Int. J. Engng Sci., 1976, Vol. 14, pp. 797–809. Pergamon Press. Printed in Great Britain

ELASTODYNAMIC NEAR-TIP FIELDS FOR A RAPIDLY PROPAGATING INTERFACE CRACK†

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Abstract—The elastodynamic stress field near a crack tip rapidly propagating along the interface between two dissimilar isotropic elastic solids is investigated. Both anti-plane and in-plane motions are considered. The anti-plane displacements and the in-plane displacement potentials are sought in the separated forms $r^qF(\theta)$, r and θ being polar coordinates centered at the moving tip. The mathematical statement of the problem reduces to a second-order linear ordinary differential equation in θ, which can be solved analytically. Formulation of the boundary and interface conditions leads to an eigenvalue problem for the singularity exponent q. For the in-plane problem, root q is found to be complex. Thus, the stresses exhibit violent oscillations within a small region around the crack tip, and the solutions have physical significance only outside this region. The angular stress distributions are plotted for various crack speeds, and it is found that at a high enough speeds the direction θ of maximum stress moves out of the interface. This result indicates that a running interface crack may move into one of the adjoining materials.

1. INTRODUCTION

A NOTEWORTHY feature of elastodynamic stress fields near a crack tip propagating in a homogeneous linearly elastic solid is that maximums of stress intensity factors, which are in the plane of crack propagation at low speeds of the crack tip, move out of this plane when the speed of crack propagation exceeds a certain value. For the transient elastodynamic problem of a crack tip moving with a time-varying velocity along a rather arbitrary but smooth two-dimensional trajectory in an isotropic material, and along a principal plane in an orthotropic material, this effect was recently analyzed in Ref. [1].

The purpose of this paper is to investigate analogous effects for a crack tip which moves rapidly along an interface between two elastic solids of different mechanical properties. Shifts of maximum values of stress intensity factors out of an interface are of interest, because of the potential relation to skewing of a crack from an interface into one of the adjoining materials. In this paper the variations with polar angle (centered at the moving crack tip) of the singular parts of the stresses are investigated for transient propagation of a crack tip along an interface.

For *elastostatic* problems the nature of the singularity near the tip of a crack along the interface of two bonded dissimilar half planes was analyzed by Williams [2]. He was apparently the first to discover that the singular parts of the stresses show intense oscillations. More complete elastostatic solutions were subsequently presented in Refs. [3–6].

The number of studies on *elastodynamic* fields generated by crack propagation along an interface is much smaller. The two-dimensional transient in-plane problem has not yet been solved in complete detail for perfect bond ahead of the crack tip. A case of smooth contact was analyzed in Ref. [7]. Antiplane motions were considered in Ref. [8]. A steady-state elastodynamic analysis of the in-plane problems, based on the assumption that fields may be assumed stationary relative to a coordinate system moving with the crack tip, was given by Gol'dshtein [9]. In Gol'dshtein's paper particular attention was devoted to stress singularities in the interface ahead of the crack tip. The steady-state problem of a crack propagating along the interface of an elastic half-space and a rigid body was studied by Jahanshahi [10]. Some general results on steady-state elastodynamic problems were included in Refs. [5, 11].

In this paper the nature of singularities near the tips of interface flaws is examined by using an extension to elastodynamic problems of the technique employed by Williams in Ref. [2]. An analogous study for a propagating crack tip in a homogeneous material was presented in Ref. [1].

It is shown that the instantaneous value of the speed of crack propagation enters as a parameter into the functions governing the variation with polar angle of the near-tip fields. This

†The efforts of two of the authors (J.D.A. and R.P.K.) were sponsored by the U.S. Army Office Durham under Grant DAHC04-75-G-0200.

implies that the dependence on polar angle is the same for steady-state and transient problems of crack propagation.

For a propagating flaw at the interface of two isotropic half-spaces of different mechanical properties, the dependence on polar angle θ of the stress fields can be determined analytically in explicit form, as shown in this paper. For a flaw at the interface of two anisotropic half-spaces a numerical procedure is required, which is presented in a sequel to this paper.

A two-dimensional geometry will be considered. The velocity of the crack tip along the interface is $c(t)$, where $c(t)$ is an arbitrary function of time, subject to the conditions that $c(t)$ and dc/dt are continuous. A system of moving Cartesian coordinates (x, y) is centered at the crack tip, such that the x-axis is in the interface. Moving polar coordinates (r, θ) are attached to the moving crack tip. The geometry is shown in Fig. 1.

In a plane two-dimensional geometry, the system of displacement equations of motion governing linearized elasticity separates into two uncoupled systems, for in-plane and anti-plane displacements, respectively; see, e.g. Ref. [12].

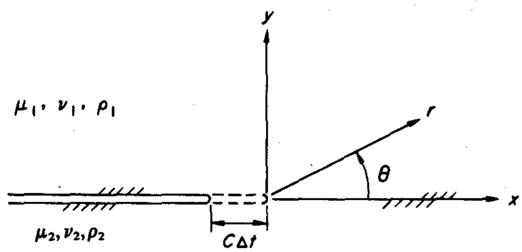

Fig. 1. Propagation of a two dimensional crack along an interface.

2. SINGULARITIES FOR ANTI-PLANE MOTIONS

Let the anti-plane displacement be instantaneously defined in terms of the moving coordinate system (x, y); i.e. $w = w(x, y, t)$. The second material derivative with respect to time, which is indicated by a superscript double dot, is then of the form

$$\ddot{w} = \frac{\partial^2 w}{\partial t^2} - \dot{c}(t)\frac{\partial w}{\partial x} - 2c(t)\frac{\partial^2 w}{\partial t \partial x} + [c(t)]^2\frac{\partial^2 w}{\partial x^2}. \tag{1}$$

Relative to the moving system of Cartesian coordinates the displacement equation of motion is

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = \frac{1}{c_T^2}\ddot{w} \tag{2}$$

where $\ddot{w}$ is defined by eqn (1) and $c_T = (\mu/\rho)^{1/2}$ is the velocity of transverse waves.

