may now be written as

$$\delta_{X_i}(t) - \delta_{X_i}(t_1^i) = \delta_{X_i}^0 \quad \text{(16)}$$

in which $\delta_{X_i}^0$ = const. = deformation enforced at time $t = t_1^i$. In our numerical calculations we will consider $\delta_{X_i}^0 = 0$. However, $\delta_{X_i}^0$ could be used to allow for jacking up of the support at the instant of prestressing ($t = t_1^i$) in order to adjust the deflections or bending moments. If the i-th support is jacked up by distance u_i, we have $\delta_{X_i}^1 = u_i/L$ in which L = span length.

Substitution of Eqs. 12–15 into Eq. 11, and of Eq. 11 into Eq. 16, results in a system of Volterra's integral equations for the history of $X_i(t)$ after time t_1^i. The method of numerical solution is well known (3,14) and can be implemented with a computer. However, the problem is "messy"—many imput data, extensive output, and no generally applicable results.

For a beam of many equal spans we may, however, expect that the histories, $X_i(t)$, for the interior supports, except those of the terminal spans, will be nearly the same, one lagging after the other by the duration, Δ, of the cycle. So we may assume $X_{i+1}(t') = X_i(t' - \Delta)$. Denoting $X_2(t') = X(t')$ (Fig. 1), and realizing that for the statically determinate period $X_i(t')$ is defined according to Eq. 8, we may write

$$\text{for} \quad t' \geq \tau_0 - \Delta: \quad X_1(t') = X(t' + \Delta); \quad \text{for} \quad t' \geq \tau_0: \quad X_2(t') = X(t');$$
$$\text{for} \quad t' \geq \tau_0 + \Delta: \quad X_3(t') = X(t' - \Delta) \quad \text{(17)}$$

These conditions, which represent periodicity conditions, are exact for a beam of infinitely many equal spans. They reduce the number of unknowns, and of integral equations, to one. Considering $i = 2$, making the substitution according to Eq. 16, and limiting consideration to one distributed uniform load, q, that represents the beam's own weight (Fig. 1), we may reduce Eq. 16 with Eqs. 11–15 to the following equation:

$$\left[\int_{t'=\tau_0^{(-)}}^{t'=t} F(t,t')dX(t'+\Delta) + \int_{t'=\tau_0+\Delta^{(-)}}^{t'=t} G(t,t'dX(t') + \int_{t'=\tau_0+2\Delta^{(-)}}^{t'=t} H(t,t')dX(t'-\Delta)\right] + f(t) = 0 \quad \text{(18)}$$

Here the label $(-)$ on Δ indicates that the integration should begin just before time $t' = \tau_0 + \Delta$ to include the step of X at this time, and $F(t,t')$, $G(t,t')$, $H(t,t')$ are functions given in Appendix I. Note that the values of $X(t' - \Delta)$ from $t' = \tau_0 + \Delta$ to $t' = \tau_0 + 2\Delta$ are known in advance, being equal to the statically determinate moment from the overhang, Eq. 8.

Equation 18 is not just an integral equation, but due to the appearance of $X(t' \pm \Delta)$ it may be characterized as an integral-difference equation. As for a general theory of such equations, nothing seems to be available in the mathematics literature. The equation can, however, be solved numerically.

Since, in a continuous beam, a change in one X_j carries over to all other X_i, we must expect an abrupt jump in X_i (Fig. 3(*b*)) whenever load q is applied at the frontal span, which occurs repeatedly at times $t' = \tau_0, \tau_0 + \Delta, \tau_0 + 2\Delta, \tau_0 + 3\Delta, \ldots$. These jumps decrease as the frontal span moves farther away, and since the elastic (sudden) changes of the support moment spread along our continuous beam (Fig. 3(*c*)) roughly as $\Delta X_{i-1} \simeq -0.268\, \Delta X_i$ (Appendix I), the jumps in $X(t')$ decay with time approximately as a geometric progression with quotient -0.268, i.e., as

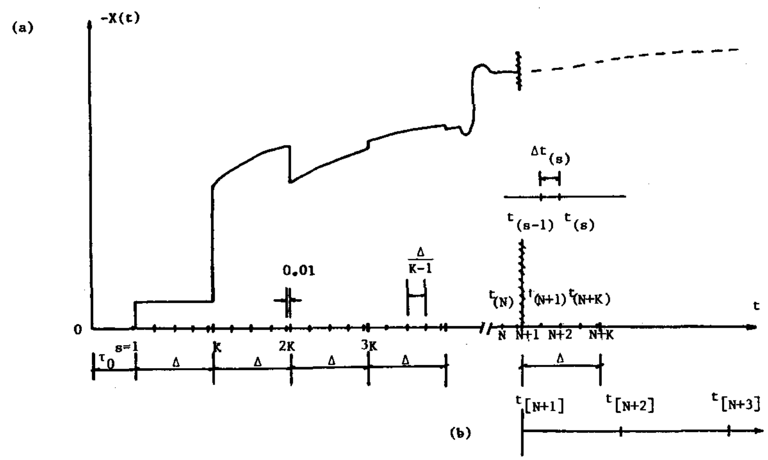

FIG. 3.—(*a*, *b*) Support Moment History and Time Divisions

$\Delta X_{\langle J+1 \rangle} \simeq -0.268\, \Delta X_{\langle J \rangle}$ in which subscript $\langle J \rangle$ refers to jump at time $t = \tau_0 + J\Delta$. ($J = 1, 2, 3, \ldots$.)

Numerical Solution

We subdivide reference time, t, by discrete times $t_{(r)}$ ($r = 1, 2, 3, \ldots$) into time steps $\Delta t_{(r+1)} = t_{(r+1)} - t_{(r)}$. In view of the sudden jumps in X at intervals Δ, it is convenient to subdivide each cycle, Δ, into the same number of steps, K. We set $t_{(1)} = \tau_0$ and the sudden jumps then occur at $t_{(1)}, t_{(1+K)}, t_{(1+2K)}, \ldots$ or $t_{(1+JK)}$ in which J = number of cycles ($J = 0, 1, 2, 3, \ldots$).

