

THREE-DIMENSIONAL HARMONIC FUNCTIONS NEAR TERMINATION OR INTERSECTION OF GRADIENT SINGULARITY LINES: A GENERAL NUMERICAL METHOD

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Abstract—The harmonic function u near point 0 from which a single singularity ray emanates is assumed to be dominated by the term $r^{\lambda} \rho^p U$ where r =distance from point 0, p =known constant and ρ =chosen function of angular spherical coordinates θ, φ , for which a partial differential equation with boundary conditions, especially those at the singularity rays, and a variational principle, are derived. Because $\text{grad } U$ is nonsingular, a numerical solution is possible, using, e.g. the finite difference or finite element methods. This reduces the problem to finding λ of the smallest real part satisfying the equation $\text{Det}(A_{ij})=0$ where A_{ij} is a large matrix whose coefficients depend linearly on $\mu = \lambda(\lambda + 1)$. In general λ and A_{ij} are complex. Solutions can be obtained either by reduction to a standard matrix eigenvalue problem for μ , or by successive conversions to nonhomogeneous linear equation systems. Computer studies have confirmed the feasibility of the method and have shown that highly accurate results can be obtained. Solutions for cracks and notches ending at a plane or conical surface, and for cracks ending obliquely at a halfspace surface, are presented. In these cases, λ is real and the singularity is always weaker ($\lambda > p$) than on the singularity line and may even disappear ($\lambda > 1$). Furthermore, elastic stresses under a wedge-shaped rigid sliding stamp or at a corner of a crack edge, and also harmonic functions at three-sided pyramidal notches, have been analyzed. Here $\lambda < p$ was found to occur. A simple analytical solution for one class of special cases has also been found and used to check some of the numerical results.

INTRODUCTION

IN THREE-dimensional potential theory there exists a great number of problems in which one or several lines of singularity of the potential gradient terminate or intersect. The singularity at the point of termination or intersection can naturally be expected to differ from the singularity on the line or lines entering this point. Solutions to these problems are of considerable interest, e.g. for scattering of waves at a corner of a screen, for distribution of electric charge, for problems of heat conduction, potential flow, seepage etc., and perhaps most importantly, for fracture mechanics of elastic bodies. At present, however, most of these problems remain unsolved.

One problem of this type is the distribution of pressure below the tip of a rigid frictionless stamp of wedge-shaped contact, pressed upon an elastic halfspace. The problem can be reduced to a potential theory problem[4]. It was discussed in 1947 by Galin[5, 4] and an approximate solution restricted to non-oscillating singularities and based on mapping of a hemisphere upon a circle was presented in 1957 by Rvachev[11]. He applied the Galerkin method in the mapped domain and, using up to six terms, made rough estimates of the real values of the singularity exponent λ . He concluded that the pressure near the tip behaves as $r^{\lambda-1}$ where λ monotonously increases from 0 to 1 as the wedge angle 2α grows from 0 to 2π (r =distance from the tip). Recently Aleksandrov and Babeshko[1] used Green's function approach to study analytically the limiting case of the wedge angle 2α tending to zero. They concluded that near the tip the pressure exhibits an oscillating singularity of the type $r^{-1.5} \cos(k \ln r)$ (k =parameter), which obviously prevails over the non-oscillating singularity r^{-1} found for $\alpha \rightarrow 0$ by Rvachev[11].

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But it seems that no complete solutions of the near singularity fields, including the singularity on the edge of the stamp near the tip, have been presented so far.

A mathematically equivalent problem for the Helmholtz reduced wave equation, consisting in wave scattering near the tip of a right-angled corner of a screen, was studied analytically by Radlow [9]. He found that near the tip the harmonic wave function is of the type $r^{0.25}$, without checking for a possible imaginary part of the exponent, and without deducing the near-tip field itself. Diffraction of elastic waves by a right-angled wedge was also studied analytically by Kraut [16].

The intent of the present study is to propose a general method for problems of this type. The method will consist of reduction to a two-dimensional eigenvalue problem for a function without gradient singularity, which can presumably be solved numerically with any desired accuracy.

TYPE OF PROBLEMS TO BE DISCUSSED

Using spherical coordinates θ, ϕ, r , the Laplace equation for the potential $u(\theta, \phi, r)$ reads

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 u}{\partial \phi^2} + \frac{2}{r} \frac{\partial u}{\partial r} + \frac{\cot \theta}{r^2} \frac{\partial u}{\partial \theta} = 0. \quad (1)$$

The problem to be discussed herein consists in finding the limiting form of the solution of this equation as $r \rightarrow 0$, assuming that the domain of the solution is an infinite space bounded by a certain surface $B(\theta, \phi) = 0$ formed by rays emanating from point 0. The boundary conditions are $u = 0$ on part of the surface and $\partial u / \partial n = 0$ on the remaining part, n being a normal to the surface. (The case of the boundary conditions $u = \text{constant}$ on part of the surface is easily reduced to the present case.) No boundary conditions will be prescribed at any point on radial rays that do not lie entirely on $B = 0$, so that an infinite number of solutions can be expected. Assuming these solutions to comprise a complete system of linearly independent functions, a certain linear combination of these solutions should satisfy the additional boundary conditions on the radial rays occurring in real problems. As will be seen, a class of admissible solutions is of the form $r^\lambda F(\theta, \phi)$ and consequently, regardless of the type of the afore-mentioned linear combination, the solution for any boundary conditions on the radial rays will be dominated in a sufficiently small neighborhood of point 0 by that solution of the present problem for which $Re(\lambda)$ is *smallest*. When $Re(\lambda) < 1$, $\text{grad } u$ near point 0 becomes unbounded, i.e. the solution exhibits a singularity at point 0. However, for the given boundary conditions on the radial rays the singularity need not necessarily occur because the solutions with $Re(\lambda) < 1$ need not be present in the afore-mentioned linear combination (as in the case of a homogeneous normal stress field parallel to a crack). The sufficient conditions of singular behavior, as well as the uniqueness aspects, cannot be included in the present study.

If u is required to be finite, only $Re(\lambda) > 0$ is admissible. (However, in some physically meaningful problems, such as the case of a line load on an elastic half-space, u is not finite at point 0.)

The surface formed by radial rays and the corresponding boundary conditions are assumed to be of such a form that the surface contains a finite number of rays on which $\text{grad } u$ is singular (unbounded). Further, it is assumed that at any finite distance L from point 0, the field within a neighborhood of the singularity ray which is sufficiently small-

ler than L is known, and is of the same type at any point of the ray except at point 0. This field may be determined by solving the plane problem in a plane normal to the ray. (The latter property is verified in some special cases by the analytical solutions reported in the sequel.)

Consider the plane section normal to a singularity ray passing through point O' at a finite distance L from point 0. It is assumed that within a neighborhood of point O' which is sufficiently smaller than L , the normal plane section of the body is bounded by two straight rays $O'A, O'B$ from point O' , forming angle α (Fig. 1). The boundary condition on the rays may be either $u = 0$ or $\partial u / \partial \phi = 0$ where $\phi =$ angular coordinate in Fig. 1. The solution near point O' in the normal plane section may be sought in the form $u \sim r^p f_1(\phi)$ where r , is the distance from point O' , and p and $f_1(\phi)$ are assumed to be the same at any finite distance L from point 0.

To give an example of the determination of p and $f_1(\phi)$, consider that the boundary conditions are $\partial u / \partial \phi = 0$ on ray $O'A$, and $u = 0$ on ray $O'B$. Substitution of $u = r^p f_1(\phi)$ into the Laplace equation yields a homogeneous ordinary differential equation for $f_1(\phi)$, whose general solution is $A \cos p\phi + B \sin p\phi$. The boundary conditions reduce to $df_1(0)/d\phi = 0, f_1(\pi - \alpha) = 0$, which yield

$$u \sim r^p f_1(\phi) \quad \text{where} \quad f_1(\phi) = \cos p\phi, \quad p = \frac{1}{2}\pi / (\pi - \alpha). \tag{2}$$

The same result is obtained when rays $O'A, O'B$ form angle 2α and $u = 0$ on both rays.

This technique of the determination of the near-singularity fields has been used in plane elasticity (where it is not as trivial as here) by Knein [8] (who thanked T. von Kármán for suggesting the basic approach), and later independently by Williams [14] and Karp and Karal [7]. In analogy with this technique, the variables may be separated in the three-dimensional problem as follows:

$$u(\theta, \phi, r) = r^n f(\theta, \phi) \tag{3}$$

where n is, in general, complex and f is a function of θ and ϕ only. It may be imagined as a function defined in a certain domain Ω on a unit sphere about point 0. Substitution of (3) into the Laplace equation (1) allows r to be completely eliminated from the equation and results in a differential equation for $f(\theta, \phi)$.

It should be noted that in the case of the Helmholtz reduced wave equation, $\nabla^2 u + k^2 u = 0$ (arising in wave scattering problems), or the Poisson equation, $\nabla^2 u = k$, the variable r is not eliminated since the additional term $k^2 r^2 f$ or kr^{2-n} , respectively, appears in the equation for f . However, in a sufficiently small neighborhood of point 0, this term becomes negligible, as compared with other terms in the equation (provided that f is

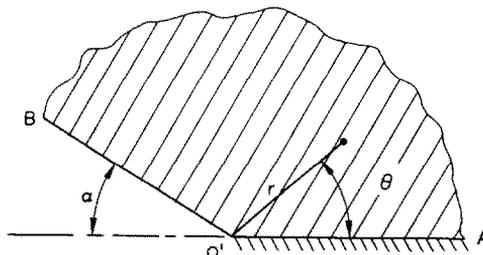


Fig. 1. Planar case giving rise to singularity at point O' .

bounded and, in the case of the Poisson equation, $n < 2$). Thus, the solutions of the reduced wave equation and the Poisson equation exhibit the same type of singularity as the harmonic functions.