In this paper we seek displacements in solids 1 and 2 in the immediate vicinity of the moving crack tip in the general forms

$$w_j(r, \theta, t) = \left(\frac{r}{d}\right)^q T(t) W_j(\beta_j, \theta) d, \tag{3}$$

where d is a length parameter, r and θ are the polar coordinates shown in Fig. 1, and β_j is defined as

$$\beta_j = c(t)/c_{Tj}. \tag{4}$$

Here $j = 1$ and $j = 2$ refer to solid 1 and solid 2, respectively.

Since the shear stresses vanish on the crack surfaces we have

$$\theta = \pm\pi: \quad \partial W_j/\partial\theta = 0. \tag{5}$$

Continuity of displacements and stresses ahead of the moving crack tip implies

$$\theta = 0; \quad W = W_2; \quad \mu_j \partial W_j / \partial\theta = \sigma_{\theta z} \tag{6a, b}$$

where $j = 1, 2$, and $\sigma_{\theta z}$ is a stress-intensity-factor.

Substituting eqn (3) into eqn (2), with w according to eqn (1), multiplying the result by r^{2-q}, and considering the limit $r \to 0$, the following equation for W_j is obtained

$$(1-\beta_j^2 \sin^2\theta)\frac{d^2 W_j}{d\theta^2} - \beta_j^2(1-q)\sin(2\theta)\frac{dW_j}{d\theta} + q\{q+\beta_j^2[(2-q)\cos^2\theta - 1)\}W_j = 0. \tag{7}$$

Following Ref. [1], a solution of eqn (7) is sought of the form

$$W_j(\beta_j, \theta) = (1-\beta_j^2\sin^2\theta)^{q/2} W_j^*(\beta_j, \theta). \tag{8}$$

Substitution of eqn (8) into eqn (7) yields a much simpler equation for W_j^*, which can, however, be further simplified by introducing the variable ω_j by

$$\tan\omega_j = (1-\beta_j^2)^{1/2}\tan\theta. \tag{9}$$

The resulting equation for W_j^* is

$$\frac{d^2 W_j^*}{d\omega_j^2} + q^2 W_j^* = 0. \tag{10}$$

Thus,

$$W_j^* = A_j \sin q\omega_j + B_j \cos q\omega_j. \tag{11}$$

The interface conditions at $\theta = 0$, given by eqn (6), yield

$$B_1 = B_2, \quad \mu_j q(1-\beta_j^2)^{1/2} A_j = \sigma_{\theta z}. \tag{12a, b}$$

The conditions at $\theta = \pm\pi$ require

$$A_1 \cos q\pi - B_1 \sin q\pi = 0; \quad A_2 \cos q\pi + B_2 \sin q\pi = 0. \tag{13a, b}$$

Equations (12) and (13) imply

$$\left[1 + \frac{\mu_1 (1-\beta_1^2)^{1/2}}{\mu_2 (1-\beta_2^2)^{1/2}}\right] \sin 2q\pi = 0. \tag{14}$$

This is an equation for q, whose lowest non-trivial root is $q = 0.5$. Thus, the singularity is of the square-root type. In view of eqn (13) it follows that B_1 and B_2 vanish identically, and the anti-plane near-tip displacements in the two half planes are of the same general forms as for a homogeneous material, which were discussed in detail in Ref. [1]. Writing

$$(\tau_{\theta z})_j = \sigma_{\theta z} T(t)(F_{\theta z})_j (r/d)^{-1/2} \tag{15}$$

where $\sigma_{\theta z}$ is defined by eqn (12b), we obtain by employing eqn (11)

$$(F_{\theta z})_j = [(-1)^{j+1}(1-\beta_j^2)^{-1/2}\Psi_{j1}\sin\theta + \Psi_{j2}\cos\theta]/\sqrt{2}. \tag{16}$$

Here ω_j has been eliminated by means of eqn (9). The functions Ψ_{j1} and Ψ_{j2} are defined as

$$\Psi_{j1} = \left[\frac{(1-\beta_j^2\sin^2\theta)^{1/2} - \cos\theta}{1-\beta_j^2\sin^2\theta}\right]^{1/2} \tag{17a}$$

$$\Psi_{j2} = \left[\frac{(1-\beta_j^2\sin^2\theta)^{1/2} + \cos\theta}{1-\beta_j^2\sin^2\theta}\right]^{1/2}. \tag{17b}$$

For $c_{T2}/c_{T1} = 1.4$, the functions $(F_{\theta z})_j$, which govern the angular variations of the stress-intensity factors are plotted in Fig. 2 versus θ, and for various values of $\beta_1 = c/c_{T1}$. It is noted that the maxima of $F_{\theta z}$ move out of the interface as the speed of the crack tip increases. The maxima shift first towards the medium for which β attains the largest value. These results suggest the possibility that the crack tip may leave the interface and move into the lower modulus material.

To compute the function $T(t)$ in eqn (16), the asymptotic considerations presented here are not sufficient, and for a particular problem a more complete mathematical analysis is required. For an example we refer to Ref. [8]. In Ref. [8] expressions are also presented for the flux of energy into the moving crack tip.

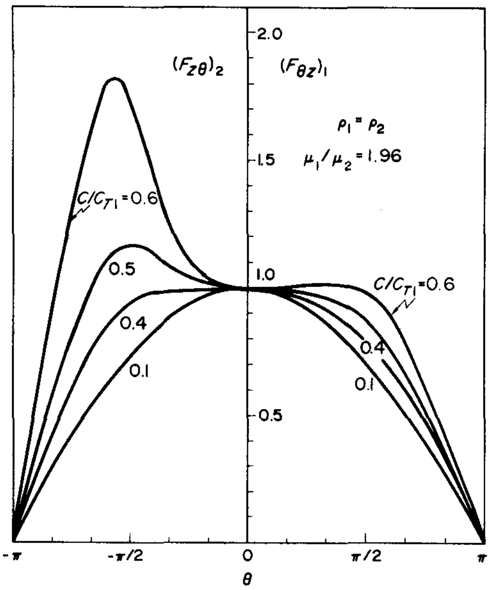

Fig. 2. Angular variation of the stress-intensity function $F_{\theta z}$ for various crack propagation speeds, and for antiplane motions.

