Int. J. Solids Structures Vol. 16, pp. 873-901
Pergamon Press Ltd., 1980. Printed in Great Britain

WORK INEQUALITIES FOR PLASTIC FRACTURING MATERIALS

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(*Received 30 April 1979; in revised form 17 December 1979*)

Abstract—Studied are second-order work inequalities for stability of plastic strain increments and fracturing stress decrements. It is found that the positiveness of the second-order work during small loading cycles does not necessarily require normality; it also allows non-normal plastic strains or fracturing relaxations which do no work, as well as non-normal ones which always do non-negative work. The latter ones include plastic strains and fracturing relaxations that are tangential to the loading surface. It is shown that the endochronic theory follows from Drucker's postulate by the same arguments as classical plasticity. The endochronic loading surface has the significance of separating the directions for which Drucker's postulate is satisfied from those for which it is not, whereas in classical plasticity it separates the stress increment directions for which the plastic strain increment vector points outside the loading surface from those for which it would point inward. The incremental linearity of classical plasticity is shown to be a tacitly implied hypothesis which does not follow from Drucker's postulate and the existence of the loading surface. Various incrementally nonlinear stress–strain relations satisfying Drucker's postulate, both such that do and do not obey normality, are demonstrated.

Furthermore, it is found that for frictional materials there exists, in addition to Drucker's (or Il'yushin's) postulate, another inequality that also suffices for stability and reflects the fact that a release of elastic energy blocked by friction or by resistance to fracturing due to compression cannot cause instability. This enlarges the domain of all stable stress increment vectors from a halfspace to a reentrant wedge. The corresponding plastic strain increment vectors have no unique direction and occupy a fan, one boundary of which is the normal vector. Dependence of the second-order work in loading cycle upon the angle between the strain increment vector and the normal is useful for comparing various theories. For the incrementally linear vertex model one needs to introduce at the tangential direction a discontinuity in this dependence.

Finally, some related questions of uniqueness or continuity of response are discussed, particularly for the case of a staircase path in strain space approaching a straight path as the number of stairs tends to infinity. For endochronic theory as well as some vertex models and for plasticity with a corner on the loading surface, the response to the staircase path in the limit does not approach the response to the straight path. Although this is not physically unreasonable, it is nevertheless possible to slightly adjust the definition of intrinsic time so that continuity (uniqueness) is achieved.

1. INTRODUCTION

The inelastic response of geomaterials such as sands, clays, concretes and rocks, arises not only from plastic slip but also from internal micro-fracturing (cracking) or grain separations which are associated with a degradation of material stiffness, and it depends on internal forces which do no work, such as internal friction. Thus, a realistic constitutive modeling of geomaterials requires various unconventional formulations which deviate from the framework of classical incremental plasticity. This is true of the new plastic-fracturing material theory[1] and the endochronic theory[2–15]. These modern theories, which have been very successful in describing the available test results, do not completely satisfy the second-order work inequality represented by Drucker's stability postulate[16–25]. Moreover, some of these theories may better be based on work inequalities of a different type.

Therefore, the purpose of this work is to explore the second-order work inequalities and normality rules (or flow rules) which may serve as a logical basis of these modern theories and allow us to derive the tensorial form of the inelastic constitutive equations by differentiation of scalar functions of stresses as well as strains (the loading functions). In addition, this will lead us to carefully examine which properties necessarily follow from the work inequality, and which do not; we will see, for example, that neither normality nor incremental linearity of classical plasticity are necessary consequences; they represent, therefore, implied simplifying assumptions. Further we will analyze some phenomena which disturb the normality rule, such as internal friction, and we will propose some less restrictive work inequalities sufficient for material stability. Finally, we will briefly consider some related questions of continuity and uniqueness which refer to a recent criticism of the endochronic theory.

Thermodynamics of deformation will not be analyzed because it is already known that inequalities such as Drucker's postulate do not follow from the basic laws of thermodynamics[19, 20]. Continuum thermodynamics, which is at present reasonably well understood, gives only a very limited information on material behavior, and investigations of the special tensorial aspects of the inelastic behavior, on which thermodynamics can yield no information, are more profitable than efforts to refine the rigorosity of the thermodynamic treatment.

2. LOADING CRITERION

We may distinguish two characteristic types of inelastic phenomena:

1. Plastic strain, which results from dislocation motions and is caused by yield on crystal slip planes. Ideally this phenomenon does not cause degradation of elastic moduli (see the parallel unloading slopes in Fig. la) and does not lead to a decline of stress at increasing strain. The plastic deformation is irreversible (Fig. la).

2. Fracturing relaxation, which results from microcracking. This phenomenon obviously causes an irreversible degradation of elastic moduli (see the unloading slopes in Fig. lb) and may lead to a decline of stress at increasing strain. Ideally, the deformation is reversible upon complete unloading[26, 27] (Fig. lb). A simple example of fracturing relaxation is given in Fig. 2; an elastic plate is extended (path $\overline{01}$) and then, while holding the length constant, a crack is cut, which causes reaction σ to drop or relax (from point 1 to point 2, Fig. 2). Then, while the length is reduced to the original value the crack closes and the strain and stress both return to zero (point 0); i.e. the strain is reversible.

We will assume that inelastic phenomena are produced by loading and are absent at unloading. To achieve a tensorially invariant formulation, inelastic phenomena must be characterized, as is well known, in terms of scalar loading functions. In case of isotropic materials, the loading functions may depend on the stress and strain tensors only through their invariants.

The plastic strains are conveniently characterized in terms of a loading function, F, which depends on the stress tensor components, σ_{ij}. This is because of the physical nature of yield as a stress-dependent phenomenon, and also because a dependence on strain would cause some regimes of increasing stress at decreasing strain to be considered as loading even though no yielding can take place at decreasing stress. The strains, though, may appear in function F as state parameters.

The fracturing relaxations should, on the other hand, be properly characterized in terms of a loading function, Φ, which depends on the strain tensor components, ϵ_{ij}. There are two reasons: First, microcracking may lead to a decline of stress at increasing strain, called strain-softening, which must be considered as loading. Although strain-softening can be modeled with the help of stress-dependent loading functions[21, 28, 29], it is more suitable to use strain-dependent loading functions, since in terms of stress we can not easily distinguish loading from unloading. This is because the stress decreases for both of them (Fig. 1c), whereas the strain decreases only for unloading. Second, the fracturing relaxations must be related to the degradation of elastic moduli, while the plastic strain must be unrelated to the degradation, and since the latter depends on stress, the former should not depend on stress. The stresses, however, may appear in function Φ as state parameters.

We must also reject, except as a special case, the possibility of a common loading function depending on both the stress and the strain. This is because plastic strain and fracturing relaxation would have to occur always simultaneously, whereas a realistic theory must admit plastic strain without fracturing or fracturing relaxation without plastic strain. Therefore, we need two independent loading functions[1]:

$$F(\sigma_{ij}, H_k) = 0 \text{ (plastic)}; \quad \Phi(\epsilon_{ij}, H_k') = 0 \text{ (fracturing)} \tag{1}$$

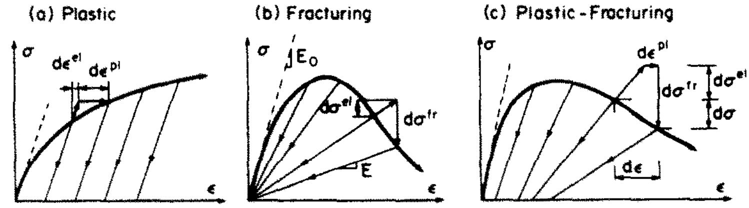

Fig. 1. Characteristic uniaxial responses of plastic, fracturing and plastic-fracturing materials.

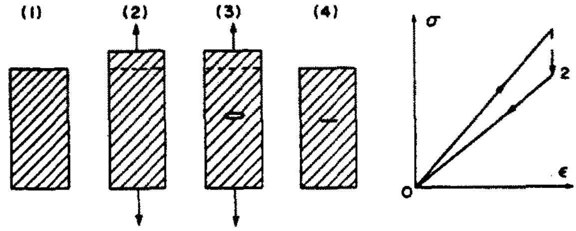

Fig. 2. Physical justification of reversibility of fracturing material with elastic matrix.

where σ_{ij} and ϵ_{ij} are the stress and strain tensors in cartesian coordinates referred to by latin subscripts ($i = 1,2,3$), and $H_k (k = 1,2,\ldots,N_0)$ and H'_k are some parameters for inelastic behavior (e.g. hardening parameters of plasticity). As remarked, ϵ_{ij} may appear in F as some of parameters H_k, and σ_{ij} may appear in Φ as some of parameters H'_k.

The purpose of the loading functions is to distinguish between loading and unloading. During loading the material remains in the plastic or fracturing state and so we always have $dF = (\partial F/\partial\sigma_{ij})d\sigma_{ij} + (\partial F/\partial H_k)dH_k = 0$, and similarly for $d\Phi$. Now, choosing the second term to be negative for loading, we may introduce the loading criteria as follows:

$$\frac{\partial F}{\partial \sigma_{ij}} d\sigma_{ij} > 0 \quad \text{(plastic loading)} \tag{2}$$

$$\frac{\partial \Phi}{\partial \epsilon_{ij}} d\epsilon_{ij} > 0 \quad \text{(fracturing loading)} \tag{3}$$

where repeated indices imply summation. Equations (1) and (3) for Φ were introduced by Dougill for a purely fracturing material[26, 27].

3. INFINITESIMAL LOADING CYCLES AND STABILITY IN THE SMALL

From plasticity theory we recall the Drucker's postulate[16–25] which may be written as:

$$\Delta W = \frac{1}{2} d\sigma_{ij}\, d\epsilon_{ij}^{pl} > 0 \quad \text{(for plastic loading)} \tag{4}$$

where $d\epsilon_{ij}^{pl}$ are the plastic strain increments. This expression, which equals area 123 in Fig. 3(a), represents the second-order work (Helmholtz's free energy in case of isothermal conditions) done on a unit material element during an infinitesimal cycle in which stress increment $d\sigma_{ij}$ is applied and removed. The first order work, represented by area 1345 (Fig. 3a), need not be

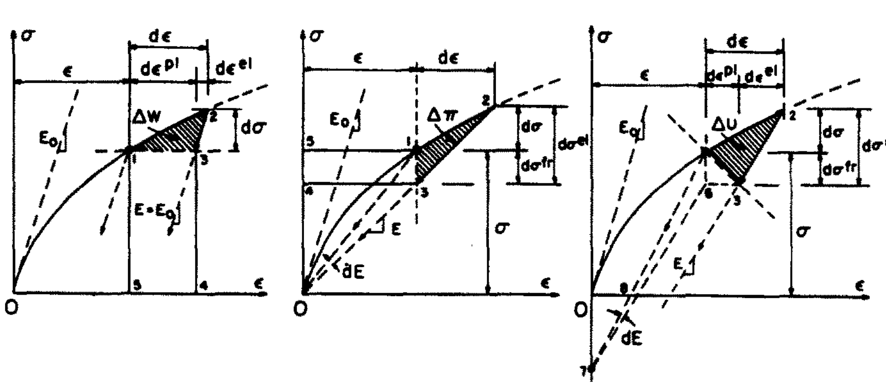

Fig. 3. Work inequalities, and distinction between plastic strain increments and fracturing stress decrements.

considered if the body is initially in equilibrium because, according to the principle of virtual work, the first-order work is canceled by the work of the loads which equilibrate σ_{ij}.

If $\Delta W > 0$, work must be supplied to produce the deformation and so the deformation would not happen if the work is not supplied, which is a stable situation. Therefore, fulfillment of Drucker's postulate (4) ensures stability of the material (stability in the small or local). If $\Delta W < 0$, work is released during the deformation, which may be, but not necessarily is, an unstable situation. Thus, condition (4) is a sufficient, albeit not a necessary condition of material stability[18, 20, 30, 31].

An analogous condition for inelastic behavior governed by a strain-dependent loading function is represented by Il'yushin's postulate[26, 27]:

$$\Delta\Pi = \frac{1}{2}\,\mathrm{d}\sigma_{ij}^{fr}\,\mathrm{d}\epsilon_{ij} > 0 \quad \text{(for fracturing loading)} \tag{5}$$

where $\mathrm{d}\sigma_{ij}^{fr}$ are the inelastic (fracturing) stress decrements. This expression, which equals area 123 in Fig. 3(b), represents the second-order complementary work (Gibbs' free energy) done on a unit element of the material during an infinitesimal cycle in which strain increment $\mathrm{d}\epsilon_{ij}$ is superimposed and removed; see Fig. 3(b). The first-order work, represented by area 1345 in Fig. 3(b), need not be considered because in equilibrium, according to the principle of virtual work, it is canceled by the work of loads. If $\Delta\Pi > 0$, work must be supplied to effect the deformation, and so the inelastic increment would not happen under controlled strain conditions if the work is not supplied[32, 33]; this indicates stability under controlled strain conditions. If $\Delta\Pi < 0$, the material could (but not necessarily will) be unstable even under controlled strain.

When plastic strain increments $\mathrm{d}\epsilon_{ij}^{pl}$ and fracturing stress relaxations (decrements) $\mathrm{d}\sigma_{ij}^{fr}$ are produced simultaneously, i.e., when there is loading for both the plastic and fracturing behaviors, we could also base our theory on the inequality:

$$\Delta U = \frac{1}{2}(\mathrm{d}\sigma_{ij}\,\mathrm{d}\epsilon_{ij}^{pl} + \mathrm{d}\sigma_{ij}^{fr}\,\mathrm{d}\epsilon_{ij}) > 0. \tag{6}$$

This expression equals area 123 in Fig. 3(c). Inequalities (4) and (5) imply (6). But they are not implied by (6) unless we make a further assumption, namely that (6) must hold even when either $\mathrm{d}\epsilon_{ij}^{pl}$ or $\mathrm{d}\sigma_{ij}^{fr}$ is imagined to be separately held zero (frozen). This assumption is in fact implied when we apply inequalities (4) or (5) to a plastic-fracturing material.

The cycle 123 in Fig. 3(c) terminates neither at the initial σ_{ij} (as in Fig. 3a) nor at the initial ϵ_{ij} (as in Fig. 3b), but at the line $\overline{13}$ that has a certain slope $\partial\sigma_{ij}/\partial\epsilon_{km}$. To be able to distinguish between $\mathrm{d}\epsilon_{ij}^{pl}$ and $\mathrm{d}\sigma_{ij}^{fr}$, this slope must be determined from an independent argument, which has nothing to do with work inequalities and is explained in the Appendix.