The gradient singularity at point 0 is only partly interpreted by the term r^n in equation (3) since along certain radial rays there is an additional singularity which must be embedded in function f . Consequently, function f is not suitable for determination by approximate numerical methods.

REMOVING SINGULARITY ON RADIAL SURFACE RAYS

It is desired to separate the variables in such a manner that, in any plane normal to the singularity ray, function u near the ray be of the same form as equation (3) in which f_1 has no gradient singularity. Considering, for the sake of brevity, the case of a single singularity ray, this may be achieved by writing

$$u(\theta, \phi, r) = r^n r_1^p U(\theta, \phi) \quad (4)$$

where r_1 = chosen continuous smooth function of θ , ϕ , and r , which is identical to the distance from the singularity ray in its vicinity and is nonzero everywhere except on the ray. Obviously, if p is the correct exponent for the gradient singularity on the ray, as determined from (2), $U(\theta, \phi)$ must be a continuous smooth function of θ and ϕ which has (a) no gradient singularity on the singularity ray, and (b) in its vicinity is proportional to the function f_1 from equation (2).

Function r_1 may always be introduced in the form

$$r_1 = r\rho(\theta, \phi) \quad (5)$$

where ρ = chosen continuous smooth function of θ and ϕ which is nonzero everywhere except on the singularity ray in whose vicinity it is identical with the distance from the ray measured on the unit sphere. Thus

$$u(\theta, \phi, r) = r^\lambda F(\theta, \phi) = r^\lambda \rho^p U(\theta, \phi) \quad (6)$$

where

$$\lambda = n + p. \quad (7)$$

If the singularity ray is located at the pole, $\theta = 0$, a very simple choice of ρ is possible:

$$\rho = \theta \quad \text{or} \quad \rho = \sin \theta. \quad (8)$$

With the latter choice, r_1 represents the exact distance from the ray not only in its vicinity but everywhere in the domain. It must be warned, however, that $\sin \theta$ cannot be used when $\theta = \pi$ is part of the domain and no singularity ray exists at $\theta = \pi$. The function $\rho = \sin k\theta$ may be suitable when symmetric boundary conditions at $\theta = \pi/2k$ are desired.

If the singularity ray is located at (θ_1, ϕ_1) , it is possible to choose, e.g.

$$\rho = \{(\theta - \theta_1)^2 + [(\phi - \phi_1) \sin \theta_1]^2\}^{1/2}. \tag{9}$$

Generalization to the case of several singularity rays contained within the domain on the unit sphere is obvious. For three rays, e.g.

$$u(\theta, \phi, r) = r^\lambda F(\theta, \phi) = r^\lambda \rho_1^{p_1} \rho_2^{p_2} \rho_3^{p_3} U(\theta, \phi) \tag{10}$$

where

$$\lambda = n + p_1 + p_2 + p_3 \tag{11}$$

p_1, p_2, p_3 being known exponents for the three rays and ρ_1, ρ_2, ρ_3 functions of the same type as $\rho(\theta, \phi)$ was before.

Now consider a singularity ray which is placed into the pole of a spherical coordinate system (e.g. ray $00'$ in Fig. 2). In the plane (θ, ϕ) the pole appears as a straight line segment $\theta = 0$ with end points given by the angles ϕ_1, ϕ_2 forming the crack or notch that creates the singularity (Fig. 3). This line segment is a part of the boundary of domain Ω in the plane (θ, ϕ) on which function $U(\theta, \phi)$ has to be solved. It is obvious that a certain boundary condition must be imposed at this line segment ($0\bar{0}'$ in Fig. 3). The boundary condition may be deduced from the fact that U must be proportional to the function f_1 from equation (2) in the vicinity of $\vartheta = 0$. In equation (2) f_1 depends only on ϕ , i.e. is independent of θ . Hence, the condition

$$\partial U / \partial \theta = 0 \quad \text{at } \theta = 0 \tag{12}$$

must be satisfied.

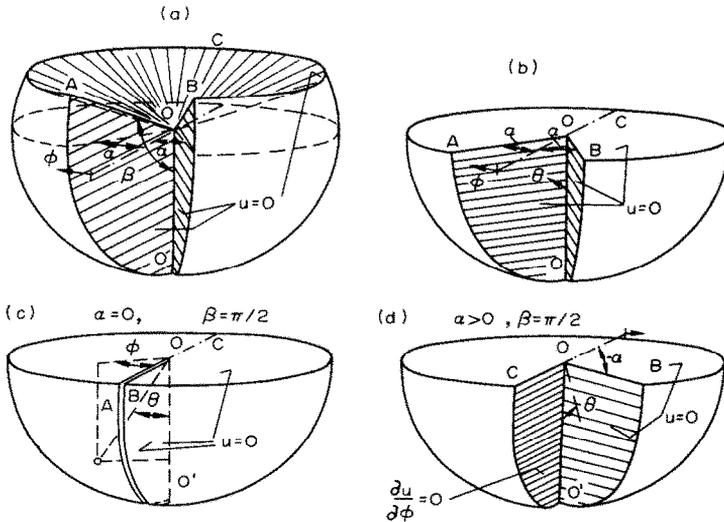


Fig. 2. Various cases of a notch or crack ending into a conical or planar surface. (The sphere is not the body surface and serves only for visualizing the situation in spherical coordinates.)

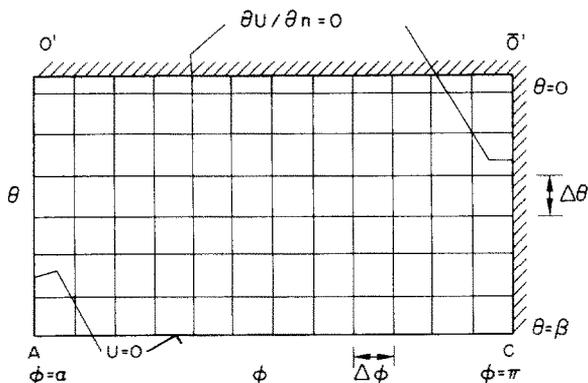


Fig. 3. Domain $O'AC'O'$ from Fig. 2 visualized in (θ, ϕ) plane, with a nodal net used for numerical solution.

DIFFERENTIAL EQUATION OF THE PROBLEM AND VARIATIONAL FORMULATION

Substitution of equation (6) into equation (1) yields, after rearrangement and simplification, the partial differential equation:

$$\rho^2 \frac{\partial^2 U}{\partial \theta^2} + \frac{\rho^2}{\sin^2 \theta} \frac{\partial^2 U}{\partial \phi^2} + \left(2p\rho \frac{\partial \rho}{\partial \theta} + \rho^2 \cot \theta \right) \frac{\partial U}{\partial \theta} + \left\{ \lambda(\lambda + 1)\rho^2 + p \left[\rho \frac{\partial \rho}{\partial \theta} \cot \theta + (p-1) \left(\frac{\partial \rho}{\partial \theta} \right)^2 + \rho \frac{\partial^2 \rho}{\partial \theta^2} \right] \right\} U = 0. \quad (13)$$

In the special case where $\rho = \sin \theta$, equation (13) becomes

$$\sin^2 \theta \frac{\partial^2 U}{\partial \theta^2} + \frac{\partial^2 U}{\partial \phi^2} + \left(p + \frac{1}{2} \right) \sin 2\theta \frac{\partial U}{\partial \theta} + \{ [\lambda(\lambda + 1) - p(p + 1)] \sin^2 \theta + p^2 \} U = 0. \quad (14)$$

The problem consists in finding a nonzero solution of equation (13) or (14) which satisfies the boundary conditions at a surface formed by the radial rays and gives the smallest positive real part of λ .

It should be noted that for $\theta \rightarrow 0$, equation (14) becomes $\partial^2 U / \partial \phi^2 + p^2 U = 0$, which gives $U \sim \cos p\phi$, as is required by equation (2).

For numerical solution, the variational formulation is of particular interest. The functional associated with the Laplace equation is known to be

$$\mathcal{W} = \int_V \frac{1}{2} (\text{grad } u)^2 dV + \mathcal{W}_1 = \frac{1}{2} \int_V \left[\left(\frac{\partial u}{\partial r} \right)^2 + \left(\frac{1}{r} \frac{\partial u}{\partial \theta} \right)^2 + \left(\frac{1}{r \sin \theta} \frac{\partial u}{\partial \phi} \right)^2 \right] dV + \mathcal{W}_1 \quad (15)$$

where V = volume, \mathcal{W}_1 = certain integral over the boundary surface. The functional which is associated with equation (13) or (14) may be obtained by reducing functional \mathcal{W} to a two-dimensional one. In conformity with the separation of variables by substitution (6), the minimizing condition for \mathcal{W} may be considered to be of the form $\delta(\delta, \mathcal{W}) = 0$ where δ_r is the variation with regard to functions of r , and δ is the variation with regard to functions of θ and ϕ . Thus, according to (15) and (7),

$$\delta \int \int \int_v \left[R' F^2 (\delta_r R)' + \frac{R}{r^2} \left(\frac{\partial F}{\partial \theta} \right)^2 \delta_r R + \frac{R}{r^2 \sin^2 \theta} \left(\frac{\partial F}{\partial \phi} \right)^2 \delta_r R \right] r^2 \sin \theta \, d\theta \, d\phi \, dr + \delta(\delta_r \mathcal{W}_1) = 0 \quad (16)$$

where $R = r^\lambda$, $R' = dR/dr$. Integration by parts with respect to $(\delta_r R)'$ provides

$$\int_r r^\lambda \left\{ \delta \int \int_\Omega \left[-\lambda(\lambda + 1)F^2 + \left(\frac{\partial F}{\partial \theta} \right)^2 + \frac{1}{\sin^2 \theta} \left(\frac{\partial F}{\partial \phi} \right)^2 \right] \sin \theta \, d\theta \, d\phi \right\} \delta_r R \, dr + \delta(\delta_r \mathcal{W}_1) = 0 \quad (17)$$

where Ω is the given domain on the unit sphere. Since this equation must be satisfied for any variation $\delta_r R$ as a function of r , the minimizing condition reduces to $\delta W = 0$, where W is the following functional

$$W = \int \int_\Omega \frac{1}{2} \left[\left(\frac{\partial F}{\partial \theta} \right)^2 + \frac{1}{\sin^2 \theta} \left(\frac{\partial F}{\partial \phi} \right)^2 - \lambda(\lambda + 1)F^2 \right] \sin \theta \, d\theta \, d\phi + W_1 \quad (18)$$

in which $F = \rho^p U$; W_1 is a certain integral arising from \mathcal{W}_1 . Because the surface of the body in three dimensions consists of radial rays, W_1 represents a certain contour integral along the boundary of the domain Ω . The foregoing simple derivation is, of course, not rigorous, but it is easy to verify using standard procedures that equation (13) is indeed associated with the variational principle $\delta W = 0$.