To be able to capture the sudden jumps in X, the first step of each cycle should be of small duration, i.e., $\Delta t_{JK+1} = 0.01$ ($J = 1, 2, \ldots$). The remaining steps in the cycle are chosen as constant, i.e., $\Delta t_s = \Delta/(K-1)$ Fig. 3(b)). Accordingly, the discrete times are generated as

$$\text{for } r = 1: \quad t_{(1)} = \tau_0; \quad \text{for } r = 1 + JK: \quad t_{(r+1)} = \tau_{(r)} + 0.01 \ (J = 1,2 \ldots);$$

$$\text{for other } r: \quad t_{(r+1)} = t_{(r)} + \frac{\Delta}{(K-1)} \qquad (19)$$

The integrals in Eq. 18 may now be approximated by finite sums using the trapezoidal rule, the accuracy of which is of the order of Δt^2. This yields

$$\sum_{s=1}^{r-1} F(\bar{t}_{[r]}, t_{(s)}) \Delta X_{(s+K)} + \sum_{s=K+1}^{r-1} G\{\bar{t}_{[r]}, t_{(s)}\} \Delta X_{(s)}$$

$$+ \sum_{s=2K+1}^{r-1} H\{\bar{t}_{[r]}, t_{(s)}\} \Delta X_{(s-K)} + f(\bar{t}_{[r]}) = 0$$

$$\text{in which} \quad \Delta X_{(s+1)} = X_{(s+1)} - X_{(s)}; \quad \bar{t}_{[s]} = \frac{t_{(s)} + t_{(s+1)}}{2} \qquad (20)$$

and $X_{(s)} = X[t_{(s)}]$. To distinguish between X_i and $X_{(s)}$ we place the time subscripts in parentheses.

Equation 20 represents an equation system for the values of $\Delta X_{(s)}$ beginning with $\Delta X_{(K+1)}$ (up to $\Delta X_{(K)}$ the values of $\Delta X_{(s)}$ are known, being statically determinate). Writing Eq. 21 for $r = (K + 2), (K + 3), \ldots (N + 1)$ we obtain $(N - K)$ equations which, however, involve N unknowns: $\Delta X_{(K+1)}, \ldots \Delta X_{(K+N)}$. The excess of unknowns is due to our use of the periodicity condition.

To be able to solve this system of equations, we must therefore add K further relations of the excess unknowns, $\Delta X_{(N+1)}, \ldots \Delta X_{(N+K)}$, to the preceding values of $\Delta X_{(s)}$. If number N is sufficiently large, we may assume that a steady pattern of time variation gets established at the end. We must observe in this regard that the sudden jumps in $\Delta X_{(s)}$ alternate between positive and negative signs, and so a similar situation is repeated after time 2Δ rather than Δ. Thus we may consider that the change of $\Delta X_{(s)}$ from t to $(t + \Delta)$ is the same as that from $(t - 2\Delta)$ to $(t - \Delta)$. So we may use $\Delta X_{(s+K)} = \Delta X_{(s-K)} + [\Delta X_{(s-K)} - \Delta X_{(s-3K)}]$ or

$$\Delta X_{(N+1)} = 2\Delta X_{(N+1-2K)} - \Delta X_{(N+1-4K)}; \quad \Delta X_{(N+2)} = 2\Delta X_{(N+2-2K)}$$

$$- \Delta X_{(N+2-4K)}; \quad \ldots; \quad \Delta X_{(N+K)} = 2\Delta X_{(N-K)} - \Delta X_{(N-3K)} \qquad (21)$$

Writing Eqs. 20 for all $r = K + 1, K + 2, \ldots N$, and including Eqs. 21, we thus get a system of $N - K$ linear algebraic equations for $N - K$ unknowns, $\Delta X_{(K+1)}, \ldots \Delta X_{(N)}$.

In contrast to the usual creep problems without a periodicity condition, this system cannot be solved recursively (step-by-step in time). Rather it must be solved simultaneously, and since the equation system is not banded, an equation solver for a square matrix must be used. This limits the number of time steps for which this method of solution is practical. Even with a large computer we can hardly go over $N = 100$, and if we use eight steps per cycle and the cycle duration is one week, we cannot get beyond $t \simeq 3$ months in this manner.

After time $T_{(N)}$, we must therefore increase the time steps. Fortunately, this can be done after only about six construction cycles because the jumps in $X(t)$ decrease at $t = \tau_0 + 7\Delta$ to about 0.0014 of their largest value, which occurs at $t = \tau_0 + \Delta$ (since $0.268^6 < 0.001$). So, using e.g., $K = 9$ steps per cycle, we may choose $t_{(N)} = t_{(JK)} = t_{(63)}$, in which case the matrix of Eqs. 20–21 is of the size 54×54, which can be handled easily. After $t_{(N)}$, we use discrete times $t_{[s]}$ which correspond to time steps $\Delta t_{[s+1]} = t_{[s+1]} - t_{[s]}$ that are kept constant in the scale of log θ, with $\theta = t - t_1$, $t_1 = \tau_0 + \Delta$; so, $(t_{[s+1]} - t_1) = (t_{[s]} - t_1)10^{1/N_d}$ in which N_d = number of steps per decade ($N_d = 16$ suffices).