4. NORMALITY RULE

Let us now try to determine the most general expression for inelastic strain permitted by the preceding scalar inequalities. We consider first the case of plastic strains alone. According to (2) and (4) for plastic loading, we require that:

$$\text{For } \frac{\partial F}{\partial\sigma_{ij}}\,\mathrm{d}\sigma_{ij} > 0\text{: } \Delta W = \frac{1}{2}\,\mathrm{d}\sigma_{ij}\,\mathrm{d}\epsilon_{ij}^{pl} > 0. \tag{7}$$

We must, however, also pay attention to the limiting case of neutral loading, and for this case we require that

$$\text{For } \frac{\partial F}{\partial_{ij}}\,\mathrm{d}\sigma_{ij} = 0\text{:} \quad \text{either } \Delta W = \frac{1}{2}\,\mathrm{d}\sigma_{ij}\,\mathrm{d}\epsilon_{ij}^{pl} = 0 \tag{8a}$$

$$\text{or } \Delta W > 0. \tag{8b}$$

This condition has been hitherto invariably considered as an equality (eqn 8a), tacitly excluding

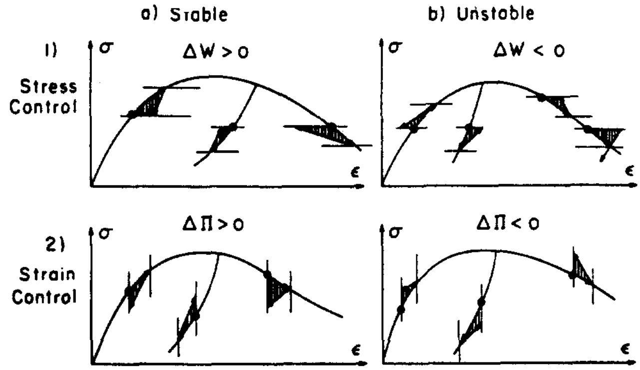

Fig. 4. Responses that are stable (a) and not necessarily stable (b).

the possibility of $\Delta W > 0$ for neutral loading (i.e. for loading increments tangential to the loading surface). There exists, however, no good reason other than convenience to exclude this possibility. Nevertheless, for the purpose of exposition we at first restrict attention to the case of equality, eqn (8a).

Since both expressions in eqn (7) are positive, their ratio must be positive, and so we may set

$$\frac{\mathrm{d}\epsilon_{ij}^{pl}\,\mathrm{d}\sigma_{ij}}{\dfrac{\partial F}{\partial \sigma_{km}}\,\mathrm{d}\sigma_{km}} = \mathrm{d}\mu > 0 \quad \text{(plastic loading).} \tag{9}$$

Conversely, if the denominator is positive and if $\mathrm{d}\mu > 0$, it follows that the numerator is positive, too. Thus, eqn (9) is equivalent to inequalities (7). Now, multiplying (9) with the denominator, we get for plastic loading:

$$\mathrm{d}\epsilon_{ij}^{t}\,\mathrm{d}\sigma_{ij} = 0, \quad \text{where } \mathrm{d}\epsilon_{ij}^{t} = \mathrm{d}\epsilon_{ij}^{pl} - \frac{\partial F}{\partial \sigma_{ij}}\,\mathrm{d}\mu. \tag{10}$$

We should note at the same time that this equation is also satisfied for neutral plastic loading with $\Delta W = 0$ (eqn 8a). Conversely, eqn (10) requires that $\Delta W = 0$ for neutral loading. So, eqn (10) is equivalent to conditions (7) and (8a) combined. On the other hand, for neutral loading with $\Delta W > 0$ (eqn 8b) eqn (10) would not be satisfied.

Equation (10) must hold for *all* stress increments $\mathrm{d}\sigma_{ij}$ which represent plastic loading. This can be achieved in two ways: (a) Either $\mathrm{d}\epsilon_{ij}^{t} = 0$, (b) or $\mathrm{d}\epsilon_{ij}^{t} \neq 0$. We consider the case $\mathrm{d}\epsilon_{ij}^{t} = 0$ first and we have

$$\mathrm{d}\epsilon_{ij}^{pl} = \mathrm{d}\epsilon_{ij}^{n}, \quad \mathrm{d}\epsilon_{ij}^{n} = \frac{\partial F}{\partial \sigma_{ij}}\,\mathrm{d}\mu. \tag{11}$$

This is the famous plastic flow rule of Prandtl and Reuss, also known as the normality rule[16, 17, 22–25]. We introduce the notation $\mathrm{d}\epsilon_{ij}^{n}$ to indicate that this type of plastic strain increment is normal to the current loading surface. The expression for $\mathrm{d}\mu$ we will discuss later.

The second way to satisfy eqn (10) is to require that the vector which represents $\mathrm{d}\epsilon_{ij}^{t}$ in the nine-dimensional space be normal to $\mathrm{d}\sigma_{ij}$. Then

$$\mathrm{d}\epsilon_{ij}^{pl} = \mathrm{d}\epsilon_{ij}^{n} + \mathrm{d}\epsilon_{ij}^{tr} \tag{12}$$

where $\mathrm{d}\epsilon_{ij}^{tr}$, which we will call transversal plastic strain, can be represented by any vector that is normal to $\mathrm{d}\sigma_{ij}$ (Fig. 5). It is a plastic strain which does no work. It seems that the possibility of such plastic strain has passed unnoticed so far, and only the case of normality, obtained for $\mathrm{d}\epsilon_{ij}^{tr} = 0$, has been considered.

In case of normality rule, the direction of $\mathrm{d}\epsilon_{ij}^{pl}$ is totally determined by the current loading

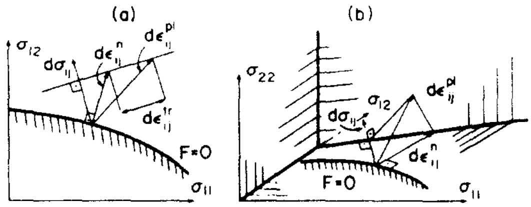

Fig. 5. Inelastic strain increments that do no work.

surface which corresponds to the current initial stress state. This situation, which ceases to be true in the general case of eqn (12), as well as all other expressions for $d\epsilon_{ij}^{pl}$ considered in the sequel, is by no means necessary. This may be illustrated by Mandel's example (Fig. 6a)[28] of a frictional block loaded by shear stress τ and also by a spring which causes that the slip limit depends on the slip and thus models plastic hardening. We assume the block to be at the point of sliding under vertical stress σ (<0) and spring force S. A small vertical stress increment $d\sigma_{ij}$ obviously causes the block to slide horizontally, in which case the inelastic deformation is normal to the applied force increment and does no work.

The question whether normality should hold can be decided from the microstructural mechanism of inelastic strain. Theoretically, if normality holds on the microstructural level, as is true of perfectly plastic slip, it must hold on the macroscopic level[34–36]. If, however, normality does not apply on the microstructural level, which is the case when we have frictional slip, microcracking or formation of voids and some hardening processes in polycrystals, we must expect that it does not apply on the macroscopic level. This is typical especially for geomaterials (rocks, concretes, soils).

A completely analogous analysis can now be made with regard to the fracturing stress decrements $d\sigma_{ij}^{fr}$. According to (3) and (5), for fracturing loading we require that:

$$\text{For}\ \frac{\partial\Phi}{\partial\epsilon_{ij}}\,d\epsilon_{ij}>0:\quad \Delta\Pi=\frac{1}{2}\,d\sigma_{ij}^{fr}\,d\epsilon_{ij}>0 \tag{13}$$

and for the limiting case of neutral loading we require that

$$\text{For}\ \frac{\partial\Phi}{\partial\sigma_{ij}}\,d\sigma_{ij}=0:\quad \text{either}\ \Delta\Pi=\frac{1}{2}\,d\sigma_{ij}^{fr}\,d\epsilon_{ij}=0 \tag{14a}$$

$$\text{or}\quad \Delta\Pi>0. \tag{14b}$$

We again at first restrict attention to the case when $\Delta\Pi=0$ (eqn 14a). Since both expressions in eqn (13) are positive, their ratio must be positive as well, and so we may set

$$\frac{d\sigma_{ij}^{fr}\,d\epsilon_{ij}}{\dfrac{\partial\Phi}{\partial\epsilon_{km}}\,d\epsilon_{km}}=d\kappa>0\quad\text{(for fracturing loading).} \tag{15}$$

Conversely, if the denominator is positive and if $d\kappa>0$, it follows that the numerator is positive as

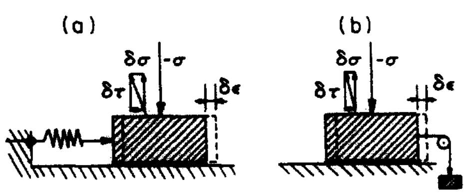

Fig. 6. Frictional block at the point of sliding, loaded by a spring (a) or by a constant force (b).

well. So, eqn (15) is equivalent to inequalities (13). We may transform it, by multiplying it with the denominator, to the form

$$d\sigma_{ij}^{*}\, d\epsilon_{ij} = 0, \quad \text{where } d\sigma_{ij}^{*} = d\sigma_{ij}^{fr} - \frac{\partial \Phi}{\partial \epsilon_{ij}}\, d\kappa. \tag{16}$$

We should note that this equation is also satisfied for neutral fracturing loading with $\Delta\Pi = 0$ (eqn 14a), whereas it would not be satisfied for neutral loading with $\Delta\Pi > 0$ (eqn 14b).

Equation (15) must be satisfied for all possible $d\epsilon_{ij}$. One way to achieve it is to require that $d\sigma_{ij}^{*}$ vanish; thus,

$$d\sigma_{ij}^{fr} = d\sigma_{ij}^{n}, \quad d\sigma_{ij}^{n} = \frac{\partial \Phi}{\partial \epsilon_{ij}}\, d\kappa. \tag{17}$$

This is the fracturing rule or normality rule in the strain space[26, 27], analogous to eqn (11).

Similarly as for plastic strains, a more general way to satisfy eqn (16) is to require that $d\sigma_{ij}^{*}$ be normal to $d\epsilon_{ij}$. Thus we have, more generally,

$$d\sigma_{ij}^{fr} = d\sigma_{ij}^{n} + d\sigma_{ij}^{tr} \tag{18}$$

where $d\sigma_{ij}^{tr}$, which we will call transversal fracturing stress decrement, can be represented by any vector that is normal to $d\epsilon_{ij}$ in the nine-dimensional space. Obviously, $d\sigma_{ij}^{tr}$ does no work.

Remember that the normality rules, eqns (11) and (15), were obtained under the assumption that $\Delta W = 0$ and $\Delta\Pi = 0$ for tangential (neutral) loading. The cases where this is not true (eqns 8b, 14b) will be analyzed later.

5. TYPES OF STRESS–STRAIN RELATIONS BASED ON NORMALITY RULE

We must now relate the proportionality coefficients $d\mu$ and $d\kappa$ to $d\epsilon_{ij}$ and $d\sigma_{ij}$. It will be expedient to first rewrite eqns (11) and (15) in the rate form

$$\dot{\epsilon}_{ij}^{n} = \frac{\partial F}{\partial \sigma_{ij}}\, \dot{\mu}, \quad \dot{\sigma}_{ij}^{n} = \frac{\partial \Phi}{\partial \epsilon_{ij}}\, \dot{\kappa} \tag{19a,b}$$

where $\dot{\epsilon}_{ij}^{n}$ and $\dot{\sigma}_{ij}^{n}$ are the rates of normal plastic strain and of normal fracturing stress. Because we still consider $\Delta W = 0$ and $\Delta\Pi = 0$ for neutral loading, these rates must vanish for such loading. This occurs when, according to (2) and (3),

$$X = \frac{\partial F}{\partial \sigma_{km}}\, \dot{\sigma}_{km}, \quad Y = \frac{\partial \Phi}{\partial \epsilon_{km}}\, \dot{\epsilon}_{km}, \tag{20}$$

respectively, vanish. When these expressions are positive, $\dot{\epsilon}_{ij}^{pl}$ and $\dot{\sigma}_{ij}^{fr}$, respectively, must be nonzero. This condition can generally be satisfied if and only if $\dot{\mu}$ and $\dot{\kappa}$, which determine the magnitudes of $\dot{\epsilon}_{ij}^{pl}$ and $\dot{\sigma}_{ij}^{fr}$, depend on X and Y, respectively:

$$\dot{\mu} = \Phi_1(X), \quad \dot{\kappa} = \Phi_2(Y) \tag{21}$$

where functions Φ_1 and Φ_2 must be continuous, smooth and monotonic functions, such that

$$\begin{gathered} \Phi_1(0) = 0, \quad \Phi_2(0) = 0 \\ \Phi_1(X) > 0 \text{ for } X > 0, \quad \Phi_2(Y) > 0 \text{ for } Y > 0. \end{gathered} \tag{22}$$

Alternatively, either $\Phi_1(X)$ or $\Phi_2(Y)$ could be zero for all X or Y, but in that case there would be no inelastic stress and strain.

Let us now consider some important special cases.

I. *Classical plasticity*

Assume that functions Φ_1 and Φ_2 are linear and that the rates are time rates, i.e. $\dot{\mu} = \mathrm{d}\mu/\mathrm{d}t$, $\epsilon^{pl} = \mathrm{d}\epsilon^{pl}/\mathrm{d}t$, etc. where t = time. We may then set $\Phi_1(X) = X/h$ and $\Phi_2(Y) = \phi Y$, where h and ϕ are some scalar coefficients depending on σ_{ij} and ϵ_{ij}. Then, substituting (20) into (21) and multiplying with $\mathrm{d}t$, we obtain the well-known expressions [37, 26]

$$\mathrm{d}\mu = \frac{1}{h}\frac{\partial F}{\partial \sigma_{km}}\mathrm{d}\sigma_{km}, \quad \mathrm{d}\kappa = \phi\frac{\partial \Phi}{\partial \epsilon_{km}}\mathrm{d}\epsilon_{km} \tag{23}$$

where $h > 0$, $\phi > 0$. Function F may always be chosen so as to have a dimension of stress and h then has also a dimension of stress and may be called the normal plastic modulus. Furthermore, function ϕ may always be chosen as non-dimensional and ϕ must then have a dimension of stress and may be called the normal fracturing modulus.

The first of eqns (23) is the same as in the classical theory of incremental plasticity [37, 22–25]. We see that $\mathrm{d}\sigma_{km}$ and $\mathrm{d}\epsilon_{km}$ are involved linearly, wherefore the stress–strain relations are incrementally linear.

From eqn (23) we see that $\mathrm{d}\mu$ becomes negative when $\mathrm{d}\sigma_{ij}$ is directed inside the loading surface (unloading). Nevertheless, the work $\Delta W = \mathrm{d}\sigma_{ij}\,\mathrm{d}\epsilon_{ij}''/2$ remains, according to eqns (11) and (23), positive, because not only $\mathrm{d}\sigma_{ij}$ but also $\mathrm{d}\epsilon_{ij}''$ is directed inside the loading surface. This last property is impossible because it would imply full reversibility (Fig. 7d). Thus, eqns (11) and (23) must be discarded in case of unloading and a purely elastic unloading has to be assumed. This is at the same time expedient for being able to define the inelastic strain in terms of a load-unload cycle.