SOME CASES ADMITTING SIMPLE ANALYTICAL SOLUTIONS

Consider solutions of equation (14) which have the form

$$U(\theta, \phi) = Y(\theta) \sin p\phi. \quad (19)$$

This obviously satisfies the boundary conditions

$$U = 0 \text{ at } \phi = 0 \text{ and at } \phi = 2(\pi - \alpha) = \pi/p \quad (20a)$$

or

$$U = 0 \text{ at } \phi = 0 \text{ and } \partial U / \partial \phi = 0 \text{ at } \phi = \pi - \alpha = \pi/2p \quad (20b)$$

that is, $U = 0$ at the planes $00'A$ and $00'B$ in Fig. 2 where α and p are related as given by equation (2). Thus, if domain Ω on the unit sphere is rectangular in θ and ϕ (Fig. 3), i.e.

$$0 \leq \phi \leq \pi/p, \quad 0 \leq \theta \leq \beta \quad (21)$$

function $Y(\theta)$ in (19) needs to satisfy only the boundary conditions on $\theta = 0$ and $\theta = \beta$. Considering that there is a free surface at $\theta = \beta$ (a cone, and for $\beta = \pi/2$ a plane; Fig. 2), and taking equation (12) into account, one may write

$$\partial U / \partial \theta = 0 \text{ at } \theta = 0, \quad U = 0 \text{ at } \theta = \beta. \quad (22)$$

Thus, it is necessary and sufficient that $Y(\theta)$ fulfill the boundary conditions:

$$\partial Y / \partial \theta = 0 \text{ at } \theta = 0, \quad Y = 0 \text{ at } \theta = \beta. \quad (23)$$

For $\alpha < \pi/2$ there is a single singularity ray in this problem, the ray $00'$ in Fig. 2, and for $\alpha \geq \pi/2$ there is none. (On rays $0A$, $0B$ there is no singularity.) Substituting equation (19) into (14), equation (14) is found to be satisfied if

$$\frac{d^2 Y}{d\theta^2} + (2p+1) \cot \theta \frac{dY}{d\theta} + [\lambda(\lambda+1) - p(p+1)] Y = 0. \quad (24)$$

Upon introduction of a new independent variable $x = \sin^2 \theta$, equation (24) may be transformed to the form

$$x(1-x) \frac{d^2 Y}{dx^2} + \left[p+1 - \left(p + \frac{3}{2} \right) x \right] \frac{dY}{dx} + \frac{\lambda(\lambda+1) - p(p+1)}{4} Y = 0 \quad (25)$$

which is seen to be a special form of the hypergeometric (Gauss) differential equation. The solution of this equation [12] leads to hypergeometric functions and it can be shown that it may be expressed in terms of elementary functions only if $\beta = \pi/2$, that is if the conical surface degenerates into a plane (Fig. 2b). Then the root λ having the smallest real part is obtained as

$$\lambda = p+1 \quad (26)$$

and the corresponding solution is

$$u(\theta, \phi) = Cr^{p+1} (\sin \theta)^p \cos \theta \sin p\phi \quad (27)$$

where C = an arbitrary constant. Equation (27) can be proved by substitution into (14). Note that, for any angle α , λ is greater than 1.

An interesting result is thus obtained: the gradient singularity at the ray (edge of a crack or notch, e.g.) disappears at the surface of the body (cases in Fig. 2b, c, d).

One important case is the half-space $-\pi \leq \phi \leq 0$ whose boundary conditions at the surface $\phi = 0$ are $u = 0$ on the infinite wedge $0 \leq \theta \leq 2\beta$ and $\partial u / \partial n = \partial u / \partial \phi = 0$ on the rest of the surface (Fig. 11a). Simple analytical solutions exist [11] for two limiting cases:

$$u = C \ln \left(\tan \frac{\theta}{2} \right) \quad \text{for } \beta \rightarrow 0 \quad (\lambda = 0) \quad (28)$$

$$u = Cr \sin \theta \sin \phi \quad \text{for } \beta \rightarrow \pi \quad (\lambda = 1) \quad (29)$$

and, of course, for the case $\alpha = \pi/2$ in which the problem reduces to a plane one and has a well known solution yielding $\lambda = \frac{1}{2}$.

REDUCTION TO A NONLINEAR EIGENVALUE PROBLEM FOR A MATRIX

Because function $U(\theta, \phi)$ has no gradient singularities, standard numerical methods may be applied to its determination. For complicated geometries of the domain Ω on the unit sphere, the most versatile approach will certainly be the finite element method. However, development of finite elements on the basis of the functional (18) exceeds the scope of this paper and will be treated separately. For domains of simpler geometries,

the use of the finite difference method appears to be easier and has been adopted for all computer studies reported herein. The differential equation (14) or (13) is replaced by its finite difference approximation in terms of nodal values U_i ($i = 1, 2, \dots$) and is written for all nodes of the chosen network inside the domain Ω . Some of the equations involve nodal points outside the boundary for which further equations expressing the boundary conditions must be added.

Application of the finite difference method, as well as the finite element method, yields a large system of N homogeneous linear algebraic equations for values U_j of function U in the nodes $j = 1, 2, \dots, N$,

$$\sum_{j=1}^N A_{ij}(\lambda)U_j = 0 \quad (i = 1, 2, \dots, N). \quad (30)$$

The determinant of matrix A_{ij} , in general complex, must vanish for a non-zero solution to exist. The problem is thus reduced to finding a root λ having the smallest positive real part. The solution is complicated by the large size of matrix $[A_{ij}]$ needed for accurate results, and also by the fact that A_{ij} are nonlinear functions of λ (polynomials). Various methods of solution are possible.

Method A. Reduction to a matrix eigenvalue problem

Noticing that equation (13) or (14) involves λ solely through the term $\lambda(\lambda + 1) = \mu$, matrix A_{ij} is a linear function of μ . Furthermore, if the finite difference method is used, μ appears only in the diagonal coefficients of A_{ij} , and if the equations expressing the boundary conditions are eliminated, μ appears in all diagonal coefficients. Mere division of each equation by the coefficient at μ reduces the matrix A_{ij} to the form:

$$A_{ij}(\lambda) = K_{ij} - \mu\delta_{ij} \quad (31)$$

where K_{ij} is a matrix which is independent of λ , and $\delta_{ij} = 1$ for $i = j$, $\delta_{ij} = 0$ for $i \neq j$. Thus, a standard eigenvalue problem in μ for a real, general (nonsymmetrical) matrix is obtained. There are two roots λ which correspond to each root μ ,

$$\lambda = -\frac{1}{2} \pm \sqrt{\frac{1}{4} + \mu}. \quad (32)$$

If the roots λ are all real, the smallest positive λ evidently corresponds to the smallest μ (\pm being replaced by $+$). But this is not true in the case of complex roots and all roots μ must be checked to determine the smallest $Re(\lambda)$.

For the solution of the eigenvalue problem obtained, standard subroutines are available [3, 6, 10, 15]. Nevertheless, even with the contemporary electronic computers it is difficult and costly to exceed 150 equations, approximately, which usually suffices for accurate but not highly accurate results. The cost is also substantially increased by the fact that for the finite difference method, matrix A_{ij} is non-symmetric.

The finite element method, which may be based on minimizing the functional (18), yields symmetric matrices A_{ij} and K_{ij} . However, in contrast with the finite difference method, μ no longer appears exclusively in the diagonal terms of A_{ij} and inversion of a matrix of the size $N \times N$ is necessary to obtain a standard eigenvalue problem in μ .

Taking advantage of the sparsity of matrix A_{ij} , a much larger matrix can be stored in

rapid access memory and the eigenvalue could be obtained by iteration [3, 6, 10]. However, this would be economical only for the eigenvalue of largest absolute value. Attempts have been made to recast the problem in terms of the highest eigenvalue, e.g. replacing λ with $1/\gamma$ and linearizing with respect to γ . It appears, however, that this approach is impossible because there are always some spurious negative roots whose absolute value is larger than the largest positive γ .

Method B. Conversion to nonhomogeneous equation systems

(a) *The case of real root.* In this method, which has been used with success [2] to solve buckling problems for very large elastic structural systems leading to equations of type (30), a certain value λ_0 is first chosen and all coefficients $A_{ij}(\lambda_0)$ are evaluated. One equation, whose number will be referred to as k , is replaced by the equation $U_k = 1$, which makes the system nonhomogeneous. Normally the matrix of the new system is nonsingular and all U_i ($i = 1, 2, \dots, N$) may be solved using the standard subroutines for systems of linear equations. The obtained values of U_i are then substituted into the original k^{th} equation, previously replaced, and the corresponding right-hand side $\sum_j A_{kj} U_j = Q$ is computed. In general Q will be nonzero and the aim is to find the smallest value of λ for which $Q = 0$. This may be done by the regula falsi method. Several values of λ_0 are selected and the plot of Q versus λ_0 is imagined as shown in Fig. 6. Subsequent choices of λ_0 are made by linear interpolation. Good accuracy and convergence is achieved only when the k^{th} equation is chosen in such a manner that no $|U_i|$ is much larger than U_k .

