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Constitutive Law for Nonlinear Creep of Concrete

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Objective of Study

When the stress exceeds about one half of the strength, f'_c, creep of concrete exhibits strong nonlinear dependence on stress and the linear principle of superposition ceases to apply. A number of investigations have been devoted to this subject (1-4,7,9,12,14,16-35), and various approximate uniaxial constitutive laws have been proposed. However, they have been cast in a form that is not valid without further generalizations for various extreme special cases, including: (a) Linear (low-stress) creep with aging and with memory properties, applicable over a broad range of load durations; (b) uniaxial and multiaxial short-time stress-strain behavior and failure conditions; (c) long-time strength, i.e., decrease of strength with load duration when stress is very high (over $0.8\ f'_c$), and also increase of strength as a result of low sustained compression; and (d) cyclic creep, i.e., acceleration of creep of concrete due to cyclic loading in the low as well as high stress range. By developing a formulation that is "supported" on all these extreme special cases in addition to the data for the uniaxial nonlinear creep, achievement of realistic representation of the material, possibly even for multiaxial stress, would be more likely. This is particularly important as adequate test data on nonlinear creep under multiaxial stress seem to be lacking.

Presented herein is a constitutive law including the foregoing four extreme special cases. Formulation of this law is accomplished by combining the endochronic theory for short-time deformations and failure of concrete (11), which satisfies extreme special cases (b), (c), and (d), with the Maxwell chain formulation (9,10,13), which satisfies extreme special case (a). Such a combination has already been suggested (11) but has not been developed in detail and verified.

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The nonlinearity of creep in the high stress range has its physical source in progressive microcracking, which occurs primarily in the interface between aggregate and cement mortar matrix. The nonlinearity of short-time deformations of concrete has its source also in microcracking; therefore, the formulation for nonlinearity at short-time deformations should logically be the special case of that for long-time deformations.

Endochronic Law for Concrete

In previous papers (5,6,11) the following constitutive relation for nonlinear short-time deformations as well as nonlinear creep has been proposed:

$$2G_\mu\, de_{ij} = ds_{ij_\mu} + s_{ij_\mu}\, dz_\mu, \quad s_{ij} = \sum_{\mu=1}^{n} s_{ij_\mu} \quad \text{(1a)}$$

$$3K_\mu (d\epsilon^V - d\lambda - d\epsilon^{V°}) = d\sigma_\mu^V + \sigma_\mu^V \frac{dt}{\tau_\mu}, \quad \sigma^V = \sum_{\mu=1}^{n} \sigma_\mu^V \quad \text{(1b)}$$

in which s_{ij}, e_{ij} = deviators of stress tensor σ_{ij} and (linearized) strain tensor ϵ_{ij} in cartesian coordinates $x_i (i = 1, 2, 3)$; $\sigma^V = \sigma_{kk}/3$ = volumetric stress, $\epsilon^V = \epsilon_{kk}/3$ = volumetric strain; s_{ij_μ} and σ_μ^V = associated hidden stresses corresponding to the μth unit of a Maxwell chain model ($\mu = 1, 2, \dots n$); G_μ and K_μ = shear modulus and bulk modulus associated with the μth unit ($G = \Sigma_\mu G_\mu$, $K = \Sigma_\mu K_\mu$ being the actual shear modulus and bulk modulus); both G_μ and K_μ depend on the age, t, of concrete; $\epsilon^{V°}$ = inelastic stress-independent volumetric strain, such as thermal dilatation; $d\lambda$ = inelastic dilatancy; t = time; τ_μ = relaxation time of the μth unit; and z_μ = so-called "intrinsic" time for the μth unit whose increments dz_μ depend both on actual time increments dt and on strain increments $d\epsilon_{ij}$ (11). For the special case of short-time deformations, the Maxwell chain model may be reduced to a single Maxwell unit, and the subscript, μ, may be dropped. For this case, the stress-strain law in Eqs. 1 has been identified from extensive test data (11) and the following expression for intrinsic time z has been derived:

$$dz = \left[\left(\frac{d\zeta}{Z_1}\right)^2 + \left(\frac{dt}{\tau_1}\right)^2\right]^{1/2}, \quad d\zeta = \frac{d\eta}{f(\eta)} \quad \text{(2a)}$$