II. *Endochronic inelasticity*

Assume again that functions Φ_1 and Φ_2 are linear, but the rates, rather than being the time rates, are rates with respect to the length of the path of the material states traced in the strain space. This length may be in general defined as [3]

$$\xi = \int (p_{ijkm}\,\mathrm{d}\epsilon_{ij}\,\mathrm{d}\epsilon_{km})^{1/2} \tag{24}$$

where p_{ijkm} are some coefficients defining the proper strain space metric (which is assumed to exist, as an approximation, although it is unlikely to have a general validity). Thus, our assumption is that

$$\dot{\mu} = \frac{\mathrm{d}\mu}{\mathrm{d}\xi}, \quad \dot{\kappa} = \frac{\mathrm{d}\kappa}{\mathrm{d}\xi}, \quad \dot{\epsilon}_{ij}^{pl} = \frac{\mathrm{d}\epsilon_{ij}^{pl}}{\mathrm{d}\xi}, \quad \dot{\sigma}_{ij}^{fr} = \frac{\mathrm{d}\sigma_{ij}^{fr}}{\mathrm{d}\xi}. \tag{25}$$

Now, setting $\Phi_1(X) = X/h$ and $\Phi_2(Y) = \phi Y$, as before, substituting (20) into (21), and multiplying by $\mathrm{d}\xi$, we obtain

$$\mathrm{d}\mu = \frac{1}{h}\frac{\partial F}{\partial \sigma_{km}}\frac{\mathrm{d}\sigma_{km}}{\mathrm{d}\xi}\mathrm{d}\xi, \quad \mathrm{d}\kappa = \phi\frac{\partial \Phi}{\partial \epsilon_{km}}\frac{\mathrm{d}\epsilon_{km}}{\mathrm{d}\xi}\mathrm{d}\xi \tag{26}$$

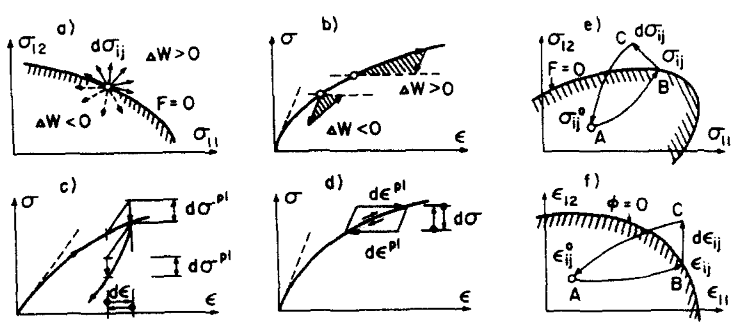

Fig. 7. Illustrations of various behavior at loading and unloading.

or

$$d\mu = \frac{d\xi}{f_1}, \quad d\kappa = \frac{d\xi}{f_2} \tag{27}$$

in which

$$\frac{1}{f_1} = \frac{1}{h}\frac{\partial F}{\partial \sigma_{km}}\frac{d\sigma_{km}}{d\xi}, \quad \frac{1}{f_2} = \phi\frac{\partial \Phi}{\partial \epsilon_{km}}\frac{d\epsilon_{km}}{d\xi}. \tag{28}$$

Expression (27) for $d\mu$ along with eqns (24) and (11) have the form of the incremental stress–strain relations for what is known as the endochronic theory[2, 3, 7], and path length ξ is the intrinsic time[3]. Adding expression (27) for $d\kappa$ along with eqn (17), one has what has recently been proposed as the fracturing endochronic theory[11].

Since F and Φ are scalar functions of σ_{ij} and ϵ_{ij} (and possibly ξ) and so are h and ϕ coefficients f_1 and f_2 are also such functions. The endochronic theory, however, makes one important deviation from our logical framework; rather than determining f_1 and f_2 from F and Φ according to eqn (28), coefficients f_1 and f_2 are *directly* introduced as empirical functions of σ_{ij}, ϵ_{ij} and ξ. Because eqn (28) is not a total derivative $dF/d\xi$, F being also dependent on ϵ_{ij} and ξ functions f_1 and f_2 do not in general allow us to solve F and Φ from eqns (28). If they did, one would be able to pass from eqns (28) backward to eqns (26) for $d\mu$ and $d\kappa$, for which the stress-strain relations are linear in terms of $d\sigma_{ij}$ and $d\epsilon_{ij}$. This, however, cannot be achieved (because tangential moduli would depend on $d\sigma_{ij}/d\xi$ and $d\epsilon_{ij}/d\xi$, which is not allowed). For this reason, and because $d\xi$ is a nonlinear function of $d\epsilon_{ij}$, the stress–strain relations of endochronic theory are incrementally nonlinear, which marks the most fundamental difference from classical incremental plasticity[7].

The expressions for $d\mu$ and $d\kappa$ are, however, unimportant for the fulfillment of the work inequalities (Drucker's and Il'yushin's postulates). So we may conclude that the time-independent endochronic theory can be regarded as a consequence of Drucker's postulate if the stress and strain rates are taken with regard to the path length rather than time. This is a perfectly rational assumption when a time-independent behavior is of interest.

Due to the fact that f_1 (and f_2) are in the ordinary endochronic theory chosen to be always positive, while $(\partial F/\partial \sigma_{km})\,(d\sigma_{km}/d\xi)$ is negative for unloading (in violation of eqn 28), the theory gives $\Delta W < 0$ for unloading, whereas the plasticity expressions, if extended to unloading, would give $\Delta W > 0$, as already remarked. Thus, the role of the loading surface in the ordinary endochronic theory is that it separates the stress increments which satisfy the Drucker's postulate from those which do not (Fig. 7a,b).

Indeed, if the same equations of endochronic theory are assumed to hold for both loading and unloading (as used in the original endochronic formulations[3, 4]) and if a stress increment directed inside the loading surface is applied and removed (Fig. 7c), then for such a cycle the theory yields $\Delta W = d\sigma_{ij}\, d\epsilon_{ij}^{pl}/2 < 0$. Consequently, the reloading slope for uniaxial stress–strain diagrams is obtained always smaller than the unloading slope (Fig. 4b), which is usually at variance with observations and requires further correction[7].

The cases $\Delta W < 0$ can be simply eliminated by stipulating that $d\epsilon_{ij}^{pl} = 0$ for unloading (eqn 2), just like one does it in classical plasticity. Then, however, we could not model inelastic behavior during unloading and reloading, typical especially of geomaterials[1, 7, 11, 12, 39, 40]. Nevertheless, a refinement of endochronic theory which gives inelastic strain at unloading yet ensures fulfillment of Drucker's postulate for small unload-reload cycles is possible; see Ref. [7]. This requires introduction of inequalities that distinguish loading, unloading and reloading and the use of jump-kinematic hardening[7]. The same refinement is necessary for plasticity to represent the inelastic behavior at unloading and reloading[1].

In classical plasticity the role of the loading surface is different but hardly more appealing; it separates the stress increment directions for which the plastic strain increment vector points outside the loading surface from those for which it would point inward. The latter case, leading to a complete reversibility in small load-unload cycles (Fig. 7d), would be unacceptable and, for this reason, a condition requiring that $d\epsilon_{ij}^{pl} = 0$ at unloading has to be imposed in plasticity. In endochronic theory, by contrast, the inelastic strain increment direction is always outward and it is by virtue of this fact that the endochronic theory, unlike plasticity, can model irreversibility at unloading if the same equations as for loading are used. This may be illustrated in the uniaxial load-unload diagram of Fig. 7(c)[4] where $d\epsilon_{11}^{pl}$ is first imposed, then removed. Because $d\sigma_{11}^{pl} = D\, d\epsilon_{11}^{pl}$ is for unloading of the same direction as for loading, a steeper slope is obtained for unloading.

III. *Non-endochronic incremental nonlinearity*

Let us now examine whether incremental nonlinearity is a necessity when we leave aside the endochronic theory (understood as a theory where we choose f_1 and f_2 rather than F and Φ, and not as a theory which makes use of the path length). As an example, consider that

$$\dot{\mu} = \Phi_1(X) = \left(\frac{X}{h}\right)^{1/3} = \left(\frac{1}{h}\frac{\partial F}{\partial \sigma_{km}}\dot{\sigma}_{km}\right)^{1/3} \tag{29}$$

and $\dot{\kappa} = \Phi_2(Y) = 0$. If the rate is considered as a time rate, we find that multiplication of the expression for $\dot{\epsilon}_{ij}^{pl}$ by dt does not render it time-independent. Consequently, this case is not of interest when our attention is restricted to plasticity.

Consider now, however, that the rates are with respect to the path length, ξ, such that $d\xi = (d\epsilon_{ij}\, d\epsilon_{ij})^{1/2}$. Then we obtain

$$d\epsilon_{ij}^{pl} = \frac{\partial F}{\partial \sigma_{ij}}\left(\frac{1}{h}\frac{\partial F}{\partial \sigma_{km}}\, d\sigma_{km}\, d\epsilon_{pq}\, d\epsilon_{pq}\right)^{1/3}. \tag{30}$$

This is a time-independent expression, which does not involve $d\sigma_{ij}$ and $d\epsilon_{ij}$ linearly, yet satisfies Drucker's postulate as well as normality rule. So, we see that incremental linearity is not a necessary consequence of the assumptions normally spelled out in an exposition of classical plasticity. The assumption of incremental linearity is tacitly implied.

Hypotheses of classical plasticity

As we saw, the incremental linearity does not follow from Drucker's postulate[16–25], not even from the normality rule. It is usually derived by starting with the loading function of the form ([22], p. 148):

$$F(\sigma_{ij}, \epsilon_{ij}^{pl}, H_p) = 0. \tag{31}$$

One differentiates this equation, i.e.

$$\dot{F} = \frac{\partial F}{\partial \sigma_{km}}\dot{\sigma}_{km} + \frac{\partial F}{\partial \epsilon_{km}^{pl}}\dot{\epsilon}_{km}^{pl} + \frac{\partial F}{\partial H_p}\frac{\partial H_p}{\partial \epsilon_{km}^{pl}}\dot{\epsilon}_{km}^{pl} = 0 \tag{32}$$

and substitutes eqn (19a) for $\dot{\epsilon}_{km}^{pl}$. This yields an equation, called the consistency condition (due to Prager) [37, 38, 22], from which $\dot{\mu}$ may be solved:

$$\dot{\mu} = \frac{1}{h}\frac{\partial F}{\partial \sigma_{km}}\dot{\sigma}_{km}, \quad \text{with } h = \left(\frac{\partial F}{\partial \epsilon_{ij}^{pl}} + \frac{\partial F}{\partial H_p}\frac{\partial H_p}{\partial \epsilon_{ij}^{pl}}\right)\frac{\partial F}{\partial \sigma_{ij}}. \tag{33}$$

This expression for $\dot{\mu}$ is linear in terms of $\dot{\sigma}_{km}$ and, therefore, the resulting incremental stress–strain relations are linear in terms of $d\sigma_{ij}$ (as well as $d\epsilon_{ij}$).

The foregoing line of reasoning rests, however, on one tacit premise, namely that (at least near the current state, i.e. for plastic strains $\epsilon_{ij}^{pl} + \Delta\epsilon_{ij}^{pl}$ where $\|\Delta\epsilon_{ij}^{pl}\|$ is sufficiently small) there exists a one-to-one (or functional) dependence of the loading function on the *total* plastic strains ϵ_{ij}^{pl}. In other words, one implies the assumption that the dependence of F upon ϵ_{ij}^{pl} is ***path-independent***, at least locally (in the small). There is no good reason for this to hold and in fact the recent formulation of the plastic-fracturing theory for concrete[1] does not satisfy this premise.

From the preceding considerations it appears that classical incremental plasticity is a consequence of as many as seven hypotheses:

(1) The stress–strain relation is time-independent.
(2) Loading function F exists.
(3) The elastic moduli are constant (no degradation).
(4) Drucker's postulate holds (eqn 7).
(5) For neutral loading $\Delta W = 0$ (eqn 8a).
(6) There is no inelastic strain increment normal to $d\sigma_{ij}$ ($d\epsilon^t_{ij} = 0$, eqn 12).
(7) The resulting stress–strain relation is incrementally linear (i.e. linear in terms of $d\sigma_{ij}$ and $d\epsilon_{ij}$).

Normally only hypotheses 1, 2 and 4 are listed, and the necessity to spell out the hypotheses 3 and 5–7 is often overlooked.

Hypothesis 5 is equivalent to a requirement of continuity of ΔW or $d\epsilon^{pl}_{ij}$ between elastic and plastic regions of the stress space (Prager's continuity condition[37, 38]). Hypotheses 4–6 and the hypothesis that F has the form of eqn (31), equivalent to hypothesis 7, may alternatively be replaced by Drucker's postulate of stability in the large[16–24] (see eqn 34 in the sequel).

Development of complete incremental stress–strain relations requires further arguments which have nothing to do with work inequalities (e.g. hardening rules or the degradation of elastic moduli due to fracturing strains). An exposition of this task for a certain plastic-fracturing material may be found in Ref. [1].

6. FINITE LOADING CYCLES AND STABILITY IN THE LARGE

Consider a unit material element in which given loads $\mathbf{P}^0$ produce homogeneous stress σ^0_{ij} such that the corresponding point A in the stress space lies anywhere within the current loading surface $F = 0$ (Fig. 7e). We subject the element to a finite loading cycle by applying additional loads $\Delta\mathbf{P}(s)$ producing additional stresses $\Delta\sigma_{ij}(s)$ that gradually move the state point along a cyclic path s from point A to a point B on the current loading surface (Fig. 7e), then to a point C outside this surface and finally back to point A(s = path length). We assume that $\overline{AB}$ and $\overline{CA}$ are finite, and $BC = d\sigma_{ij}$ is infinitesimal.

If the work δW that must be done by $\Delta\mathbf{P}(s)$ during this cycle is positive, the cycle cannot occur if the work is not supplied, and so the material elements is stable. If δW is not positive, the cycle may occur spontaneously (without our added loads) and so the element may be (but not necessarily is) unstable. For the entire cycle, $\delta W = \oint \Delta\sigma_{ij}(s)\, d\epsilon_{ij}(s)$ where $\Delta\sigma_{ij}(s) = \sigma_{ij}(s) - \sigma^0_{ij}$. For plastic-fracturing materials we have $d\epsilon_{ij} = d\epsilon^{el}_{ij} + d\epsilon^{pl}_{ij}$ and $\sigma_{ij} = \sigma^{el}_{ij} - \sigma^{fr}_{ij}$ where ϵ^{el}_{ij} and σ^{el}_{ij} are the elastic strains and stresses (see Appendix I), and we get $\delta W = \delta W^{el} + \delta W^{pl} + \delta W^{fr}$ in which $\delta W^{pl} = \oint \Delta\sigma_{ij}(s)\ d\epsilon^{pl}_{ij}\ (s)$, $\delta W^{fr} = -\oint \sigma^{fr}_{ij}(s)\ d\epsilon^{el}_{ij}(s)$ and $\delta W^{el} = \oint \Delta\sigma^{el}_{ij}(s)\ d\epsilon^{el}_{ij}(s)$ with $\Delta\sigma^{el}_{ij} = \sigma^{el}_{ij} - \sigma^0_{ij}$. Because $d\epsilon^{pl}_{ij}$ (unlike $d\sigma^{fr}_{ij}$) can be non-zero only on the path segment $\overline{BC}$ ($\overline{BC} = d\epsilon_{ij}$), we have $\delta W^{pl} = (\sigma_{ij} - \sigma^0_{ij}) d\epsilon^{pl}_{ij}$ where σ_{ij} denotes the stress at point B at the loading surface $F = 0$. Therefore, the sufficient condition of stability in the large may be written as

$$\delta W = (\sigma_{ij} - \sigma^0_{ij})\ d\epsilon^{pl}_{ij} + \delta W^{el} + \delta W^{fr} > 0. \tag{34}$$

When there is no fracturing, we have $\delta W^{fr} = 0$; furthermore $\delta W^{el} = 0$ because elastic behavior does not dissipate energy if the elastic moduli are constant. Then $(\sigma_{ij} - \sigma^0_{ij}) d\epsilon^{pl}_{ij} > 0$, which is the well-known Drucker's postulate of stability in the large[16–24] (equivalent to Hill's principle of maximum plastic work[41]) and implies convexity of the loading surface $F = 0$ as a sufficient (but not necessary) condition for stability. However, if there is fracturing, δW^{fr} and δW^{el} are generally nonzero and can have either sign, and so $(\sigma_{ij} - \sigma^0_{ij}) d\epsilon^{pl}_{ij}$ can be positive or negative. So, if fracturing (degradation, damage) takes place, nothing indicates that the plastic loading surface $F = 0$ should be convex. However, even in absence of fracturing the loading surface need not be convex if there is friction, and one can construct a finite cycle similar to that illustrated by Fig. 6 (Section 8) such that $(\sigma_{ij} - \sigma^0_{ij})\ d\epsilon_{ij} < 0$.