From computer studies it has been found that the radius of convergence of this method is sometimes quite small. The interval of interest, such as $0 \leq \lambda \leq 2$, must be scanned in small steps of λ , about 0.05, to obtain a good initial guess of λ . Only then may the automatic improvements of λ by the regula falsi method be started. This is obvious from the sharply varying slope of the plot b in Fig. 6 (although there exists cases of slowly varying slope, as is shown by curve c in Fig. 6). Much care is needed to avoid missing the smallest root. Nevertheless, if a great number of problems with the given data varying in small steps are being solved, as in the studies reported herein, an approximate value of λ is usually known in advance and scanning of the large interval may be skipped. It is because of this fact that this method is usually much more economical than the general programs for eigenvalues. Moreover, in contrast with method A, this method allows handling up to one or two thousand equations without great difficulties (using either various elimination methods for banded matrices with tape storage or the sparse matrix storage in rapid access memory combined with an iterative solution).

If no estimate of λ is available, computationally the most efficient procedure is a combination of methods A and B. First, method A with a low number of nodes is used to obtain an approximate value of λ and the solution is then refined to a high accuracy using method B with a large number of nodes.

As an alternative to method B, it is also possible to apply the regula falsi method to the plot of the determinant of A_{ij} versus λ . But the slope of this plot usually varies even more rapidly and the convergence properties were found to be poorer than in method B.

(b) *The case of complex root.* When root λ of the smallest real part happens to be complex, a similar method may be used. In this case U_i , as well as matrix A_{ij} depending on λ , must be considered as complex. The objective is to find complex λ_0 for which

$Q = 0$. First, an arbitrary value of $Im(\lambda_0) = X$ is chosen and, keeping it fixed, the plot of $Re(Q)$ versus $Re(\lambda_0)$ is examined, choosing various values of $Re(\lambda_0)$. The point at which $Re(Q) = 0$ may be found by the regula falsi method and the corresponding $Im(Q)$ denoted as Y . The foregoing procedure is then repeated for various $Im(\lambda_0) = X$ and, considering the plot of Y versus X , the point at which $Y = 0$ (and $Q = 0$) is again found by the regula falsi method.

The fact that the regula falsi method must be applied in two dimensions makes the computations much more expensive than in the case of a real λ . Moreover, because the matrix is complex, the running time of each equation system is longer. Nevertheless, using tape storage or compact storage of the sparse matrix, much larger systems can be handled than with method *A*.

Method C. Successive linearizations with respect to λ

If the technique presented here is applied in elasticity, A_{ij} will usually consist of general polynomials in λ , complex or real. Method *B* will still be applicable, but instead of Method *A* successive solutions of eigenvalue problems will be required. Since A_{ij} are, in general, analytic functions of a complex variable λ , expansion in a Taylor series about a chosen complex λ_0 is possible. The series may be truncated after the second term to achieve linearization. Equations (30) are then reduced to the form

$$\sum_{j=1}^N (A_{ij}^0 - \lambda' A'_{ij}) U_j = 0 \quad (i = 1, 2, \dots, N) \quad (33)$$

where $\lambda' = \lambda_0 - \lambda$, $A_{ij}^0 = A_{ij}(\lambda_0)$, $A'_{ij} = -\partial A_{ij}(\lambda_0)/\partial \lambda$. Multiplication of matrix equation (33) by the inverse of A'_{ij} then produces a standard eigenvalue problem, in general with a complex matrix. The objective is to find λ_0 for which $\lambda' = 0$. This can be achieved by application of the regula falsi method, in the same procedure as the condition $Q = 0$ is achieved in method *B*.

This method has so far been tested only in the problems with real λ as reported in the sequel. The convergence was rapid and the radius of convergence appeared to be quite large, that is the method automatically converged even if the initial guess of λ_0 was very poor (e.g. 0.2 in the case of plot (a) in Fig. 6). This is due to the slow change of the slope in this plot.

APPLICATION AND RESULTS OF NUMERICAL STUDIES

Cracks and notches ending at a plane or conical surface

To examine the numerical methods described above, the cases whose exact solution is known were solved first. These are the cases shown in Fig. 2 and characterized by boundary conditions (20a) and (22). There is only one singularity ray $00'$ in these problems. The domain Ω is given by equation (21) and is rectangular in terms of θ and ϕ , as shown in Fig. 3. The domain was subdivided by a rectangular network with steps $\Delta\phi = (2\pi - \alpha)/2n$, $\Delta\theta = \frac{1}{2}\pi/(n + \frac{1}{2})$. It is noteworthy that no interior nodes of the network may be placed on the boundary line $\theta = 0$ because the finite difference equations for the nodes on this line do not involve any node with $\theta \neq 0$. However, these nodes may be considered as exterior nodes of the network used in formulating the boundary condition (as has been done in computing Table 2).

The solutions were carried out for various sizes of the step. The results for $\alpha = 0$, given in Fig. 7, indicate an excellent convergence of λ to the exact value 1.5 resulting

from equations (26) and (2). For 72 equations the error is 0.4 per cent and for 1152 equations it is 0.04 per cent. The convergence is monotonous and from the slope of the line in Fig. 7 it may be inferred that the error \approx (step size)^{1.9}, that is, the convergence is almost quadratic. A similar picture is obtained for other values of α .

The solutions were also computed for non-zero values of β , that is, the cases of cracks or notches ending at the apex of a conical surface, which do not admit simple analytical solutions. The results are plotted in Fig. 4 and some of the results are tabulated in Table 1 (where three decimals are believed to be correct).

The results for $\beta \neq 0$ were compared and agreed with one-dimensional finite difference solutions of the ordinary differential equation (24). (For $\alpha = 0$, $\beta = \pi/2$, step $\Delta\theta = \beta/50$ gave $\lambda = 1.4998$, which is only 0.013 per cent in error.) For several cases, the eigenstates, as characterized by function $Y(\theta)$ in equation (19), are plotted in Fig. 5. As a check, some solutions have also been run with $\rho^p = \sqrt{\vartheta}$ and the results were identical.

It is interesting to note that at a certain value $\beta > \pi/2$, depending on α , λ becomes less than 1.0 so that grad u is singular at point 0. But the singularity is always weaker than at the edge 00' (see Fig. 4).

Cracks ending obliquely at a halfspace surface

The general method presented herein was further applied to the case shown in Fig. 8b, which is equivalent to the case in Fig. 8a where the boundary conditions are also given. There is only one singularity ray 00' in this problem, with $p = \frac{1}{2}$ from equation

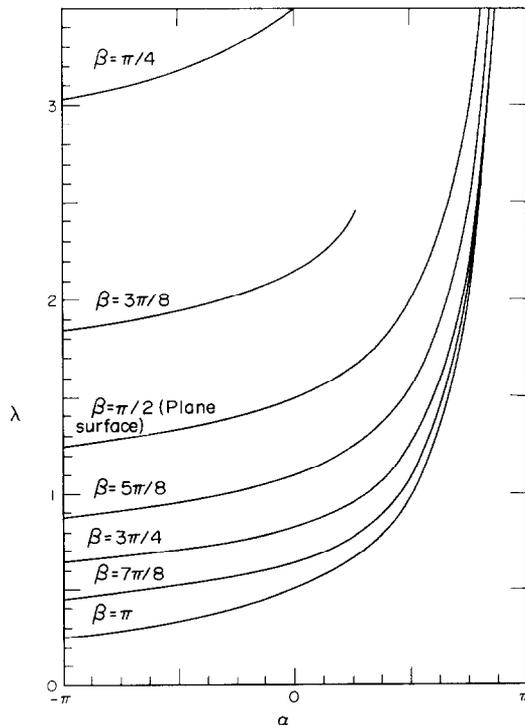


Fig. 4. Singularity strength for cases shown in Fig. 2.

Table 1. Values of λ for various α and β in the problem shown in Fig. 2

β	$\pi/4$	$3\pi/8$	$\pi/2$	$5\pi/8$	$3\pi/4$	$7\pi/8$	π
α							
$-\pi$	3.031	1.846	1.250	0.888	0.641	0.455	0.250
$-\pi/2$	3.189	1.953	1.333	0.958	0.704	0.515	0.333
$-\pi/4$	3.314	2.039	1.400	1.015	0.755	0.565	0.400
0	3.500	2.167	1.500	1.100	0.833	0.642	0.500
$\pi/8$	3.631	2.257	1.571	1.162	0.890	0.699	0.571
$\pi/4$	3.806	2.449	1.667	1.244	0.967	0.778	0.667

(2). The pole of the spherical coordinates is conveniently placed into this ray. Then, using $\rho = \sin \vartheta$, the problem is again governed by the differential equation (14). Ω is the domain $O'ABCD O'$ (or $O'AECD O'$) in Fig. 8 as is shown in plane (θ, φ) in Fig. 10. The boundary conditions are $\partial u / \partial \varphi = 0$ on side $DC O'$ (plane of symmetry), $\partial u / \partial \vartheta = 0$ at the pole O' (side $O' \bar{O}'$ in Fig. 10), $u = 0$ on side $O'A$ (crack surface) and $u = 0$ on side ABC (or AEC). The latter side appears in the plane (θ, φ) as the curve defined as follows:

$$\theta = \arctan (\tan \beta / \cos \varphi); \quad \text{if } \theta < 0 \text{ then } \theta \leftarrow -\theta + \pi. \tag{34}$$

Formulation of the finite difference equivalents of the boundary condition $u = 0$ in terms of the nodal values near boundary ABC requires locating point (θ_c, φ_c) which is symmetrical with respect to plane ABC to a given outside node (θ_a, φ_a) . To this end, the value of θ which corresponds to $\varphi = \varphi_a$ according to (34) is first calculated and, using spherical trigonometry, the following expressions are successively evaluated

$$\left. \begin{aligned} \cos \psi &= 1 - 2(\cos \beta \sin \varphi_a)^2 \\ \theta_c &= \arccos [\cos \theta_a \cos (\theta_a - \theta) + \sin \theta_a \sin (\theta_a - \theta) \cos \psi] \\ \varphi_c &= \varphi_a + \arcsin [\sin (\theta_a - \theta) \sin \psi / \sin \theta_c]. \end{aligned} \right\} \tag{35}$$

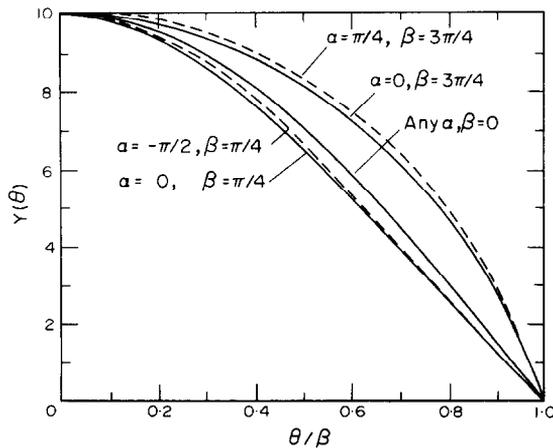


Fig. 5. Plot of function $Y(\theta)$ from equation (19) for some cases shown in Fig. 2.