3. SINGULARITIES FOR IN-PLANE MOTIONS

It is convenient to express the in-plane displacements $u(x, y, t)$ and $v(x, y, t)$ in terms of the displacement potentials φ and ψ by

$$u = \frac{\partial\varphi}{\partial x} + \frac{\partial\psi}{\partial y}; \quad v = \frac{\partial\varphi}{\partial y} - \frac{\partial\psi}{\partial x}. \tag{18a, b}$$

Equations (18a, b) satisfy the displacement equations of motion provided that φ and ψ are solutions of

$$\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} = \frac{1}{\kappa^2 c_T^2}\ddot{\varphi}; \quad \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = \frac{1}{c_T^2}\ddot{\psi}. \tag{19a, b}$$

Here $\ddot{\varphi}$ and $\ddot{\psi}$ are defined analogously to eqn (1), and

$$\kappa^2 = \frac{c_L^2}{c_T^2} = \frac{\lambda + 2\mu}{\mu} = \frac{2(1-\nu)}{1-2\nu}. \tag{20}$$

In eqn (20) λ and μ are Lamé's elastic constants, and ν is Poisson's ratio.

In the vicinity of the crack tip expressions for the potentials φ and ψ are sought in the forms

$$\varphi_j = d^2T(t)Re\left[\left(\frac{r}{d}\right)^p \Phi_j(\alpha_j, \theta)\right] \tag{21}$$

and

$$\psi_j = d^2T(t)\,Re\left[\left(\frac{r}{d}\right)^p \Psi_j(\beta_j, \theta)\right], \tag{22}$$

respectively, where d is a length parameter, and

$$\alpha_j = \frac{c(t)}{c_{Lj}} = \frac{c(t)}{\kappa_j c_{Tj}} = \frac{\beta_j}{\kappa_j}. \tag{23}$$

Here β_j and κ_j are defined by eqns (4) and (20), respectively. Since we must include the possibility that p is complex, it is necessary to work with complex algebra. In order that the strain energy density be integrable, we require $Re(p) > 1$.

The method of solution for $\Phi_j(\alpha_j, \theta)$ and $\Psi_j(\beta_j, \theta)$ is completely analogous to the method used for solving $W_j(\beta_j, \theta)$. Similarly to the development of eqns (7)–(11) the pertinent solutions are obtained in complex notation as

$$\Phi_j = (1-\alpha_j^2 \sin^2\theta)^{p/2}[A_j e^{ip\epsilon_j} + B_j e^{-ip\epsilon_j}] \tag{24}$$

$$\Psi_j = (1-\beta_j^2 \sin^2\theta)^{p/2}[C_j e^{ip\omega_j} + D_j e^{ip\omega_j}], \tag{25}$$

where ω_j is defined by eqn (9), and ϵ_j is defined analogously as

$$\tan\epsilon_j = (1-\alpha_j^2)^{1/2}\tan\theta. \tag{26}$$

The constant p is obtained from the boundary conditions.

The displacement potentials (21) and (22) may be used to obtain the stresses $(\tau_\theta)_j$ and $(\tau_{\theta r})_j$ in the system of polar coordinates. The results are

$$(\tau_\theta)_j = \mu_j T(t)\left(\frac{r}{d}\right)^{p-2}\left\{[\kappa_j^2 p^2 - 2p(p-1)]\Phi_j + \kappa_j^2\frac{\partial^2\Phi_j}{\partial\theta^2} + 2(1-p)\frac{\partial\Psi_j}{\partial\theta}\right\} \tag{27}$$

$$(\tau_{\theta r})_j = \mu_j T(t)\left(\frac{r}{d}\right)^{p-2}\left\{2(p-1)\frac{\partial\Phi_j}{\partial\theta} - p(p-2)\Psi_j + \frac{\partial^2\Psi_j}{\partial\theta^2}\right\}. \tag{28}$$

In the system of polar coordinates the boundary conditions on the surface of the propagating crack are

$$\tau_\theta = 0 \quad \text{and} \quad \tau_{\theta r} = 0, \qquad \text{for} \qquad \theta = \pm\pi, \quad r > 0. \tag{29}$$

Ahead of the propagating crack tip we wish to consider various interface conditions, depending on the type of contact at the interface. These various interface conditions will be examined in the sequel. Without loss of generality we may at this stage prescribe continuity of the tractions, i.e.

$$\theta = 0, \quad r > 0: \qquad (\tau_\theta)_1 = (\tau_\theta)_2 = \left(\frac{r}{d}\right)^{p-2} T(t)\sigma_\theta \tag{30}$$

$$(\tau_{\theta r})_1 = (\tau_{\theta r})_2 = \left(\frac{r}{d}\right)^{p-2} T(t)\sigma_{\theta r}. \tag{31}$$

Here σ_θ and $\sigma_{\theta r}$ define the interface tractions. To examine various specific interface conditions, eqns (30) and (31) may be supplemented by additional conditions, as shown in the sequel.