Considering in the strain space a similar finite loading cycle (Fig. 7f) which begins and ends at point A inside the loading surface and includes an infinitesimal increment $\mathrm{d}\epsilon_{ij}$ beyond the loading surface $\Phi = 0$, we can derive an inequality analogous to (34). There is one difference, however. Whereas δW^{el} vanishes if the behavior is purely plastic, it does not vanish if the material is purely fracturing because the elastic moduli are not constant (degradation, damage). Thus, convexity of loading surface $\Phi = 0$ is not required for plastic-fracturing materials even if the work inequality for stability in the large is postulated. (For purely fracturing materials this was already pointed out by J. W. Dougill.)

7. TANGENTIAL INELASTIC STRAIN

Expressions (11), (12) for plastic strain increments and expressions (17), (18) for fracturing stress decrements give $\Delta W = 0$ and $\Delta\Pi = 0$ for increments $\mathrm{d}\sigma_{ij}$ and $\mathrm{d}\epsilon_{ij}$ that are tangential to the loading surface. In general we can have, however, $\Delta W \geq 0$ and $\Delta\Pi \geq 0$ for such loading increments (eqns 8a,b and 14a,b), and we will now explore such formulations.

Let us first consider plastic strains and the loading surface in the stress space. The second-order work of plastic strain increment according to eqns (11) and (23) is $\Delta W = (1/2)\,\mathrm{d}\sigma_{ij}\,\mathrm{d}\epsilon^{n}_{ij}$. A generalization which ensures that $\Delta W \geq 0$ is

$$\Delta W = \frac{1}{2}\,\mathrm{d}\sigma_{ij}\,\mathrm{d}\epsilon^{n}_{ij} + \frac{h_t}{2}\,\mathrm{d}\epsilon^{t}_{ij}\,\mathrm{d}\epsilon^{t}_{ij} \tag{35}$$

because the second term is always non-negative and h_t is assumed to be positive. It is no restriction on generality if the vector representing $\mathrm{d}\epsilon^{t}_{ij}$ is assumed to lie in the tangent plane of the current loading surface F. We may then call $\mathrm{d}\epsilon^{t}_{ij}$ the tangential plastic strain increment, and we may call h_t the tangential plastic modulus. By introducing $\mathrm{d}\epsilon^{t}_{ij}$, the formerly defined transversal plastic strain increment $\mathrm{d}\epsilon^{tr}_{ij}$ can now obviously be nonzero; but it would add nothing new to consider that $\mathrm{d}\epsilon^{tr}_{ij}$ occurs simultaneously with $\mathrm{d}\epsilon^{t}_{ij}$ and $\mathrm{d}\epsilon^{n}_{ij}$. So, $\mathrm{d}\epsilon^{tr}_{ij}$ is dropped from now on.

The tangent plane of F is eight-dimensional and $\mathrm{d}\epsilon^{t}_{ij}$ would in general be given by five independent components. However, as far as the work of $\mathrm{d}\sigma_{ij}$ is concerned, only the component along the normal projection of $\mathrm{d}\sigma_{ij}$ onto the tangent plane matters. So, we will assume, for the purpose of simplification, that $\mathrm{d}\epsilon^{t}_{ij}$ has such a direction.

The unit normal of F is expressed as $n_{ij} = (\partial F/\partial\sigma_{ij})/\|\partial F/\partial\sigma\|$ where $\|\partial F/\partial\sigma\| = [(\partial F/\partial\sigma_{km})(\partial F/\partial\sigma_{km})]^{1/2}$ = magnitude of $\partial F/\partial\sigma_{ij}$. The length of the projection of $\mathrm{d}\sigma_{ij}$ onto the normal to F is, therefore, $\mathrm{d}\sigma_{ij}n_{ij}$, and the vector of this projection is $n_{ij}(\mathrm{d}\sigma_{pq}\,n_{pq})$. So, the vector of the projection of $\mathrm{d}\sigma_{ij}$ onto the tangent plane is $\mathrm{d}\sigma_{ij} - n_{ij}\,(\mathrm{d}\sigma_{pq}\,n_{pq})$, and $\mathrm{d}\epsilon^{t}_{ij}$ may be taken as $(1/h_t)$ times this expression. Substituting for n_{ij} and n_{pq}, we thus obtain:

$$\mathrm{d}\epsilon^{t}_{ij} = \frac{1}{h_t}(\mathrm{d}\sigma_{ij} - n_{ij}\,n_{km}\,\mathrm{d}\sigma_{km}) = \frac{1}{h_t}\left(\mathrm{d}\sigma_{ij} - \frac{\partial F}{\partial\sigma_{ij}}\,\frac{\dfrac{\partial F}{\partial\sigma_{km}}\,\mathrm{d}\sigma_{km}}{\dfrac{\partial F}{\partial\sigma_{pq}}\,\dfrac{\partial F}{\partial\sigma_{pq}}}\right). \tag{36}$$

We may check that indeed $\mathrm{d}\epsilon^{t}_{ij}\,(\partial F/\partial\sigma_{ij}) = 0$ or $\mathrm{d}\epsilon^{t}_{ij}\,\mathrm{d}\epsilon^{n}_{ij} = 0$, and we also easily verify that always

$$\Delta W_t = \frac{1}{2}\,\mathrm{d}\sigma_{ij}\,\mathrm{d}\epsilon^{t}_{ij} = \frac{h_t}{2}\,\mathrm{d}\epsilon^{t}_{ij}\,\mathrm{d}\epsilon^{t}_{ij} \geq 0. \tag{37}$$

Moreover, we should note that $\Delta W > 0$ when $\mathrm{d}\sigma^{t}_{ij}$ is parallel to the loading surface (eqn 8b).

For the purpose of illustration, consider a von Mises type loading surface, $F = \bar{\tau} - H_1 = 0$ where $\bar{\tau} = (1/2\ s_{ij}\,s_{ij})^{1/2}$, $s_{ij} = \sigma_{ij} - \delta_{ij}\sigma$ = stress deviator, $\sigma = \sigma_{kk}/3$, δ_{ij} = Kronecker delta. Substituting into eqn (36), we find that the deviator part of the tangential plastic strain increment is

$$\mathrm{d}e^{t}_{ij} = \frac{1}{h_t}\left(\mathrm{d}s_{ij} - s_{ij}\,\frac{\mathrm{d}\bar{\tau}}{\bar{\tau}}\right). \tag{38}$$

We now observe that this is an expression that was implied in Budianski's work[42–44] proposed by Rudnicki and Rice[45, 46] as a simple form of vertex-hardening. They used this expression in a study of material instabilities of strain-localization type, and they found that the presence of such inelastic strains has a profound destablizing effect[45]. We should note that this is in spite of the fact that the presence of $d\epsilon^t_{ij}$ does not cause violation of Drucker's postulate, because ΔW is never less than it would be without $d\epsilon^t_{ij}$.

Since the use of $d\epsilon^t_{ij}$ makes ΔW positive for increments $d\sigma_{ij}$ parallel to the loading surface, and since we want to achieve continuity of ΔW as a function of $d\sigma_{ij}$ direction, we must assume that the expressions for $d\epsilon^t_{ij}$ and $d\epsilon^n_{ij}$ (eqns 11 and 23) for all unloading directions for which vector $d\sigma_{ij}$ points inside the loading surface are valid as long as $\Delta W \geq 0$. For this purpose it is, however, necessary to assume that $d\epsilon^n_{ij}$ is never directed inside the loading surface, and because $d\mu$ according to eqn (23) becomes negative for unloading, one must redefine $d\mu$. This may be done for example as follows

$$\begin{array}{l}\text{For } DF \geq 0: \quad d\mu = DF/h_n; \\ \text{For } DF \leq 0: \quad d\mu = -DF/h'_n;\end{array} \quad DF = \frac{\partial F}{\partial \sigma_{km}} d\sigma_{km} \tag{39}$$

where h_n is the normal plastic hardening modulus for forward normal loading and h'_n is the same for backward normal loading (unloading); both h_n and h'_n are positive.

Note that if the expression $d\mu = DF/h_n$ were used also for $dF \leq 0$ (unloading), then ΔW would never become zero; rather, it would remain positive for all unloading, thus leading to $d\epsilon^{pl}_{ij}$ that is of opposite direction than $d\sigma_{ij}$, which is impossible. This would contrast with the endochronic theory, where the direction of $d\epsilon^{pl}_{ij}$ for unloading is the same as that of $d\epsilon_{ij}$. It is because of this that negative ΔW for unloading (Fig. 8b) appears to be the lesser evil, tolerated in the simplest endochronic theory.

The fact that the plastic hardening modulus for normal loading must change its sign (eqn 39) when the direction parallel to the loading surface is crossed (i.e. when an outward direction changes to an inward one) takes away some of the appeal of the incrementally linear expressions (36) and (38) for tangential inelastic strain increments.

For geometric interpretation it is helpful to characterize the magnitude of the deviation of the $d\sigma_{ij}$ direction from the direction of the outward normal of the loading surface by an angle, θ(Fig 8e). This angle may, for example, be defined by

$$\cos\theta = n_{ij}\frac{d\sigma_{ij}}{\|d\boldsymbol{\sigma}\|} = \frac{\partial F}{\partial \sigma_{ij}} d\sigma_{ij}\left(\frac{\partial F}{\partial \sigma_{km}}\frac{\partial F}{\partial \sigma_{km}} d\sigma_{pq}\, d\sigma_{pq}\right)^{-1/2} \tag{40}$$

where $\|d\boldsymbol{\sigma}\| = (d\sigma_{ij}\, d\sigma_{ij})^{1/2}$ = magnitude of $d\sigma_{ij}$; $0 \leq \theta \leq \pi$. Substituting now eqn (39) into eqn (11) for $d\epsilon^n_{ij}$, and using $n_{ij}\, d\sigma_{ij} = \cos\theta\|d\boldsymbol{\sigma}\|$, we obtain

$$\Delta W_n = \frac{1}{2} d\sigma_{ij}\, d\epsilon^n_{ij} = \begin{cases} \dfrac{1}{2h_n}\left\|\dfrac{\partial F}{\partial \boldsymbol{\sigma}}\right\|^2 \|d\boldsymbol{\sigma}\|^2\cos^2\theta & \text{for } 0 \leq \theta \leq \pi/2 \\ -\dfrac{1}{2h'_n}\left\|\dfrac{\partial F}{\partial \boldsymbol{\sigma}}\right\|^2 \|d\boldsymbol{\sigma}\|^2\cos^2\theta & \text{for } \pi/2 \leq \theta \leq \pi \end{cases} \tag{41}$$

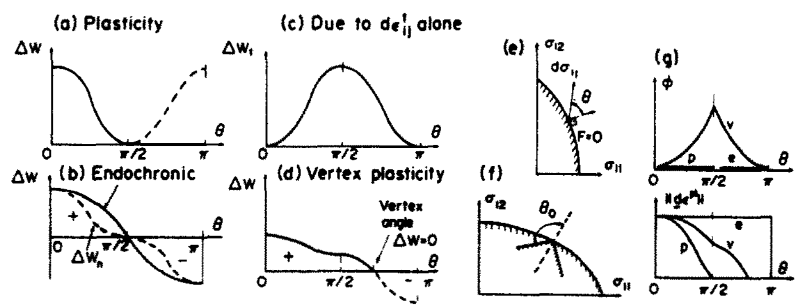

Fig. 8. Dependence of second-order work in load-unload cycles upon direction angle θ of applied stress increment for various theories.

Using eqn (36) we may calculate for the tangential inelastic strains:

$$\Delta W_t = \frac{1}{2h_t} \|\mathrm{d}\sigma\|^2 \sin^2 \theta \tag{42}$$

The dependence of ΔW_t and ΔW_n on θ at constant $\|\mathrm{d}\boldsymbol{\sigma}\|$ is sketched in Fig. 8(c) and (b). We may now determine that $\Delta W = \Delta W_n + \Delta W_t = 0$ occurs for angle $\theta = \theta_0 \geq \pi/2$ (Fig. 8d) such that

$$\tan \theta_0 = -\left(\frac{h_t}{h'_n}\right)^{1/2} \left\|\frac{\partial F}{\partial \boldsymbol{\sigma}}\right\| = -\left(\frac{h_t}{h'_n}\right)^{1/2} \frac{\partial F}{\partial \sigma_{ij}} \frac{\partial F}{\partial \sigma_{ij}} \tag{43}$$

Note that $\theta_0 = \pi/2$ for $h_t \to \infty$ and $\theta_0 = \pi$ for $h_t \to 0$. Also, note that ΔW would never vanish if $\mathrm{d}\mu$ were given by eqn (11) for all θ. To avoid negative ΔW (eqn 4), one may then impose the condition $\Delta W = 0$ and $\mathrm{d}\epsilon_{ij}^{pl} = 0$ for $\theta \geq \theta_0$. The limiting directions $\theta = \theta_0$ are represented by two straight lines (Fig. 8) which form an outward pointing vertex. In this light, the tangential plastic strain increments, $\mathrm{d}\epsilon_{ij}^t$, may be regarded as a manifestation of vertex-hardening. This terminology has been used already by Rudnicki and Rice[45] for eqn (38), and our observation lends an explanation.

For comparison, the variation of ΔW with angle θ at constant $\|\mathrm{d}\boldsymbol{\epsilon}\|$, rather than constant $\|\mathrm{d}\boldsymbol{\sigma}\|$, is sketched in Fig. 8(b) for the ordinary endochronic theory ($\Delta W \sim \cos\theta$). Obviously, this theory would have to be enhanced by tangential inelastic strain increments in order to shift the point $\Delta W = 0$ to an angle $\theta > \pi/2$ and create a vertex effect.

To avoid a discontinuous jump in h_n (eqn 39), a continuous dependence of h upon angle θ would have to be introduced into Rudnicki–Rice's vertex model[45], i.e.

$$\mathrm{d}\epsilon_{ij}^{pl} = \frac{DF}{h(\boldsymbol{\sigma}, \boldsymbol{\epsilon}, \theta)} + \frac{1}{h_t}(\mathrm{d}\sigma_{ij} - n_{ij}\, n_{km}\, \mathrm{d}\sigma_{km}). \tag{44}$$

However, this would deprive the incremental stress–strain relations of their linearity because θ depends on $\mathrm{d}\sigma_{ij}$. (Note that h_t could also depend on θ.)

For isotropic materials, it may be also useful to introduce, instead of θ, two independent angles θ and θ', one for the deviatoric stress space, s_{ij}, and one for the $(\sigma_{kk}, \bar{\tau})$ space, and for each of them separately develop equations analogous to eqns (41)–(44).

An analogous expression can be derived for tangential fracturing stress decrements:

$$\mathrm{d}\sigma_{ij}^t = \phi_t \left(\mathrm{d}\epsilon_{ij} - \frac{\partial \Phi}{\partial \epsilon_{ij}} \frac{\dfrac{\partial \Phi}{\partial \epsilon_{km}} \mathrm{d}\epsilon_{km}}{\dfrac{\partial \Phi}{\partial \epsilon_{pq}} \dfrac{\partial \Phi}{\partial \epsilon_{pq}}} \right). \tag{45}$$

We can again show that $\Delta\Pi$ then becomes augmented by the term $\mathrm{d}\sigma_{ij}^t\, \mathrm{d}\sigma_{ij}^t/2\phi_t$ which is always positive, and draw analogy to eqns (39)–(44).