A crucial point now is that, due to the jumps in $X(t)$ becoming negligible after $t_{(N)}$, the variation of $X(t)$ may be considered as smooth and slow. We may exploit it to get rid of the difference aspect of the integral equation (Eq. 18), relating $dX(t' + \Delta)/dt'$ and $dX(t' - \Delta)/dt'$ to $dX(t')/dt'$. For this purpose we may assume that, over a short interval, $X(t)$ varies similarly as the relaxation curve $R(t, t_1)$ for strain imposed at age $t_1 = \tau_0 + \Delta$ (which may be approximately obtained according to Ref. 6). Approximately $R(t, t_1) = [J(t, t_1)]^{-1}$, and so we have $dX(t)/d \ln \theta = \theta[dX(t)/dt] = A\psi(t)$ in which A = some constant, and function $\psi(t)$ is defined as

$$\psi(t) = (t - t_1) \frac{d}{dt} [J(t, t_1)]^{-1} \qquad (22)$$

Similarly, $dX(t_{[s]})/d \ln \theta = A\psi(t_{[s]})$. So, if Δt is kept constant in the log θ scale, we may introduce in Eq. 19 the approximation $\Delta X(t_{[s]} - \Delta) \simeq R_o \Delta X_{[s]}$, $\Delta X(t_{[s]} + \Delta) = R_1 \Delta X_{[s]}$ in which

$$R_0 = \frac{\psi(t_{[s]} - \Delta)}{\psi(t_{[s]})}; \quad R_1 = \frac{\psi(t_{[s]} + \Delta)}{\psi(t_{[s]})} \qquad (23)$$

Thus, instead of Eq. 20 we obtain for Eq. 18 the approximation

$$\sum_{s=1}^{N} F\{\bar{t}_{[r]}, t_{(s)}\} \Delta X_{(s+K)} + \sum_{s=K+1}^{N} G\{\bar{t}_{[r]}, t_{(s)}\} \Delta X_{(s)}$$

$$+ \sum_{s=2K+1}^{N} H\{\bar{t}_{[r]}, t_{(s)}\} \Delta X_{(s-K)} + \sum_{N+1}^{r-1} \phi_{[r,s]} \Delta X_{[s]} + f(\bar{t}_{[r]}) = 0 \qquad (24)$$

in which $\phi_{[r,s]} = R_1F(\bar{t}_{[r]}, t_{[s]}) + G(\bar{t}_{[r]}, t_{[s]}) + R_0H(\bar{t}_{[r]}, t_{[s]})\bar{t}_{[s]}$

$$= \frac{t_{[s]} + t_{[s+1]}}{2} \quad \text{(25)}$$

As a cruder but still acceptable alternative, we would assume that $X(t)$ varies from $t_{[s]} - \Delta$ to $t_{[s]} + \Delta$ as a straight line in log θ scale.

Equations 24–25 are valid only when Δt is kept constant in log $(t - t_1)$ scale after time $t_{(N+1)}$. Note that the discrete times for subdivision before $t_{(N+1)}$ and after $t_{(N+1)}$ are distinguished by parentheses (.) and brackets [.]. Especially note that the values $\Delta X_{(N+2)}$ and $\Delta X_{(N+3)}$ are the extrapolated values from the implicit solution (Eq. 21) and are not in-

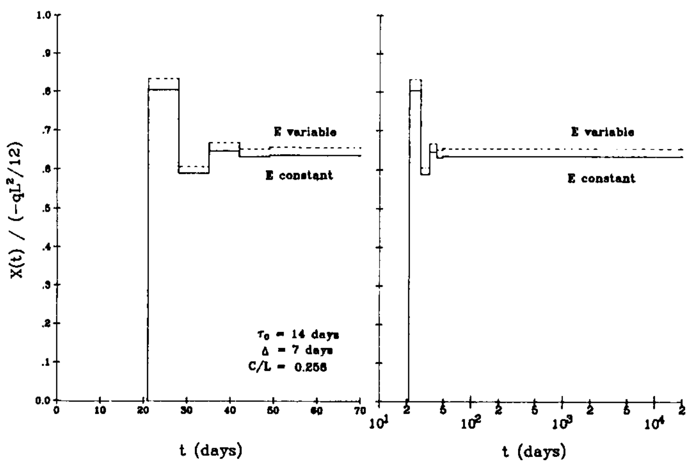

FIG. 4.—Solutions for No Creep: (*a*) E = Constant; and (*b*) E = Variable

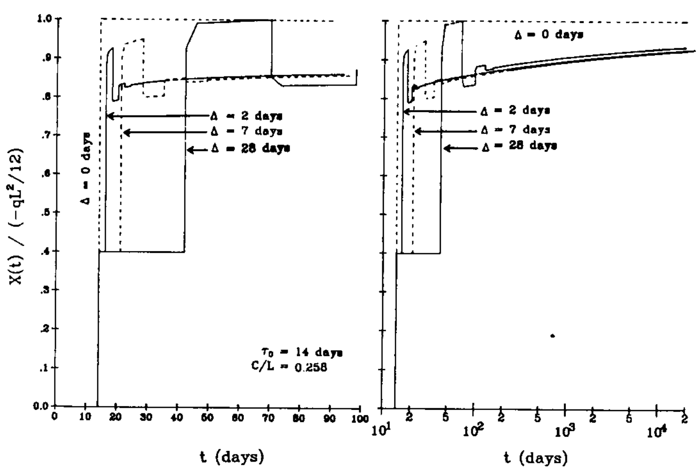

FIG. 5.—Solutions for Various Construction Cycles Δ

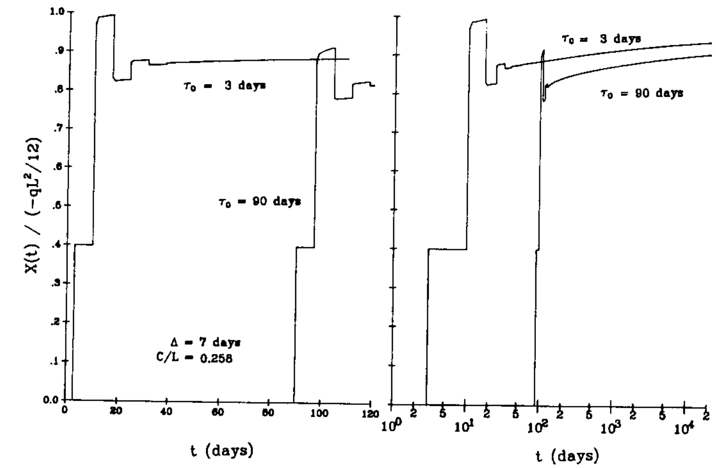

FIG. 6.—Solutions for Various Age, τ_0, at Removal of Traveler Truss

terchangeable with $\Delta X_{[N+2]}$ and $\Delta X_{[N+3]}$ which belong to a different time subdivision (Fig. 3(*b*)).