It easily follows from eqns (27) and (30) that

$$(\beta_j^2 - 2)(A_j + B_j) - 2i(1-\beta_j^2)^{1/2}(C_j - D_j) = \sigma_\theta / p(p-1)\mu_j, \tag{32}$$

while eqns (31) and (28) yield

$$2i(1-\alpha_j^2)^{1/2}(A_j - B_j) + (\beta_j^2 - 2)(C_j + D_j) = \sigma_{\theta r} / p(p-1)\mu_j. \tag{33}$$

The conditions of vanishing surface tractions at $\theta = +\pi$ yields

$$(\beta_1^2 - 2)(A_1 + \gamma B_1) - 2i(1-\beta_1^2)^{1/2}(C_1 - \gamma D_1) = 0 \tag{34}$$

$$2i(1-\alpha_1^2)^{1/2}(A_1 - \gamma B_1) + (\beta_1^2 - 2)(C_1 + \gamma D_1) = 0 \tag{35}$$

where

$$\gamma = e^{-2ip\pi}. \tag{36}$$

Similarly, the condition that the surface tractions vanish at $\theta = -\pi$ yields

$$(\beta_2^2 - 2)(\gamma A_2 + B_2) - 2i(1-\beta_2^2)^{1/2}(\gamma C_2 - D_2) = 0 \tag{37}$$

$$2i(1-\alpha_2^2)^{1/2}(\gamma A_2 - B_2) + (\beta_2^2 - 2)(\gamma C_2 + D_2) = 0. \tag{38}$$

By combining eqns (32) for $j = 1$ with eqn (34), and eqn (33) for $j = 1$ with eqn (35), we find in matrix notation

$$\begin{bmatrix} B_1 \\ D_1 \end{bmatrix} = \frac{1}{p(p-1)}\frac{1}{1-\gamma}[L_1]\begin{bmatrix} \sigma_\theta \\ \sigma_{\theta r} \end{bmatrix}. \tag{39}$$

By combining eqns (32) and (33) for $j = 2$ with eqns (37) and (38), we obtain

$$\begin{bmatrix} B_2 \\ D_2 \end{bmatrix} = -\frac{1}{p(p-1)}\frac{\gamma}{1-\gamma}[L_2]\begin{bmatrix} \sigma_\theta \\ \sigma_{\theta r} \end{bmatrix} \tag{40}$$

where

$$L_j = \frac{1}{\mu_j R_j}\begin{bmatrix} \beta_j^2 - 2 & -2i(1-\beta_j^2)^{1/2} \\ 2i(1-\alpha_j^2)^{1/2} & \beta_j^2 - 2 \end{bmatrix}. \tag{41}$$

Here R_j is the Rayleigh function which is defined as

$$R_j = (\beta_j^2 - 2)^2 - 4(1-\alpha_j^2)^{1/2}(1-\beta_j^2)^{1/2}. \tag{42}$$

From eqns (34) and (35) we find

$$\begin{bmatrix} A_1 \\ C_1 \end{bmatrix} = \gamma[P_1]\begin{bmatrix} B_1 \\ D_1 \end{bmatrix}, \tag{43}$$

while eqns (37) and (38) yield

$$\begin{bmatrix} A_2 \\ C_2 \end{bmatrix} = \frac{1}{\gamma}[P_2]\begin{bmatrix} B_2 \\ D_2 \end{bmatrix}. \tag{44}$$

The matrices P_1 and P_2 are defined by

$$P_j = \frac{1}{R_j}\begin{bmatrix} -(\beta_j^2-2)^2 - 4(1-\alpha_j^2)^{1/2}(1-\beta_j^2)^{1/2} & -4i(\beta_j^2-2)(1-\beta_j^2)^{1/2} \\ 4i(\beta_j^2-2)(1-\alpha_j^2)^{1/2} & -(\beta_j^2-2)^2 - 4(1-\alpha_j^2)^{1/2}(1-\beta_j^2)^{1/2} \end{bmatrix}. \tag{45}$$

By means of eqns (39)–(45), A_j, B_j, C_j and D_j, which appear in the expressions for Φ_j and Ψ_j, have now been expressed in terms of the interface tractions σ_θ and $\sigma_{\theta r}$.

For future reference it remains to express the displacement components in terms of the displacement potentials. The results are

$$(u_r)_j = T(t)d\left(\frac{r}{d}\right)^{p-1}\left[p\Phi_j + \frac{\partial \Psi_j}{\partial \theta}\right] \tag{46}$$

$$(u_\theta)_j = T(t)d\left(\frac{r}{d}\right)^{p-1}\left[\frac{\partial \Phi_j}{\partial \theta} - p\Psi_j\right]. \tag{47}$$

At this stage we are ready to determine p for various interface conditions ahead of the moving crack tip.

The case of *perfect contact* is of greatest interest. For perfect contact eqns (30) and (31) must be supplemented by conditions requiring continuity of the displacements ahead of the crack tip, i.e.,

$$\theta = 0, \quad r > 0 \qquad (u_r)_1 = (u_r)_2 = \left(\frac{r}{d}\right)^{p-1} dT(t)u_{r0} \tag{48a}$$

$$(u_\theta)_1 = (u_\theta)_2 = \left(\frac{r}{d}\right)^{p-1} dT(t)u_{\theta 0}. \tag{48b}$$

By employing eqns (46) and (47) these conditions yield

$$[Q_1]\begin{bmatrix} A_1 \\ C_1 \end{bmatrix} + [S_1]\begin{bmatrix} B_1 \\ D_1 \end{bmatrix} = [Q_2]\begin{bmatrix} A_2 \\ C_2 \end{bmatrix} + [S_2]\begin{bmatrix} B_2 \\ D_2 \end{bmatrix} \tag{49}$$

where

$$[Q_j] = \begin{bmatrix} 1 & i(1-\beta_j^2)^{1/2} \\ i(1-\alpha_j^2)^{1/2} & -1 \end{bmatrix} \tag{50}$$

and

$$[S_j] = \begin{bmatrix} 1 & -i(1-\beta_j^2)^{1/2} \\ -i(1-\alpha_j^2)^{1/2} & -1 \end{bmatrix}. \tag{51}$$