The normal and tangential plastic strain increments do not exhaust all possibilities. As far as the work inequalities are concerned, any further plastic strain increment $\mathrm{d}\epsilon_{ij}^l$ which does no work on $\mathrm{d}\sigma_{ij}$ is a possibility. This includes increments $\mathrm{d}\epsilon_{ij}^l$ in lateral directions that are normal to $\mathrm{d}\sigma_{ij}$ and to $\mathrm{d}\epsilon_{ij}^n$ (as well as $\mathrm{d}\epsilon_{ij}^t$) (Fig. 5b), i.e. which satisfy the conditions

$$\mathrm{d}\sigma_{ij}\, \mathrm{d}\epsilon_{ij}^l = 0, \quad \frac{\partial F}{\partial \sigma_{ij}} \mathrm{d}\epsilon_{ij}^l = 0. \tag{46}$$

Since there are six components of $\mathrm{d}\epsilon_{ij}^l$, and we have two conditions, four independent tensors $\mathrm{d}\epsilon_{ij}^l$ are possible (but in plane strain only one). We will not pursue the question of $\mathrm{d}\epsilon_{ij}^l$ further because there are no test data on this phenomenon.

As observed, the stress increments that are parallel to the loading surface and give $\Delta W > 0$, as well as the inelastic strains normal to the stress increments, represent responses which are stable in the small according to Drucker's postulate ($\Delta W \geq 0$). However, such responses, real as they undoubtedly are, do not necessarily satisfy inequality (34), i.e. $(\sigma_{ij} - \sigma^*_{ij})\, d\epsilon^{pl}_{ij}$ can be negative for some σ^*_{ij}, even though $d\sigma_{ij}\, d\epsilon^{pl}_{ij}$ is non-negative. So, these responses are not necessarily stable in the large in Drucker's sense, and they violate Hill's principle of maximum plastic work[41, 36].

It is, of course, also possible to write nonlinear incremental stress–strain relations for which $d\epsilon^{pl}_{ij}$ is not normal to the loading surface yet Drucker's postulate (eqn 4) is satisfied for all $d\sigma_{ij}$ that do not point inward. An example is an expression discussed by Mróz[47]

$$d\epsilon^{pl}_{ij} = \frac{1}{h_2}\frac{\partial F}{\partial \sigma_{km}}\, d\sigma_{km}\, \frac{d\sigma_{ij}}{(d\sigma_{rs}\, d\sigma_{rs})^{1/2}} \ (\text{if } DF \geq 0) \tag{47}$$

for which $\Delta W \sim \cos\theta$ at constant $\|d\sigma\|$. So the curve of ΔW vs angle θ looks the same as shown in Fig. 8(b) for the endochronic theory.

To understand the differences between various possible expressions for $d\epsilon^{pl}_{ij}$, some other characteristics may be also useful; for instance the variations of the direction angle φ of $d\epsilon^{pl}_{ij}$, defined by $\cos\varphi = n_{ij}\, d\epsilon^{pl}_{ij}/\|d\epsilon^{pl}_{ij}\|$, and of the magnitude $\|d\epsilon^{pl}_{ij}\|$ as a function of angle θ. These dependences are exemplified in Fig. 8(g) for classical plasticity with normality (p), endochronic theory (e) and vertex-hardening (v) according to eqn (38).

In this work we concentrate on analyzing implications of work inequalities for the expressions for inelastic stresses and strains. The more serious (and more difficult) question is, however, that of the actual expression to be used for a given material. This question can be answered only on the basis of experiments or micromechanics models, which we do not consider here. We should at least point out that both of these suggest the existence of vertex effects and of inelastic strain or stress increments that do not obey normality. This is true of many materials, including plastic polycrystalline metals[42, 48–52, 43] as well as frictional geomaterials[1, 11, 12].

8. FRICTIONALLY BLOCKED ELASTIC ENERGY

Spring-loaded block

We have already remarked that internal friction can cause ΔW to be negative yet the material remains stable. This was mentioned by Drucker[16, 19, 20] and an instructive example we have already used (Fig. 6) was given by Mandel[28] (for further discussion see Meier[31]). The characteristic property which allows the material to be stable even when it releases energy ($\Delta W < 0$) is that the released energy is an elastic energy which has been blocked by friction and is released due to a decrease in the compressive force that produces the friction. We will illustrate it first by an example which is more general than that of Mandel[28] as it involves shear dilatancy.

Consider that the stress–strain relation is modeled by a block which slides on a rough surface and is loaded by a horizontal spring of spring constant C (Fig. 6a). The slip corresponds to plastic shear angle γ^{pl}, the horizontal applied force to shear stress τ and the vertical applied force (positive for tension) to normal stress σ. The roughness of the surface would normally cause the slip $d\gamma^{pl}$ to be accompanied by a certain vertical displacement or dilatancy $d\epsilon^{pl} = \beta d\gamma^{pl}$, where β represents the dilatancy factor and $d\gamma^{pl} \geq 0$.

First we recall the well-known fact that sliding of the block violates the normality rule. In the (τ, σ) space the slip condition is $F = \tau + \beta'\sigma - H_1 = 0$, where β' = friction coefficient and H_1 = hardening parameter = current cohesion limit. The slip condition is graphically represented by the line in Fig. 9. The normal to this slip surface has the inclination $1/\beta'$. The vector $\vec{d\epsilon}$ of slip and vertical displacement, plotted in the same diagram, gives a line of inclination $1/\beta$. Obviously, normality exists only for $\beta = \beta'$, but the value of β is independent of β' and in particular β may be zero.

The presence of the spring causes a change in the limit value of $\tau + \beta'\sigma$, which represent hardening. The current cohesion limit is $H_1 = \tau_p + C\gamma^{pl}$, where τ_p = initial cohesion limit. Assume now that the spring force S is such that sliding of the block is imminent. Consequently,

$$F = \tau + \beta'\sigma - \tau_p - C\gamma^{pl} = 0. \tag{48}$$

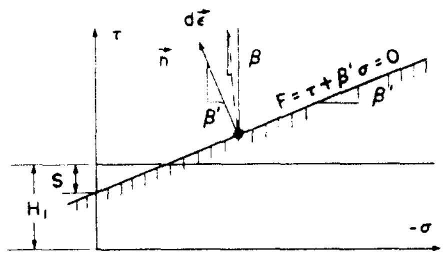

Fig. 9. Yield condition for frictional block of Fig. 6.

Now we apply load increments $d\tau$ and $d\sigma$ such that $d\tau$ is opposite to the spring force and $|d\tau| < k|d\sigma|$ (Fig. 6a), and we realize that this causes the block to slide to the right by $d\gamma^{pl}$, with corresponding normal displacement $\beta d\gamma^{pl}$. This is so because $d\sigma$ reduces the friction capacity more than $d\tau$ increases it by relieving the spring. Equilibrium after sliding requires that $F + dF = (\tau + d\tau) + \beta'(\sigma + d\sigma) - \tau_p - C(\gamma^{pl} + d\gamma^{pl}) = 0$. Subtracting eqn (48), we get the condition of continuing equilibrium

$$dF = d\tau + \beta' d\sigma - C d\gamma^{pl} = 0. \tag{49}$$

Thus, we have

$$d\gamma^{pl} = \frac{1}{C}(d\tau + \beta' d\sigma), \quad d\epsilon^{pl} = \beta d\gamma^{pl} \tag{50}$$

and the second-order work done by $d\tau$ and $d\sigma$ on the block will be

$$\Delta W = \frac{1}{2}(d\tau d\gamma^{pl} + d\sigma d\epsilon^{pl}) = \frac{1}{2C}(d\tau + \beta d\sigma)(d\tau + \beta' d\sigma). \tag{51}$$

If $\beta = \beta'$, which is the case of normality (Fig. 9), the expression for ΔW is symmetric and we always have $\Delta W > 0$. However, if $\beta = 0$ (flat surface, no dilatancy) and if we choose $d\tau < 0$ and $d\sigma > -d\tau/\beta'$ we get $\Delta W < 0$, i.e. energy is released by the block. Nevertheless, the block is stable because infinitely small loads $d\tau$ and $d\sigma$ cause an infinitely small deformation $d\gamma^{pl}$. More generally, if $\beta' > \beta > 0$ and if we choose $d\sigma > 0$ and $-\beta' d\sigma < d\tau < -\beta d\sigma$ (< 0), we always get $\Delta W < 0$, and the block is still stable. Note, however, that if the spring force were replaced with a constant force (e.g. weight), no new equilibrium would exist, i.e. the system would be unstable. Thus, the stability is obviously due to the fact that the driving force decreases with increasing displacements, as is true for the release of elastic energy.

From the finding that $\Delta W < 0$ is not an unstable situation in these cases we may conclude that a release of frictionally blocked elastic energy is harmless for stability. We have seen that this can occur only if $\beta \neq \beta'$ (lack of normality) and thus it is expedient to rewrite eqn (51) in the form

$$\Delta W = \Delta W_n + \Delta W_f$$

where

$$\Delta W_n = \frac{1}{2C}(d\tau + \beta d\sigma)^2, \; \Delta W_f = \frac{\beta' - \beta}{2C} d\sigma(d\tau + \beta d\sigma). \tag{53}$$

Here ΔW_n is always positive, and it is solely ΔW_f which may cause ΔW to become negative.

Frictional continuum

To establish continuum analogy to the preceding example, we must: (a) express ΔW and F by means of differentials of the same variables (b) express ΔW in terms of invariants because F must be given in terms of invariants; (c) express ΔW by means of only two stress variables and two strain variables, and in such a manner that (d) cross products be absent, just as $d\sigma\, d\tau^{pl}$ or $d\tau\, d\epsilon^{pl}$ is absent from eqn (51). This last condition is the salient property which defines friction, namely, the friction-producing force (such as σ in Fig. 6) is a force that does no work on some displacement (on $d\gamma^{pl}$ in Fig. 6) yet affects this displacement. These conditions can be met by writing

$$\Delta W = \frac{1}{2} d\sigma\, d\epsilon^{pl}_{kk} + \frac{1}{2} ds_{ij}\, de^{pl}_{ij} = \frac{1}{2} d\sigma(3d\epsilon^{pl}) + \frac{1}{2} p\, d\bar{\tau}\, d\hat{\gamma}^{pl} \tag{54}$$

In which

$$p = p_{ij} q_{ij}, \quad p_{ij} = \frac{ds_{ij}}{d\bar{\tau}}; \quad q_{ij} = \frac{de^{pl}_{ij}}{d\hat{\gamma}^{pl}}, \quad \bar{\tau} = \left(\frac{1}{2} s_{ij} s_{ij}\right)^{1/2} \quad d\hat{\gamma}^{pl} = \left(\frac{1}{2} de^{pl}_{ij}\, de^{pl}_{ij}\right)^{1/2} \tag{55}$$

and $d\bar{\tau} = s_{ij}\, ds_{ij}/2\bar{\tau}$, $\sigma = \sigma_{kk}/3$. Coefficients p_{ij} and q_{ij} characterize the direction of vectors ds_{ij} and de^{pl}_{ij} in the stress space and coefficient p is a function of the angle between these two vectors and of the angle between ds_{ij} and s_{ij}. We chose to normalize these vectors in different ways. Instead of the plastic path length $\hat{\gamma}^{pl}$ (Odquist's hardening parameter) one might think of using $\bar{\gamma}^{pl} = (e^{pl}_{ij}\, e^{pl}_{ij}/2)^{1/2}$ and write $q_{ij} = de^{pl}_{ij}/d\bar{\gamma}^{pl}$ (where $d\bar{\gamma}^{pl} = e^{pl}_{km}\, de^{pl}_{km}/2\bar{\gamma}^{pl}$); but this would be inconvenient since it is $\hat{\gamma}^{pl}$, rather than $\bar{\gamma}^{pl}$, which is suitable as a hardening parameter in the loading function. One might alternatively think of using $p_{ij} = ds_{ij}/d\hat{\tau} = (ds_{ij}\, ds_{ij}/2)^{1/2}$ = stress path length; but this would again be inconvenient because the loading function depends on $\bar{\tau}$ rather than $\hat{\tau}$.

Comparison of eqn (54) with eqn (51) indicates that the variables $d\sigma$, $d\tau$, $d\epsilon^{pl}$, $d\gamma^{pl}$ for the block corresponds to continuum variables $d\sigma$, $d\bar{\tau}$, $3d\epsilon^{pl}$ and $p\ d\hat{\gamma}^{pl}$, respectively. A general loading function for isotropic materials may be considered in the form

$$F(\sigma, \bar{\tau}, J_3, \epsilon^{pl}, \hat{\gamma}^{pl}, H_k) = 0 \tag{56}$$

where $J_3 = s_{ik}s_{km}s_{mi}/3$ = third invariant of s_{ij}; H_k are possible further hardening parameters in addition to ϵ^{pl} and $\hat{\gamma}^{pl}$. Differentiating F, we get

$$dF = \frac{\partial F}{\partial \sigma} d\sigma + \frac{DF}{D\bar{\tau}} d\bar{\tau} + \frac{DF}{D\hat{\gamma}^{pl}} \frac{p d\hat{\gamma}^{pl}}{p} = 0 \tag{57}$$

where

$$\frac{DF}{D\bar{\tau}} = \frac{\partial F}{\partial \bar{\tau}} + \frac{\partial J_3}{\partial \bar{\tau}} \frac{\partial F}{\partial J_3}, \quad \frac{DF}{D\hat{\gamma}^{pl}} = \frac{\partial F}{\partial \hat{\gamma}^{pl}} + \frac{2\beta}{3} \frac{\partial F}{\partial \epsilon^{pl}} + \frac{\partial F}{\partial H_k} \frac{\partial H_k}{\partial \hat{\gamma}^{pl}} \tag{58}$$

$$\beta = 3d\epsilon^{pl}/2d\hat{\gamma}^{pl}. \tag{59}$$

The last expression is chosen to define the dilatancy factor, in which $2d\hat{\gamma}^{pl}$ is used because for pure shear it equals $2de^{pl}_{12}$ = plastic shear angle increment. Note that if F depends on J_3, $DF/D\bar{\tau}$ depends on the direction of vector ds_{ij} and if $\beta \neq 0$ or $\partial F/\partial H_k \neq 0$, then $DF/D\hat{\gamma}^{pl}$ depends on the direction of vector de^{pl}_{ij}. Dividing eqn (57) by $DF/D\bar{\tau}$ and keeping in mind the proper correspondence of variables with the frictional block, comparison of eqn (57) with eqn (49) furnishes us

$$C = -\frac{1}{p} \frac{DF/D\hat{\gamma}^{pl}}{DF/D\bar{\tau}}, \quad \beta' = \frac{\partial F/\partial \sigma}{DF/D\bar{\tau}} \quad (\text{if } C, \beta' \geq 0). \tag{60}$$

The dilatancy factor for the block, $d\epsilon^{pl}/d\gamma^{pl}$, corresponds according to definition (50), to the ratio $3\mathrm{d}\epsilon^{pl}/p\,\mathrm{d}\bar{\gamma}^{pl}$ which equals $2\beta/p$ where β is given by (59). Thus, according to (53), the frictionally blocked second-order elastic energy is

$$\Delta W_f = \frac{\beta' - \beta^*}{2C}\,\mathrm{d}\sigma(\mathrm{d}\bar{\tau} + \beta^*\,\mathrm{d}\sigma), \quad \beta^* = \frac{2}{p}\beta. \tag{61}$$

This expression is general, applicable to any loading function. Note that the equivalent spring constant C for the frictionally blocked elastic energy as well as dilatancy factor β^* depends on the directions of $\mathrm{d}s_{ij}$ and $\mathrm{d}\epsilon_{ij}^{pl}$. So does friction coefficient β' if $\partial F/\partial J_3 \neq 0$.

