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## VISCOELASTICITY OF SOLIDIFYING POROUS MATERIAL—CONCRETE

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### NATURE OF PROBLEM

Concrete is a porous composite with deformation properties that vary significantly with its age due to the gradual solidification of hydration products of cement and water (24). Over a broad range the creep law of concrete may be considered to be linear and obey the principle of superposition (3). Restricting this study to uniaxial stress, the linear creep law of an aging material may be written in the form (3)

$$\epsilon(t) = \int_0^t J(t, t') d\sigma(t') \quad \dots \dots \dots (1)$$

in which  $\sigma$  = stress;  $\epsilon$  = strain (including instantaneous strain);  $t$  = time measured from the instant of set of concrete; and  $J(t, t')$  = creep function of the material = strain at time  $t$  caused by a time-constant unit stress applied at time  $t'$ . Due to the age dependence of material properties, the creep function is not merely a function of load duration  $t - t'$ , but depends on both current time  $t$  and age at loading  $t'$  individually. This tremendously complicates the identification of creep function from experimental creep data (3) and is a source of confusion and controversy in defining creep properties to be used in structural analysis (6,7). The difficulties become particularly acute when dealing with very young concrete, for the time variation of its elastic and creep properties is quite significant.

As far as aging is concerned, the form of the creep function has so far been chosen empirically (3,5,6,7). However, to make a choice among many possible forms of the age dependence is difficult. The difficulties could be

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alleviated if the form of the age-dependence could be deduced by treating the solidifying material (hydrating cement paste) as a varying composite whose components have age-independent properties. Apparently, this has not been attempted yet and it is, therefore, chosen as the objective of this study. A new methodology of treating the age-dependence of creep will be proposed and a particular form of creep function will be deduced and correlated with two physical parameters of the solidification process: the volume of solidified matter,  $v$ , and the mean length of passage,  $l$ , of the particles whose migration causes creep.

Attention will be restricted to basic creep, i.e., creep at constant humidity and temperature. Many of the considerations will be applicable more generally to solidifying porous materials of any type, and in such considerations the words solidification and solidified matter will be used instead of hydration and cement gel.

### AGE-DEPENDENCE OF CREEP DUE TO VOLUME GROWTH OF SOLIDIFIED COMPONENT

The properties of the porous solidified material as a whole must be distinguished from the properties of the solidified matter between the pores. It will be assumed

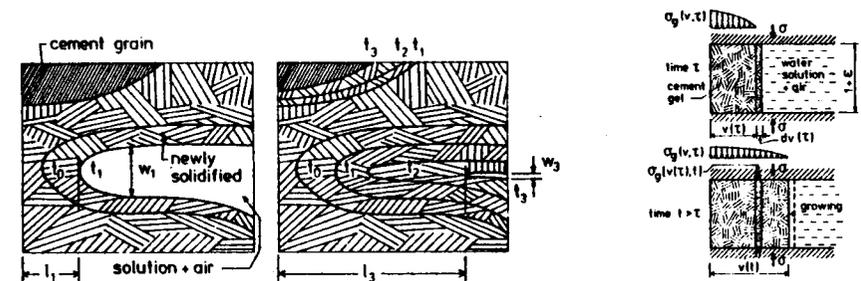


FIG. 1.—Filling of Capillary Pores by Solidification (of Cement Gel)

FIG. 2.—Parallel Coupling Model for Deformation of Solidifying Porous Material

that the creep properties of the solidified matter (cement gel) are linear, being characterized by creep function  $J_x(t, t')$ , and the effect of volume growth of the solidified matter (Fig. 1) will now be explored. Regarding the interaction of the elements solidified at various times (Fig. 1), some reasonable hypothesis must now be made.

**Hypothesis 1.**—The average strain in all microelements of the solidified matter is the same and equals the macroscopic strain,  $\epsilon$ . Then the macroscopic stress,  $\sigma$ , in the porous material as a whole is the sum of the stresses in all microelements.

This hypothesis is the familiar model of parallel coupling of components in composites (Fig. 2), which is known to yield the upper bound on the stiffness of the composite. This bound yields much better estimates for elasticity of cement paste and concrete than the lower bound based on the hypothesis of series coupling, for which the stress rather than strain is the same in all components and the strains rather than stresses are summed. In all models for strength or elastic modulus of cement paste, the stresses rather than strains in the components are being summed (10,12,14,15,16), although the parallel coupling

assumption is often not explicitly stated. However, the main reason for not assuming a series coupling model is the practical impossibility of imagining in a one-dimensional model [Fig. 3(a)] how a load could be applied on the solidified matter at the boundary of a unit macroscopic element and at the same time unobstructed volume growth in the load direction [Fig. 3(a)], rather than to the side of it (Fig. 2), could take place. Thus, Hypothesis 1, although certainly not exact, should be quite realistic.

The interaction with the aggregate as well as the dissolving anhydrous cement grains will first be neglected, for the sake of simplicity, and the mixture of liquids and gas in the pores (i.e., water solution and air with water vapor in capillary pores of cement paste) will be assumed to be relatively much more compressible, so that it does not carry any significant load.

**Creep Function in Terms of Volume Growth.**—Stress  $\sigma_x$  in the solidified matter (microstress) varies with location and depends on the time at which the matter at a given location solidified. The most convenient coordinate to characterize the location is the volume fraction,  $v(t)$ , of the solidified matter (Fig. 2). Thus,  $\sigma_x(v, t)$  will denote the stress in the solidified matter (cement gel) at time  $t$  and at the location where the solidification occurred when the volume of solidified matter was  $v$ . At that time this location was at the surface of the pore if the

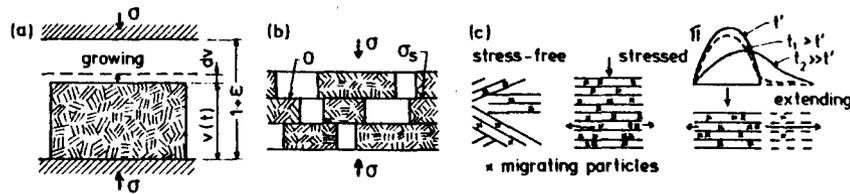


FIG. 3.—(a) Alternative Model with Series Coupling (Unrealistic); (b) Model to Explain Effect of Three-Dimensional Pore System; (c) Stressed and Stress-Free Locations and Spreading Pressure  $\pi$  of Migrating Particles for Extending Passages

solid matter precipitated from solution; however, this is not necessarily the case and growth at an interface between cement grain and cement gel is also possible (Fig. 1). Volume  $v(t)$  will refer strictly to the one-dimensional model (Fig. 2) and its equivalent for a three-dimensional pore system will be given later.