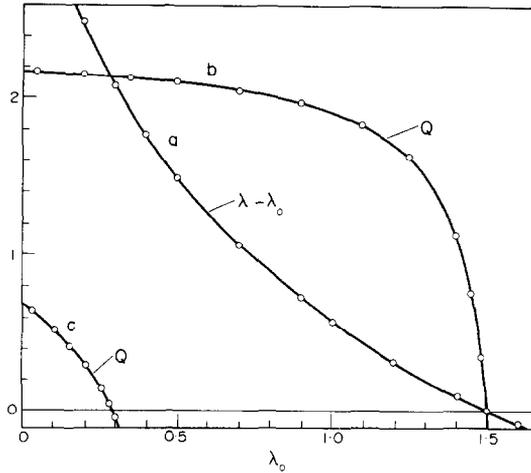


Fig. 6. Examples of dependence of λ and parameter Q upon λ_0 in methods A and B for solving equations (30). (Curves a, b correspond to Fig. 2c, curve c to Fig. 12a.)

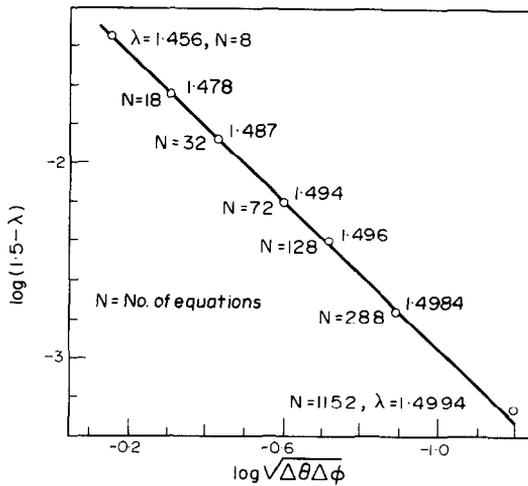


Fig. 7. Convergence of λ to the exact value 1.5 with diminishing step size of the net, for the case in Fig. 2c.

The U -value at point (θ_c, φ_c) is then related by linear interpolation to the U -values at the nearest three interior nodes p, q, r using area coordinates in the triangle pqr , i.e. $U_c = (a_{cqr}U_p + a_{cqp}U_q + a_{cpr}U_r)/a_{pqr}$ where a_{cqr} etc. are the areas of the triangles indicated by subscripts. The boundary condition $U=0$ is expressed in the form $U_a = -U_c(\sin \theta_c/\sin \theta_a)^p$.

Introducing finite difference grids with 80–100 nodes, the problem has been solved for various values of β . The values of λ of the smallest real part are all real and the computer results are shown as points on curve a in Fig. 9. The limiting values $\lambda = 1$ and 2 for $\beta \rightarrow 0$ and $\beta \rightarrow \pi$ result from equation (2) for the plane problem. The value $\lambda = 1.5$

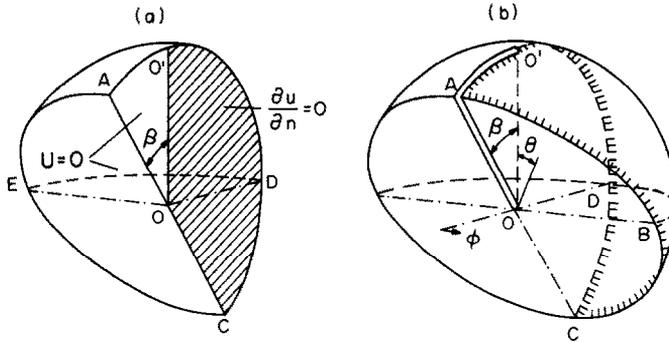


Fig. 8. (a) A quarterspace with boundary conditions giving rise to singularity on line OO' . (b) A halfspace with a crack whose edge is not perpendicular to the surface.

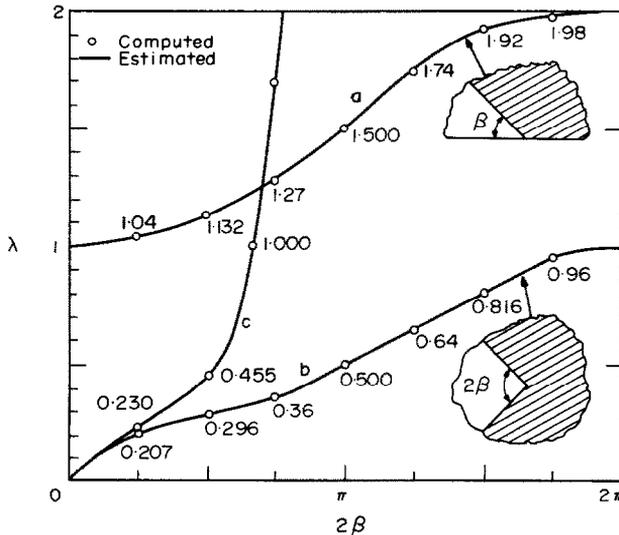


Fig. 9. Smallest real root λ for cases shown in Fig. 11 (curve b), Fig. 8 (curve a) and Fig. 13 (curve c).

for $2\beta = \pi$ agrees with the result from Fig. 4. It is noteworthy that the singularity on the ray OO' disappears at the surface of the halfspace (Fig. 8b) for any angle β . The eigenstates $U(\theta, \varphi)$ for two special cases are drawn in Fig. 10. For the case $\beta = \pi/4$ (Fig. 12a) it has been verified by method A that no complex roots exist. This case has also been analyzed by method B with grid $\Delta\varphi = \pi/24$, $\Delta\theta = \pi/26$ (359 nodes), which yielded $\lambda = 1.131$, while the coarser grid ($\Delta\varphi = \pi/12$, $\Delta\theta = \pi/13$) resulted in $\lambda = 1.128$ (the second and third smallest λ was 2.25 and 2.95). Assuming a similar convergence pattern as in Fig. 7, the correct value can be estimated as $\lambda = 1.131 + 0.001$, or $\lambda = 1.132$.

Elastic stresses under wedge-shaped stamps or at crack edge corners. Wave scattering by corners.

Furthermore, the case of a rigid sliding stamp with wedge-shaped contact of an

Table 2. Nodal values of function U; (A)—Case in Fig. 10a ($p = 0.5$); (B)—Fig. 12a ($p = 0.5$); (C)—Fig. 12b ($p = 0.5$); (D)—Fig. 14 ($p = 0.5$); (E)—Fig. 11 for $\beta = \pi/8(p = 2/3)$. Nodal coordinates are $\theta_k = K\Delta\theta$; $\varphi_i = J\Delta\varphi$ in cases (A), (B), (D) and (E); $\varphi_i = (J - 1)\Delta\varphi$ in case (C); grid $\Delta\varphi = \pi/24$, $\Delta\theta = \pi/26$. In Table (C) the left and right of Fig. 14 are interchanged. The top row of each case was obtained by extrapolation.