Obviously we must have $C \geq 0$ and $\beta' \geq 0$. All subsequent considerations are invalid if this is not so. Not only the derivatives of F, but also p must be checked for this purpose. Normally $(DF/D\bar{\gamma}^{pl})/(DF/D\bar{\tau}) < 0$, and then p must be positive; this is so if $\mathrm{d}s_{ij}\,\mathrm{d}\epsilon_{ij}^{pl} > 0$.

Work inequality

Let us now introduce the work expression

$$\Delta\bar{W} = \Delta W - \Delta W_f = \frac{1}{2}\,\mathrm{d}\sigma_{ij}\,\mathrm{d}\epsilon_{ij}^{pl} - \Delta W_f. \tag{62}$$

In case that ΔW becomes negative due to release of the frictionally blocked elastic energy, $\Delta\bar{W}$ will still remain positive, and the situation is as we learnt stable. On the other hand if ΔW becomes negative for other reasons ($\Delta W_f = 0$), so will $\Delta\bar{W}$. Thus the following proposition, which gives a less restrictive (more general) sufficient condition for material stability than Drucker's postulate, appears to be true for isotropic materials under controlled stress or strain:

$$\textit{If either } \Delta W > 0 \textit{ or } \Delta\bar{W} > 0\textit{, the material is stable.} \tag{63}$$

Note that we cannot discard the condition $\Delta W > 0$ because ΔW_f can be negative when $\beta\mathrm{d}\sigma < -\mathrm{d}\bar{\tau}$ even if $\Delta W > 0$.

In stress space $(\sigma, \bar{\tau})$, the domain of $\mathrm{d}\sigma_{ij}$ vectors that give $\Delta W > 0$ occupies the halfplane a in Fig. 10(b) and the domain of those that give $\Delta\bar{W} > 0$ occupies a certain other halfplane b. The combined domain of vectors of applied stress increments $\mathrm{d}\sigma_{ij}$ for which the response is inelastic and is assured to be stable occupies the union of these two halfplanes, i.e. the reentrant wedge (Fig. 10b), one side of which is tangent to the loading surface.

Condition (62) may equivalently be stated as follows:

$$\textit{If } \Delta W - \chi\Delta W_f > 0 \textit{ for any } \chi \in (0, 1)\textit{, the material is stable.} \tag{64}$$

Since $\Delta W - \chi\Delta W_f$ is a linear function of χ, the extremes can occur only at $\chi = 0$ and $\chi = 1$, and so condition (62) follows from (64) and vice versa.

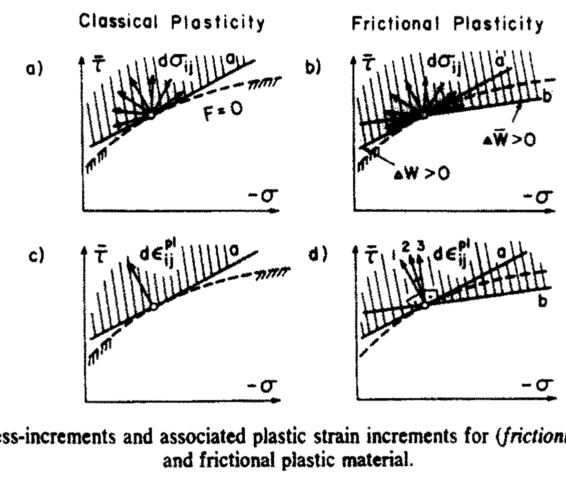

Fig. 10. Stable stress-increments and associated plastic strain increments for (*frictionless*) plastic material and frictional plastic material.

Plastic strain increment

In view of the fact that the flow rule can be derived from $\Delta W > 0$, it is interesting to see what follows by the same line of reasoning from the more general condition $\Delta W - \chi \Delta W_f > 0$. We have

$$\Delta W - \chi \Delta W_f = \frac{1}{2} \mathrm{d}s_{ij} \, \mathrm{d}\epsilon_{ij}^{pl} + \frac{3}{2} \mathrm{d}\sigma \mathrm{d}\epsilon^{pl} - \chi \frac{\beta' - \beta^*}{2C} \mathrm{d}\sigma(\mathrm{d}\bar{\tau} + \beta^* \, \mathrm{d}\sigma) > 0. \tag{65}$$

The loading criterion (4) may be written as

$$\frac{\partial F}{\partial s_{ij}} \mathrm{d}s_{ij} + \frac{\partial F}{\partial \sigma} \mathrm{d}\sigma > 0 \tag{66}$$

The ratio of expressions (65) and (66) must obviously be positive and denoting it as $\mathrm{d}\mu/2$ ($\mathrm{d}\mu > 0$), we get

$$\left(\frac{\partial F}{\partial \sigma_{ij}} \mathrm{d}\mu - \mathrm{d}\epsilon_{ij}^{pl}\right) \mathrm{d}s_{ij} + \left[\frac{\partial F}{\partial \sigma} \mathrm{d}\mu - 3\mathrm{d}\epsilon^{pl} + \chi \frac{\beta' - \beta^*}{C} (\mathrm{d}\bar{\tau} + \beta^* \mathrm{d}\sigma)\right] \mathrm{d}\sigma = 0 \tag{67}$$

This equation must hold for any $\mathrm{d}s_{ij}$ and $\mathrm{d}\sigma$. Pursuing the same line of argument as in classical plasticity, we note that this is possible only if the bracketed expressions vanish, i.e. if

$$\mathrm{d}e_{ij}^{pl} = \frac{\partial F}{\partial s_{ij}} \mathrm{d}\mu, \quad 3\mathrm{d}\epsilon^{pl} = \frac{\partial F}{\partial \sigma} \mathrm{d}\mu - \chi \frac{\beta' - \beta^*}{C} (\mathrm{d}\bar{\tau} + \beta^* \, \mathrm{d}\sigma)$$

$$= \frac{\partial F}{\partial \sigma} \mathrm{d}\mu - \frac{\chi}{C}\left(\beta' - \frac{3}{p\hat{F}} \frac{\mathrm{d}\epsilon^{pl}}{\mathrm{d}\mu}\right)\left(\mathrm{d}\bar{\tau} + \frac{3}{p\hat{F}} \frac{\mathrm{d}\epsilon^{pl}}{\mathrm{d}\mu} \mathrm{d}\sigma\right) \tag{68}$$

where we used $\beta^* = (2/p)3\mathrm{d}\epsilon^{pl}/2\mathrm{d}\bar{\gamma}^{pl}$ and substituted $\mathrm{d}\hat{\gamma}^{pl} = \hat{F}\mathrm{d}\mu$ where $\hat{F} = [(\partial F/\partial s_{ij})(\partial F/\partial s_{ij})/2]^{1/2}$, which follows from the above expression for $\mathrm{d}e_{ij}^{pl}$.

Equation (68) governs the ratio of the $\mathrm{d}\epsilon_{ij}^{pl}$ components, i.e. the direction of the vector $\mathrm{d}\epsilon_{ij}^{pl}$. Using the same logic as in classical plasticity, we could further consider the magnitude of $\mathrm{d}\epsilon_{ij}^{pl}$ to be proportional to $\Delta W - \chi \Delta W_f$.

Except for $\chi = 0$, eqn (68) is nonlinear with regard to $\mathrm{d}\epsilon_{ij}^{pl}/\mathrm{d}\mu$. Moreover, C, β' and p depend on the direction of vectors $\mathrm{d}\sigma_{ij}$ and $\mathrm{d}\epsilon_{ij}^{pl}$, which complicates its practical use. The equation is, however, instructive.

What we should observe is that, by pursuing basically the same line of reasoning as used in classical plasticity to derive the flow rule, we now obtain no unique direction of vector $\mathrm{d}\epsilon_{ij}^{pl}$ but a continuous set of infinitely many possible directions characterized by an arbitrary parameter $\chi \in (0, 1)$. In the volumetric section of stress space all the possible directions of $\mathrm{d}\epsilon_{ij}^{pl}$ fill a continuous fan of finite angle (1–2–3 in Fig. 10d). One boundary direction of the fan is the normal to the loading surface ($\chi = 0$). The other boundary direction (3 in Fig. 11d, $\chi = 1$) can be thought to be normal to some other surface (b in Fig. 10d).

This situation resembles that encountered in classical plasticity at the corner of the loading surface. It is also similar to what is assumed in nonassociated plasticity; however, the direction of fan boundary ($\chi = 1$) is not unique and is not known in advance as it is not uniquely determined by the current loading surface in the stress space. Moreover, material stability is assured for all loading directions $\mathrm{d}\sigma_{ij}$ within the fan, while in nonassociated plasticity the stability is not assured.

Stability in the large

Finally, let us mention that the spring-loaded frictional block also violates Drucker's postulate of stability in the large. Let the block be initially in equilibrium under loads σ^0 and τ^0 within the yield surface. The shear stress is then increased to the value τ at imminent sliding and, applying $\mathrm{d}\sigma$ and $\mathrm{d}\tau$ same as before the block slips to the right. The stress is then returned to σ^0 and τ^0. The work during this cycle is $\delta W = (\tau - \tau^0)\mathrm{d}\gamma^{pl}$ which is negative yet the material is stable. This illustrates that the postulate is violated due to friction.

9. ELASTIC ENERGY BLOCKED BY FRICTIONAL FRACTURING RESISTANCE

An analogous situation may arise in fracturing deformation. To illustrate it, consider first a model of the stress–strain relation in the form of a unit deformable elastic-fracturing block (Fig. 11). The horizontal and vertical displacements of the top face of the block, representing the shear strain γ and the normal strain ϵ, are controlled by pistons. The bottom face slides on rollers and is held by a horizontal spring of spring constant C. The spring force alters the shear stress in the block and thereby it modifies the limit $\gamma = \gamma_0$ at which fracturing occurs. Thus, the spring models the fracturing hardening of the material. We assume the limit condition of fracturing of the block to be

$$\Phi = \gamma + \alpha'\epsilon - H_1', \text{ with } H_1' = \gamma_0 + \frac{\tau^{fr}}{C} \tag{69}$$

where τ^{fr} = fracturing relaxation of shear stress τ, and α' = fracturing friction coefficient (see[1]), which represents the effect of normal strain on the fracturing limit in shear.

Assume the block is initially at the limit of further fracturing. Now consider that we retract the vertical piston (Fig. 11) by $d\epsilon$ and at the same time expand the horizontal piston by $d\gamma$. The initial limit state is given by eqn (69) and the new limit state is given by $\Phi + d\Phi = (\gamma + d\gamma) + \alpha'(\epsilon + d\epsilon) - \gamma_0 - (\tau^{fr} + d\tau^{fr})/C = 0$. Subtracting eqn (69) we have

$$d\Phi = d\gamma + \alpha' d\epsilon - d\tau^{fr}/C. \tag{70}$$

Therefore

$$d\tau^{fr} = C(d\gamma + \alpha' d\epsilon), \quad d\sigma^{fr} = \alpha d\tau^{fr} \tag{71}$$

where we also included the hydrostatic stress relaxation $d\sigma^{fr}$ associated with $d\tau^{fr}$; α = fracturing dilatancy factor. The complementary work is

$$\Delta\Pi = \frac{1}{2}(d\gamma d\tau^{fr} + d\epsilon d\sigma^{fr}) = \frac{1}{2} C(d\gamma + \alpha d\epsilon)(d\gamma + \alpha' d\epsilon). \tag{72}$$

If $\alpha = \alpha'$, which is the case of normality, the expression for $\Delta\Pi$ is symmetric and we always have $\Delta\Pi > 0$. However, if $\alpha = 0$ and if we choose $d\gamma < 0$ and $d\epsilon > -d\gamma/\alpha'$, we get $\Delta\Pi < 0$, i.e. energy is released by the system. Yet, the system is stable since infinitely small disturbances $d\gamma$ and $d\epsilon$ cause infinitely small changes $d\tau$ and $d\sigma$. The same result is obtained in the more general case when, for $\alpha' > \alpha > 0$, we choose $d\epsilon > 0$ and $-\alpha' d\epsilon < d\gamma < -\alpha d\epsilon (< 0)$.

What is here happening is that expansion $d\epsilon$ tends to diminish the fracturing resistance in shear, and so further fracturing is caused by the spring. The amount of fracturing is limited because the spring force decreases during $d\gamma$. The complementary work $\Delta\Pi$ is negative because $d\tau^{fr}$ is in the positive τ direction if the negative $d\gamma$ is chosen such that the ratio $|d\gamma|/d\epsilon$ is sufficiently small; i.e. the effect of $d\epsilon > 0$ (stimulation of fracturing) prevails over the effect of $d\gamma < 0$ (further hardening). Note that if a constant load (e.g. a weight) were applied instead of a

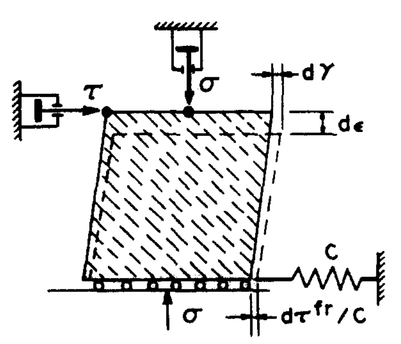

Fig. 11. Example of a fracturing block whose fracturing resistance is sensitive to volume change.

spring, no new equilibrium would exist and thus the system would be unstable. So, the fact that the released energy is of an elastic nature makes an essential difference.