$$d\eta = F(\underset{\sim}{\epsilon}, \underset{\sim}{\sigma})\, d\xi, \quad d\xi = \sqrt{\frac{1}{2} de_{ij}\, de_{ij}} \quad \text{(2b)}$$

in which Z_1 = constant; ξ = distortion measure; $f(\eta)$ = strain-hardening function; $F(\underset{\sim}{\epsilon}, \underset{\sim}{\sigma})$ = strain-softening function; and $\underset{\sim}{\epsilon}$, $\underset{\sim}{\sigma}$ = strain and stress tensors. Both $f(\eta)$ and $F(\underset{\sim}{\epsilon}, \underset{\sim}{\sigma})$ are expressed in Ref. 11. An expression for $d\lambda$ as a function of $d\xi$ has also been given (11). For conception of endochronic theory (for metals) (33).

Extension to Nonlinear Creep

To model the long-term nonlinear creep, the distortion measure, ξ, must be redefined so as to exclude the linearly viscoelastic parts of strains, denoted as $de_{ij}^{\prime}$, since otherwise ξ would increase (and would thereby cause nonlinearity) even at low-stress creep, which is known to be linear in stress and unrelated to microcracking. This may be accomplished by setting (11):

$$d\xi_\mu = \left(\frac{1}{2} de_{ij_\mu}^0\, de_{ij_\mu}^0\right)^{1/2}; \quad de_{ij_\mu}^0 = de_{ij} - de_{ij_\mu}^{\prime}; \quad de_{ij_\mu}^{\prime} = \frac{s_{ij_\mu}\, dt}{2G_\mu \tau_\mu} \quad \text{(3a)}$$

$$d\eta_\mu = F(\underset{\sim}{\epsilon}, \underset{\sim}{\sigma})\, d\xi_\mu, \quad d\zeta_\mu = \frac{d\eta_\mu}{f(\eta_\mu)} \quad \text{(3b)}$$

in which subscript μ must now be appended to ξ, η, and ζ. Furthermore, it has been shown that, for modeling creep over a broad range of load durations, different intrinsic times z_μ must be introduced for the individual Maxwell units, defined (11) as

$$dz_\mu = \left[\left(\frac{d\zeta_\mu}{Z_\mu}\right)^2 + \left(\frac{dt}{\tau_\mu}\right)^2\right]^{1/2} \quad (\mu = 1, 2, \dots n) \quad \text{(4)}$$

in which τ_μ = relaxation times, which are properly chosen as $\tau_\mu = \tau_1 10^{\mu-1}$ (9,13) and can be identified from the test data on linear creep at low stress alone (10); and Z_μ = constants which may be called relaxation strains.

The Maxwell chain formulation for nonlinear long-time creep of concrete, as just outlined, has been suggested in Ref. 11 but has not yet been compared with test data. This is the present study's aim.

Examining the test data that are available in the literature on nonlinear creep of concrete at high stress, it appears that all pertain to uniaxial loading, and no serious information is available on multiaxial creep at high stress. Therefore, attention must be restricted to the analysis of uniaxial creep data.

In view of the absence of adequate experimental information on nonlinear volume changes and lateral creep strains under high uniaxial stress, some reasonable assumption must be made on dilatancy λ and creep Poisson ratio ν. Most available data pertain to compressive stress less than 0.75 f_c' (f_c' = uniaxial strength), and few go up to 0.9 f_c'. In this range the inelastic dilatancy (volume change) is negligible and is assumed here as zero. The apparent Poisson ratio remains in this range approximately constant, $\nu_{app} \approx 0.2$ (although on approach to f_c' both λ and ν_{app} sharply increase). Therefore, assume that under uniaxial load $\epsilon_{22} = \epsilon_{33} \approx -0.2\epsilon_{11}$; this gives $\epsilon^V \approx (1 - 2 \times 0.2)\, \epsilon_{11}/3 = 0.2\, \epsilon_{11}$, $e_{11} \approx 0.8\, \epsilon_{11}$, $e_{22} = e_{33} \approx -0.4\, \epsilon_{11}$, and $[(1/2)\, de_{ij} de_{ij}]^{1/2} \approx [(0.8^2 + 0.4^2 + 0.4^2)\, \epsilon_{11}^2/2]^{1/2} = 0.693\, |\epsilon_{11}|$. These values were substituted into the expressions for $F(\underset{\sim}{\epsilon}, \underset{\sim}{\sigma})$ and $f(\eta)$ from the earlier version of endochronic theory, namely Eqs. 12-17 of Ref. 6 (since the refined expressions of Ref. 11 were not available when this work was carried out). Dropping the unnecessary subscripts, 11, for the uniaxial stresses and strains ($\epsilon_{11} = \epsilon$, $\sigma_{11} = \sigma$), this substitution provided:

$$d\eta_\mu = \left(b_0 + \frac{b_2}{b_1 - \sigma_\mu} |\epsilon|\right) d\xi_\mu, \quad d\xi_\mu = |d\epsilon - d\epsilon_\mu^{\prime}|, \quad d\epsilon_\mu^{\prime} = \frac{\sigma_\mu}{E_\mu} \frac{dt}{\tau_\mu} \quad \text{(5)}$$

in which E_μ = Young's elastic modulus associated with the μth unit of Maxwell chain, and (11)

$$d\zeta_\mu = \frac{d\eta_\mu}{1 + \beta_1 \eta_\mu} \quad \ldots\ldots (6)$$

with $b_0 = 0.693\nu$ ($\nu = 0.2$), $b_1 = f'_c/0.4$, $b_2 = 0.693^2\, b_1$, $\beta_1 = 30$.

The stress-strain law for uniaxial deformation is, according to Eq. 1, given by

$$d\epsilon = \frac{d\sigma_\mu}{E_\mu} + \frac{\sigma_\mu}{E_\mu} dz_\mu + d\epsilon^0, \quad \sigma = \sum_{\mu=1}^{n} \sigma_\mu \quad \ldots\ldots (7)$$

in which ϵ^0 = stress-independent inelastic strain (e.g., thermal dilatation); and dz_μ is expressed by Eqs. 4, 5, and 6. The values of material parameters, b_0, b_1, b_2, β_1, and Z_1, are fixed on the basis of previous data fitting for the special case of short-time multiaxial deformations (11), and the values of material parameters, E_μ, are fixed by fitting test data on the long-term linear creep at low uniaxial stress. Relaxation times τ_μ cannot be identified from test data but must be suitably chosen (9,10,13). A suitable choice is $\tau_\mu = \tau_1\, 10^{\mu-1}$; i.e., the relaxation times are spaced in log-time scale uniformly by decades. The only parameters available for fitting test data on nonlinear creep at high uniaxial stress are the parameters Z_μ ($\mu = 2, 3, \ldots n$). They have been identified from test data with the help of a computer. The creep curves (curves of elastic plus creep strain per unit stress) for different ages at loading in the linear low-stress range have been analyzed separately in advance, using the subroutine MAXWL1 from Ref. 10. This subroutine yields the coefficients of a cubic polynomial, $C_{1_\mu} + C_{2_\mu}$ (log t) + C_{3_μ} (log t)^2 + C_{4_μ} (log t)^3, expressing each of the moduli E_μ ($\mu = 1, 2, \ldots n$).

After establishing the values of E_μ it was possible to search for the values of Z_μ ($\mu = 2, \ldots n$). For this purpose a FORTRAN IV program has been written. This program integrates the stress-strain relations given by Eqs. 4-7 numerically in a step-by-step algorithm using specified values of Z_μ. The numerical algorithm is described in the subsequent section. The program has been developed by a simple generalization of subroutine CRCURV listed in Ref. 10 and was mathematically formulated in Eqs. 15-19 of Ref. 13. It has then been tried to express the values Z_μ as functions of discrete variable μ by some simple formula with only a few coefficients. Various formulas were chosen, their coefficients were varied, and the computed corresponding creep curves at high stresses were automatically plotted (using CALCOMP plotter) for a number of chosen sets of coefficients, until the best possible fit of test data was found. The trial-and-error procedure appeared to be sufficient because the unknown coefficients were few. This was then repeated for various other formulas, and finally the best overall fit was selected. In this manner it was found that among simple expressions, including linear, quadratic, cubic, rational, power-type, and exponential functions, the formula

$$Z_\mu = Z_1 \frac{1}{\mu} e^{\mu-1} \quad \ldots\ldots (8)$$

with $Z_1 = 268 \times 10^{-6}$ gave on the average the best agreement with all test data considered herein. This value of Z_1 is much higher than that obtained in Refs. 6 or 11. This is because only one Maxwell unit was used and the