Solving Eq. 24 for $\Delta X_{[r]}$, we may carry out the calculation after $t_{(N+1)}$ recursively, step-by-step, without solving simultaneous equations. In this manner we can easily reach long times, say without t = 50 yr.

The deflections can be calculated from curvatures which can be evaluated from $\Delta X_{(i)}$ approximating Eq. 3 by a sum, and substituting M according to Eq. 9.

For numerical calculations, only the case of uniform own weight (load q, Fig. 1) has been considered, even though our general formulation is applicable to arbitrary loads, including a later applied secondary load (p in Fig. 1), as well as the loads due to prestressing, represented by the anchorage reactions and the radial forces from curved tendons. The compliance function was assumed in the form of the log-double power law (5):

$$J(t,t') = \frac{1}{E_0} + \frac{\phi_0}{E_0} \ln [1 + \phi_1(t'^{-m} + \alpha)(t - t')^n] \quad \text{(26)}$$

with E_0 = 8.13 × 10^6 psi (564 MPa); $\phi_0 = 0.603$; $\phi_1 = 68.4$; $m = 1.106$; $n = 0.342$; and $\alpha = 3.6 \times 10^{-3}$. This law represents a small improvement over the double power law (9–11); the final slopes of the creep curves at long times are represented more realistically. For short times, $t - t'$, this law is essentially identical to the double power law.

The histories of the support bending moment, $X(t)$, are plotted in Figs. 4–6 for various values of construction cycle, Δ, age, τ_0, and overhang length, C. For comparison, the case of no creep and variable elastic modulus, $E(t') = 1/J(t' + 0.1, t')$ (from Eq. 26), and the case of no creep and constant elastic modulus, $J(t,t') = 1/E_0$, are also plotted in Fig. 4. The solution is given in terms of the ratio $X(t)/X_\infty$ in which $X_\infty = -qL^2/12$ = support moment for an elastic monolithic beam loaded after construction = theoretical asymptotic value of $X(t)$ in case $J(t,t')$ would tend to ∞. Figure 5 also shows the solution for simultaneous construction of all spans, $\Delta = 0$, for which the algorithm in Eqs. 20–21 does not apply; however, the algorithm in Eqs. 24–25 can be started from the moment of first loading.

The computer program by which the present numerical results have been obtained is listed in FORTRAN IV in Ref. 8, in which the input and output are also fully described. The input consists of six material parameters, E_0, n, m, α, ϕ_1, ϕ_0, three structure characteristics, I, L, and C/L and two construction time sequence parameters, τ_0 and Δ. Alternatively, the preceding material parameters can be replaced by user's subroutine to calculate $J(t,t')$ (of any type). The output consists of tables and graphs of $X(t)$ as a function of time t. For details, Ref. 27 is available on request.

Summary and Conclusions

The stress history in a continuous beam that is erected sequentially in span-length sections which have an overhang beyond the supports and are assembled on a traveler truss from precast segments made monolithic by prestressing has been analyzed. A linear aging integral-type creep law is assumed for concrete. The problem is complicated by two aspects: (1) The structural system changes from statically determinate to indeterminate, and this happens not once but repeatedly, the number of redundant bending moments growing with time; and (2) the concretes of individual sections are of different ages and are loaded at different times. A system of Volterra integral equations for the history of support bending moments is derived. The conclusions are:

1. By considering a periodicity condition due to construction cycle for the limiting case of a beam of infinitely many spans, which represents a good approximation when there are many spans, the problem may be reduced to a single integral equation for the history of one unknown support bending moment. This equation is, however, of a new type, not yet obtained in concrete creep theory, i.e., an integral-difference equation involving different time lags in the integrated unknown.
2. The new integral-difference equation can be solved by approximating integrals with sums and assuming a smooth extrapolation of the solution beyond the chosen final time. This leads to a system of simultaneous linear algebraic equations, which cannot be solved recursively, step-by-step.
3. The moment history exhibits periodic sudden jumps, the period being the construction cycle. These jumps decay roughly as a geometric progression of quotient (−0.268) and become negligible after about six cycles. Therefore, a smooth variation is later approached. This allows approximating the long-term behavior with a single Volterra integral equation which now can be solved recursively, step-by-step.
4. The numerical results may be interpreted in relation to the elastic support moment, X_{es}, and elastic midspan moment, X_{em}, for monolithic construction. Numerical solutions (Figs. 5–7) show that the effect of speeding up the construction cycle from $\Delta = 28$ days to $\Delta = 2$ days is to decrease the early maximum support moment by 8% of X_{es}, and increase the early maximum midspan moment by 16% of X_{em}, while the effect on the long-term values is negligible. The effect of decreasing concrete age at time of assembly of the span from 90 days to 3 days is small; a change of only about 5% of X_{es} in the final value and the minimum after second cycle, and 10% of X_{ms} in the final value of midspan moment. The effect of increasing the overhang length (Fig. 7) from 0 to $C = 0.258L$ is to increase both the maximum support moment and the moment at $t = 2 \times 10^4$ days by about 8% of the elastic support bending moment for monolithic construction. This causes the maximum (midspan) bending moment to increase by about 30% of X_{ms}. However, the effect of overhang length of the minimum support moment after the second cycle (for which the midspan moment is maximum) is more significant, the change being about 15%.
5. As expected, the solution tends to approach with increasing time the support bending moment for a monolithically constructed elastic continuous beam ($X_\infty = -qL^2/12$); however, at 50 yr the moment change from the initial value is only about 53% of the difference from this elastic value (X_∞). For very long periods, the solution appears to be independent of the construction sequence and is about the same as for a continuous beam erected simultaneously, a case whose solution is well known.

