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Concrete Reinforcing Net: Safe Design

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Introduction

In a previous work (2), the optimum limit design of dense regular reinforcing nets was studied with friction on the cracks, and the limit design envelope was determined as the yield locus that corresponds to yielding of all reinforcement. Subsequent reexamination of the problem revealed, however, that the domain of safe design is larger than this yield locus. This interesting property would not exist in the absence of friction.

Although for most practical cases the yield domain given in Ref. 2 coincides with the safe design domain, for cases which are far from the optimum design it does not. Therefore, we will complement the previous work (2) by presenting the complete safe design domain in this study. We will also examine its relationship to other methods.

Safe Design Domain

All basic assumptions used in Ref. 2 are retained. In particular, we assume ideal plasticity of reinforcement, neglect dowel action and bar kinking, and assume all cracks to be parallel, continuous, and densely spaced, and the reinforcing net to be orthogonal, regular and dense. The normal stress, σ^c_{nn}, and the shear stress, σ^c_{nt}, transmitted by concrete across the cracks are restricted by the inequality $|\sigma^c_{nt}| < -k\sigma^c_{nn}$ ($\sigma^c_{nn} \leq 0$) in which k is the friction coefficient for the cracks, which, according to tests of Paulay and Lieber (6), may be taken as $k = 1.7$ and, according to ACI Code (5) as $k = 1.4$.

The basic principle of design is that the given internal forces must be safely resisted for cracks of *any* angle θ. Thus one first determines the domain of safe reinforcement ratios, ρ_x and ρ_y, for cracks of one fixed angle θ, and then

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one finds the domain that is common to all angles θ between 0° and 180°. In Figs. 2–4 of Ref. 2, the results of such analysis were presented in terms of nondimensional limit stresses

$$n_x = \frac{\sigma^s_{xx}\rho_x}{\sigma_1}; \quad n_y = \frac{\sigma^s_{yy}\rho_y}{\sigma_1} \qquad (1)$$

in which σ^s_{xx} and σ^s_{yy} = the tensile stresses in the reinforcing bars of directions x and y, respectively; σ_1 = the maximum applied principal tensile stress; and ρ_x and ρ_y = the reinforcement ratios in x and y directions, respectively. $\sigma_1 = N_1/h$; N_1 = maximum principal normal force; and h = plate thickness. The safe design requires that

$$n_x \leq r_x = \frac{f_y\rho_x}{\sigma_1}; \quad n_y \leq r_y = \frac{f_y\rho_y}{\sigma_1} \qquad (2)$$

in which f_y = yield stress of reinforcement; and r_x, r_y = reinforcement parameters.

In terms of n_x and n_y at the limit state, the locus of the limit states for all crack angles θ was found to be (see Eqs. 8 and 9 of Ref. 2):

$$[(n_x - n^0_x) - \beta_1(n_y - n^0_y)]\,[(n_y - n^0_y) - \beta_1(n_x - n^0_x)] = (2\beta_2 n^0_{xy})^2 \qquad (3)$$

in which

$$n^0_x = \frac{1+m}{2} + \frac{1-m}{2}\cos 2\alpha; \quad n^0_y = \frac{1+m}{2} - \frac{1-m}{2}\cos 2\alpha;$$

$$n^0_{xy} = \frac{1-m}{2}\sin 2\alpha;$$

$$\beta_1 = \left[\tan\left(\frac{\pi}{4} - \frac{\beta}{2}\right)\right]^2; \beta_2 = \frac{1}{2}\left[\cos\left(\frac{\pi}{4} - \frac{\beta}{2}\right)\right]^{-2} \qquad (4)$$

where $m = \sigma_2/\sigma_1$ = ratio of applied principal stresses; α = angle of σ_1 with respect to reinforcement direction x; β = arctan (k) = friction angle on the cracks; and k = friction coefficient. It can be shown that $n^0_x = \sigma_{xx}/\sigma_1$, $n^0_y = \sigma_{yy}/\sigma_1$ and $n^0_{xy} = \sigma_{xy}/\sigma_1$, in which σ_{xx}, σ_{yy}, and σ_{xy} are the components of the applied stresses in the (x, y) coordinate system. The yield locus represented by Eq. 3 is a hyperbola in the (n_x, n_y) plane and is depicted in Fig. 1(*a*) for $\sigma_1 = 1$, $m = 0.5$, $\alpha = 30°$.