According to Hypothesis 1, visualized by the model in Fig. 2, equilibrium requires that

$$\int_{v=0}^{v(t)} \sigma_x(v', t) dv' = \sigma(t) \quad \dots \dots \dots (2)$$

Noting that  $v = v(t)$ , one may also write

$$\int_{\tau=0}^t \sigma_x[v(\tau), t] dv(\tau) = \sigma(t) \quad \dots \dots \dots (3)$$

at any time  $t > 0$ . Although stress  $\sigma_x[v(\tau), t]$  at times  $t < \tau$  has no physical meaning (for no matter has yet solidified at that location), it will be expedient to define this stress as zero, i.e.

$$\sigma_x[v(\tau), t'] = 0 \text{ for } v(\tau) > v(t') \text{ for } \tau > t' \quad \dots \dots \dots (4)$$

Eq. 3 may then be rewritten as

$$\int_{\tau=0}^t \sigma_x[v(\tau), t'] dv(\tau) = \sigma(t') \quad \dots \dots \dots (5)$$

Now consider the strain at the location where solidification occurred at time  $\tau$  preceding the current time  $t$ . The fact that is basic for the mathematical formulation is that the matter solidifies in a stress-free state, i.e., at the time of solidification the stress in the solidified matter is zero;  $\sigma_x[v(\tau), \tau] = 0$ . The macroscopic strain at the time of solidification is  $\epsilon(\tau)$  and the solidified matter experiences only the strain after time  $\tau$ , which equals  $\epsilon(t) - \epsilon(\tau)$  according to Hypothesis 1. Consequently, the stress-strain relation for the solidified element may be written as

$$\epsilon(t) - \epsilon(\tau) = \int_0^t J_x(t, t') \frac{\partial \sigma_x[v(\tau), t']}{\partial t'} dt' \quad (t \geq \tau) \quad \dots \dots \dots (6)$$

Eqs. 5 and 6 represent two integral equations for three unknowns  $\sigma(t)$ ,  $\epsilon(t)$ , and  $\sigma_x(v, t)$ . Unknown  $\sigma_x(v, t)$  needs to be eliminated so as to obtain a single relationship between  $\sigma(t)$  and  $\epsilon(t)$ . For this purpose, multiply Eq. 6 by  $dv(\tau)$  and integrate from  $v(0) = 0$  to  $v(t)$ ;

$$\int_{\tau=0}^t [\epsilon(t) - \epsilon(\tau)] dv(\tau) = \int_{\tau=0}^t \int_{v=0}^{v(\tau)} J_x(t, t') \frac{\partial \sigma_x[v(\tau), t']}{\partial t'} dt' dv(\tau) \quad \dots \dots \dots (7)$$

On the right-hand side, the order of integration over  $dt'$  and of differentiation with regard to  $t'$  may be switched with the integration over  $dv(\tau)$ ;

$$\int_{\tau=0}^t [\epsilon(t) - \epsilon(\tau)] \frac{dv(\tau)}{d\tau} d\tau = \int_{v=0}^{v(t)} J_x(t, t') \frac{\partial}{\partial t'} \left\{ \int_{\tau=0}^t \sigma_x[v(\tau), t'] dv(\tau) \right\} dt' \quad \dots \dots \dots (8)$$

Now,  $\sigma_x$  may be eliminated by use of Eq. 5, and at the same time the integral on the left-hand side may be integrated by parts. This yields

$$\int_{\tau=0}^t v(\tau) d\epsilon(\tau) = \int_{v=0}^{v(t)} J_x(t, t') d\sigma(t') \quad \dots \dots \dots (9)$$

This is the resulting integral equation relating stress and strain histories. It is further of interest to express the creep rate  $\dot{\epsilon}(t) = d\epsilon(t)/dt$  by differentiating Eq. 9:

$$v(t) \dot{\epsilon}(t) = J_x(t, t) \dot{\sigma}(t) + \int_{v=0}^{v(t)} \frac{\partial J_x(t, t')}{\partial t} d\sigma(t') \quad \dots \dots \dots (10)$$

in which  $J_x(t, t) = 1/E_x$  where  $E_x$  = instantaneous modulus of the solidified

matter between the pores, which must be independent of age, due to Hypothesis 2. The creep rate can be also expressed by differentiating Eq. 1

$$\dot{\epsilon}(t) = J(t, t) \dot{\sigma}(t) + \int_{t'=0}^t \frac{\partial J(t, t')}{\partial t} d\sigma(t') \quad (11)$$

in which  $1/J(t, t) = E(t) =$  elastic modulus of the porous material as a whole. Dividing Eq. 10 by  $v(t)$  and comparing it with Eq. 11, it follows that

$$E(t) = E_g v(t) \quad (12)$$

$$\text{and } \frac{\partial J(t, t')}{\partial t} = \frac{1}{v(t)} \frac{\partial J_g(t, t')}{\partial t} \quad (13)$$

the integration of which furnishes

$$J(t, t') = \frac{1}{E_g v(t')} + \int_{t'=\tau}^t \frac{1}{v(\tau)} \frac{\partial J_g(\tau, t')}{\partial \tau} d\tau \quad (14)$$

This is the final result, an expression giving the effect of volume growth of the solidified matter upon the creep function. This equation applies to any porous solidifying material, not just the hydrated cement paste.

It is also of interest to note that the elastic parts of stress and strain obey the equation  $\dot{\sigma}(t) = E(t) \dot{\epsilon}(t)$ , which has been deduced before (2,3). The equation is not the same as  $\sigma(t) = E(t) \epsilon(t)$ , which has been sometimes erroneously used.