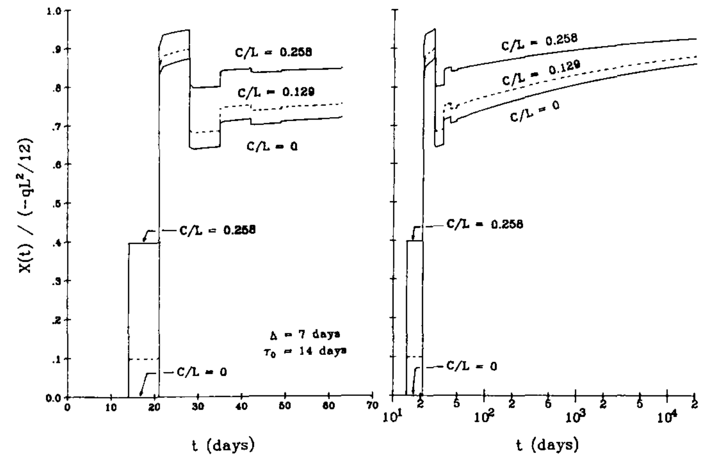

FIG. 7.—Solutions for Various Overhang-to-Span Ratios C/L

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$$\theta_{ij}^k(t) = f_{ij}^k \int_{t'=\Delta_k}^{t'=t} J(t+\Delta_\kappa,\, t'+\Delta_k)\, H[t'-(\tau_0-\Delta_k')]\, dX_j(t') \quad \text{(27)}$$

$$\theta_{21}^1(t) = f_{21}^1 \int_{t'=-\Delta}^{t'=t} J(t+\Delta,\, t'+\Delta)\, H(t'-\tau_0)\, dX_1(t')$$

$$= f_{21}^1 \int_{t'=\tau_0^-}^{t'=t} J(t+\Delta,\, t'+\Delta)\, dX(t'+\Delta) \quad \text{(28)}$$

$$\theta_{21}^2(t) = f_{21}^2 \int_{t'=0}^{t'=t} J(t,t')\, H(t'-\tau_0)\, dX_1(t') = f_{21}^2 \int_{t'=\tau_0^-}^{t'=t} J(t,t')\, dX(t'+\Delta)$$

$$\theta_{21}^3(t) = 0; \quad f_{21}^3 = 0, \quad \theta_{21}^4(t) = 0, \quad f_{21}^4 = 0 \quad \text{(29)}$$

$$\theta_{22}^1(t) = f_{22}^1 \int_{t'=\Delta}^{t'=t} J(t+\Delta,\, t'+\Delta)\, H(t'-\tau_0)\, dX_2(t')$$

$$= f_{22}^1 \int_{t'=\tau_0^-}^{t'=t} J(t+\Delta,\, t'+\Delta)\, dX(t') \quad \text{(30)}$$

$$\theta_{22}^2(t) = f_{22}^2 \int_{t'=0}^{t'=t} J(t,t')\, H(t'-\tau_0)\, dX_2(t') = f_{22}^2 \int_{t'=\tau_0^-}^{t'=t} J(t,t')\, dX(t') \quad \text{(31)}$$

$$\theta_{22}^3(t) = f_{22}^3 \int_{t'=0}^{t'=t} J(t,t')[t'-(\tau_0+\Delta)]\, dX(t') = f_{22}^3 \int_{t'=\tau_0+\Delta^-}^{t'=t} J(t,t')\, dX(t')$$

$$\theta_{23}^1(t) = 0; \quad f_{23}^1 = 0, \quad \theta_{23}^2(t) = 0; \quad f_{23}^2 = 0 \quad \text{(32)}$$

$$\theta_{22}^4(t) = f_{22}^4 \int_{t'=\Delta}^{t'=t} J(t-\Delta,\, t'-\Delta)\, H[t'-(\tau_0+\Delta)]\, dX_2(t')$$

$$= f_{22}^4 \int_{t'=\tau_0+\Delta^-}^{t'=t} J(t-\Delta,\, t'-\Delta)\, dX(t') \quad \text{(33)}$$

$$\theta_{23}^3(t) = f_{23}^3 \int_{t'=0}^{t'=t} J(t,t')\, H[t'-(\tau_0+\Delta)]\, dX_3(t')$$

$$= f_{23}^3 \int_{t'=\tau_0+\Delta^-}^{t'=t} J(t,t')\, dX(t'-\Delta) \quad \text{(34)}$$

$$\theta_{23}^4(t) = f_{23}^4 \int_{t'=\Delta}^{t'=t} J(t-\Delta,\, t'-\Delta)\, H[t'-(\tau_0+\Delta)]\, dX_3(t')$$

$$= f_{23}^4 \int_{t'=\tau_0+\Delta^-}^{t'=t} J(t-\Delta,\, t'-\Delta)\, dX(t'-\Delta) \quad \text{(35)}$$

$$\phi_i^k(t) = \sum_m a_{im}^k \int_{t'=-\Delta_k}^{t'=t} J(t+\Delta_k,\, t'+\Delta_k)\, H[t'-\tau_0-\Delta_k']\, dq_m(t') \quad \text{(36)}$$

$$\ldots + \phi_{23}^1(t) + \phi_{24}^1(t)$$

$$= a_{21}^1 \int_{t'=-\Delta}^{t'=t} J(t+\Delta,\, t'+\Delta)\, H(t'-\tau_0)\, dq_1(t')$$