It might be thought that the safe domain in terms of reinforcement is proportional to the envelope in Fig. 1(*a*). However, this is not quite so. By substituting the inequalities from Eq. 2 into Eq. 3, we obtain the hyperbola in terms of ρ_x and ρ_y [see Fig. 1(*b*)]. From any point A on the yield locus, we can obtain a safe design point by increasing ρ_x or ρ_y. Thus, point B in Fig. 1(*b*) represents a safe design because it can be obtained from the safe design point A by increasing ρ_x. Similarly, point D is safe because it can be obtained from C by increasing ρ_y.

Therefore, the safe design domain must be extended up to the horizontal and vertical tangents PP′ and QQ′ of the yield locus, as shown in Fig. 1(*b*). We may note, however, that the cases where the safe design envelope differs from the yield locus are not typical designs because they are far from the optimum design point M.

The safe design domain may now be obtained from Eq. 3 by adding the conditions resulting from straight boundaries PP′ and QQ′. The reinforcement

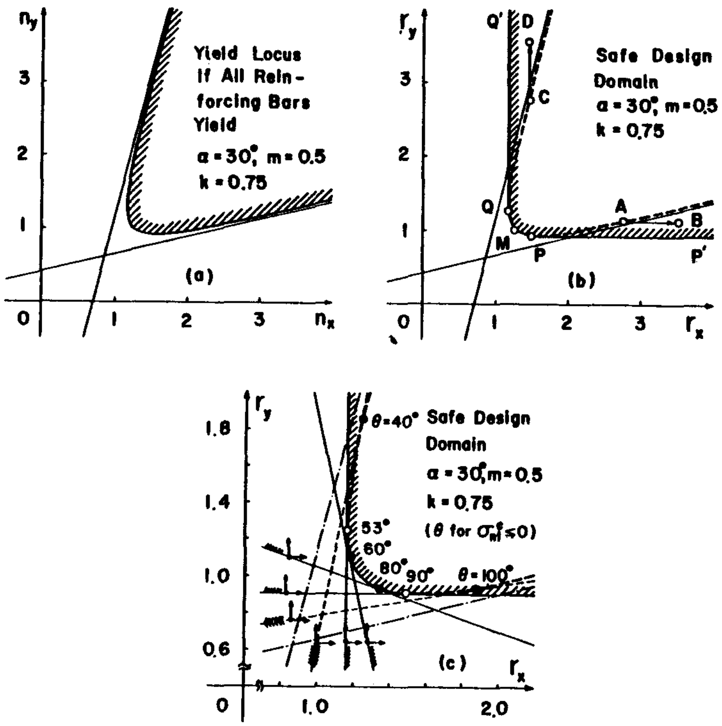

FIG. 1.—Yield Locus and Safe Design Domain

ratios ρ_x and ρ_y are safe if parameters ρ_x and ρ_y are such that at least one of the following conditions be satisfied:

$$[(r_x - n_x^0) - \beta_1(r_y - n_y^0)]\,[(r_y - n_y^0) - \beta_1(r_x - n_x^0)] \ge (2\beta_2 n_{xy}^0)^2; \quad \ldots \quad (5a)$$

$$\text{or} \quad r_x \ge r_{xP} \quad \text{and} \quad r_y \ge r_{yP} \quad \text{for} \quad \rho_x \ge \rho_y; \quad \ldots \quad (5b)$$

$$\text{or} \quad r_x \ge r_{xQ} \quad \text{and} \quad r_y \ge r_{yQ} \quad \text{for} \quad \rho_x < \rho_y \quad \ldots \quad (5c)$$

in which r_{xP}, r_{yP}, r_{xQ}, and r_{yQ} = the coordinates of transition points P and Q in Fig. 1(*b*). Points P and Q are found as the points of horizontal and vertical tangents of the curve given by Eq. 5*a*. This yields:

$$r_{xP} = 1 + n_{xy}^0\,[(\operatorname{cosec}\beta + \sin\beta)\sec\beta - \tan\alpha];$$

$$r_{yP} = m + n_{xy}^0\,[(\operatorname{cosec}\beta - \sin\beta)\sec\beta + \tan\alpha]; \quad \ldots \quad (6a)$$

$$r_{xQ} = 1 + n_{xy}^0\,[(\operatorname{cosec}\beta - \sin\beta)\sec\beta - \tan\alpha];$$

$$r_{yQ} = m + n_{xy}^0\,[(\operatorname{cosec}\beta + \sin\beta)\sec\beta + \tan\alpha] \quad \ldots \quad (6b)$$

For $\sigma_{xy} \ge 0$, the crack angles corresponding to points P and Q are:

$$\theta_P = \frac{\pi}{2} \quad \text{for} \quad \sigma_{nt}^c \le 0, \quad \text{and} \quad \theta_P = \beta \quad \text{for} \quad \sigma_{nt}^c \ge 0;$$

$$\theta_Q = \frac{\pi}{2} - \beta \quad \text{for} \quad \sigma_{nt}^c \le 0, \quad \text{and} \quad \theta_Q = 0 \quad \text{for} \quad \sigma_{nt}^c \ge 0 \quad \ldots \quad (7)$$

For $\sigma_{xy} < 0$ the crack angles corresponding to points P and Q are interchanged. Note that Eqs. 6*b* and 5*c* may be deleted if ρ_x is chosen to denote the heavier reinforcement.