**Porosity Decrease and Effective Volume Growth.**—The elastic modulus is obtained as  $E(t) = E_g v(t)$  (Eq. 14). If  $v(t)$  were interpreted as the actual volume fraction of gel in the material, this equation would be of rather limited validity. In fact, over a very broad range, the elastic modulus of cement pastes is known to be governed, according to Powers' gel-space ratio concept (19,20), by the relation  $E(t) = E_g (1 - p)^3$  where  $p =$  capillary porosity = volume fraction of pores which are emptied at relative humidity 33%, and  $1 - p =$  gel volume. To justify it physically, imagine the material to be subdivided by fictitious layers of equal thickness [Fig. 3(b)]. Assume that pore walls in each layer consist only of straight lines normal to the layer. Of the solid mass in each layer, only that part which is in contact with solid in both adjacent layers is assumed to receive stress, and this stress is assumed to be the same everywhere,  $\sigma_s$ . The probability of encountering solid at a chosen location of one given layer is  $1 - p$ . Then the joint probability of encountering solid at that location also in both adjacent layers must be  $(1 - p)^3$ . Thus, the force transmitted across a layer of unit area is not  $(1 - p)\sigma_s$  but  $(1 - p)^3\sigma_s$ , and the strain normal to the layer is  $\sigma_s/E_g$ , giving  $E = (1 - p)^3\sigma_s/(\sigma_s/E_g) = E_g(1 - p)^3$ . This justification of the equation for  $E$  is a simplified form of that given by Helmuth and Turk (16). Note that in addition to the idealizations made, the argument also implies that the pore distributions in adjacent layers are assumed to be statistically independent, which is probably correct only if the layer has certain proper thickness relative to the mean pore size.

The foregoing statistical argument about stress transmission is independent of whether the deformation is short-time or delayed, and so it becomes clear that  $v(t)$  for a three-dimensional pore system should be interpreted as  $v(t)$

$= [1 - p(t)]^3$  and may be called the effective volume of solidified matter. Note that the model employed [Fig. 3(a)] represents a combination of parallel coupling with series coupling, while the resulting formula for  $E$  is purely of the parallel coupling type.

Recently, more accurate expressions of very broad applicability have been developed by Grudemo (10,14,15); according to these,  $v$  would be a more complicated function of  $p$ , and  $p$  should include not only capillary pores, but also the volume of  $\text{Ca(OH)}_2$  and about one-half of the volume of anhydrous cement grains because all  $\text{Ca(OH)}_2$  crystals and about one-half of cement grains appear to sit loose within the matrix (except when the density of the paste is the highest attainable).

In case of mortar or concrete, it is necessary to define  $v(t)$  in accordance with the relationship of  $E(t)$  of concrete to the capillary porosity of cement paste in concrete (12,19).

#### AGE-DEPENDENCE OF PROPERTIES OF SOLIDIFIED COMPONENT (CEMENT GEL)

Although Eq. 13 indicates the manner in which the volume growth of solidified matter (cement gel) should affect creep, this equation would not be particularly useful unless the form of the creep function of the solidified matter could be determined. In this respect, a pertinent physical fact is that the microstructure of cement gel, characterized by the geometrical arrangements of layers, molecules, atoms, and their spacings, is fixed in time (20) (which is not true of cement paste as a whole, of course). This is an idealization because some time changes in the composition of cement gel are taking place, due to differences in the rates of various chemical reactions in hydration (10,24), but these will be assumed to be insignificant. Accordingly, the instantaneous elastic modulus of cement gel,  $E_g$ , must be constant, as has already been considered (Eq. 14). If creep were caused by some sort of dislocation movement, as in metals, or by sliding of molecular chains, as in polymers, the creep properties of solidified matter would be also time-invariant, and the creep function would then depend only on  $t - t'$  rather than  $t$  and  $t'$  separately, i.e.,  $J_g(t, t') = J_g(t - t')$ .

However, in cement gel the matter is not as simple. Based on the nature of the effects of varying humidity and temperature, creep must have its source in some sort of migration (or diffusion) of particles of solids (probably Ca-ions) along micropores of gel filled by strongly held water molecules (4,11). These migrations, by which the microstructure adjusts itself to the load, proceed from the locations of high stress to the stress-free locations, which could mean either to the capillary pores (macropores) in cement paste or to the cement gel pores (micropores) normal to the vector of applied stress (3). The rates of this migration must depend on the mean length of passage of migrating particles,  $l$ .

The migrations could consist either in a diffusion process governed by viscosity, or in a random walk process, and in either case the migration rate should be proportional to  $l^2$ . Apart from the effect of  $l$ , the creep properties of cement gel should be time-invariant, for there is no other distant interaction. Consequently, the following hypothesis appears to be justified.

**Hypothesis 2.**—The properties of the solidified matter (cement gel) are invariant in time, except for the effect of the length of passage,  $l$ , upon migration of solid particles due to load.

Consequently, the creep function of solidified matter should have the form  $J_s(t, t') = 1/E_s + F_s(t - t')/l^2$  where  $E_s$  = instantaneous modulus = constant, and  $F_s$  is a function of only load duration  $t - t'$ .

Now comes a difficult step: to decide how the mean passage length,  $l$ , should depend on time. Clearly,  $l$  must grow with age because the macropores become smaller, making the mean distance from micropores to macropores larger. In addition, growth of  $l$  may be caused by obstruction of migration passages caused by newly solidified particles. While  $l$  undoubtedly increases with both current time  $t$  and age at loading  $t'$ , a simplification is called for. The following simplifying hypothesis appears to be reasonable.

**Hypothesis 3.**—The mean length of migration passage,  $l$ , depends mainly upon  $t'$ , the age at load application.

In consequence,  $l = l(t')$  and

$$J_s(t, t') = \frac{1}{E_s} + \frac{F_s(t - t')}{l^2(t')} \quad (t \geq t') \quad \dots \dots \dots (15)$$

with  $F_s(0) = 0$ .

The last hypothesis (Eq. 15) is, no doubt, far weaker than the preceding ones. There are two arguments in its favor. Firstly, it must be realized that after the load application has activated migration along a certain passage, the subsequent solidification (hydration) process is unlikely to deposit new particles of cement gel in this existing migration passage because the bonds between molecules along such a passage are strained; new bonds are more likely to be established at stress-free locations. Secondly, even if some of the new particles were deposited, contrary to the preceding argument, at the end of a previously activated migration passage, the mean gradient of the spreading pressure produced previously in the passage would remain essentially unaffected for some time, until part of the stress were transferred, due to migrations, from the original passage into the new extension of the passage. This means that the migration would be taking place much of the time essentially along the passage activated initially at time  $t'$ , which again indicates that  $l = l(t')$ . The latter argument, however, is not valid for very long creep durations, which indicates that Hypothesis 3 (Eq. 15) might be unacceptable for that case and remain valid only for not too long creep durations.