Substituting the results (43), (39), (44) and (40) into eqn (49) we obtain the following system of homogeneous equations for the interface tractions σ_θ and $\sigma_{\theta r}$

$$[M]\begin{bmatrix} \sigma_\theta \\ \sigma_{\theta r} \end{bmatrix} = 0. \tag{52}$$

The matrix $[M]$ is

$$[M] = \begin{bmatrix} (1-\gamma)(a_1 - a_2) & -i(1+\gamma)(b_1 + b_2) \\ -i(1+\gamma)(c_1 + c_2) & -(1-\gamma)(a_1 - a_2) \end{bmatrix} \tag{53}$$

where

$$a_j = [2(1-\alpha_j^2)^{1/2}(1-\beta_j^2)^{1/2} - (2-\beta_j^2)]/\mu_j R_j \tag{54}$$

$$b_j = \beta_j^2(1-\beta_j^2)^{1/2}/\mu_j R_j \tag{55}$$

$$c_j = \beta_j^2(1-\alpha_j^2)^{1/2}/\mu_j R_j. \tag{56}$$

The condition that the determinant $|M|$ must vanish yields a quadratic equation for γ, whose two roots can easily be obtained as

$$\gamma_1 = \gamma \quad \text{and} \quad \gamma_2 = \gamma^{-1} \tag{57a, b}$$

where

$$\gamma = \frac{(a_1 - a_2) - (b_1 + b_2)^{1/2}(c_1 + c_2)^{1/2}}{(a_1 - a_2) + (b_1 + b_2)^{1/2}(c_1 + c_2)^{1/2}}. \tag{58}$$

The coefficient p can subsequently be solved from eqn (36) as

$$p = \pm \frac{i}{2\pi} \ln \gamma. \tag{59}$$

An additional result for perfect contact is the relation between σ_θ and $\sigma_{\theta r}$, which can be obtained from eqn (52) as

$$\sigma_{\theta r} = \pm i \frac{(c_1 + c_2)^{1/2}}{(b_1 + b_2)^{1/2}} \sigma_\theta. \tag{60}$$

It is also possible to formulate the problem in terms of dimensionless interface displacements u_{r0} and $u_{\theta 0}$, which are defined in eqns (48a, b). Following a similar procedure one can obtain a relation between u_{r0} and $u_{\theta 0}$ for perfect contact as

$$u_{\theta_0} = \pm i \frac{(a_1 c_2 + c_2 a_1)(b_1 + b_2)}{(b_1 + b_2)^{1/2}(c_1 + c_2)^{1/2}(a_2 b_1 + a_1 b_2)} u_{r0}. \tag{61}$$

The interface displacements are related to the interface stress σ_θ as follows

$$u_{\theta 0} = \pm i \frac{a_1 c_2 + c_2 a_1}{(b_1 + b_2)^{1/2}(c_1 + c_2)^{1/2}} \frac{\sigma_\theta}{p - 1} \tag{62}$$

and

$$u_{r0} = \frac{a_2 b_1 + a_1 b_2}{(b_1 + b_2)} \frac{\sigma_\theta}{p - 1}. \tag{63}$$

If the interface is between a *rigid solid* (say solid 2) and a deformable solid (solid 1), we have the following conditions instead of eqns (48a, b)

$$\theta = 0, \quad r > 0 \qquad (u_r)_1 = (u_\theta)_1 = 0. \tag{64}$$

The right hand side of eqn (49) now vanishes, and γ is obtained as

$$\gamma = \frac{a_1 - b_1^{1/2} c_1^{1/2}}{a_1 + b_1^{1/2} c_1^{1/2}}. \tag{65}$$

The results are simpler for *smooth contact*. This case is defined by

$$\theta = 0, \quad r > 0 \qquad \sigma_{\theta r} = 0; \quad (u_\theta)_1 = (u_\theta)_2. \tag{66a, b}$$

Now the second equation of (52) is operative. We immediately find

$$\gamma = -1. \tag{67}$$

The case of pressureless contact is equally simple. This case is defined by

$$\theta = 0, \quad r > 0 \qquad \sigma_\theta = 0; \quad (u_r)_1 = (u_r)_2. \tag{68a, b}$$

Equation (52) yields $\gamma = -1$.

Finally, when the mechanical properties are identical, and the half-spaces are in perfect contact, we again obtain $\nu = -1$. For this special case matrix $[M]$ in eqn (53) is a null matrix and eqns (52) are automatically satisfied. Therefore σ_θ and $\sigma_{\theta r}$ can take independent values, permitting separate consideration of symmetric (Mode I) and antisymmetric (Mode II) displacements. For a detailed discussion we refer to Ref. [1]. For identical mechanical properties eqn (60) reduces to

$$\sigma_{\theta r} = \pm i \frac{c_1^{1/2}}{b_1^{1/2}} \sigma_\theta. \tag{69}$$

If this relation is employed with $\sigma_\theta = 1$, then the real part of the solution obtained corresponds to symmetric displacements with $\sigma_\theta = 1$, and the imaginary part corresponds to antisymmetric displacements with $\sigma_{\theta r} = \pm(c_1/b_1)^{1/2}$.