The safe domain may also be deduced in a different manner. The yield hyperbola for the yield locus (Fig. 1) was derived in Ref. 2 as the envelope of the conditions of frictional yield for all crack angles θ. These conditions given by Eq. 4 of

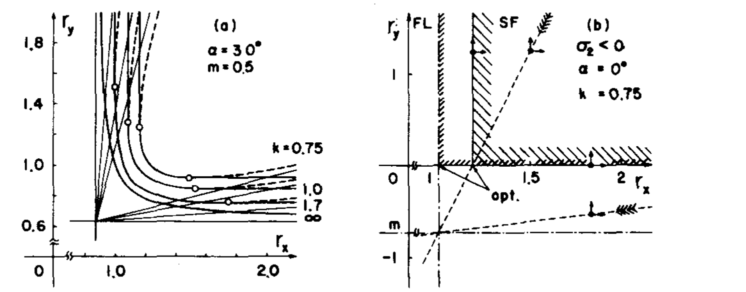

FIG. 2.—Safe Design Domains (*a*) for Typical Friction Coefficients and Biaxial Tension; (*b*) for Tension-Compression

Ref. 2 are represented by the straight lines tangent to the hyperbola in Fig. 1(*c*). From any point of each of these straight lines we get other safe points by increasing either ρ_x or ρ_y, as is indicated by the arrows in Fig. 1(*c*). Obviously, when these straight lines are of negative slope, only the halfplane on one side of the straight line represents a safe design. When, however, these lines are of positive slope, the arrows in Fig. 1(*c*) point to both sides of the line and so the halfplanes on both sides are safe, which means that the straight lines present no restriction and should be disregarded. Thus, the domain common to all halfplanes obtained for all angles θ is again seen to be the hyperbola augmented by straight tangents as shown in Fig. 1(*c*).

The safe design domains for typical friction coefficients are exemplified in Fig. 2 for the case of σ_1 = 1.0, m = 0.5, α = 30°. Fig. 2(*b*) further shows the safe domain when the second applied principal stress is compressive. This is again larger than the yield locus if all reinforcement yields, which is given by the inclined dashed line in Fig. 2(*b*) [see Fig. 3(*b*) of Ref. 2].

Similar adjustment may be made to the yield locus in terms of applied forces. The yield locus in absence of friction is exemplified on the left of Fig. 3 for the reinforcement ratios ρ_y/ρ_x = 0.5 and 1.0, giving the ratios of yield forces in bars $\sigma^s_{yy}/\sigma^s_{xx} = 0.5$ and 1.0. The locus of yield of all reinforcement is given by the surfaces that consist of hyperbolas in vertical planes, shown as dashed curves on the right of Fig. 3 for these two reinforcement cases. The yield envelopes when either both or only one reinforcement yields are obtained by augmenting these hyperbolas with the straight segments as shown on the right of Fig. 3.

Failure Mechanism

The hyperbolic segment QP [Fig. 1(*b*)] of the safe design envelope corresponds to cases where all reinforcing bars yield [Fig. 4(*b*)]. The crack represents a