The foregoing justification of Hypothesis 3 is reinforced by certain recent successes of the particle migration (diffusion) model used herein. This model has been first derived from thermodynamics of phase equilibria and of surfaces (3), as an extension of the ideas of Powers (21). Later this model has been shown to describe correctly the test data on drying creep and shrinkage, and creep of predried concrete (3,4) as well as test data on creep at various temperatures and at temperature changes (9). Also, a probabilistic model of creep as a random process, deduced from this model of creep mechanism (and the dependance on  $l^2$  in particular), gave realistic predictions of statistical distributions of creep data (11).

Functions  $v(t)$  and  $l(t)$  are both determined by the degree of solidification (degree of hydration), and so they should be mutually related. To specify this relationship, it may be assumed that the increase in length  $l$  of migration passages during  $dt$  is essentially given by the mean width,  $w(t)$ , of pores in which new

cement gel is solidifying at time  $t$ . Thus,  $-dp(t) \approx w(t)d[l^2(t)]$ , and so Eq. 15 could be rewritten as

$$J_s(t, t') = \frac{1}{E_s} + F_s(t - t') \left[ - \int_0^{t'} \frac{dp(\tau)}{w(\tau)} \right]^{-1} \dots \dots \dots (16)$$

Thickness  $w(t)$  cannot be equal to maximum pore size because solidification takes place not only in larger pores but also in small ones. Nevertheless,  $w(t)$  is perhaps in a fixed relation to maximum capillary pore size, whose dependance on age was measured by Auskern and Horn (1).

**CREEP FUNCTION FOR BASIC CREEP OF AGING CONCRETE**

Based on Eqs. 14 and 15, the creep function of concrete (in absence of moisture exchange and temperature changes) ought to have the form

$$J(t, t') = \frac{1}{E_s v(t')} + \frac{1}{l^2(t')} \int_{t'}^t \frac{1}{v(\tau)} \frac{F_s(\tau - t')}{\partial \tau} d\tau \dots \dots \dots (17)$$

which is the main result of this study. The creep rate at time  $t$  due to a constant unit stress acting since time  $t'$  ought to be

$$\dot{\epsilon}(t) = \frac{1}{l^2(t')v(t)} \frac{\partial F_s(t - t')}{\partial t} \dots \dots \dots (18)$$

The creep function is here expressed in terms of functions of one variable, which is a considerable simplification of the problem.

The creep function has often been considered in the form  $J(t, t') = (1/E_0) + \phi(t')F(t - t')$ , which is readily recognized to constitute a special case of Eq. 17, such that  $\phi(t') = [v l^2(t')]^{-1}$ ,  $F(t - t') = \int_{t'}^t (\partial F_s / \partial \tau) d\tau$ , and  $v =$  constant. A special case of this is the recently proposed double power law (3):

$$J(t, t') = \frac{1}{E_0} + \frac{\phi_1}{E_0} (t')^{-m} (t - t')^n \dots \dots \dots (19)$$

which has been shown to fit the basic creep data available in the literature much closer than older formulations (5,6,7); its extension to cover the case of drying creep has been also quite successful (8). Eq. 19 is a special case of Eqs. 18 and 17 for  $1/l^2 = (t')^{-m} cv$ ,  $F_s(t - t') = (t - t')^n cE_0/\phi_1$ ,  $E_0 = E_s v$ , in which  $c$  is an arbitrary constant and  $v =$  constant. It seems that the relative success of this simplification, as evidenced by data fits (5,6,7), is attributable to the fact that  $v$  does not vary substantially except in very young concrete, and only such data have been fitted so far. It has not been clear before why the fits with double power law were better for constant  $E_0$  rather than time-variable  $E_0$ . From Eq. 17 it is seen that a variable  $E_0$  does not match the creep term if  $v =$  constant. Further study of double power law (Eq. 19) indicated that a distinct improvement of fits and more systematic values of coefficients for various concretes can be obtained if  $(t')^{-m}$  is replaced by  $(t')^{-m} + \alpha$ , where  $\alpha =$  positive constant.

Another extreme case is the very young concrete, for which the increase

of  $v$  is the dominant factor, so that the increase of  $l(t')$  may be negligible in comparison. Setting  $l = \text{constant}$ , and choosing again power functions, which appear to be generally best suited for describing the time variation of properties of concrete, Eq. 18 may be chosen in the form

$$\dot{\epsilon} = \frac{\partial J(t, t')}{\partial t'} = \frac{\phi_0}{E_0} t^{-p} (t - t')^{n+p-1} \dots \dots \dots (20)$$

in which  $v = c l^2 t^{-p}$ ; and  $F_{\epsilon}(t - t') = (t - t')^{n+p} \phi_0 / [E_0 c (n + p)]$  where  $c$  is an arbitrary constant. Substitution in Eq. 17 yields

$$J(t, t') = \frac{A}{E_0} (t')^{-p} + \frac{\phi_0}{E_0} B(t, t'), \quad \text{with}$$

$$B(t, t') = \int_{t'}^t \tau^{-p} (\tau - t')^{n+p-1} d\tau \dots \dots \dots (21)$$

in which  $A = c l^2 / E_{\epsilon} = \text{constant}$ . This integral is recognized to be the binomial integral, and so Eq. 20 may be called binomial creep law. For realistic exponent values ( $0 < p < 1$ ,  $0 < n < 1$ ), the binomial integral cannot be integrated with elementary functions, but the integration is easily done numerically by a computer or by Dirichlet series approximations developed for this purpose in Appendix I.

As a digression, it is rather interesting to note that, by contrast to Eq. 19 and most other creep functions for concrete, Eq. 21 allows the Laplace transform to be used for solving the stress history from given strain history. This is because the kernel  $J_{\epsilon}(t, t')$  of the integral becomes a convolution integral. The Laplace transform can also be applied to solve structural analysis problems for this creep function, but only if creep properties, including function  $v(t)$ , are the same for all parts of the structure.

Consider now two limiting cases of Eq. 20. One is when  $t - t' \ll t'$ , in which case  $t \approx t'$ ; then Eq. 21 provides  $J(t, t') = (t')^{-p} (t - t')^{n+p} \phi_0 / E_0 (n + p) + \text{constant}$ . Another limiting case is when  $t - t' \gg t'$ , in which case  $t \approx t - t'$ ; then Eq. 20 yields  $\dot{\epsilon} = (t - t')^{n-1} \phi_0 / E_0$  which gives  $J(t, t') = (t - t')^n \phi_0 / E_0 n + \text{constant}$ . The first limiting case, representing short-time creep, is equivalent to double power law. The second case, representing long-time creep, represents also a power curve, but of a smaller exponent. Thus, the binomial creep law (Eqs. 20 and 21) represents a transition from a power curve of a larger exponent ( $n + p$ ), to a power curve of a smaller exponent ( $n$ ). Indeed, it can be seen from creep data in Ref. 5 that such a transition should improve the fits.