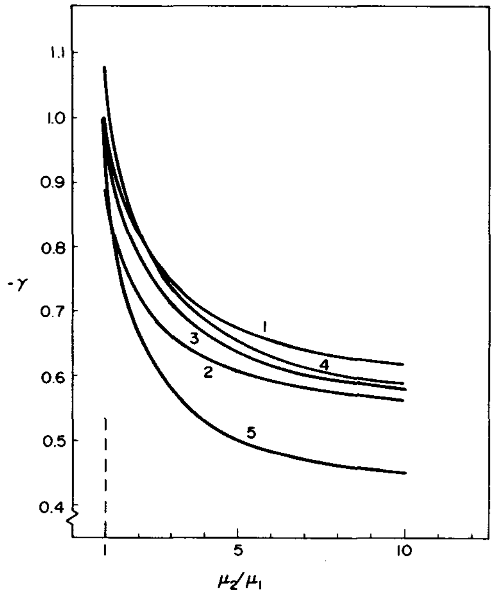

Fig. 3. Variation of root γ with the ratio of the shear moduli for various crack propagation speeds and Poisson ratios. $\nu_1 = 0.3$; $\rho_1 = \rho_2$; 1...$c/c_{R1} = 0.1$; $\nu_1/\nu_2 = 1$; 2...$c/c_{R1} = 0.4$; $\nu_1/\nu_2 = 0.8$; 3...$c/c_{R1} = 0.4$; $\nu_1/\nu_2 = 1$; 4...$c/c_{R1} = 0.4$; $\nu_1/\nu_2 = 1.2$; 5...$c/c_{R1} = 0.7$; $\nu_1/\nu_2 = 1$.

For the case of *perfect contact* the root γ was computed from eqn (58), and plotted in Fig. 3. The ratio of the shear moduli μ_2/μ_1 and the relative speed of the crack tip $c(t)/c_{R1}$, where c_{R1} is the velocity of Rayleigh surface waves in medium 1, were taken as the primary variables. The computations were carried out for Poisson's ratio $\nu_1 = 0.3$, and $\rho_1 = \rho_2$. For $c(t)/c_{R1} = 0.4$, three values of the ratio ν_1/ν_2 were considered. For all these cases γ was found as a negative number. It is noted that the variation of γ with μ_2/μ_1 is quite pronounced in the range $1 < \mu_2/\mu_1 < 10$.

If γ is negative, the constant p given by eqn (59) as $p = \pm(i/2\pi)\ln\gamma$ is a complex quantity of the form $p = \pm(i/2\pi)\ln(-1)\pm(i/2\pi)\ln|\gamma|$. It follows that p_1 must equal $\pm n/2$, n being an odd integer. The smallest value of p which is compatible with the requirement that the strain energy be intergrable near the propagating crack tip, is $p_1 = 3/2$. Thus,

$$p_{\pm} = p_1 \pm ip_2 = \frac{3}{2} \pm \frac{i}{2\pi}\ln|\gamma|. \tag{70}$$

Since p is complex, the term r^{p-2} gives rise to terms of the form $\cos[p_2\ln(r/d)]$ and $\sin[p_2\ln(r/d)]$ where d is a length parameter. The corresponding stresses show violent oscillations near the tip of the interface flaw, while the displacements at the free surfaces show interpenetration. From the physical point of view this behavior is impossible. Within the context of static problems it has, however, been noted by several authors, e.g. [3]–[6], that the singular behavior is confined to a small region around the crack tip. It may be assumed that at a not too small distance from the crack tip the oscillating stresses do have physical interest. It should also be noted that the oscillatory distributions of stresses and displacements still give rise to a very simple expression for the flux of energy into the propagating crack tip.

We will consider the angular variation of τ_θ and $\tau_{\theta r}$, in some detail. Equations (30) and (31) imply for $\theta = 0$, $r > 0$,

$$\tau_\theta = Re\left\{\left(\frac{r}{d}\right)^{p-2} T(t)\sigma_\theta\right\} \tag{71a}$$

$$\tau_{\theta r} = Re\left\{\left(\frac{r}{d}\right)^{p-2} T(t)\sigma_{\theta r}\right\}. \tag{71b}$$

In general σ_θ can be a complex quantity, i.e.,

$$\sigma_\theta = \sigma_{\theta 1} + i\sigma_{\theta 2}. \tag{72a}$$

Thus, from eqn (60) we obtain

$$\sigma_{\theta r} = \mp\frac{(c_1+c_2)^{1/2}}{(b_1+b_2)^{1/2}}\left\{\sigma_{\theta 2} - i\sigma_{\theta 1}\right\}. \tag{72b}$$

Substituting for σ_θ and $\sigma_{\theta r}$, eqns (71a) and (71b) yield for $\theta = 0$

$$\tau_\theta = \left(\frac{d}{r}\right)^{1/2} T(t)\left\{\sigma_{\theta 1}\cos\left(p_2\ln\frac{r}{d}\right) \mp \sigma_{\theta 2}\sin\left(p_2\ln\frac{r}{d}\right)\right\} \tag{73a}$$

and

$$\tau_{\theta r} = \left(\frac{d}{r}\right)^{1/2} T(t)\left(\frac{c_1+c_2}{b_1+b_2}\right)^{1/2}\left\{\mp\sigma_{\theta 2}\cos\left(p_2\ln\frac{r}{d}\right) + \sigma_{\theta 1}\sin\left(p_2\ln\frac{r}{d}\right)\right\}. \tag{73b}$$

It is not difficult to express τ_θ and $\tau_{\theta r}$ for $\theta \neq 0$ in terms of σ_θ. Substituting eqn (60) in eqns (39) and (40), we obtain B_1, D_1, B_2 and D_2 in terms of σ_θ, which in turn can be substituted in eqns (43) and (44) to yield A_1, C_1, A_2 and C_2 also in terms of σ_θ. These results are substituted in eqns (24) and (25), and the resulting expressions for the potentials are subsequently employed in eqns (27) and (28) to yield expressions for $(\tau_\theta)_j$ and $(\tau_{\theta r})_j$ of the following general forms

$$(\tau_\theta)_j = Re\left[\left(\frac{r}{d}\right)^{p_\pm - 2} T(t)\left\{F_{1j}(\theta) \pm iF_{2j}(\theta)\right\}\sigma_\theta\right] \tag{74a}$$

$$(\tau_{\theta r})_j = Re\left[\left(\frac{r}{d}\right)^{p_\pm - 2} T(t)\left\{G_{1j}(\theta) \pm G_{2j}(\theta)\right\}\sigma_\theta\right]. \tag{74b}$$