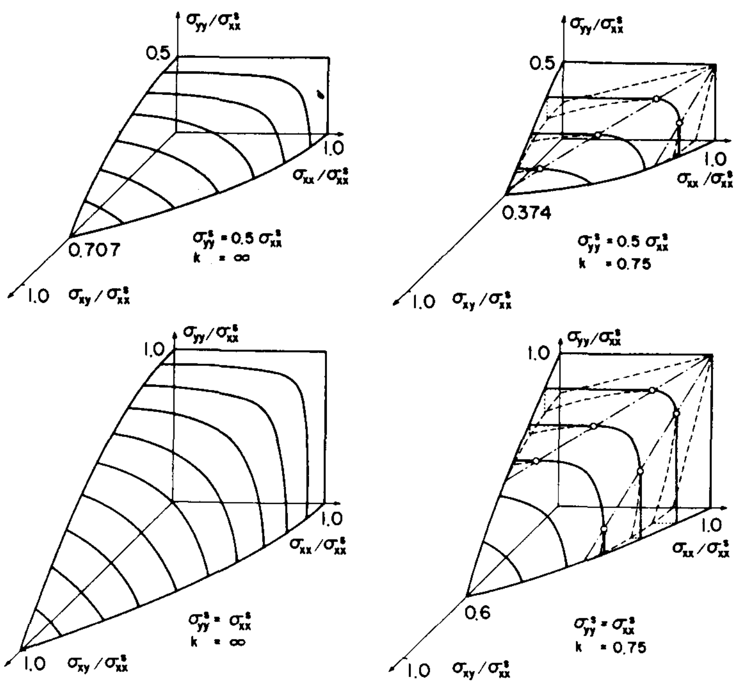

FIG. 3.—Yield Surfaces in Absence of Friction (Left) and with Account of Friction (Right)

collapse mechanism with two degrees of freedom, because both components of the displacement across the crack can grow arbitrarily. The ratio of normal displacement, δ_n, to tangential displacement, δ_t, called the dilatancy ratio, is obviously indeterminate according to our assumptions. Therefore, if all reinforcement yields, the bars are capable of deforming according to the crack dilatancy ratio, α_d, that is characteristic of the rough crack surfaces.

The straight segments PP′ and QQ′ in Fig. 1(*b*) correspond to cases where only bars of one direction yield. The crack becomes a mechanism with a single degree of freedom: the component of the displacement across the crack in

the direction of the bars which are not yielding remains constant and only the displacement component transverse to these bars can grow [see Fig. 4(*c*)]. Thus, the limit state condition for this case implies a certain ratio δ_n/δ_t depending on the angles of reinforcing bars.

Although it was not necessary to consider it for our analysis, the minimum admissible ratio δ_n/δ_t, called the crack dilatancy ratio, α_d, is a property of the crack surfaces depending on the stresses transmitted across the crack (see Ref. 1). The dilatancy ratio δ_n/δ_t permitted by the reinforcement must equal or exceed in magnitude the value of α_d; $|\delta_n/\delta_t| \geq \alpha_d$. This condition need not be always satisfied if only reinforcement of one direction yields. Therefore, in such cases the deformation growth at constant load is in reality impossible, even though the assumptions of our equilibrium analysis do not present such a restriction. In such cases of yielding of bars of a single direction, the load

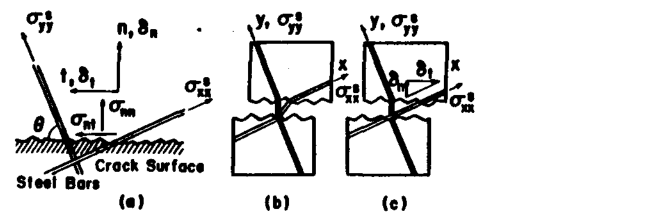

FIG. 4.—Failure Mechanism: (*a*) Stresses on Crack Surface; (*b*) Two-Directional Yielding; (*c*) One-Directional Yielding

cannot remain constant during deformation and must further change to bring the stress state onto the hyperbola so as to allow yielding in both directions.

Note on Derivation from Work

The limit state conditions can be also obtained from the work equation for the failure mechanism. The works of external and internal forces per unit length of the crack surface are:

$$W_{ext} = T_{nn}\delta_n + T_{nt}\delta_t; \quad W_{int} = (T^c_{nn} + T^s_{nn})\,\delta_n + (T^c_{nt} + T^s_{nt})\,\delta_t \quad \text{. . . . . . . (8)}$$

in which T_{nn} and T_{nt} = the normal and tangential components of the resulting forces on the crack surface of angle θ, respectively; T^c_{nn} and T^c_{nt} = the resultants of the stresses transmitted by concrete; and T^s_{nn} and T^s_{nt} = the resultants of the stresses transmitted by reinforcement [Fig. 4(*a*)]. Setting $W_{ext} = W_{int}$, we get:

$$(T^c_{nn} + T^s_{nn} - T_{nn})\,\delta_n + (T^c_{nt} + T^s_{nt} - T_{nt})\,\delta_t = 0 \quad \text{. . . . . (9)}$$

Now, if bars of both directions yield [Fig. 4(*b*)], we may substitute $\delta_n \neq 0$, $\delta_t = 0$, or $\delta_n = 0$, $\delta_t \neq 0$, which gives:

$$T^c_{nn} + T^y_{nn} = T_{nn}; \quad T^c_{nt} + T^y_{nt} = T_{nt} \quad \text{. . . . . (10)}$$

in which T^y_{nn} and T^y_{nt} = the resultants due to the yield stresses in steel reinforcement, $\sigma^s_{xx} = f_y$, $\sigma^s_{yy} = f_y$.