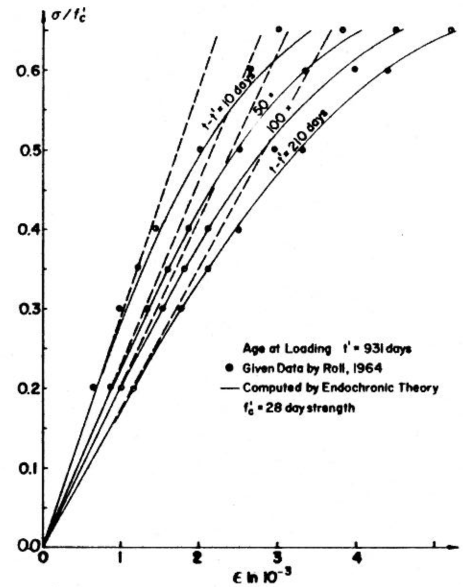

FIG. 1.—Dependence of Uniaxial Creep Strain on Sustained Stress (Creep Isochrones)

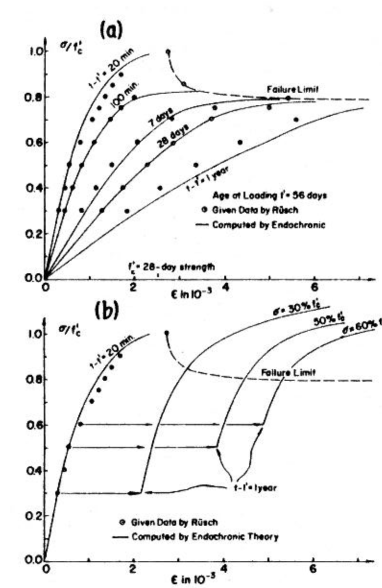

FIG. 2.—(a) Uniaxial Creep Isochrones and Dependence of Strength on Load Duration; (b) Effect of Previous Sustained Load on Short-Time Strength

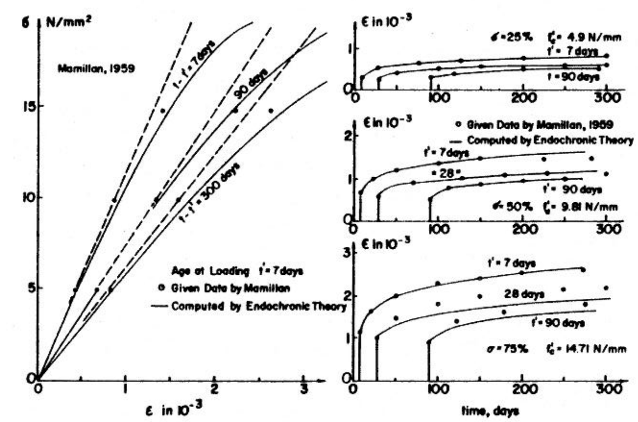

FIG. 3.—Dependence on Uniaxial Creep on Sustained Stress, σ, and on Age at Loading, t'

strains in the short-time data analyzed in Refs. 6 or 11 actually included a short-time creep strain. The fact that Z_μ increases with μ (Eq. 8) means that the dependence of the intrinsic time upon the strain weakens at later stages of creep, which is not surprising. The formula

$$dz_\mu = \left[\left(\frac{d\zeta_\mu}{Z_\mu} \right)^2 + \left(\psi_\mu \frac{dt}{\tau_\mu} \right)^2 \right]^{1/2} \quad (\mu = 1, 2, \ldots n) \quad \ldots\ldots (9)$$

which is slightly more general than Eq. 2, has also been considered, ψ_μ being restricted to functions that become unity for small stress and strain, e.g., $\psi_\mu = 1 + c_\mu \zeta_\mu$. However, no improvement of data fits could be discerned.