K \ J	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	
0	501	1001	1499	1995	2487	2974	3455	3927	4390	4841	5278	5697	6098	6474	6828	7157	7447	7708	7931	8120	8287	8433	8568	8684	
1	478	954	1434	1912	2389	2865	3339	3809	4273	4729	5176	5604	6022	6414	6785	7128	7440	7717	7958	8165	8317	8431	8509	8563	
2	408	821	1238	1653	2068	2480	2893	3294	3684	4062	4430	4788	5134	5468	5785	6085	6368	6625	6857	7065	7240	7384	7499	7583	
3	333	672	1023	1390	1776	2183	2613	3064	3536	4020	4516	5016	5514	6002	6474	6921	7336	7712	8063	8382	8664	8901	9100	9269	
4	250	510	789	1092	1427	1797	2202	2647	3114	3612	4130	4662	5197	5728	6245	6738	7199	7618	7998	8351	8673	8964	9214	9424	
5	160	334	533	770	1051	1380	1760	2189	2661	3171	3710	4271	4841	5411	5969	6504	7005	7468	7891	8281	8638	8964	9259	9514	
6	59	141	256	420	644	934	1288	1705	2178	2699	3258	3844	4446	5050	5645	6218	6755	7249	7698	8105	8475	8809	9107	9368	
7			-47	40	296	457	787	1197	1697	2287	2967	3737	4587	5507	6487	7527	8627	9787	10927	12047	13147	14227	15287	16327	
8					-50	256	655	1129	1669	2261	2911	3621	4391	5221	6111	7051	8041	9081	10171	11301	12461	13641	14841	16061	
9																									
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0	461	925	1379	1832	2271	2704	3121	3524	3912	4283	4632	4962	5269	5551	5813	6048	6257	6439	6594	6722	6822	6892	6936	6950	
1	448	903	1359	1814	2251	2673	3082	3469	3833	4173	4490	4782	5048	5289	5504	5692	5852	6004	6148	6284	6402	6492	6554	6588	
2	404	809	1229	1652	2064	2452	2816	3156	3473	3767	4038	4285	4508	4707	4882	5032	5166	5284	5386	5472	5542	5598	5642	5676	
3	344	697	1062	1438	1824	2210	2596	2972	3338	3694	4040	4376	4692	4988	5264	5520	5756	5972	6168	6344	6492	6612	6702	6764	
4	284	577	882	1208	1554	1910	2276	2652	3028	3404	3780	4146	4502	4848	5174	5480	5766	6032	6278	6494	6682	6842	6972	7072	
5	224	457	702	968	1254	1560	1886	2232	2598	2974	3360	3746	4122	4488	4844	5190	5526	5852	6158	6434	6682	6902	7092	7252	
6	164	337	522	728	954	1200	1466	1752	2058	2384	2730	3096	3472	3858	4244	4630	5016	5402	5788	6174	6560	6946	7332	7718	
7	104	217	342	488	654	840	1046	1272	1518	1784	2070	2376	2692	3018	3354	3700	4056	4422	4798	5184	5580	5986	6402	6828	
8	64	137	212	308	424	560	726	912	1118	1344	1590	1856	2142	2448	2774	3120	3486	3872	4278	4704	5150	5606	6072	6548	
9	44	91	146	211	286	382	508	664	850	1066	1302	1558	1834	2130	2446	2782	3138	3514	3910	4326	4762	5218	5694	6180	
10	24	51	86	121	166	221	296	392	518	674	860	1076	1322	1598	1904	2240	2606	2992	3398	3824	4270	4736	5222	5728	
11	14	31	56	81	106	141	186	251	337	453	609	805	1041	1317	1633	1999	2395	2821	3267	3733	4229	4745	5281	5837	
12	8	17	32	47	62	87	112	147	192	257	343	469	625	811	1027	1273	1559	1885	2251	2647	3073	3529	4015	4521	
13	4	9	16	23	30	47	64	81	108	145	192	259	346	464	612	790	1008	1256	1534	1842	2180	2548	2946	3374	
14	2	4	8	12	16	24	32	40	58	76	104	142	190	258	346	464	612	790	1008	1256	1534	1842	2180	2548	
15	1	2	4	6	8	12	16	20	28	36	54	72	90	118	156	204	272	360	468	606	774	982	1230	1518	
16		1	2	3	4	6	8	10	14	18	26	34	42	60	78	106	144	192	250	328	426	544	692	870	
17			1	2	3	4	6	8	10	12	16	20	24	36	48	64	84	108	136	176	228	292	368	456	
18				1	2	3	4	6	8	10	12	14	16	24	32	42	54	72	96	126	162	210	268	336	
19					1	2	3	4	6	8	10	12	14	16	24	32	42	54	72	96	126	162	210	268	
0	401	802	1203	1604	2005	2406	2807	3208	3609	4010	4411	4812	5213	5614	6015	6416	6817	7218	7619	8020	8421	8822	9223	9624	
1	381	762	1143	1524	1905	2286	2667	3048	3429	3810	4191	4572	4953	5334	5715	6096	6477	6858	7239	7620	8001	8382	8763	9144	
2	361	722	1083	1444	1805	2166	2527	2888	3249	3610	3971	4332	4693	5054	5415	5776	6137	6498	6859	7220	7581	7942	8303	8664	
3	341	682	1023	1364	1705	2046	2387	2728	3069	3410	3751	4092	4433	4774	5115	5456	5797	6138	6479	6820	7161	7502	7843	8184	
4	321	642	963	1284	1605	1926	2247	2568	2889	3210	3531	3852	4173	4494	4815	5136	5457	5778	6099	6420	6741	7062	7383	7704	
5	301	602	903	1204	1505	1806	2107	2408	2709	3010	3311	3612	3913	4214	4515	4816	5117	5418	5719	6020	6321	6622	6923	7224	
6	281	562	843	1124	1405	1686	1967	2248	2529	2810	3091	3372	3653	3934	4215	4496	4777	5058	5339	5620	5901	6182	6463	6744	
7	261	522	783	1044	1305	1566	1827	2088	2349	2610	2871	3132	3393	3654	3915	4176	4437	4698	4959	5220	5481	5742	6003	6264	
8	241	482	723	964	1205	1446	1687	1928	2169	2410	2651	2892	3133	3374	3615	3856	4097	4338	4579	4820	5061	5302	5543	5784	
9	221	442	663	884	1105	1326	1547	1768	1989	2210	2431	2652	2873	3094	3315	3536	3757	3978	4199	4420	4641	4862	5083	5304	
10	201	402	603	804	1005	1206	1407	1608	1809	2010	2211	2412	2613	2814	3015	3216	3417	3618	3819	4020	4221	4422	4623	4824	
11	181	362	543	724	905	1086	1267	1448	1629	1810	2001	2182	2363	2544	2725	2906	3087	3268	3449	3630	3811	4002	4183	4364	
12	161	322	483	644	805	966	1127	1288	1449	1610	1771	1932	2093	2254	2415	2576	2737	2898	3059	3220	3381	3542	3703	3864	
13	141	282	423	564	705	846	987	1128	1269	1410	1551	1692	1833	1974	2115	2256	2397	2538	2679	2820	2961	3102	3243	3384	
14	121	242	363	484	605	726	847	968	1089	1210	1331	1452	1573	1694	1815	1936	2057	2178	2299	2420	2541	2662	2783	2904	
15	101	202	303	404	505	606	707	808	909	1010	1111	1212	1313	1414	1515	1616	1717	1818	1919	2020	2121	2222	2323	2424	
16	81	162	243	324	405	486	567	648	729	810	891	972	1053	1134	1215	1296	1377	1458	1539	1620	1701	1782	1863	1944	
17	61	122	183	244	305	366	427	488	549	610	671	732	793	854	915	976	1037	1098	1159	1220	1281	1342	1403	1464	
18	41	82	123	164	205	246	287	328	369	410	451	492	533	574	615	656	697	738	779	820	861	902	943	984	
19	21	42	63	84	105	126	147	168	189	210	231	252	273	294	315	336	357	378	399	420	441	462			

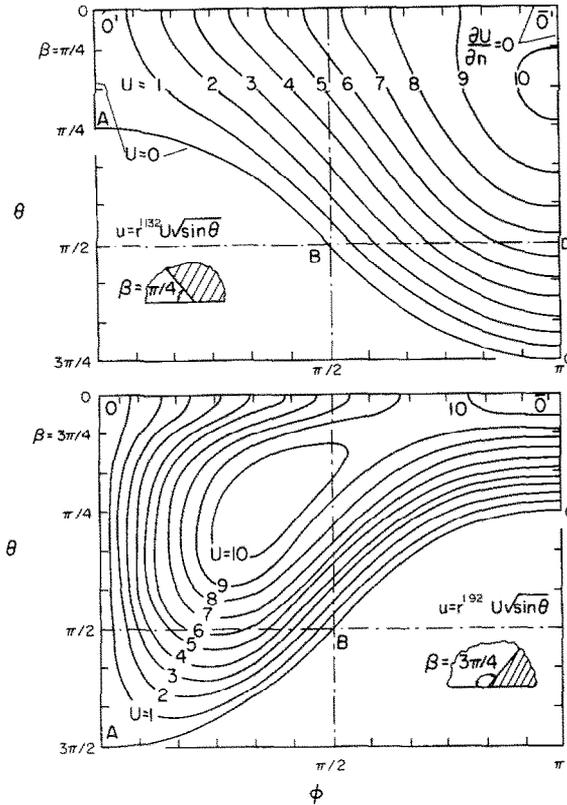


Fig. 10. Function U computed for the cases shown in Fig. 8 ($\rho^p = \sqrt{\sin \theta}$).

arbitrary angle, 2β , pressed upon the surface of an isotropic elastic half-space (Fig. 11a), has been studied. As is well known[4], this problem of elasticity may be reduced, with the help of Papkovitch–Neuber potentials, to a potential theory problem, $\nabla^2 u = 0$. Introducing cartesian coordinates x, y, z and considering the half-space $z \geq 0$ or $0 \leq \varphi \leq \pi$, the boundary conditions at the halfspace surface are

$$u = 0 \text{ on } \mathcal{T} \text{ and } \partial u / \partial z = 0 \text{ outside } \mathcal{T} \tag{36}$$

where \mathcal{T} = the wedge-shaped domain below the stamp ($00'A0''$ in Fig. 11a). Also, at infinity, $\text{grad } u$ is required to vanish. After solving for u , the displacements u_x, u_y, u_z in cartesian coordinates x, y, z at any point of the half-space $z \geq 0$ are given as[4]:

$$u_x = (1 - 2\nu) \int_z^\infty \frac{\partial u}{\partial x} dz - z \frac{\partial u}{\partial x}, \quad u_y = (1 - 2\nu) \int_z^\infty \frac{\partial u}{\partial y} dz - z \frac{\partial u}{\partial y}, \quad u_z = 2(1 - \nu)u - z \frac{\partial u}{\partial z}, \tag{37}$$

where ν = Poisson's ratio of the material. The normal stress below the stamp is $\sigma_z = (\partial u / \partial z)E / (1 + \nu)$ where E = Young's modulus.