As far as the physical mechanism of creep is concerned, the time variation of  $l(t)$  explains a phenomenon that has often been regarded as puzzling: The creep of very old concrete still continues to decline strongly with age, even though almost no hydration is taking place, i.e.,  $p(t)$  or  $v(t)$  is almost constant. Explanation is possible if the following hypothesis is true.

**Hypothesis 4.**—With advancing age, the solidification is taking place in narrower and narrower pores. Their width  $w(t)$  declines drastically, by orders of magnitude, as high age is reached.

This hypothesis seems logical, since solidification must first occur in the

easiest accessible spaces, which are the large ones, and only very late in the thinnest, least accessible pores, when no large pores are left. Even though solidification at any given time need not be taking place in the largest pores available at that time (because the distance from the remnant of the anhydrous cement grain also matters, especially at advanced hydration), it is pertinent to note that the maximum size of capillary pores indeed strongly decreases with age, as has been demonstrated by Auskern and Horn (1). Consequently,

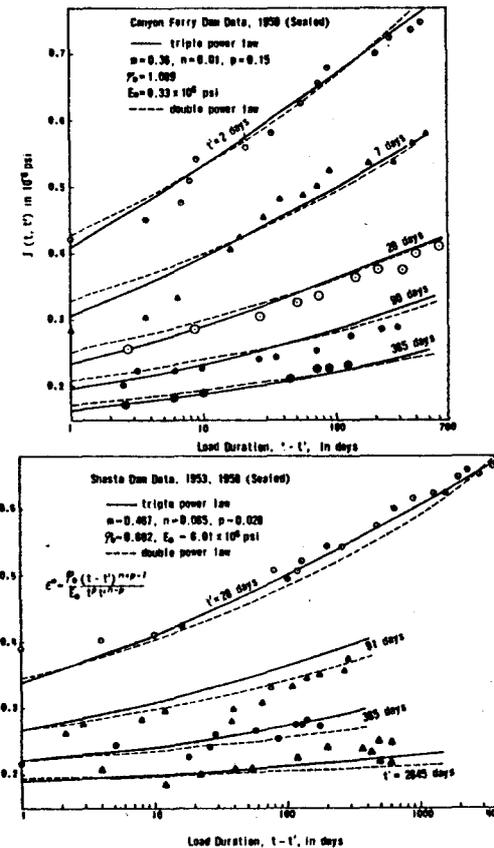


FIG. 4.—Fit of Creep Data of Hanson and Harboe (Ref. 6) for Canyon Ferry Dam and for Shasta Dam

noting that  $d[l^2(t)] = d v(t) / w(t)$ , it is clear that  $l^2(t)$  can change very much even if  $v(t)$  changes very little, provided that  $w(t)$  becomes sufficiently small. This explains the continuation of age effect into long creep durations.

The data fits obtained with the double power law, although quite good, show some deviations from measurements that appear to be systematic rather than random. One of them is the fact that for concrete loaded at high age, the upward curvature and final slope of creep curves at long load durations [in the  $\log(t - t')$  scale] is much too small, compared with most of the test data

(5). This should be improved if both  $l$  and  $v$  were considered as variables. Thus, preserving the power-type functional dependences, it seems appropriate to assume  $v \sim t^p$ ,  $l^2 \sim t'^{m-p}$ , and  $F_x(t-t') \sim (t-t')^{n+p}$ . Accordingly, in view of Eq. 18, it is proposed to characterize the basic creep of concrete by means of the following expression for the creep rate under unit stress sustained since time  $t'$ :

$$\dot{\epsilon}(t) = \frac{\phi_0}{E_0} \frac{(t-t')^{n+p-1}}{(t')^{m-p} t^p} \dots \dots \dots (22)$$

with four material parameters:  $m$ ,  $n$ ,  $p$ , and  $\phi_0/E_0$ . The creep function is then

$$J(t, t') = \frac{A}{E_0} (t')^p + \frac{\phi_0}{E_0} (t')^{-m+p} B(t, t') \dots \dots \dots (23)$$

in which  $A$  is a fifth material constant. Eq. 22 may be called triple power law for creep rate.

Using optimization techniques of the type described in Refs. 5 and 7, Eq. 23 has been used to fit two well-known creep data sets, i.e., those of Hanson and Harboe, which were described and fitted by double power law in Ref. 5. The results, shown in Fig. 4, indicate a slight improvement brought about by Eq. 23, consisting in the fact the curve for low  $t'$  is more straight and the curve for high  $t'$  curves more strongly upwards and has a steeper final slope. Further improvement might again be expected if  $(t')^{-m+p}$  is replaced with  $(t')^{-m+p} + \alpha$ .

**Note on Reversible Creep Component.**—In an effort to reduce the structural analysis problem to first or second-order differential equations in time, the creep function was considered by some researchers to be a sum of a "reversible" (delayed elastic) component and an "irreversible" (flow) component. Examples of such creep functions are those for the rate-of-creep method, known also as Dischinger's method, the rate-of-flow method, and the improved Dischinger method (23) (which now forms the basis of German DIN specifications and is proposed to be adopted for the new CEB International Recommendations). In the context of this study, it is interesting to note that Eq. 14 or Eq. 17 cannot be brought to a form in which the "reversible" and "irreversible" terms would be separated. The intrinsic creep function of cement gel might perhaps allow separating the "reversible" and "irreversible" components, but due to the variation of  $v$  and  $l$  this is impossible for  $J(t, t')$ . In fact, no thermodynamic definition of such components has yet been presented, and it is impossible to do so, although this separation is possible for small time intervals  $dt$  in which the changes of  $v$  and  $l$  are negligible. Consequently, the separation of "reversible" and "irreversible" creep components in the creep function appears to be theoretically unjustified, as far as the total creep strains are concerned.

This conclusion is not surprising; creep functions based on separating reversible and irreversible total creep strains have been shown to fit creep test data much poorer than other creep functions (5,7), and they have also been found to compare rather poorly with creep recovery data of not too limited time range, which can be seen in log-time plots (see reply in Vol. 7, Ref. 6). Aside from that, from the point of view of creep structural analysis, the separation of irreversible and reversible total creep components has now lost its practical usefulness because

of the recent development of new quasi-elastic methods of creep structural analysis which are both simpler and more accurate than the solutions of differential equations resulting from a creep function with separate reversible and irreversible terms (6,3).