Comparison with Test Data

The fits of a few most pertinent test data available in the literature (22,29,30,31) are shown by solid lines in Figs. 1-3. Obviously, agreement is satisfactory both for the stress level effect (Figs. 1-3) and the age effect (Fig. 3). It is seen that the theory also predicts the decrease of strength with the duration of load (Fig. 2). This phenomenon, called long-time strength, is observed only for loads exceeding about 0.85 of the strength. For smaller sustained stress, the opposite is true; i.e., when concrete is loaded for a long time in the working stress range and then the load is suddenly increased to failure, the strength is higher than that for short-time loading (8,15). A consistent and complete data set for this effect seems unavailable; however, Fig. 2 demonstrates that the present theory is capable of modeling this phenomenon.

An important check for any theory of nonlinear creep is the test data for the stress relaxation at high stress and for the creep recovery after unloading from a previous sustained high stress. Such test data have been given, e.g., by Mamillan (Figs. 4, 5) and Roll (Fig. 6) (22,29). The linear theory of creep is known to substantially overestimate the recovery after high stress creep (and to underestimate stress relaxation at high strain). From Figs. 5 and 6 it is seen that this deficiency is improved by the present theory. However, the improvement is not sufficient; i.e., the recovery is still overestimated (and the relaxation is underestimated); see the solid lines in Figs. 5 and 6. The reason possibly is that simultaneous drying, taking place during all of these tests, increases creep more strongly than recovery. To fit these data, the effect of simultaneous drying would have to be taken into account. This could be done by combining the present theory with that from Ref. 12. This step, however, lies beyond the scope of this study and, consequently, close fits of the data on creep recovery and stress relaxation of drying concrete cannot be expected herein.

Algorithm of Numerical Time-Step Integration

For calculation of a structure's response, time t may be divided by discrete times $t_r (r = 1, 2, 3, \dots N)$ in time-steps $\Delta t = t_r - t_{r-1}$. To be able to reach long-term response, the time step must be increased with creep duration, for which a constant division in log t-scale is most convenient (8,9). Then, however, special algorithms of the type described in Refs. 10 and 13 are needed in order to avoid numerical instability when Δt becomes larger than τ_1, the smallest relaxation time considered.

The time-step formulas may be obtained by writing the exact integral of differential equations in Eq. 7, obtained under the assumption that the derivatives $d\epsilon_{11}/dz_\mu$ and $d\epsilon^0_{11}/dz_\mu$ are constant within the interval (t_{r-1}, t_r) and vary discontinuously at times t_{r-1} and t_r. The integration yields $\sigma_\mu(t) = Ae^{-z_\mu}$ (in which A = constant) and application of the initial condition at $t = t_{r-1}$ provides

$$\sigma_{\mu_r} = \sigma_{\mu_{r-1}} e^{-\Delta z_\mu} + \kappa_{\mu_r} E_{\mu_{r-(1/2)}} (\Delta\epsilon_r - \Delta\epsilon^0_r) \quad (\mu = 1, \dots n) \quad \dots\dots\dots\dots \quad (10)$$

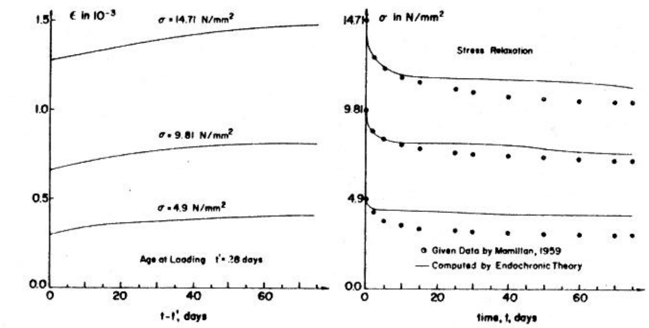

FIG. 4.—Associated Histories of Uniaxial Strain and Stress (Relaxation Tests in Flexible Frame)