$$+ a_{22}^1 \int_{t'=-\Delta}^{t'=t} J(t+\Delta,\, t'+\Delta)\, H(t'-\tau_0)\, dq_2(t') + 0 + 0$$

$$= a_{21}^1 \int_{t'=\tau_0^-}^{t'=t} J(t+\Delta,\, t'+\Delta)\, dq_1(t')$$

$$+ a_{22}^1 \int_{t'=\tau_0^-}^{t'=t} J(t+\Delta,\, t'+\Delta)\, dq_2(t') \quad (a_{23}^1 = 0,\, a_{24}^1 = 0)$$

$$= a_{21}^1 \int_{t'=\tau_0^-}^{t'=t} J(t+\Delta,\, t'+\Delta)\, H(t'-\tau_0)\, dq(t')$$

$$+ a_{22}^1 \int_{t'=\tau_0^-}^{t'=t} J(t+\Delta,\, t'+\Delta)\, H(t'-\tau_0)\, dq(t')$$

$$= a_{21}^1 qJ(t+\Delta,\, \tau_0+\Delta) + a_{22}^1 qJ(t+\Delta,\, \tau_0+\Delta) = a_2^1 qJ(t+\Delta,\, \tau_0+\Delta) \quad \text{(37)}$$

$$\phi_2^2(t) = \phi_{21}^2(t) + \phi_{22}^2(t) + \phi_{23}^2(t) + \phi_{24}^2(t)$$

$$= a_{21}^2 \int_{t'=0}^{t'=t} J(t,t')\, H(t'-\tau_0)\, dq_1(t') + a_{22}^2 \int_{t'=0}^{t'=t} J(t,t')\, H(t-\tau_0)\, dq_2(t')$$

$$= a_{21}^2 \int_{t'=\tau_0^-}^{t'=t} J(t,t')\, dq_1(t') + a_{22}^2 \int_{t'=\tau_0^-}^{t'=t} J(t,t')\, dq_2(t')$$

$$= a_{21}^2 \int_{t'=\tau_0^-}^{t'=t} J(t,t')\, H(t'-\tau_0)\, dq(t') + a_{22}^2 \int_{t'=\tau_0^-}^{t'=t} J(t,t')\, H(t'-\tau_0)\, dq(t')$$

$$= a_{21}^2 qJ(t,\tau_0) + a_{22}^2 qJ(t,\tau_0) = a_2^2 qJ(t,\tau_0) \quad (a_{23}^2 = 0,\, a_{24}^2 = 0) \quad \text{(38)}$$

$$\phi_2^3(t) = \phi_{21}^3(t) + \phi_{22}^3(t) + \phi_{23}^3(t) + \phi_{24}^3(t)$$

$$= 0 + 0 + a_{23}^3 \int_{t'=0}^{t'=t} J(t,t')\, H[t'-(\tau_0+\Delta)]\, dq_3(t')$$

$$+ a_{24}^3 \int_{t'=0}^{t'=0} J(t,t')\, H[t'-(\tau_0+\Delta)]\, dq_4(t') \quad (\text{for} \quad a_{21}^3 = 0,\, a_{22}^3 = 0)$$

$$= a_{23}^3 \int_{t'=\tau_0+\Delta^-}^{t'=t} J(t,t')\, dq_3(t') + a_{24}^3 \int_{t'=\tau_0+\Delta^-}^{t'=t} J(t,t')\, dq_4(t')$$

$$= a_{23}^3 \int_{t'=\tau_0+\Delta^-}^{t'=t} J(t,t')\, H[t'-(\tau_0+\Delta)]\, dq(t')$$

$$+ a_{24}^{3} \int_{t'=\tau_0+\Delta^-}^{t'=t} J(t,t')\, H[t'-(\tau_0+\Delta)]\,dq(t')$$

$$= a_{23}^{3} qJ(t,\tau_0+\Delta) + a_{24}^{3} qJ(t,\tau_0+\Delta) = a_2^{3} qJ(t,\tau_0+\Delta) \quad \text{(39)}$$

$$\phi_2^{4}(t) = \phi_{21}^{4}(t) + \phi_{22}^{4}(t) + \phi_{23}^{4}(t) + \phi_{24}^{4}(t)$$

$$= 0 + 0 + a_{23}^{4} \int_{t'=\Delta}^{t'=t} J(t-\Delta,\, t'-\Delta)\, H[t'-(\tau_0+\Delta)]\,dq_3(t')$$

$$+ a_{24}^{4} \int_{t'=\Delta}^{t'=t} J(t-\Delta, t'-\Delta) H\,[t'-(\tau_0+\Delta)]\,dq_4\,(t')$$

$$= a_{23}^{4} \int_{t'=\tau_0+\Delta^-}^{t'=t} J(t-\Delta, t'-\Delta)\,dq_3\,(t') + a_{24}^{4} \int_{t'=\tau_0+\Delta^-}^{t'=t} J(t-\Delta, t'-\Delta)\,dq_4\,(t')$$

$$= a_{23}^{4} \int_{t'=\tau_0+\Delta^-}^{t'=t} J(t-\Delta, t'-\Delta) H\,[t'-(\tau_0+\Delta)]\,dq(t')$$

$$+ a_{24}^{4} \int_{t'=\tau_0+\Delta^-}^{t'=t} J(t-\Delta, t'-\Delta) H\,[t'-(\tau_0+\Delta)]\,dq(t')$$

$$= a_{23}^{4} J(t-\Delta,\tau_0)q + a_{24}^{4} qJ(t-\Delta,\tau_0) = a_2^{4} qJ(t-\Delta,\tau_0) \quad \text{(40)}$$

$$\phi_i^{kC}(t) = \sum_j a_{ij}^{kC} \int_{t'=-\Delta_k}^{t'=t} J(t+\Delta_k, t'+\Delta_k)\,\{H[t'-(\tau_0-\Delta_k)]$$