**CHANGE OF CREEP RATE DUE TO TEMPERATURE OR DRYING**

**Effect of Temperature.**—The effect of temperature is twofold (3,9). First, hydration is faster at higher temperature, which can be described by replacing actual time  $t$  with equivalent hydration period  $t_e$  when determining  $v$  and  $l$ , i.e.,  $v = v(t_e)$  and  $l = l(t_e)$ . The definition of  $t_e$  is analogous to the concept of maturity, except that it is defined with the help of the (mean) activation energy of hydration,  $Q$ ;  $t_e = \int \beta_\tau dt$ ,  $\beta_\tau = \exp[(1/T_0 - 1/T)Q/R]$  in which  $T_0 =$  reference temperature,  $R =$  gas constant (3,9). Second, the rate of particle migrations must be greater at higher temperature (3,9). This may be accounted for by multiplying  $\partial F_x(\tau - t')/\partial \tau$  by a function of temperature which is of the form  $f_\tau = \exp\{(1/T_0 - 1/T)U/R\}$ , with  $U$  being the activation energy of particle migrations. Thus, Eq. 17 may be extended as follows:

$$J(t, t') = \frac{1}{E_x v(t'_e)} + \frac{1}{l^2(t'_e)} \int_{t'}^{t_e} \frac{f_\tau}{v(\tau_e)} \frac{\partial F_x(\tau - t')}{\partial \tau} dt \dots \dots \dots (24)$$

in which  $t'_e = \int_0^{t'} \beta_\tau dt$ ; and  $\tau_e = \int_0^\tau \beta_\tau dt$ .

**Effect of Simultaneous Drying.**—It has been well demonstrated that the large increase in creep during drying is a rate-type phenomenon (3,4), consisting essentially in the increase in creep rate as a function of the drying rate, or of the rate of shrinkage strain  $S(t)$ , which is related to the drying rate. Therefore, the present formulation, which gives a relatively simple expression (Eq. 18) for the creep rate, is suitable for adaptation to the drying effect. The increase in creep rate is due to the increase of mobility of the migrating particles in the micropores and interlayer spaces of cement gel caused by the migration of water molecules. Drying has no effect on  $l$  and  $v$ , except that it slows down the growth of  $l$  and  $v$ , as indicated by the definition of the equivalent hydration period,  $t_e$ , for variable humidity (3,4), i.e., by  $l = l(t_e)$ ,  $v = v(t_e)$ . Consequently, the shrinkage rate  $dS(t)/dt$  should affect only the intrinsic creep rate of the cement gel (at fixed  $l$ ),  $\partial F_x(t-t')/\partial t$ . Thus, Eq. 18 may be generalized as follows:

$$\dot{\epsilon}(t) = \frac{1}{l^2(t'_e)v(t_e)} \frac{\partial F_x(t-t')}{\partial t} F_x \left( \frac{dS(t)}{dt} \right) \dots \dots \dots (25)$$

in which  $F_x$  is a function to be identified either empirically or by comparison with the more accurate model of drying creep given in Ref. 4. The creep function in the presence of drying is obtained by integration of Eq. 25. A similar comment could be made on the creep acceleration due to the rate,  $dT/dt$ , of temperature change (transitional thermal creep). Then, Eq. 25 must be multiplied by function  $F_\tau(dT/dt)$ . Obviously,  $F_x \geq 1$ ,  $F_\tau \geq 1$ .

EFFECT OF ELASTIC AGGREGATE AND REFINED INTERACTION MODELS

The creep curves of concrete and cement paste are known to have similar shapes. Although the aggregate strongly reduces creep, it does not affect significantly the shape of creep curves. Thus, the creep function derived here (Eq. 17) can be applied to concrete, provided that it is obtained solely by fitting creep data for concrete. The quantitative physical interpretation of  $v(t)$  must then be discarded.

It would certainly be useful if the effect of aggregate could be calculated from composition of concrete. As far as the elastic modulus is concerned, formulas giving good predictions are available (e.g., the formula of Hashin) (12). They are generally of the form  $E = f_E(v_a, E_a, v_a, v_s, E_s, v_s, E_p, v_p)$  in which  $v_a, v_s$  = volume fractions of coarse aggregate and fine aggregate (sand) in concrete,  $E_a, E_s, v_a, v_s$  = associated elastic moduli and Poisson ratios,  $E_p, v_p$  = elastic modulus and Poisson ratio of cement paste determined with the account of its capillary porosity  $p(t)$ . To impress upon Eq. 17 the correct value of the elastic modulus, one may define  $v(t) = f_E[v_a, E_a, v_a, v_s, E_s, v_s, E_p(t), v_p] / E_p$  and call it the effective solidified volume. However, this adjustment alone could

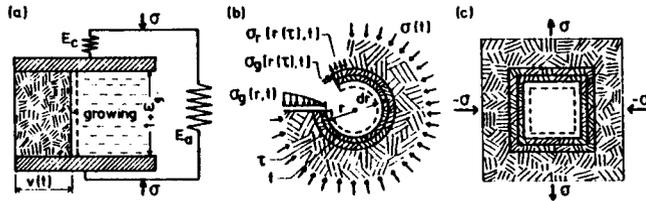


FIG. 5.—(a) Possible Refined Model When Interaction with Elastic Components is Considered Explicitly; (b) Cylindrical or Spherical Pore Model Analogous to Fig. 2; (c) Model for Deviatoric (Shear) Stress Loading

not be sufficient because the ratio of creep strain to elastic strain would remain unchanged, while in reality this ratio is much reduced by the presence of aggregate. The reduction is due to the gradual transfer of stress from the creeping cement paste upon the aggregate, which does not creep, and this is different from the effect of porosity in cement paste (Eq. 15), where no transfer of stress upon the pores can occur. Since the presence of aggregate does not affect the migration passage length,  $l$ , it is only the function  $F_R(t - t')$  that must be modified to reflect the aggregate content, i.e.,  $F_R(t - t')$  must be replaced by  $k_a F_R(t - t')$  where coefficient  $k_a$  as a function of  $v_a, E_a, v_a, v_s, E_s, v_s, v_p$  would have to be determined empirically. There is a difficulty, however, in that theoretically  $k_a$  should not be a constant but a function of  $t$  and  $t'$ , and this would destroy the simple form of Eq. 17. As an approximation, the effective modulus (better yet, the age-adjusted effective modulus) of cement paste for long time loading ( $t = 10^4$  days) and for some typical  $t'$  (28 days) could be used in function  $f_E$  to estimate  $k_a$  as a constant. It remains to be seen whether this brings some advantages against direct fitting of Eq. 17 to the test data for concrete.