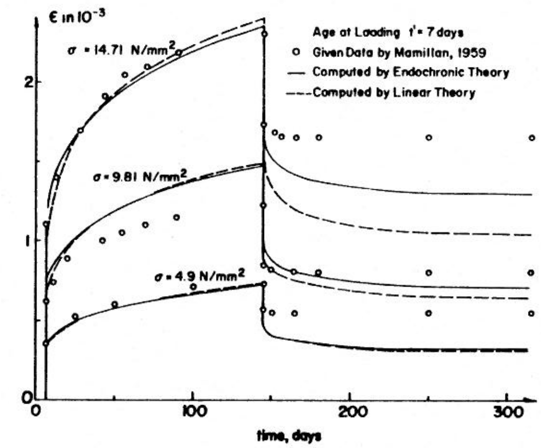

FIG. 5.—Mamillan's Data on Uniaxial Creep and Creep Recovery at Various Sustained Stress Levels (Unloaded to Zero Stress)

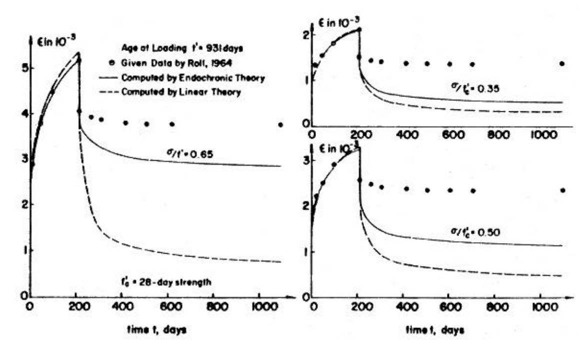

FIG. 6.—Roll's Data on Uniaxial Creep and Creep Recovery at Various Sustained Stress Levels (Unloaded to Zero Stress)

in which $\kappa_{\mu_r} = (1 - e^{-\Delta z_\mu})/\Delta z_\mu$ . . . . (11)

Here subscript r refers to time t_r and subscript $r - 1/2$ refers to the average value in the time-step, e.g., $E_{\mu_{r-1/2}} = 1/2\,(E_{\mu_{r-1}} + E_{\mu_r})$ and $\Delta z_\mu = z_{\mu_r} - z_{\mu_{r-1}}$. Eq. 10 represents a recurrent formula for hidden stresses. Furthermore, substituting Eq. 10 into the relation $\Delta\sigma_r = \Sigma_\mu\, \Delta\sigma_{\mu_r}$, one obtains

$$\Delta\sigma_r = E''_r\,(\Delta\epsilon_r - \Delta\epsilon''_r) \qquad (12)$$

in which $$E''_r = \sum_{\mu=1}^{n} \kappa_{\mu_r} E_{\mu_{r-1/2}} \qquad (13)$$

$$E''_r\,\Delta\epsilon''_r = \sum_{\mu=1}^{n} (1 - e^{-\Delta z_\mu})\,\sigma_{\mu_{r-1}} + E''_r\,\Delta\epsilon^0_r \qquad (14)$$

The standard step-by-step formulas are of the same form, but instead of $1 - e^{-\Delta z_\mu}$ they have Δz_μ and instead of κ_{μ_r} they have 1. Obviously, for very small Δz_μ this is equivalent, but not so for large Δz_μ. For Δz_μ much larger than one (which can happen when the time-step equals, say, 100 days and $\tau_1 = 0.001$ day) the inelastic strain, $\Delta\epsilon''_\mu = (\sigma_\mu/E_\mu)\,\Delta z_\mu$, would then become extremely large even if the stress in the μth Maxwell unit has almost dissipated ($\sigma_\mu \approx 0$). This would cause numerical instability (which was mathematically demonstrated for a similar case in Ref. 8). On the other hand, $1 - e^{-\Delta z_\mu}$ is bounded regardless of the value of Δz_μ, and so numerical instability cannnot occur for larger Δz_μ (for a proof in a similar case see again Ref. 8). Physically, the necessity of basing the formulas on the assumption of imposed (linear) history of ϵ within each time step may be explained by the fact that in a Maxwell solid subjected to imposed constant strain (relaxation), stress is always bounded, while strain at imposed constant stress grows without bounds. (By the same argument, for the Kelvin chain model, stress rather than strain would have to be assumed to vary linearly with the time step.)