The functions F_{1j}, F_{2j}, G_{1j} and G_{2j} have been plotted as solid lines in Fig. 4 for $\nu_1 = \nu_2 = 0.3$,

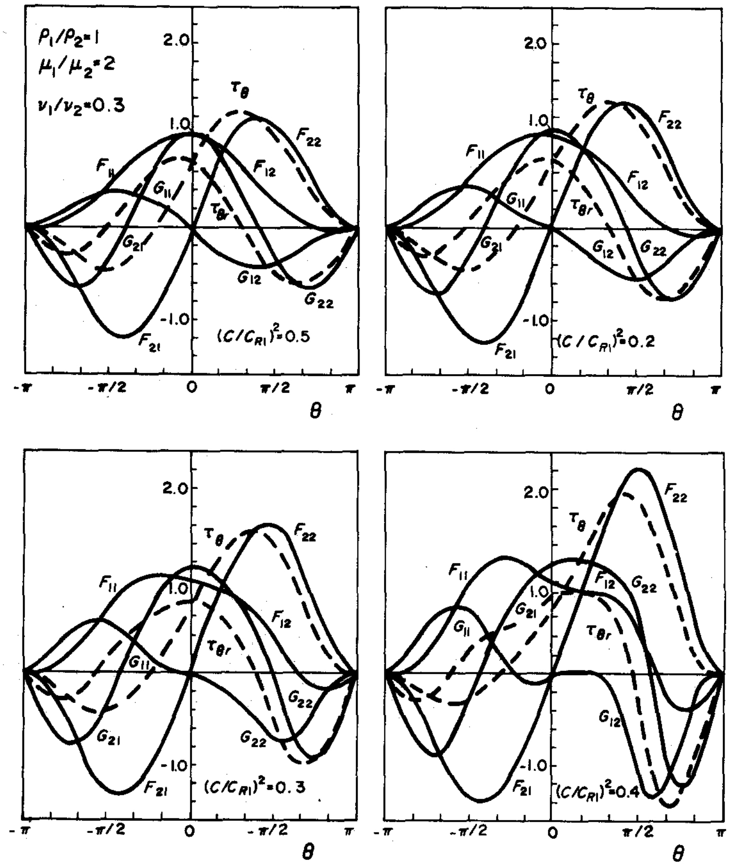

Fig. 4. Angular variations of functions F and G for a crack propagating at the interface of two dissimilar isotropic solids.

$\rho_1 = \rho_2$, $\mu_1/\mu_2 = 2$, and for four values of (c/c_{R1}). It can be seen that maximum values of these functions increase as the speed of crack propagation increases. It is also noted that the maximums of F_{11} and G_{22} move out of the interface.

In the limit as $\mu_1/\mu_2 \to 1$, the value of γ approaches -1 and we find $p_1 = 3/2$, $p_2 \equiv 0$. The angular variations of $(\tau_\theta)_j$ and $(\tau_{\theta r})_j$ are given by F_{1j} and G_{1j}, respectively, for Mode I fracture, and by F_{2j} and G_{2j} for Mode II fracture. For comparison these functions have been plotted in Fig. 5.

For an interface crack the dependence of the stresses on the circumferential angle θ is apparently complicated, and depends also on the distance from the crack tip. This is best exhibited by writing eqns (74a, b) as

$$(\tau_\theta)_j = T(t)\left(\frac{d}{r}\right)^{1/2}\left[\cos\left(p_2 \ln\frac{r}{d}\right)\{F_{1j}(\theta)\sigma_{\theta 1} \mp F_{2j}(\theta)\sigma_{\theta 2}\}\right.$$
$$\left. - \sin\left(p_2 \ln\frac{r}{d}\right)\{F_{2j}(\theta)\sigma_{\theta 1} \pm F_{1j}(\theta)\sigma_{\theta 2}\}\right] \tag{75a}$$

and

$$(\tau_{\theta r})_j = T(t)\left(\frac{d}{r}\right)^{1/2}\left[\cos\left(p_2 \ln\frac{r}{d}\right)\{G_{1j}(\theta)\sigma_{\theta 1} \mp G_{2j}(\theta)\sigma_{\theta 2}\}\right.$$
$$\left. - \sin\left(p_2 \ln\frac{r}{d}\right)\{G_{2j}(\theta)\sigma_{\theta 1} \pm G_{1j}(\theta)\sigma_{\theta 2}\}\right]. \tag{75b}$$

A good idea of the dependence on θ and r can be obtained by considering specific distances from the crack tip, namely distances such that

$$p_2 \ln(r/d) = \pi/2, \quad \text{or} \quad p_2 \ln(r/d) = 3\pi/4, \quad \text{or} \quad p_2 \ln(r/d) = \pi. \tag{76a, b, c}$$

Still, to explicitly determine the variation of stresses with θ we need to know the ratio $\sigma_{\theta 2}/\sigma_{\theta 1}$,

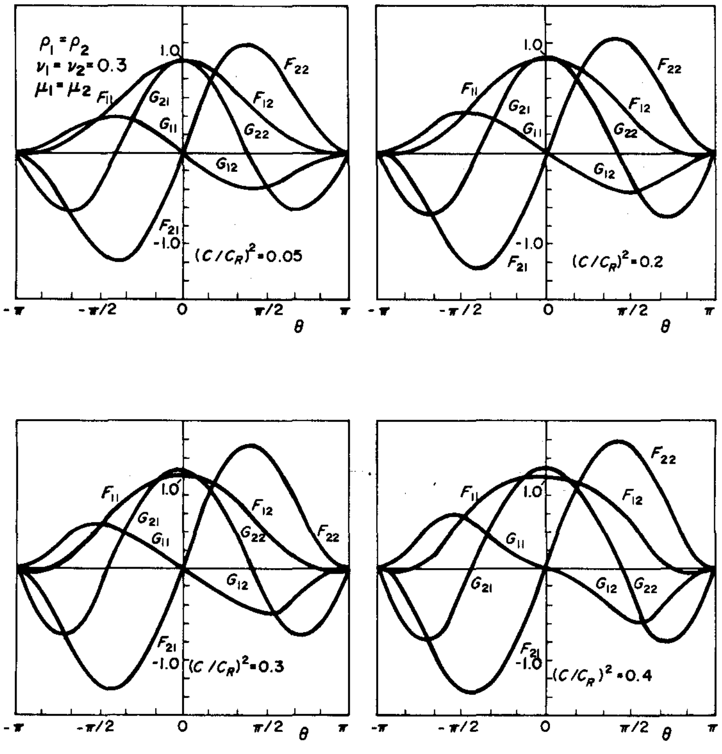

Fig. 5. Angular variations of functions F and G in a homogeneous isotropic solid for four crack propaga speeds.