A more realistic though more involved model can be set up by considering

the transfer of stress from cement paste or aggregate in terms of integral equations for creep. The parallel coupling and the series coupling are known to give the upper and lower bounds on elastic moduli. One way of getting a closer estimate is to introduce a model that has both parallel and series coupling, with the proportion of their stiffnesses to be determined only empirically (as has been done for the elastic modulus by Counto). With this in mind, consider, for the sake of greatest tractable generality, that the model from Fig. 2 is coupled in series with elastic spring  $E_c$  and in parallel with elastic spring  $E_a$ , as shown in Fig. 5(a). Denoting the deformation and the load on the growing cement gel block [Fig. 5(a)] by  $\epsilon_x$  and  $\sigma_x$ , the equilibrium conditions provide  $\sigma_c = \sigma - E_a \epsilon$ ,  $\epsilon_x = \epsilon - \sigma_c / E_c = a \epsilon - \sigma / E_c$ , in which  $a = 1 + E_a / E_c$ . Consequently, Eq. 9 may be generalized to the present case simply by replacing  $\sigma$  and  $\epsilon$  with  $\sigma_c$  and  $\epsilon_x$ . This yields

$$\int_{\tau=0}^{t'} v(\tau) \left[ a d\epsilon(\tau) - \frac{1}{E_c} d\sigma(\tau) \right] = \int_{\tau=0}^{t'} J_x(t, \tau) [d\sigma(\tau) - E_a d\epsilon(\tau)] \dots \dots (26)$$

When unit sustained stress  $\sigma = 1$  applied instantaneously at time  $t'$  is considered, Eq. 26 becomes

$$\int_{\tau=t'}^{t'} [E_a J_x(t, \tau) + a v(\tau)] d\epsilon(\tau) = \left[ 1 - \frac{E_a}{E(t')} \right] J_x(t, t') - a \left[ \frac{1}{E(t')} - \frac{1}{E_c} \right] v(t') \dots \dots (27)$$

with  $\epsilon(t') = 1 / E(t')$ . It is not possible to express the unit creep curves explicitly; they must be solved from Eq. 27, which represents an integral equation for  $\epsilon(t)$ . Solving  $\epsilon(t)$  for various values of  $t'$ , creep function  $J(t, t')$  can be determined. Although the numerical computer solution of Eq. 27 is easy, the formulation becomes perhaps unnecessarily complicated for practical purposes.

It is also possible to generalize Eq. 27 to account for time variation of the volume of the elastic component due to the growth of calcium hydroxide crystals, as well as the dissolution of cement grains in water [Fig. 5(a)]. However this would complicate matters since the cement grains are dissolving, and at dissolution the matter is not removed in a stress-free state, as for solidification, but in a stressed state. Fortunately, this need not be done because it appears that cement grains do not receive much stress and at least half of them sit loose within the paste matrix (14,15). Similarly,  $\text{Ca(OH)}_2$  crystals are also loose within the paste matrix, except perhaps for the densest possible pastes. Thus, cement grains and  $\text{Ca(OH)}_2$  crystals are more properly treated as voids.

Because the coarse aggregate forms a rigid skeleton in which the hard pieces of aggregate are essentially in contact, it seems appropriate to consider that most of the coarse aggregate volume acts as a parallel spring,  $E_a$ . The spring coupled in series,  $E_c$ , would then represent only the remaining small portion of aggregate (sand) and possibly also that part of cement grains and those calcium hydroxide crystals that are not loose. It remains to decide, however, whether the sheets of hydrated silicates which are parallel to the applied normal stress should not better be considered as an elastic rather than viscoelastic component.

**Hydrostatic Stress.**—For better approximation of the actual geometry of capillary pores, it might be appropriate to consider for hydrostatic applied stress  $\sigma$  an axisymmetric cylindrical pore model or a spherical pore model, whose cross section is shown in Fig. 5(b). In this model, the tangential normal strains in individual layers of radius  $r = r(\tau)$ , solidified at various times, must obey the relation  $\epsilon_r(r, t) = u(r, t)/r$ , in which  $u$  = radial displacement and  $u(t, t) = \epsilon(t)$  = the macroscopic strain. The tangential normal stresses,  $\sigma_r(r, t)$ , in all layers of radius  $r = r(\tau)$  must be in equilibrium with the macroscopic stress  $\sigma(t)$ ; then, either  $\sigma(t) = \int \sigma_r [r(\tau), t] dr(\tau)$  for cylindrical pores, or  $\sigma(t) = \int 2\pi r(\tau) \sigma_r [r(\tau), t] dr(\tau)$  for spherical pores. In either case, the conditions are analogous to Hypothesis 1, except for the factor  $r$  and for the fact that radial stresses  $\sigma_r [r(\tau), t]$  and radial strains  $\epsilon_r = \partial u / \partial r$  should also be considered and a creep Poisson ratio of solidified matter must be introduced [Fig. 5(b)]. Therefore, the resulting expression for  $J(t, t')$  must be of the same form as Eq. 17, although the functions  $l(t)$  and  $v(t) = (4/3)\pi [1 - r^3(t)]$  have more complicated factors. The main difference will be only in quantitative correlation of  $v(t)$  and  $w(t)$  to the actually measured values, and the cylindrical or spherical model might be preferable for this purpose.

As for deviatoric stresses, note that they can always be represented by equal compressive and tensile stresses in two orthogonal directions. Using the idea of parallel coupling, the model shown in Fig. 5(c) can then be introduced and analyzed similarly as before.

**CONCLUSIONS**

1. Assuming that the average strains in the microelements that solidified at various times are equal, the effect of solidification upon the creep function can be expressed by means of the volume of solidified matter at time  $t$  (Eq. 14). By virtue of this assumption it is possible to eliminate the microstresses from a system of two integral equations relating the stress and strain histories.

2. Assuming that the properties of cement gel are time-invariant and that the creep rate depends on the mean length  $l$  of the passage of migrating solid particles at the time of load application, the creep function of concrete can be expressed (Eq. 17) in terms of three functions of one variable: the passage length, the effective solidified volume, which is a function of capillary porosity, and the creep function of cement gel at fixed passage length.