In the endochronic theory for creep, numerical instability at large Δt would also be caused by the term $\Delta\epsilon^t_\mu$ in the expression for $\Delta\xi_\mu$ based on Eq. 5, because this term has the form of creep strain of Maxwell solid at constant stress and is unbounded. By the same argument as before, remedy may be achieved by calculating $\Delta\epsilon^t_\mu$ not for constant σ_μ but for σ_μ varying in correspondence to a prescribed linear variation of ϵ with t. Integrating the linearly viscoelastic relation $d\epsilon - d\epsilon^0 = d\sigma_\mu/E_\mu + \sigma_\mu\, dt/E_\mu\tau_\mu$ at constant $d\epsilon/dt$ and constant $d\epsilon^0/dt$ and imposing the initial condition at t_{r-1}, gives $\sigma_\mu(t) = \sigma_{\mu_{r-1}} \exp\,[(t_{r-1} - t)/\tau_\mu] + E_\mu\tau_\mu\,\Delta\epsilon/\Delta t$ which, after substitution into $d\epsilon^t_\mu = \sigma_\mu\, dt/(E_\mu\tau_\mu)$ and integration from t_{r-1} to t_r, provides

$$\Delta\epsilon^t_\mu = \frac{\sigma_{\mu_{r-1}}}{E_{\mu_{r-1/2}}}(1 - e^{-\Delta t/\tau_\mu}) + (\Delta\epsilon - \Delta\epsilon^0)\left[1 - \frac{\tau_\mu}{\Delta t}(1 - e^{-\Delta t/\tau_\mu})\right] \qquad (15)$$

Note that the second term is second-order small when $\Delta t \rightarrow 0$ and $\Delta\epsilon \rightarrow 0$ and the first term gives $\Delta\epsilon^t_\mu \approx \sigma_{\mu_{r-1}}\,\Delta t/(E_\mu\tau_\mu)$ when Δt is small.

The computation in each step $\Delta t = t_r - t_{r-1}$ may now proceed as follows:

1. The increments, $\Delta\epsilon\ \Delta\sigma_\mu$, are estimated, e.g., assuming that they have the same values as those in the previous step Δt. This is done for the purpose of evaluating mean values in the rth step, $\sigma_{\mu_{r-(1/2)}} = \sigma_{\mu_r} + \Delta\sigma_\mu/2$ and $\epsilon_{r-(1/2)} = \epsilon_{r-1} + \Delta\epsilon/2$.

2. The linear viscoelastic part of strain increment $\Delta\epsilon^t$ is evaluated from Eq. 15, and the $\Delta\xi_\mu$ values are then obtained from $|\Delta\epsilon - \Delta\epsilon^t|$.

3. The $\Delta\eta_\mu$ and $\Delta\zeta_\mu$ are evaluated from a central difference approximation of Eqs. 5 and 6, in which $\epsilon_\mu = \epsilon_{\mu_{r-(1/2)}}$, $\sigma_\mu = \sigma_{\mu_{r-(1/2)}}$, $\eta_\mu = \eta_{\mu_{r-1}} + (1/2)\,\Delta\eta_\mu$; and the intrinsic time increments, Δz_μ, are calculated from a finite difference form of Eq. 4.

4. The pseudo-inelastic strain increment, $\Delta\epsilon''_r$, and the pseudo-instantaneous elastic modulus, E''_r, for the rth step are computed from Eqs. 12 and 14.

5. In case of prescribed strain increment, $\Delta\sigma_r$ is then calculated from the quasi-elastic relation in Eq. 12. (When $\Delta\sigma_r$ is prescribed, $\Delta\epsilon_r$ is calculated from Eq. 12, and when neither $\Delta\sigma_r$ nor $\Delta\epsilon_r$ is specified, Eq. 12 is used as a fictitious elastic stress-strain law to solve the structure according to the theory of elasticity.)

6. The stresses, σ_{μ_r}, in individual Maxwell units at the end of interval are then determined from Eq. 10.

7. Steps 1-3 are iterated several times using the values of $\Delta\epsilon_r$ and $\Delta\sigma_r$ from the previous iteration. If the values of z_μ from the last two iterations differ by more than 0.1%, smaller time step Δt should be chosen.