$$- H[t'-(\tau_0-\Delta_k']\}\,dq_j^C(t') \quad \text{(41)}$$

$$\phi_2^{1C}(t) = a_{21}^{1C} \int_{t'=0}^{t'=t} J(t+\Delta, t'+\Delta)\{H[t'-(\tau_0-\Delta)]$$

$$- H(t'-\tau_0)\}\,dq_1^C(t') + a_{22}^{1C} \int_{t'=\Delta}^{t'=t} J(t+\Delta, t'+\Delta)\{H[t'-(\tau_0-\Delta)]$$

$$- H(t'-\tau_0)\}\,dq_2^C(t')\ (q_2^C(t)=0) + a_{23}^{1C} \int_{t'=\Delta}^{t'=t} J(t+\Delta, t'+\Delta)\{H[t'-(\tau_0-\Delta)]$$

$$- H(t'-\tau_0)\}\,dq_3^C(t)\ (a_{23}^{1C}=0) + a_{24}^{1C} \int_{t'=\Delta}^{t'=t} J(t+\Delta, t'+\Delta)\{H[t'-(\tau_0-\Delta)]$$

$$- H(t'-\tau_0)\}\,dq_4^C(t)\ (q_4^C(t)=0) = a_{21}^{1C} \int_{t'=\tau-\Delta^-}^{t'=\tau_0^+} J(t+\Delta, t'+\Delta)\,dq_1^C(t')$$

$$= a_{21}^{1C} \int_{t'=\tau_0-\Delta^-}^{t'=\tau_0^+} J(t+\Delta, t'+\Delta)\{\delta[t'-(\tau_0-\Delta)]$$

$$- \delta(t'-\tau_0)\}\,q(t')\,dt'\ (q(t')=q)$$

$$= a_{21}^{1C}\, q\,[J(t+\Delta,\tau_0) - J(t+\Delta,\tau_0+\Delta)] \quad \text{(42)}$$

$$\phi_2^{2C}(t) = \sum_j a_{2j}^{2C} \int_{t'=0}^{t'=t} J(t,t')\,[H(t'-\tau_0) - H(t'-\tau_0)]\,dq_j^C(t') = 0 \quad \text{(43)}$$

$$\phi_2^{4C}(t) = \sum_j a_{2j}^{4C} \int_{t'=\Delta}^{t'=t} J(t-\Delta, t'-\Delta)\{H[t'-(\tau_0+\Delta)]$$

$$- H(\tau_0+\Delta)\}\,dq_j^C(t') = 0 \quad \text{(44)}$$

$$\phi_2^{3C}(t) = a_{21}^{3C} \int_{t'=0}^{t'=t} J(t,t')\{H(t'-\tau_0) - H[t'-(\tau_0+\Delta)]\}\,dq_1^C(t')(a_{21}^{3C}=0)$$

$$+ a_{22}^{3C} \int_{t'=0}^{t'=t} J(t,t')\{H(t'-\tau_0) - H[t'-(\tau_0+\Delta)]\}\,dq_2^C(t')(q_2^C(t')=0)$$

$$+ a_{23}^{3C} \int_{t'=0}^{t'=t} J(t,t')\{H(t'-\tau_0) - H[t'-(\tau_0+\Delta)]\}\,dq_3^C\,(t')$$

$$+ a_{24}^{3C} \int_{t'=0}^{t'=t} J(t,t')\{H(t'-\tau_0) - H[t'-(\tau_0+\Delta)]\}\,dq_4^C(t')(q_4^C(t')=0)$$

$$= a_2^{3C} \int_{t'=\tau_0^-}^{t'=\tau_0+\Delta^+} J(t,t')\{\delta(t'-\tau_0) - \delta[t'-(\tau_0+\Delta)]\}\,q(t')\,dt'$$

$$(\text{for}\quad q(t')=q) = a_2^{3C} q[J(t,\tau_0+\Delta)] \quad \text{(45)}$$

$$t' \geq \tau_0+\Delta\!: \quad F(t,t') = f_{21}^{1} J(t+\Delta, t'+\Delta) + f_{21}^{2} J(t,t') \quad \text{(46)}$$

$$\tau_0+\Delta > t' \geq \tau_0\!: \quad F(t,t') = f_{21}^{1}\,[J(t+\Delta, t'+\Delta) - J(\tau_0+2\Delta, t'+\Delta)]$$

$$+ f_{21}^{2}\,[J(t,t') - J(\tau_0+\Delta, t')] \quad \text{(47)}$$

$$t' \geq \tau_0+\Delta\!: \quad G(t,t') = f_{22}^{1} J(t+\Delta, t'+\Delta) + (f_{22}^{2} + f_{22}^{3})\, J(t,t')$$

$$+ f_{22}^{4} J(t-\Delta, t'-\Delta) \quad \text{(48)}$$

$$t' \geq \tau_0+2\Delta\!: \quad H(t,t') = f_{23}^{3} J(t,t') + f_{23}^{4} J(t-\Delta, t'-\Delta) \quad \text{(49)}$$

$$f(t) = f_{21}^{1}\,\Delta X(\tau_0)[J\,(t+\Delta,\tau_0+\Delta) - J(\tau_0+2\Delta,\tau_0+\Delta)] + f_{22}^{3}\,\Delta X(\tau_0)\, J(t,\tau_0$$

$$+ \Delta) + f_{21}^{2}\,\Delta X\,(\tau_0)[J(t,\tau_0) - J(\tau_0+\Delta,\tau_0)] + f_{22}^{4}\,\Delta X\,(\tau_0)\, J(t-\Delta,\tau_0)$$