If only bars of the x-direction yield, the displacement (δ_n, δ_t) is normal to the y-direction [Fig. 4(c)] and $\sigma^s_{xx} = f_y$, $\sigma^s_{yy} < f_y$. If we choose the dilatancy ratio as $\alpha_d = |\delta_n/\delta_t| = \tan\beta = k$, we have the case of normality, which corresponds to the segment QQ′ [Fig. 1(b)] for which $\theta = 90° - \beta$ and σ^c_{nt} $\sigma_{xy} \leq 0$. In this case, no actual deformation with $\delta_n/\delta_t \neq \tan\beta$ is permitted by reinforcing bars. However, for the purpose of obtaining the equilibrium condition for a direction other than normal to the y-axis, we may consider the virtual work due to δ_n and δ_t of any ratio, provided that the condition $\sigma^s_{yy} < f_y$ is satisfied. We thus obtain Eq. 10 again. The locus of all states $\sigma^s_{yy} \leq f_y$ is, in the (r_x, r_y) plane, given by the point which satisfies Eq. 10 for a given θ; and a vertical ray emanating upward from that point. A similar argument holds for the case in which only bars in the y-direction yield.

Therefore consideration may first be limited to Eq. 10 and, after finding the corresponding yield envelope, the domain may subsequently be expanded by rays emanating upward and to the right from all points on the envelope. The rest of the analysis is the same, because Eq. 10 is equivalent to the starting equation of the previous analysis (Eq. 1 of Ref. 2).

Conceptual Differences from Other Theories

So far, there exist three different limit design theories for net-reinforced concrete:

1. Classical frictionless design.
2. Perfect plasticity.
3. Present slip-free (frictional) design.

The differences with regard to perfect plasticity solutions were analyzed in Ref. 3. The main difference is that the Mohr circle envelope approach, used as the basis of the frictional yield criterion of perfect plasticity (modified Coulomb criterion), implies concrete to remain isotropic up until the final collapse state. The yield criterion then implies isotropy, which would be correct only if the crack would not exist before the final collapse and would be created only during the collapse. This is a less conservative assumption than our assumption that there may be a preexisting crack of any direction. Due to the existence of cracks, concrete is, of course, not isotropic from the beginning. The Mohr's envelope approach and the isotropic yield criteria are then inapplicable.

The choice of crack direction is the only significant difference between various theories, because everything else follows from equilibrium once this direction is known. In plasticity, the crack direction is determined (with the help of the normality rule) from the stress state in concrete at collapse, but this crack direction normally does not give the maximum stress in reinforcement. (The same is true of frictionless design in which one assumes cracks in the direction of principal strain.) By contrast, the present approach is based on considering the stresses in reinforcement for all possible crack directions, which guarantees that the crack direction is the critical one with respect to the reinforcement. It is thus clear that the present solution can never give less reinforcement than the plasticity solution, and that the plasticity solution does not, in general, give the extreme possible effects in reinforcement, which detracts from the

safety margin (in detail, see Ref. 3). This is similar to classical frictionless design, in which the crack direction is also determined from conditions other than those critical for the reinforcement.

Reinforcing nets can also be designed on the basis of service stresses and deformations, and here the concept of rough cracks exhibiting friction and dilatancy due to slip also provides a useful enhancement of the classical approach (4).

Summary and Conclusions

The limit design of a dense regular reinforcing net in concrete panels or walls under in-plane loading with friction on the cracks due to aggregate interlock is studied. The previously established hyperbolic yield locus, which corresponds to simultaneous yielding of reinforcing bars of both directions, does not give the complete domain of safe reinforcing ratios. The safe domain is larger and the enlargement corresponds to cases when only bars of one direction yield. Those cases are far from the optimum design. Deformations for these cases do not generally conform to the dilatancy ratio that is characteristic of the normal and tangential components on the crack and they do not allow an increase of deformation at constant load.

Acknowledgment

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Appendix.—References

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ABSTRACT: The limit design of a dence regular reinforcing net in concrete panels or walls under in-plane loading is studied with consideration of friction on the cracks due to aggregate interlock. The previously established hyperbolic yield locus, which corresponds to simultaneous yielding of reinforcing bars of both directions, does not give the complete domain of safe reinforcing ratios. The safe domain is larger and the enlargement corresponds to cases when only bars of one direction yield. Those cases are far from the optimum design. Deformations for these cases do not generally conform to the dilatancy ratio that is characteristic of the normal and tangential components on the crack and they do not allow an increase of deformation at constant load.