This problem is equivalent to the case of an infinite elastic body containing a sharp crack whose edge forms a sharp corner of an arbitrary angle 2β (Fig. 11b). The body is subjected at infinity to uniformly distributed normal stresses perpendicular to the crack

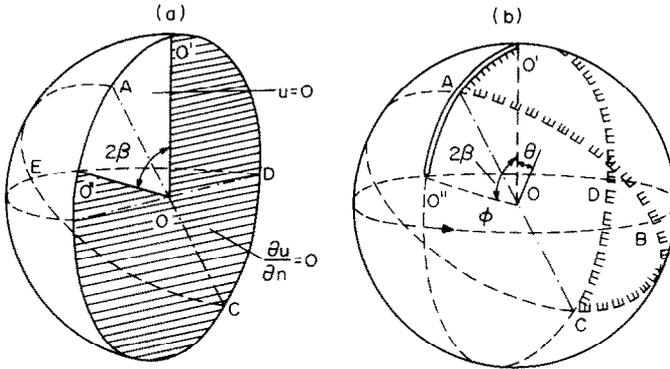


Fig. 11. (a) Wedge stamp contact (unshaded) upon elastic halfspace or, alternatively, a screen, scattering waves. (b) Crack corner in an infinite elastic space. (The sphere is not the body surface and serves only for visualizing the situation in spherical coordinates.)

plane. Furthermore, the near field in this problem happens to be the same as in the scattering of waves by a sharp corner of a screen[9].

Although in Fig. 11 there are two singularity rays, OO' and OO'' , each with $p = \frac{1}{2}$ from equation (2), the problem may be reduced to one with a single singularity ray, taking advantage of the fact that the solution u or F must be symmetrical with regard to plane $OABC$ in Fig. 11b. The formulation of the problem then differs from the previous case (Fig. 8) solely by the boundary condition on side ABC (or AEC) which reads:

$$\partial\{(\sin \theta)^p U\}/\partial n = 0 \quad (38)$$

where $p = \frac{1}{2}$ and n is the normal to the side AEC (normal on the sphere, not in the plane (θ, φ)). The finite difference formulation of the boundary condition is the same as in the previous case (Fig. 8, equations 34, 35), except that $U_a = U_c (\sin \theta_c / \sin \theta_a)^p$.

Computer results for grids with 80–100 nodes are shown by the points on curve b in Fig. 9. Note that the points adjacent to $\lambda = 0.5$ fit a smooth curve passed through the well-known value of $\lambda = 0.5$ for $2\beta = \pi$. The point $\lambda = 0$ for $\beta \rightarrow 0$ follows from equation (28), and $\lambda = 1$ for $\beta \rightarrow \pi$ results from equation (29). The eigenstates $U(\theta, \varphi)$ are shown for two special cases by lines of equal U -value in Fig. 12. It is worth noting that the computed variation of U along the top side $O'O'$ of Fig. 12a or b is proportional to $\sin(\theta/2)$, as is required by equation (2).

The results prove that the singularity at the sharp concave ($2\beta < \pi$) corner O of a crack edge (Fig. 11b) in an elastic body is more severe ($\lambda < p$) than at a straight crack edge ($p = \frac{1}{2}$). This fact is of interest for fracture mechanics of brittle elastic materials. It shows that at a concave corner of the crack edge of a planar crack the crack will propagate faster or at a lower stress than at a straight edge. Obviously, this would tend to straighten the crack edge. A plane crack with a straight edge is thus more stable than a plane crack with zig-zag edge.

All roots have been computed by method B , restricting consideration to real λ . However, the case $2\beta = \pi/2$ has also been computed by method A ; the result was identical and it was found that no complex roots λ exist in this case (similar to cases $2\beta = \pi$ and $\beta = \pi$).

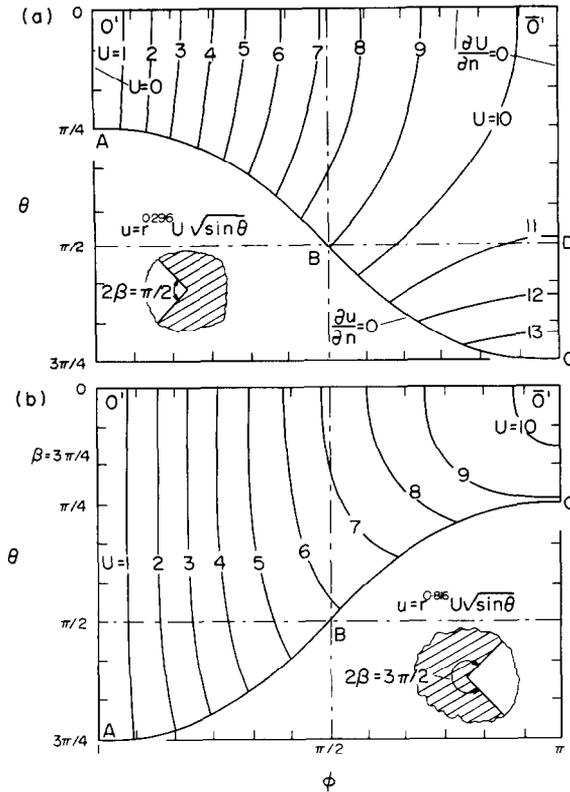


Fig. 12. Function U computed for the case shown in Fig. 11 ($\rho^p = \sqrt{\sin \theta}$).

All points on curve b in Fig. 9 have first been computed with grids $\Delta\phi = \pi/12, \Delta\theta = \pi/13$ (80–100 nodes). For the case of a right-angled corner ($2\beta = \pi/2$; Fig. 12a) this grid (having 86 nodes, not counting those outside the domain) yielded $\lambda = 0.2905$. This case has then also been computed with a grid $\Delta\phi = \pi/24, \Delta\theta = \pi/26$ (having 359 nodes), which has resulted in $\lambda = 0.2947$ (using method B). This differs by 0.0043 (i.e. 1.46 per cent) from the coarser grid. Nearly the same difference, namely 0.0044, is found between the grids with the same spacing in Fig. 7. Because the result for a grid with 359 nodes would be in Fig. 7 about 0.0015 below the exact value, it can be inferred that the correct value probably is $\lambda = 0.2947 + 0.0015$, i.e.

$$\lambda = 0.296 \quad (\text{for } 2\beta = \pi/2). \tag{39}$$

(However, Radlow [9] deduced for this case analytically the value $Re(\lambda) = 0.25$, whose difference from the present result is obviously much greater than the probable value of numerical error, and so reexamination of the analysis in [9] seems to be in order.) The second and third smallest λ , as found by method A with 86 nodes for $p = \frac{1}{2}$, are 1.43 and 2.03. It was also checked that higher p does not give smaller λ .

The case $2\beta = 3\pi/2$ (Fig. 12b) has also been computed with the finer grid, which has furnished $\lambda = 0.8148$; assuming similar convergence behavior as before, the correct

value is $\lambda = 0.815 + 0.001 = 0.816$. For $2\beta = \pi/4$, the finer grid yielded $\lambda = 0.2057 + 0.0015 = 0.207$.

Rvachev's results for real roots [11], obtained by a mapping technique and the Galerkin method, differ considerably (by as much as 0.2) from curve b in Fig. 9. This is probably due to numerical error because Rvachev used only six unknown parameters.

For $\beta \rightarrow 0$, Aleksandrov and Babeshko [1] found λ to be complex, with $\text{Re}(\lambda) = -0.5$, which overrides the value $\lambda = 0$ indicated by equation (28) and shown in Fig. 9. However, no cases with very small β have yet been run to verify this result. This will be postponed until the finite element formulation is completed (because refinement of the nodal network near side $0A$ in Fig. 10 will probably be desirable for high accuracy).

Singularity at the apex of pyramidal notches with three equal sides

The cases of pyramidal notches shown in Fig. 13 have 3 singularity rays and possess one plane of symmetry through each of the three edges, so that only the domain $O'ABCD0'$ needs to be considered. This domain includes only one singularity ray and the formulation of the problem is the same as for the cases shown in Fig. 11, discussed previously, except for two differences. First, angle 2α of the planes forming each radial edge depends on the apex angle 2β of each side, namely $\sin \alpha = 1/(2 \cos \beta)$, and the value of p is then given by equation (2). Second, the location of the right boundary in the (θ, φ) plane is different, as is shown in Fig. 14 for the case of the right-angled pyramidal notch ($2\beta = \pi/2$).

This case was analyzed only by method B, assuming λ to be real. The smallest roots computed with networks of about 80 nodes are given by the points on curve c in Fig. 9 and the eigenstate U for one case is shown in Fig. 14. Note that the cases $2\beta > 2\pi/3$ represent a convex solid pyramid rather than a concave notch. For $2\beta = 2\pi/3$ the pyramid degenerates into a planar surface ($\alpha = 0$), for which $\lambda = 1$. The case $\beta \rightarrow 0$ should give the same value as for curve b. (According to the result in [1], the values in Fig. 9 will possibly be overridden by some complex λ at $\beta \rightarrow 0$.) The right-angled pyramidal notch (Figs. 14 and 13a) has also been computed with a finer grid $\Delta\varphi = \pi/24$, $\Delta\theta = \pi/26$, which has yielded $\lambda = 0.4533$. Assuming similar convergence behavior as before, the correct value is $\lambda = 0.4533 + 0.0015 = 0.455$. For $2\beta = \pi/4$, the finer grid yielded $\lambda = 0.2288 + 0.0015 = 0.230$ ($\alpha = 32.765^\circ$, $p = 0.61127$).

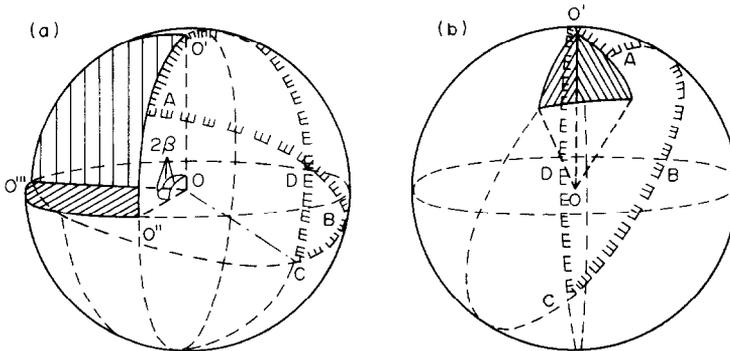


Fig. 13. Pyramidal notches with three sides of equal apex angle. (The sphere is not the body surface and serves only for visualizing the situation in spherical coordinates.)