$$+ f_{22}^{1}\,\Delta X\,(\tau_0)[J(t+\Delta,\tau_0+\Delta) - J(\tau_0+2\Delta,\tau_0+\Delta)] + f_{23}^{3}\,\Delta X\,(\tau_0)\, J(t,\tau_0+\Delta)$$

$$+ f_{22}^{2}\,\Delta X\,(\tau_0)[J(t,\tau_0) - J(\tau_0+\Delta,\tau_0)] + f_{23}^{4}\,\Delta X(\tau_0)\, J(t-\Delta,\tau_0)$$

$$+ a_2^{1} q[J\,(t+\Delta,\tau_0+\Delta) - J(\tau_0+2\Delta,\tau_0+\Delta)] + a_2^{3} q\, J(t,\tau_0+\Delta)$$

$$+ a_2^{2} q[J(t,\tau_0) - J(\tau_0+\Delta,\tau_0)] + a_2^{4} q\, J(t-\Delta,\tau_0)$$

$$+ a_2^{1C} q[J(t+\Delta,\tau_0) - J(t+\Delta,\tau_0+\Delta) - J(\tau_0+2\Delta,\tau_0) + J(\tau_0+2\Delta,\tau_0+\Delta)]$$

$$+ a_2^{3C} q[J(t,\tau_0) - J(t,\tau_0+\Delta) - J(\tau_0+\Delta,\tau_0) + J(\tau_0+\Delta,\tau_0+\Delta)] \quad \text{.......} \quad (50)$$

Appendix II.—Sudden Moment Increments

The instantaneous deformation changes at applications of load q are elastic and, if we consider the elastic modulus as uniform, their spread along the continuous beam to the left is governed by the three-moment equation $\Delta X_{i-1} + 4\Delta X_i + \Delta X_{i+1} = 0$. The general solution of this difference equation has the form $\Delta X_i = \Delta X_1 r^{i-1}$, and substituting this into the equation we get $r^2 + 4r + 1 = 0$, from which the root $|r| < 1$ is $r = \sqrt{3} - 2 = -0.268$ or $\Delta X_{i-1} = -0.268\Delta X_i$. If we take into account the differences in the elastic modulus between the sections, the actual ratios are slightly higher than -0.268 and are not the same for all spans, but they tend to -0.268 as one moves away from the frontal span.

The first two sudden jumps in $X(t)$ in the frontal span do not follow this rule. The first jump is, for a uniform q, $\Delta X_3 = -qC^2/2$ (in which C = overhang length). The second jump follows from the relation

$$(F_{21}^1 E_1^{-1} + f_{21}^2 E_2^{-1})\,\Delta X_1 + (f_{22}^1 E_1^{-1} + f_{22}^2 E_2^{-1} + f_{22}^3 E_3^{-1}$$

$$+ f_{22}^4 E_4^{-1})\,\Delta X_2 + (f_{23}^3 E_3^{-1} + f_{23}^4 E_4^{-1}\,\Delta X_3) + a_2^3 E_3^{-1} + a_2^4 E_4^{-1} = 0 \quad \text{.......} \quad (51)$$

in which $E_1 = E(\tau_0 + 2\Delta)$, $E_2 = E_3 = E(\tau_0 + \Delta)$, $E_4 = E(\tau_0)$ [with $E(\tau) = 1/J(\tau + 0.1,\tau)$]. This relation can be solved for ΔX_2 if we substitute $\Delta X_3 = -qC^2/2$ and $\Delta X_1 = -0.268\,\Delta X_2$. For span numbers 3, 4, 5, and 15, the jump (ΔX_2) which is due to loading on the frontal span is 0.8, 0.8036, 0.8038, and 0.8038, respectively, by the slope-deflection method.

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Appendix IV.—Notation

The following symbols are used in this paper:

a_{ij}^k = deformation (rotation) in sense of X at joint i, due to curvature changes in segment k caused by $q_j = 1$;
a_i^{kC} = deformation (rotation) in sense of X at joint i, due to curvature changes in overhang segment k caused by $q = 1$ on overhang;
C = overhang length (Fig. 1);
f_{ij}^k = deformation (rotation) in sense of X at joint i, due to curvature changes in segment k caused by $X = 1$ at joint j;
$H(\ldots)$ = Heaviside function;
$I(x)$ = centroidal moment of inertia of cross section at location x;
$J(t,t')$ = given compliance function of concrete = strain at concrete age t, caused by unit sustained stress acting since age t';
K = number of steps per cycle;
L = span length = constant (Fig. 1);
$M(x,t')$ = bending moment in beam at reference time, t', and location x;
$\bar{M}_j(x)$ = bending moments in primary system caused by $X_j = 1$;
$q_i(t)$ = uniform distributed load in span i;
$R(t,t')$ = relaxation function = stress at age t caused by unit constant strain introduced at age t';
t, t' = reference time for whole structure;
t_1 = $\tau_0 + \Delta$;
$X_i(t)$ = bending moment at time t at joint i;
X_∞ = $-qL^2/12$;
x = length coordinate of beam;
Δ = duration of cycle of construction (Fig. 1);
$\delta_{X_i}(t)$ = deformation in sense of X_i on primary system (Fig. 1);
θ_i = slope of beam at joint i;
θ_{ij}^k = deformation (rotation) in sense of X at joint i, from time when restraint on θ_i (i.e. statically indeterminate action) begins to current time t, due to curvature changes in segment k caused by $X(t)$ at joint j;
$\kappa(x,t)$ = curvature of beam caused by load;
τ_0 = age of concrete at first loading (Fig. 1);
ϕ_{ij}^k = deformation (rotation) in sense of X at joint i, from time when restraint on θ_i (i.e. statically indeterminate action) begins to current time t, due to curvature changes in segment k caused by load history, $q_j(t)$; and
ϕ_i^{kC} = deformation (rotation) in sense of X at joint i, from time when restraint on θ_i begins to current time t, due to curvature changes in overhang segment k caused by load history $q(t)$ in overhang.