An energy release that does not have a destablizing effect is generally possible when $\alpha' \neq \alpha$. Thus, it is convenient to rewrite eqn (72) in the form

$$\Delta\Pi = \Delta\Pi_n + \Delta\Pi_f \tag{73}$$

$$\text{where } \Delta\Pi_n = \frac{1}{2}C(\mathrm{d}\gamma + \alpha\,\mathrm{d}\epsilon)^2, \quad \Delta\Pi_f = \frac{\alpha' - \alpha}{2}C\,\mathrm{d}\epsilon(\mathrm{d}\gamma + \alpha\,\mathrm{d}\epsilon). \tag{74}$$

For continuum generalization, we may write the second-order complementary work expression in the form

$$\Delta\Pi = \frac{1}{2}\mathrm{d}\sigma^{fr}(3\mathrm{d}\epsilon) + \frac{1}{2}q\,\mathrm{d}\hat{\tau}^{fr}\,\mathrm{d}\bar{\gamma} \tag{75}$$

$$q = p'_{ij}q'_{ij}, \quad p'_{ij} = \frac{\mathrm{d}s^{fr}_{ij}}{\mathrm{d}\hat{\tau}^{fr}}, \quad q'_{ij} = \frac{\mathrm{d}e_{ij}}{\mathrm{d}\bar{\gamma}},$$

$$\mathrm{d}\hat{\tau}^{fr} = \left(\frac{1}{2}\mathrm{d}s^{fr}_{ij}\,\mathrm{d}s^{fr}_{ij}\right)^{1/2}, \quad \bar{\gamma} = \left(\frac{1}{2}e_{ij}\,e_{ij}\right)^{1/2} \tag{76}$$

and $\mathrm{d}\bar{\gamma} = e_{ij}\,\mathrm{d}e_{ij}/2\bar{\gamma}$, $\epsilon = \epsilon_{kk}/3$. Coefficients p_{ij} and q_{ij} characterize the directions of vectors $\mathrm{d}s^{fr}_{ij}$ and $\mathrm{d}\bar{e}^{pl}_{ij}$ in the stress space. We do not use $\bar{\tau}^{fr}$ instead of $\hat{\tau}^{fr}$ because the latter is more appropriate for the loading function as the damage parameter, and we do not use path length $\hat{\gamma}$ instead of $\bar{\gamma}$ because the fracturing loading function depends on $\bar{\gamma}$ rather than $\hat{\gamma}$. Comparison of eqn (75) with eqn (72) indicates that the variables $\mathrm{d}\gamma$, $\mathrm{d}\epsilon$, $\mathrm{d}\tau^{fr}$ and $\mathrm{d}\sigma^{fr}$ for the block correspond to the continuum variables $\mathrm{d}\bar{\gamma}$, $3\mathrm{d}\epsilon$, $q\mathrm{d}\bar{\tau}^{fr}$ and $\mathrm{d}\sigma^{fr}$, respectively. A general loading function for isotropic materials may be considered in the form

$$\Phi(\epsilon, \bar{\gamma}, J_3^{\epsilon}, \sigma^{fr}, \hat{\tau}^{fr}, H'_k) = 0 \tag{77}$$

where $J_3^{\epsilon} = e_{ik}\,e_{km}\,e_{mi}/3$ = third invariant of e_{ij}; H'_k are possible further hardening parameters. Differentiating Φ we get

$$\mathrm{d}\Phi = \frac{1}{3}\frac{\partial\Phi}{\partial\epsilon}(3\mathrm{d}\epsilon) + \frac{D\Phi}{D\bar{\gamma}}\mathrm{d}\bar{\gamma} + \frac{DF}{D\hat{\tau}^{fr}}\frac{q\mathrm{d}\hat{\tau}^{fr}}{q} = 0 \tag{78}$$

where

$$\frac{D\Phi}{D\bar{\gamma}} = \frac{\partial\Phi}{\partial\bar{\gamma}} + \frac{\partial J_3^{\epsilon}}{\partial\bar{\gamma}}\frac{\partial\Phi}{\partial J_3^{\epsilon}}, \frac{DF}{D\hat{\tau}^{fr}} = \frac{\partial F}{\partial\hat{\tau}^{fr}} + \alpha\frac{\partial F}{\partial\sigma^{fr}} + \frac{\partial F}{\partial H'_k}\frac{\partial H'_k}{\partial\hat{\tau}^{fr}} \tag{79}$$

$$\alpha = \mathrm{d}\sigma^{fr}/\mathrm{d}\hat{\tau}^{fr}. \tag{80}$$

The last expression is chosen to define the fracturing dilatancy factor. Dividing (79) by $\partial\Phi/\partial\bar{\gamma}$ and comparing it with (70), we get

$$C' = -q\frac{D\Phi/D\bar{\gamma}}{DF/D\hat{\tau}^{fr}}, \quad \alpha' = \frac{1}{3}\frac{\partial\Phi/\partial\epsilon}{\partial\Phi/\partial\bar{\gamma}} \quad (\text{if } C, \alpha' \geq 0) \tag{81}$$

and according to (74) the elastic energy blocked by fracturing resistance is

$$\Delta\Pi_f = \frac{\alpha' - \alpha}{2}C'\,\mathrm{d}\,\epsilon(\mathrm{d}\bar{\gamma} + \alpha\,\mathrm{d}\epsilon). \tag{82}$$

Note that C', α' and α depend on the directions of the vectors of $\mathrm{d}\epsilon_{ij}$ and $\mathrm{d}\sigma^{fr}_{ij}$ in the strain space.

We may now introduce the complementary work expression

$$\Delta\bar{\Pi} = \Delta\Pi - \Delta\Pi_f = \frac{1}{2}\mathrm{d}\sigma_{ij}^{fr}\,\mathrm{d}\epsilon_{ij} - \Delta\Pi_f. \tag{83}$$

If $\Delta\Pi$ becomes negative as a result of a release of elastic energy blocked by fracturing resistance, which is a stable situation, $\Delta\bar{\Pi}$ remains positive. However, if $\Delta\Pi$ becomes negative for other reasons ($\Delta\Pi_f = 0$), so does $\Delta\bar{\Pi}$. Thus, the following generalized stability condition appears to be true:

If either $\Delta\Pi > 0$ *or* $\Delta\bar{\Pi} > 0$, *the material is stable under controlled strain conditions.* (84)

Concerning the direction of the vector $\mathrm{d}\sigma_{ij}^{fr}$, equations analogous to (64)–(68) could be written, leading to similar conclusions.

10. INVERSE FRICTION AND OTHER FRICTIONAL EFFECTS

In paragraph "Frictional Continuum" of Section 8 we stated conditions (a)–(d) for analogy with the frictional block (Fig. 6) and defined the friction-producing force as a force that does no work on some displacement yet affects that displacement. From eqn (54) we saw that $\mathrm{d}\sigma$ does no work on $\mathrm{d}\bar{\gamma}^{pl}$ but affects it if the loading function depends on both σ and $\bar{\gamma}^{pl}$ (eqn 56). We may now, however, notice from eqn (54) that, conversely, $\mathrm{d}\bar{\tau}$ does no work on $\mathrm{d}\epsilon^{pl}$ yet can affect $\mathrm{d}\epsilon^{pl}$ if the loading function depends on ϵ^{pl}, as is normal for geomaterials. So, $\mathrm{d}\bar{\tau}$ may alternatively be regarded as a friction-producing force, and may be assumed to correspond to $\mathrm{d}\sigma$ for the block (Fig. 6), while $\mathrm{d}\sigma$, $p\,\mathrm{d}\bar{\gamma}^{pl}$ and $3\mathrm{d}\epsilon^{pl}$ are assumed to correspond to $\mathrm{d}\tau$, $\mathrm{d}\epsilon^{pl}$ and $\mathrm{d}\gamma^{pl}$ for the block. In this case, which we may call inverse friction, we better write the differential of eqn (56) in the form

$$\mathrm{d}F = \frac{DF}{D\bar{\tau}}D\bar{\tau} + \frac{\partial F}{\partial\sigma}\mathrm{d}\sigma + \frac{1}{3}\frac{DF}{D\epsilon^{pl}}(3\,\mathrm{d}\epsilon^{pl}) = 0 \tag{85}$$

where

$$\frac{DF}{D\epsilon^{pl}} = \frac{\partial F}{\partial\epsilon^{pl}} + \frac{3}{2\beta}\frac{\partial F}{\partial\bar{\gamma}^{pl}} \tag{86}$$

while $DF/D\bar{\tau}$ and β are given by eqns (58) and (59). Comparison of eqn (85) with eqn (49) now yields

$$C = -\frac{1}{3}\frac{DF/D\epsilon^{pl}}{\partial F/\partial\sigma}, \quad \beta' = \frac{DF/D\bar{\tau}}{\partial F/\partial\sigma} \tag{87}$$

where β' may be called the inverse friction coefficient. The expression for ΔW_f has again the form of eqn (61) in which, though, $\beta^* = p\,\mathrm{d}\gamma^{pl}/3\mathrm{d}\epsilon^{pl} = p/2\beta$ where $\beta = 3\mathrm{d}\epsilon^{pl}/2\mathrm{d}\bar{\gamma}^{pl}$.

Note that this expression for ΔW_f cannot be reduced to the previous one (eqns 60–61). By contrast, in terms of the loading surface alone these two types of friction would be equivalent; they are both represented by Drucker–Prager loading surface.

The sufficient stability condition (eqn 63) may now be further broadened. We may define $\Delta\tilde{W} = \Delta W - \Delta\tilde{W}_f$ where $\Delta\tilde{W}_f = \Delta W_f$ as given by eqns (87) and (61) with $\beta^* = p/2\beta$, while $\Delta\bar{W}$ remains to be given by eqns (60)–(62). Then:

If $\Delta W > 0$ *or* $\Delta\bar{W} > 0$ *or* $\Delta\tilde{W} > 0$, the *material is stable.* (88)

Instead of reentrant wedge, the domain of $\mathrm{d}\sigma_{ij}$ vectors that produce inelastic strain and stable response now becomes a reentrant pyramid, one side of which is tangent to the loading surface.

Equivalently, we can state that:

If $\Delta W - \chi\Delta W_f - \psi\Delta\tilde{W}_f > 0$ *for any* $\chi \in (0, 1)$, *and* $\psi \in (0, 1)$, *the material is stable.* (89)

To make distinction, $\Delta \tilde{W}_f$ may be called the frictionally blocked volumetric elastic energy, while the previously introduced ΔW_f is the frictionally blocked deviatoric elastic energy.

The line of reasoning that is used in classical plasticity to deduce the normality rule (eqns 65–68) would now generalize eqn (68) to a form that contains two arbitrary parameters χ and ψ and indicates that all stable plastic strain increment vectors fill a pyramid (rather than just a fan). The normal to F lies on one side of this pyramid.

Obviously, it is similarly possible to introduce inverse fracturing friction, distinguish volumetric and deviatoric elastic energies blocked by fracturing resistance, and generalize eqn (86) similarly to eqn (88) or (89).

Are other types of frictional relations possible in isotropic continua? They are not. According to the aforementioned conditions (a)–(d) for friction, we would need to express ΔW as a sum of two terms, other than those in eqn (54), such that only invariants of $d\sigma_{ij}$ and $d\epsilon_{ij}$ are involved and each of these appears only in one term. To do this, we would need to express σ_{ij} as a sum of two stress states such that the nonzero invariants of one of them are zero for the other, and this can be done in only one way, namely by separating σ_{ij} into volumetric and deviatoric stress states. We would need for example, to separate σ_{ij} into two stress states such that I_1 and J_2 (but not J_3) always vanish for one and J_3 (but not I_1 and J_2) always vanishes for the other, but this cannot be done because J_2 and J_3 are nonlinear and I_1 is linear.

It is of course, possible that instead of eqn (54) we alternatively write

$$\Delta W = \frac{3}{2} d\sigma \, d\epsilon^{pl} + \frac{1}{2} p \, d\bar{\tau}_3 \, d\hat{\gamma}_3^{pl} \tag{90}$$

where $p = p_{ij} q_{ij}$, $p_{ij} = ds_{ij}/d\bar{\tau}_3$, $q_{ij} = de_{ij}^{pl}/d\hat{\gamma}_3^{pl}$, $\bar{\tau}_3 = J_3^{1/3}$, $d\gamma_3^{pl} = (J_3^{pl})^{1/3}$; J_3 and J_3^{pl} are the third invariants of s_{ij} and de_{ij}^{pl}. However, since this expression is based on separating σ_{ij} into s_{ij} and σ, just as eqn (54) is, the resulting form of ΔW_f must be equivalent.

It may be instructive to illustrate the meaning of coefficient p from eqn (54). Consider the special case when the medium principal axes of ds_{ij} and e_{ij}^{pl} coincide and let them lie in axis x_2. Also assume that the medium principal values of ds_{ij} and $d\epsilon_{ij}^{pl}$ are zero, i.e. $ds_{22} = de_{22}^{pl} = 0$. For a suitable choice of axes x_1 and x_2, the stress state in plane (x_1, x_2) can be represented as hydrostatic stress $d\sigma$ superimposed on a pure shear stress of magnitude $d\hat{\tau}$. Likewise, in some other axes x_1' and x_2' the strain state in plane (x_1, x_2) can be represented as volumetric strain $d\epsilon^{pl}$ superimposed on a pure shear strain of magnitude $d\hat{\gamma}^{pl}$. Since $ds_{22} = -ds_{11}$, we have $ds_{11} = \pm d\hat{\tau}/\sqrt{2}$. Furthermore, working in the principal axes of ds_{ij}, we have $ds_{12} = 0$, and because de_{22}^{pl} we have $de_{11}^{pl} = \pm d\hat{\gamma}^{pl} \cos 2\omega/\sqrt{2}$ where ω = angle between the maximum principal directions of $d\sigma_{ij}$ and $d\epsilon_{ij}^{pl}$. Thus, from $\Delta W = (3/2)\, d\sigma \, d\epsilon^{pl} + (1/2)\, ds_{ij} \, de_{ij}^{pl}$ we obtain

$$\Delta W = \frac{3}{2} d\sigma \, d\epsilon^{pl} + \frac{1}{2} d\hat{\tau} \,(2 d\hat{\gamma}^{pl} \cos 2\omega) \tag{91}$$

So, coefficient p from eqn (54) is simply equal to $2 \cos 2\omega$, and we see that it may vary between 2 and -2. When $d\sigma_{ij}$ and $d\epsilon_{ij}^{pl}$ are coaxial (as in a cubic triaxial test) $p = \cos 2\omega = 2$ and we have a one-to-one correspondence with the friction block example, without introduction of any further arbitrary factor.

11. QUESTIONS OF UNIQUENESS AND ENDOCHRONIC THEORY

Instead of using second-order work inequalities intimately connected with stability in the small, the theory of inelastic behavior can be also based on other plausible basic hypotheses such as the requirements of uniqueness (or continuity) of response; or convexity of the transformation from the strain space to the stress space as given by the tangential moduli matrix, or local path-independence of response, or strong ellipticity of the resulting eigenvalue problem[18–20, 27, 47, 54]. Under certain additional assumptions (e.g. the afore-mentioned assumptions of consistency, continuity between elastic and plastic regions and incremental linearity), each of these hypotheses leads to a normality rule (and some lead also to convexity of loading surfaces). Similarly as Drucker's postulate, all of these hypotheses are, however, unreasonably strong if applied to all possible situations.

The requirements of uniqueness and strong stability have been used by Sandler[55] in a criticism of the endochronic theory. These requirements, however, lack physical justification in certain respects[7], and in other respects they can be easily met by refinements of the endochronic theory[7, 11, 12].

One further interesting question of uniqueness has recently been raised by Rivlin[56]. He considers two loading paths in the strain space: one is a straight line and the other is a regular staircase path that touches the straight line as shown in Fig. 12. When the number of stairs tends to infinity, the mean distance between these two paths (norm) tends to zero, and so one might wish that the responses to these two loading paths approach each other in the limit.

If one believes in the normality rule and incremental linearity and if one assumes smooth loading surfaces (no corners), the directions and magnitudes of $d\sigma_{ij}^{fr}$ and $d\epsilon_{ij}^{pl}$ are independent of $d\epsilon_{ij}$ and $d\sigma_{ij}$ directions and are determined solely by the magnitudes of ϵ_{ij} and σ_{ij} and the projections of $d\epsilon_{ij}$ and $d\sigma_{ij}$ onto the normal of the loading surface (e.g. DF in eqn 39). In the limit the magnitudes of ϵ_{ij} in Fig. 12 and the normal projections of $d\epsilon_{ij}$ are the same for both paths and therefore the directions of $d\sigma_{ij}^{fr}$ and $d\epsilon_{ij}^{pl}$ for both paths are also the same under the foregoing assumptions. So, the responses in the limit are the same for both paths if classical plasticity without corners on the yield surface is assumed.

For the ordinary endochronic theory, however, these two responses do not approach each other in the limit because the magnitude of inelastic strain increments is proportional to the increments of path length ξ and the length of the staircase path in Fig. 11(a) is independent of the size and number of stairs and is always greater, even in the limit, than the length of the straight path. Is this an incorrect aspect of the endochronic theory? Can it be rectified?

The problem may be considered from various viewpoints:

(1) First of all, when the number of its stairs tends to infinity, the path is not differentiable. Such a path cannot be practically realized. It would not be unduly restrictive to exclude non-differentiable paths from the range of applicability.