which has to be computed by other means. If this ratio is known $(\tau_\theta)_j(r/d)^{1/2}/\sigma_{\theta 1}T(t)$ and $(\tau_{\theta r})_j(r/d)^{1/2}\sigma_{\theta 1}T(t)$ can be computed from eqns (75a, b). One special case is when σ_θ is real, i.e., $\sigma_{\theta 2}\equiv 0$. This case applies to the example discussed in the next section. When σ_θ is real, the functions F_{ij}, F_{2j}, G_{ij} and G_{2j} explicitly determine the variation with θ of the near-tip stresses. In Fig. 4 the dimensionless stresses have been plotted. The stress $(\tau_\theta)_j(r/d)^{1/2}/\sigma_{\theta 1}T(t)$ is given by F_{1j}, τ_θ, or F_{2j}, for $p_2 \ln(r/d)=\pi/2$; $p_2\ln(r/d)=3\pi/4$, or $p_2\ln(r/d)=\pi$, respectively. The shear stress $(\tau_{\theta r})_j(r/d)^{1/2}/\sigma_{\theta 1}T(t)$ is given by G_{1j}, τ_θ or G_{2j} for these three cases.

4. STEADY-STATE CRACK PROPAGATION UNDER THE INFLUENCE OF A MOVING LOAD

In Ref. [9] Gol'dshtein investigated the stress distribution in the vicinity of the tip of a semi-infinite crack, which moves with a constant velocity along the interface of two elastic materials. The crack tip propagates under the influence of two moving concentrated loads, one on each crack surface, which are equal in magnitude and opposite in direction, and which remain at a constant distance d behind the crack tip. Stationary fields were assumed relative to Cartesian coordinates (x, y) moving with the crack tip. In terms of these coordinates the boundary conditions on the crack-surface are

$$x\le 0,\quad y=0:\qquad (\tau_{yy})_j=-\tau\delta(x+d),\quad (\tau_{yx})_j=0. \tag{77}$$

For the case that the speed c is smaller than the lesser of the two Rayleigh-wave velocities the stresses σ_θ and $\sigma_{\theta r}$ in the plane of the crack, just ahead of the crack tip, were obtained as, see eqn (3.4) of Ref. [9],

$$\tau_\theta=\frac{\tau(1+|\gamma|)}{2\pi d|\gamma|^{1/2}}\left(\frac{d}{r}\right)^{1/2}\cos\left[p_2\ln\frac{r}{d}\right] \tag{78}$$

$$\tau_{\theta r}=\frac{\tau(1+|\gamma|)}{2\pi d|\gamma|^{1/2}}\left(\frac{d}{r}\right)^{1/2}\left(\frac{c_1+c_2}{b_1+b_2}\right)^{1/2}\sin\left[p_2\ln\frac{r}{d}\right]. \tag{79}$$

The analysis presented in the preceding sections is general, and includes the steady-state solution for constant crack propagation speed as a special case. Equations (78) and (79) may thus be compared with eqns (73a) and (73b). We note first that the length parameter d, introduced in (21) and (22), here is the distance the loads travel behind the crack tip. It also immediately follows that for the present case we have

$$\sigma_{\theta z}=0;\quad\text{and}\quad T(t)=\frac{\tau(1+|\gamma|)}{\sigma_{\theta 1}2\pi d|\gamma|^{1/2}}. \tag{80}$$

The stresses $(\tau_\theta)_j$ and $(\tau_{\theta r})_j$ follow from eqns (75a, b) as

$$(\tau_\theta)_j=\frac{\tau(1+|\gamma|)}{2\pi d|\gamma|^{1/2}}\left(\frac{d}{r}\right)^{1/2}\left\{F_{1j}(\theta)\cos\left(p_2\ln\frac{r}{d}\right)-F_{2j}(\theta)\sin\left(p_2\ln\frac{r}{d}\right)\right\} \tag{81a}$$

$$(\tau_{\theta r})_j=\frac{\tau(1+|\gamma|)}{2\pi d|\gamma|^{1/2}}\left(\frac{d}{r}\right)^{1/2}\left\{G_{1j}(\theta)\cos\left(p_2\ln\frac{r}{d}\right)-G_{2j}(\theta)\sin\left(p_2\ln\frac{r}{d}\right)\right\}. \tag{81b}$$

5. CONCLUDING REMARKS

In this paper we have examined the variations with polar angle of the elastodynamic near-tip stress fields, for a crack which moves with a non-uniform velocity along an interface between two elastic solids of different material properties. In the immediate vicinity of the propagating crack tip the stresses show violent oscillations. The radius over which these oscillations are significant depends on the applied loads, and on the speed of the crack tip. In general it may be assumed that the region of violent oscillations is small.

The results show the magnitudes of stress intensity factors for an arbitrary angle θ, relative to the corresponding factors in the interface. In conjunction with a fracture criterion based on stress magnitudes and/or stored strain energy, the results presented here can in principle be used to examine the proclivity of the crack tip to move out of the interface into one of the adjoining materials.

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(*Received* 17 *November* 1975)