Limitations of Present Formulation

In addition to microcracking, the nonlinear dependence of concrete creep on stress has a second physical source in moisture movement due to drying or wetting. This effect takes place not only at high stress but also in the working stress range (below $0.5\,f'_c$), in which the creep is linear if there is no simultaneous drying or wetting. Formulation of this nonlinear effect for the working stress range has already been accomplished (12). To model the nonlinear creep of drying concrete in the high stress range, the formulation from Ref. 12 could be combined with the present one, which is restricted to concrete in which moisture movement occurs. However, this objective is beyond the scope of this study. As an approximation, the present formulation can be used even for drying concrete members, provided that they are so massive that drying is sufficiently slow for nonlinearity due to drying to be insignificant.

Finally, a third type of nonlinearity arises in creep recovery, which is manifested by deviations from the principle of superposition at low stress and without drying whenever strain (not stress) is reversed. This effect is also not covered by the present model.

The most severe limitation of the present formulation is due to the lack of data on nonlinear creep at multiaxial stress. What has in effect been done is to assume that the effects of multiaxial stress on creep and on short-time deformations are similar. No doubt, this is a simplication and further extensions will be needed when appropriate data become available.

Conclusions

1. The endochronic theory for nonlinear creep of concrete gives a correct formulation in the basic extreme special cases, such as: (a) Rate-type creep law for linear creep of aging concrete modeled by Maxwell chain; (b) multiaxial

stress-strain behavior and failure conditions for short-time deformations; (c) decrease of strength with load duration; and (d) inelastic strain accumulation under cyclic load.

2. The theory presented is capable of accurately describing uniaxial compression creep tests for various ages at loading and for various stress levels.

3. The predicted decrease of uniaxial compression strength with load duration agrees with test data.

4. In case of drying concrete, the present theory does not describe nonlinear creep recovery and nonlinear stress relaxation sufficiently well, but still it is distinctly better than the linear theory based on the principle of superposition.

5. Structural creep problems can be analyzed in time steps by the algorithm presented.

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Appendix II.—Notation

The following symbols are used in this paper:

E, E_μ = Young's modulus and hidden modulus for σ_μ;
$e_{ij}, e^t_{ij_\mu}, e_{ij_\mu}$ = strain deviator, its linear viscoelastic part, and μth hidden strain deviator;
G, G_μ = shear modulus, and hidden shear modulus for σ_μ;
K, K_μ = bulk modulus, and hidden bulk modulus for σ_μ;
s_{ij}, s_{ij_μ} = stress deviator, and μth hidden stress deviator;
t = time;
Z_μ = relaxation strain parameter for μth hidden strain (Eqs. 2, 8);
z, z_μ = intrinsic time, and intrinsic time for μth hidden strain;
ϵ, ϵ_μ = uniaxial strain, and μth hidden uniaxial strain;
$\zeta, \zeta_\mu, \eta, \eta_\mu$ = time-independent intrinsic times and intrinsic time measures;
ξ, ξ_μ = distortion measure, and distortion measure for μth hidden strain; and
τ_μ = relaxation times of Maxwell chain model;

Subscripts

i, j = cartesian coordinates x_i, $i = 1, 2, 3$;
r = rth time step; and
μ = variables associated with τ_μ.

12729 LAW FOR NONLINEAR CREEP OF CONCRETE

KEY WORDS: **Aging tests (materials); Concrete; Concrete structures; Creep; Creep recovery;** Integration (mathematics); Numerical analysis; **Repeated loading;** Strength; Viscoelasticity

ABSTRACT: Endochronic theory, previously proposed and verified for multiaxial time-independent experimental data, is extended to nonlinear long-time creep at high stress and is compared with available uniaxial creep data. The extension is based on a Maxwell chain model, each unit of which is characterized by its own intrinsic time, an independent variable whose increments depend both on time and strain increments. The dependence on the latter involves the previously determined hardening and softening functions. Aging is included and the previously established Maxwell chain model for low-stress creep with aging is obtained as a special case. The theory also describes the decrease of strength with load duration when the compression is high, gives an increase when the compression is low, and yields the additional inelastic strain accumulation due to cyclic load. An effective and numerically stable algorithm for timestep integration of structural response, permitting the time steps to be increased with the load duration, is presented.

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