(2) Is it, however, physically realistic to expect the same limiting response for the staircase path and the straight path? It is not. This must be concluded by considering the microscopic mechanism of inelastic behavior.

The mechanism may typically consist of microcracking and plastic slip. If we assume the normality rule and absence of corners, then the direction of microcracks and of plastic slips can depend only on ϵ_{ij} and σ_{ij} and be independent of the direction of $d\epsilon_{ij}$ and $d\sigma_{ij}$. However, if we admit that the normality rule need not hold (i.e. vertex effects may exist), then $d\sigma_{ij}^{fr}$ and $d\epsilon_{ij}^{pl}$ can be decomposed into normal components and components in the direction of $d\epsilon_{ij}$ or $d\sigma_{ij}$. The latter components are caused by microcracking and plastic slips whose directions depend on $d\epsilon_{ij}$ and $d\sigma_{ij}$; this type of microcracks are predominantly normal to the principal direction of $d\sigma_{ij}$, and the predominant plastic slips occur at 45° angle. Thus, for the staircase path in Fig. 11(a), the directions of the latter type of microcracking alternate between the directions sketched in Fig. 12(b) and (c) and the directions of plastic slip alternate between those sketched in Fig. 12(c) and (f). By contrast, for the straight path the directions of prevalent microcracking and plastic slips are always those shown in Fig. 12(d) and (g).

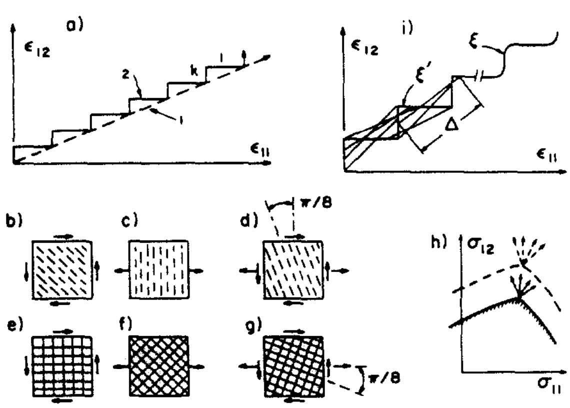

Fig. 12. Staircase loading path (a), Smoothed path length (i), and associated behavior (b–g, h).

We now see that, unless normality and absence of corner is assumed, the microcracking and plastic slips for the two paths occur in different directions even when the number of stairs tends to infinity. Therefore, the responses must be different and it may well be a shortcoming of classical plasticity that it cannot model this behavior.

The example demonstrates that insistance on uniqueness in all situations is tantamount to assuming normality and denying the possibility of vertex effects. Rudnicki–Rice's vertex model (eqn 38) gives also a non-unique response for the case considered (and so does Mróz's expression, eqn 47).

(3) Even in classical plasticity, if there is a corner on the loading surface (Tresca criterion) the direction of $d\epsilon_{ij}^{pl}$ is not unique as far as the flow rule is concerned, and depends on the imposed direction of $d\epsilon_{ij}$. Thus, when we have a staircase path infinitely close to a straight path, such that the stress point remains at the corner of the loading surface (Fig. 12), as the surface expands due to hardening the $d\epsilon_{ij}^{pl}$ vectors for the two paths will have different directions even when the number of stairs tends to infinity. Thus, even in the limit, the plastic slips are occurring on different planes, and so different hardenings and different responses must be expected. For example, if Odquist's path length $\bar{\gamma}^{pl}$ (eqn 55) is used as a parameter in the loading surface, the values of $\bar{\gamma}^{pl}$ will be different for these paths even in the limit.

(4) It is nevertheless possible to slightly adjust the definition of intrinsic time so that both paths give in the limit the same response. To this end we may replace eqn (24) by

$$\xi = \Delta + \frac{1}{\Delta}\int_{\Delta}^{\xi} \left\{ p_{ijkm}[\epsilon_{ij}(s) - \epsilon_{ij}(s-\Delta)][\epsilon_{ij}(s) - \epsilon_{ij}(s-\Delta)] \right\}^{1/2} ds \tag{92}$$

(for $\xi \geq \Delta$) where Δ is a certain small positive number and s is either the actual path length, $s = (p_{ijkm}\, d\epsilon_{ij}\, d\epsilon_{km})^{1/2}$, or time or loading parameter if ϵ_{ij} are given as its continuous functions. Note that for $\Delta \to 0$ we have lim ξ = path length s. Introduction of a finite value for Δ has the effect of rounding all sharp corners of the strain path with a radius of the order of magnitude of Δ, and so lim ξ = straight path length when the number of stairs in Fig. 12 tends to infinity. We may imagine ξ as the sum of all possible segments of length Δ between two points on the path (Fig. 12h), averaged over the length of the path. If we choose $\Delta = 10^{-8}$ (and p_{ijkm} is of the order of unity and s = path length), then the effect of Δ on practical fitting of all test data modeled so far in the literature with the endochronic theories is undetectable; the intrinsic time can be evaluated on the computer as usual, yet the theory gives the same limit for both paths in Fig. 12.

For the case when a uniaxial cyclic strain $a \sin \omega t$ is superimposed on a constant strain (t = time, a, ω = constants) we have $s \sim at$. We see that definition of ξ by eqn (92) increases the rate of convergence of ξ with $a \to 0$ at fixed t (or fixed number of cycles) to the value $\xi = 0$ which corresponds to $a = 0$. This means that, often in accordance with the actual behavior, the response to minute oscillations ($a \to 0$) would be much less inelastic or essentially elastic if eqn (92) is used.

Due to micro-inhomogeneity of the material, it is actually impossible to induce in the material a sequence of extremely (infinitely) small plastic slips or microcrack advances such that each two subsequent ones are of different directions. In this light, Δ has a physical justification and corresponds to the limit of continuum modeling. Thus, ξ according to eqn (92) may be called the *intrinsic time with finite resolution.*

In closing, one should not feel too disappointed, since the inelastic theories are not the only ones where one has to tolerate some unappealing, paradoxical limiting behavior. Elasticity or plate bending theory, which are as perfect as any theory could be, are replete with instances of such behavior. For example, the deflection of a simply supported regular polygonal plate does not tend to the deflection of a simply supported circular plate as the number of corners tends to infinity (Babuška's paradox). One has to take a positive view of the endochronic theory as long as it affords us for some materials and phenomena a much better description of the experimentally observed behavior[4, 6, 11, 10] than the classical theories.

12. CONCLUSIONS

1. In inelastic behavior one can distinguish the plastic strains, which are associated with no

change in elastic moduli and the fracturing stress relaxations, which cause a decrease of elastic moduli.

2. The positiveness of the second-order work ΔW of plastic strains or complementary work $\Delta\Pi$ of fracturing stress decrements corresponding to load-unload cycles guarantees material stability under stress or strain controlled conditions and may be used in constructing the stress–strain relations.

3. The condition that $\Delta W > 0$ or $\Delta\Pi > 0$ during loading does not necessarily require normality of plastic strain increments and of fracturing stress decrements. It allows them to have also such direction that no work is done. Moreover, certain kinds of plastic strain increments and fracturing stress decrements that are tangential to the loading surface and always do non-negative work are also allowed by these conditions.

4. The endochronic theory can be derived from Drucker's postulate just as logically as classical plasticity. The role of the loading surface in endochronic theory is that it separates the stress increment directions for which Drucker's postulate is satisfied from those for which it is not, whereas, in classical plasticity its role is that it separates the stress increment directions for which the plastic strain increment vector points outside loading surface from those for which it would point inward. None of these roles can be regarded as more fundamental; but the former role is advantageous by making it possible for the endochronic theory to exhibit irreversibility at unloading if the same equations as for loading are used, whereas the latter role would cause classical plasticity to exhibit full reversibility in small load-unload cycles if the same equations were used.

5. The incremental linearity of the stress–strain relations of classical plasticity is a tacitly implied hypothesis and does not necessarily follow from Drucker's postulate and the existence of the loading surface. Various incrementally nonlinear stress–strain relations satisfying Drucker's postulate, both such that do and do not obey normality, have been demonstrated.

6. Dependence of ΔW upon the angle θ of the stress increment vector with the normal of loading surface is useful for comparing various theories. So is the dependence of plastic strain increment direction angle and the magnitude upon θ.

7. Rudnicki–Rice's vertex model, for which there is an additional plastic strain increment parallel to the loading surface, has the advantages of incremental linearity and never leads to violation of Drucker's postulate. However, it has one undesirable feature, namely that the plastic hardening modulus h_n for the normal component of inelastic strain must be considered to change its sign when the loading direction parallel to the loading surface is crossed (i.e. when an outward direction changes to an inward one). To avoid a discontinuous jump in h_n, a continuous dependence of h_n upon angle θ would have to be introduced; this would, however, take away the advantage of incremental linearity.

8. For frictional materials (as well as materials whose fracturing is sensitive to volume change), there exists, in addition to Drucker's postulate (or Il'yushin's postulate) another inequality that also suffices for stability. It differs by a term that represents the second-order elastic energy blocked by friction (or by resistance to fracturing due to volume compression). This term enlarges the domain (in the stress space) of all stable stress increment vectors from a halfspace to a reentrant wedge. The same argument as that used to derive the normality rule in classical plasticity shows that the corresponding plastic strain increment vector has no unique direction but can have many directions which occupy a fan, one boundary of which is the normal vector. There are similarities but also important differences with regard to non-associated plasticity and the situation at a corner of the loading surface.

9. Apart from friction in deviator strains due to hydrostatic stress there exists friction in volumetric strain due to deviator stress intensity. The elastic energy blocked by this inverse friction leads to still another sufficient stability condition. The set of stress increment vectors that produce inelastic strain and stable responses gets enlarged from a reentrant wedge to a reentrant pyramid one side of which is tangent to the loading surface and the set of stable plastic strain increment vectors gets enlarged from a fan to a pyramid one side of which contains the normal.

10. Theories of inelastic behavior can be alternatively based on various other hypotheses, e.g. the requirement of uniqueness (or continuity) of response. However, it would be unreasonable to expect uniqueness in all situations, for example cyclic loading of vanishing

amplitude superimposed on a static load[7] or a regular staircase path[56] (in the stress space) approaching a straight-line path. Good physical reasons exist for the response to the staircase path not to approach the response to the straight-line path in the limit. This is actually so for the endochronic theory, Rudnicki–Rice's vertex model as well as some types of plasticity with a corner on the loading surface and non-endochronic incrementally nonlinear models. Although such behavior is not unreasonable from the physical view point, it is nevertheless possible to define a "smoothed" intrinsic time (eqn 92), such that uniqueness (or response continuity) for the staircase path is assured yet none of the previously published fits of experimental data is affected.

Acknowledgements—Financial support under U.S. National Science Foundation Grant ENG 75-14848-A01 to Northwestern University as well as a 12 month Guggenheim Fellowship awarded to the author for 1978–79 is gratefully acknowledged. Thanks are due to John W. Hutchinson of Harvard University and Zenon Mróz of Polish Academy of Sciences for their valuable critical comments.

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APPENDIX—DISTINCTION BETWEEN PLASTIC AND FRACTURING PHENOMENA

The difference between plastic and fracturing phenomena consists in the fact that the latter cause degradation of elastic moduli while the former do not. The law of changes of elastic moduli C_{ijkm} as a function of stress and strain increments must be given to be able to define $\mathrm{d}\epsilon_{ij}^{pl}$ and $\mathrm{d}\sigma_{ij}^{fr}$ in eqn (6). For the pure fracturing material the condition of full reversibility (Fig. 3b) requires that $\sigma_{ij} = C_{ijkm}\epsilon_{km}$. Differentiating we get $\mathrm{d}\sigma_{ij} = C_{ijkm}\mathrm{d}\epsilon_{km} - \mathrm{d}\sigma_{ij}^{fr}$ where $\mathrm{d}\sigma_{ij}^{fr} = -\mathrm{d}C_{ijkm}\epsilon_{km}$. The uniaxial counterpart of this relation is $\sigma = E\epsilon$ which yields $\mathrm{d}\sigma_{11} = E\mathrm{d}\epsilon_{11} - \mathrm{d}\sigma_{11}^{fr}$ where $\mathrm{d}\sigma_{11}^{fr} - \epsilon_{11}\,\mathrm{d}E$; this is graphically illustrated in Fig. 3(b). If we set $\mathrm{d}\sigma_{11} + \mathrm{d}\sigma_{11}^{fr} = \mathrm{d}\sigma_{11}^{el}$ and $\mathrm{d}\sigma_{ij} + \mathrm{d}\sigma_{ij}^{fr} = \mathrm{d}\sigma_{ij}^{el}$, we may write $\mathrm{d}\sigma_{11}^{el} = E\mathrm{d}\epsilon_{11}$ and $\mathrm{d}\sigma_{ij}^{el} = C_{ijkm}\mathrm{d}\epsilon_{km}$ for the pure fracturing material.

To generalize this to a plastic-fracturing material, we may now retain the preceding relations but with elastic strain increments instead of the total ones, i.e.

$$\mathrm{d}\sigma_{ij}^{el} = C_{ijkm}\,\mathrm{d}\epsilon_{km}^{el}, \quad \mathrm{d}\sigma_{11}^{el} = E\mathrm{d}\epsilon_{11}^{el} \tag{93}$$

where $\mathrm{d}\epsilon_{km}^{el} - \mathrm{d}\epsilon_{km} - \mathrm{d}\epsilon_{km}^{pl}$, $\mathrm{d}\sigma_{ij}^{el} = \mathrm{d}\sigma_{ij} + \mathrm{d}\sigma_{ij}^{fr}$ in which again $\mathrm{d}\sigma_{ij}^{fr} = \mathrm{d}C_{ijkm}\epsilon_{km}$ or $\mathrm{d}\sigma_{11}^{fr} = -\epsilon_{11}\,\mathrm{d}E$. Thus, eqn (93) yields

$$\mathrm{d}\sigma_{ij} - \mathrm{d}C_{ijkm}\,\epsilon_{km} = C_{ijkm}\,(\mathrm{d}\epsilon_{km} - \mathrm{d}\epsilon_{km}^{pl}) \tag{94}$$

or

$$d\sigma_{11} - \epsilon_{11}\, dE = E(d\epsilon_{11} - d\epsilon_{11}^{pl}). \quad (95)$$

The last relation is graphically illustrated in Fig. 3(c) and indicates the following graphical construction to determine $d\sigma_{11}^{fr}$ and $d\epsilon_{11}^{pl}$ if the unloading slopes E for various points of the stress–strain curve are known.

Using unloading slope E, pass from point 1 to point 7 on the stress axis (Fig. 3c); then, decreasing the slope by dE, plot line 76 and locate point 6 on the vertical line through point 1. Passing a horizontal line through point 6, and the line of the unloading slope through point 2, find intersection 3. Then, $\overline{16} = -d\sigma_{11}^{fr}$, $\overline{63} = d\epsilon_{11}^{pl}$.

The increments of dC_{ijkm} or dE must further be related to the change of the loading surface Φ in the strain space; see Ref.[2]. If the unloading is inelastic, $\overline{17}$ = *initial* unloading slope.

Although the foregoing method of superimposing $d\sigma_{ij}^{fr}$ and $d\epsilon_{ij}^{pl}$ appears to be logical, it remains to be confirmed experimentally. One could make some other hypothesis[1], for example, such that the unloading lines emanating from points 1 and 6 in Fig. 3(c) intersect at point 8 rather than 7, or at some point between 8 and 7. One such possibility has been tried in fitting test data for concrete[1] but did not perform as well as the method outlined above (eqns 94, 95).