3. Dependence of passage length on current time and age at loading, as assumed in Hypothesis 3 (Eq. 16), is the weakest part of the present model and future work should be aimed at refining it.

4. The double power law represents a special case for which the volume growth of cement gel is negligible, which is an acceptable assumption for not too young concrete. The theory confirms that for double power law the theoretical instantaneous strain should be indeed age-independent, as found by data fits. The theory indicates the triple-power law for creep rate to be one simple plausible generalization consistent with growing gel volume.

5. The effects of temperature and drying can be introduced into the present theory as modifications of creep rate.

6. According to the present theory, a separation of total creep in reversible

and irreversible components, proposed for CEB International Recommendations, lacks justification.

7. Further refinements by including interaction with elastic components explicitly are tractable.

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**APPENDIX I.—DIRICHLET SERIES EXPANSION OF POWER FUNCTIONS**

The binomial integral  $B(t, t')$  does not possess an exact closed-form expression. It has been found, however, that an approximate but very accurate closed-form

**TABLE 1.—Coefficients of Dirichlet Series Expansion of Power Function**

$n$ (1)	$\alpha(n)$ (2)	$\beta(n)$ (3)	$n$ (4)	$\alpha(n)$ (5)	$\beta(n)$ (6)
-1.00	-3.6284	-0.03901	-0.10	-0.2426	0.8844
-0.90	-2.6593	-0.03502	-0.01	-0.02315	0.9897
-0.75	-2.1489	-0.03311	0.01	0.02287	1.0101
-0.60	-1.6679	-0.02540	0.05	0.1114	1.0478
-0.50	-1.3650	-0.02040	0.10	0.2151	1.0893
-0.45	-1.2156	0.01437	0.20	0.3963	1.1527
-0.35	-0.9250	0.10867	0.35	0.5992	1.2010
-0.29	-0.7550	0.33143	0.50	0.7240	1.2005
-0.24	-0.6154	0.5988	0.60	0.7700	1.1793
-0.22	-0.5603	0.6637	0.75	0.7947	1.1207
-0.20	-0.5057	0.7154	0.90	0.7804	1.0434
-0.15	-0.3720	0.8117			

expression can be obtained if the power functions in Eq. 22 for the creep rate are approximated by sums of exponentials, i.e., by Dirichlet series. It is generally true that within the interval

$$0.3\tau_1 \leq t \leq 0.5\tau_N \dots \dots \dots (28)$$

the following approximation holds:

$$t^n \approx \sum_{n=1}^{N+1} A_n e^{-t/\tau_n} \dots \dots \dots (29)$$

in which for  $\mu = 1, 2, \dots, N-1$ :  $A_\mu = -\alpha(n)\tau_\mu^n$ ,  $\tau_\mu = 10^{\mu-N}\tau_N = 10^{\mu-1}\tau_1$  (30a)

for  $\mu = N$ :  $A_N = -1.2\alpha(n)\tau_N^n$  . . . . . (30b)

for  $\mu = N+1$ :  $A_{N+1} = \beta(n)\tau_N^n$  . . . . . (30c)

and  $\tau_{N+1} = 10^s\tau_N$ ;  $\alpha(n)$  and  $\beta(n)$  are coefficients that have been determined by least-square fitting. They are given in Table I for various values of power function exponent  $n$ , both positive and negative. (For positive  $n$ , this type of formula has already been given in Ref. 6.)

Using Eq. 29, it is possible to make the following substitutions in Eqs. 21 or 22, the accuracy of which is within the thickness of the line;

$$(t-t')^{n+p-1} \approx \sum_{\mu=1}^{N+1} A_\mu e^{-(t-t')/\tau_\mu} \dots \dots \dots (31)$$

$$t^{-p} \approx \sum_{\nu=1}^{N+1} B_\nu e^{-t/\tau_\nu} \dots \dots \dots (32)$$

in which  $A_\mu$  and  $B_\nu$  are given by Eqs. 30a-30c, with  $n$  replaced by  $n+p-1$  or by  $-p$ , respectively. Substituting in the integral in Eq. 21, one can easily integrate, obtaining

$$B(t, t') \approx \sum_{\mu=1}^{N+1} \sum_{\nu=1}^{N+1} \frac{A_\mu B_\nu}{T_{\mu\nu}} e^{-t'/\tau_\nu} (1 - e^{-T_{\mu\nu}(t-t')}),$$

with  $T_{\mu\nu} = \frac{1}{\tau_\mu} + \frac{1}{\tau_\nu}$  . . . . . (33)

Expansions in Eqs. 31 and 32 also allow linear creep problems for concrete structures to be reduced to linear algebraic equation systems.

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#### APPENDIX III.—NOTATION

The following symbols are used in this paper:

- $E$  = instantaneous (Young's) modulus of elasticity;  
 $E_g$  = instantaneous modulus of cement gel;  
 $F_g(t - t')$  = theoretical creep function of cement gel at constant  $l$ ;  
 $l(t)$  = mean length of passage of particles migrating due to load;  
 $v(t)$  = effective volume of cement gel (Fig. 2);  
 $J(t, t')$  = creep function of cement paste or concrete (Eq. 1);  
 $J_g(t, t')$  = creep function of cement gel;  
 $t$  = time measured from set of concrete;  
 $t'$  = time at load application;  
 $\epsilon, \sigma$  = uniaxial strain and stress; and  
 $\tau$  = time as integration variable.

**Closing Remark (Added in Proof).**—The change of cement paste is also caused by silicate polymerization, which consists in formation of new bonds. For new bonds to form, certain atoms or their groups must migrate into positions where the attractive and repulsive forces between them are in equilibrium, which means that at the instant the new bonds form they are in an *unstressed* state. However, this is again the same crucial fact underlying the present model (Eq. 6). To accommodate polymerization, the solidifying matter must be understood as the components of polymer chains, and the increments of  $v$  (atoms involved in new bonds) must be considered extremely small (see Hypothesis 4). While the present mathematical structure appears to be quite broadly applicable as the simplest possible model, the precise microstructural interpretation is an open question and further research might lead to refinements which would require more sophisticated mathematics.

The fact that the present model decomposes the material system into *invariant* components (substances) is also needed as the basis for thermodynamic treatment of aging creep. As is well known, such decomposition is generally required to formulate thermodynamic relations for chemically reacting systems (thermodynamics of mixtures). It remains to be seen, of course, whether the present decomposition is the most proper one.