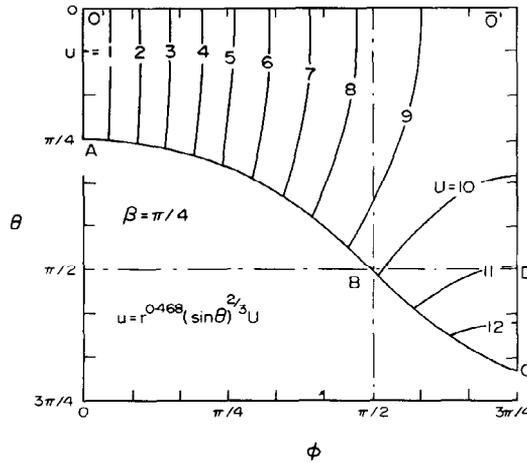


Fig. 14. Function U computed for the case shown in Fig. 13a ($\rho^n = (\sin \theta)^{2/3}$).

Development of singular finite elements and extensions to elasticity

It should be noted that knowledge of the dominant near-singularity field is all that is needed for the development of finite elements comprising the singularity, following the path shown, e.g. by Walsh [13]. Table 2 gives sufficient information for formulation of such elements for the most important cases. This brings within reach the numerical solutions by the finite element method for three-dimensional domains with arbitrary boundaries.

The general method presented can also be applied to problems of three-dimensional elasticity which are not reducible to potential theory problems. Here expressions of the form (6) or (10) augmented by rigid-body displacement terms for the neighborhood of the ray must be introduced for each of the three displacement components. The analysis may be based either on Navier's differential equations of equilibrium in spherical coordinates or the Papkovitch-Neuber potentials. In either case the problem is reduced to three simultaneous second order partial differential equations in θ, ϕ for three functions with nonsingular gradients. Applying a numerical method, the problem is again reduced to the system of algebraic equations (30). But its matrix is about three times larger than for the same nodal network in a potential theory problem, has a wider band, and is less sparse. This will make computations more costly.

Application of the numerical method to singularities in plane elasticity

There is a number of unanalyzed singular problems in plane elasticity, especially in combinations of cracks, notches and bonded dissimilar materials. Analytical solutions are complicated and a great effort is being made to solve various cases one by one, as can be seen in the recent literature. The present method of numerical solution can be useful in all these problems, too. The technique of Knein [8] and Williams [13] reduces the plane problem to an ordinary rather than a partial differential equation. The system of equations (30) arising in numerical solution is therefore of a much smaller size, and usually also narrowly banded, which substantially simplifies computations according to methods A, B or C outlined previously.

CONCLUSION

Three-dimensional harmonic functions near termination or intersection of gradient singularity lines can be determined by the general numerical method presented herein. Knowledge of the near field will enable the development of finite elements comprising the singularity.

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Résumé—La fonction harmonique U au voisinage du point 0, à partir duquel part un rayon à singularité, est supposée être dominée par le terme $r^\lambda \rho^p U$ où r = distance du point 0, p = constante connue, et ρ une fonction choisie des coordonnées sphériques angulaires θ, φ telles que $u \sim (\rho r)^p$ au voisinage du rayon pour tout r fini. U est une fonction inconnue de θ, φ , pour laquelle une équation différentielle partielle avec les conditions aux limites, en particulier celles sur les rayons singuliers, et un principe variationnel sont déduits. Parce que grad U est non singulier, une solution numérique est possible, utilisant, par exemple, les méthodes des différences finies ou des éléments finis. Ceci ramène le problème à celui de trouver un λ de partie réelle la plus petite et satisfaisant à l'équation $\text{Det}(A_{ij}) = 0$, où A_{ij} est une grande matrice dont les coefficients dépendent linéairement du $\mu = \lambda(\lambda + 1)$. En général λ et A_{ij} sont complexes. Des solutions peuvent être obtenues soit en se ramenant à un problème ordinaire de valeur propre de matrice pour μ , soit par des conversions successives à des systèmes d'équations linéaires non homogènes. Des études par ordinateur ont confirmées la faisabilité de cette méthode et ont montrées que des résultats de grande précision peuvent être obtenus. Des solutions pour des fissures et des entailles aboutissant à un plan, ou à une surface conique, et pour des fissures aboutissant obliquement à la surface d'un demi-espace, sont présentées. Dans ces cas, λ est réel et la singularité est toujours plus faible ($\lambda > p$) que sur la ligne singulière et peut même disparaître ($\lambda > 1$).

De plus, des contraintes élastiques sous l'effet d'un poinçon rigide, en forme de coin et glissant, ou à l'angle du bord d'une fissure ont été analysées, ainsi que des fonctions harmoniques pour des entailles pyramidales à trois côtés. Ici on avait $\lambda < p$.

Une solution analytique simple pour une classe de cas particuliers a également été trouvée. On l'a utilisé pour vérifier quelques uns des résultats numériques.

Zusammenfassung—Es wird angenommen, dass die harmonische Funktion u nahe dem Punkt 0, von dem ein einzelner Singularitätsstrahl ausgeht, durch den Ausdruck $r^\lambda \rho^p U$ beherrscht wird, wo r = Abstand vom Punkt 0, p = bekannte Konstante und ρ = gewählte Funktion von winkligen Kugelkoordinaten θ, φ

so dass $u \sim (\rho r)^p$ nahe dem Strahl für jedes endliche r . U ist eine unbekannte Funktion von θ, φ , für die eine partielle Differentialgleichung mit Grenzbedingungen, besonders die an den Singularitätsstrahlen, und ein Variationsprinzip abgeleitet werden. Weil Gradient U nicht-singulär ist, ist eine numerische Lösung möglich, verwendend, zum Beispiel, die den Netzverfahren oder die Methode der endlichen Elementen. Dies reduziert das Problem auf das Finden von λ des kleinsten reellen Teiles, der die Gleichung $\text{Det}(A_{ij}) = 0$ befriedigt, wo A_{ij} eine grosse Matrize ist, deren Koeffizienten linear von $\mu = \lambda(\lambda + 1)$ abhängen. Im Allgemeinen sind λ und A_{ij} komplex.

Lösungen können entweder durch Reduktion auf ein gewöhnlicher Matrizeigenwertproblem für μ erhalten werden, oder durch aufeinanderfolgende Umwandlungen auf nicht-homogene lineare Gleichungssysteme. Untersuchungen an Rechenanlagen bestätigten die Durchführbarkeit der Methode und zeigten, dass sehr genaue Resultate erhalten werden können. Es werden Lösungen für Risse und Kerben vorgelegt, die an einer Ebene oder Kegelfläche enden, und für Risse, die quer an einer Halbraumoberfläche enden. In diesen Fällen ist λ reell und die Singularität ist immer schwächer ($\lambda > p$) als an der die Singularitätslinie und kann sogar verschwinden ($\lambda > 1$). Weiter wurden elastische Spannungen unter einem keilförmigen starren gleitenden Stempel oder an der Ecke einer Risskante analysiert, wie auch harmonische Funktionen an dreiseitigen pyramidalen Kerben. Es wurde gefunden, dass $\lambda > p$ hier vorkommt. Es wurde auch eine einfache analytische Lösung für eine Klasse von Spezialfällen gefunden und wurde verwendet, um einige der numerischen Resultate zu prüfen.

Sommario—Si assume che la funzione armonica u , vicino al punto 0 da cui emana un raggio con singolarità singola, sia dominata dal termine $r^\lambda \rho^p U$ dove r è la distanza dal punto 0, p è una costante nota e ρ è una funzione scelta delle coordinate sferiche angolari θ e φ tale che $u \sim (\rho r)^p$ vicino al raggio per ogni valore finito di r . U è una funzione non nota di θ e φ per la quale vengono derivati un'equazione a differenziali parziali con condizioni allo strato limite, specialmente quelle sui raggi con singolarità, ed un principio variazionale. Poiché $\text{grad } U$ non è singolare, una soluzione numerica è possibile, usando per esempio il metodo delle differenze finite o quello degli elementi finiti. Ciò riduce il problema a trovare il valore di λ della più piccola parte reale che soddisfa l'equazione $\text{Det}(A_{ij}) = 0$ dove A_{ij} è una grande matrice i cui coefficienti dipendono linearmente da $\mu = \lambda(\lambda + 1)$. In generale, λ et A_{ij} sono numeri complessi. Le soluzioni possono venir ottenute o per riduzione al problema del valore speciale di una matrice normale per μ , o per conversioni successive a sistemi di equazioni lineari non omogenee. Degli studi effettuati su calcolatori hanno confermato la praticabilità di questo metodo e hanno dimostrato che si possono ottenere risultati molto accurati. Vengono presentate soluzioni per fenditure e intagli terminanti su una superficie piana o conica, e per fenditure terminanti obliquamente sulla superficie di un semispazio. In questi casi λ è reale e la singolarità è sempre più debole ($\lambda > p$) che sulla linea di singolarità e può perfino scomparire ($\lambda > 1$). Sono state analizzate inoltre le sollecitazioni elastiche sotto uno stampo scorrevole rigido a forma di cuneo o su un angolo del bordo della fenditura, come pure le funzioni armoniche su intagli piramidali a tre facce. È stata inoltre trovata una semplice soluzione analitica per una classe di casi speciali, e la soluzione è stata usata per controllare alcuni dei risultati numerici.

Абстракт — Принято, что гармоническая функция u вблизи точки 0, от которого излучается один особенный луч, подвержена к господству члена $r^\lambda \rho^p U$, где r = расстояние от точки 0, p = известное постоянное, ρ = выбор функции от угловых сферических координат θ, ϕ так, что $u \sim (\rho r)^p$ вблизи луча для любого конечного значения r . При этом U есть неизвестная функция от θ, ϕ , для которого дан вывод частного дифференциального уравнения с граничными условиями, в частности, для тех на особенных лучах, а также выводится вариационный принцип. Так как $\text{grad } U$ есть неособенная, возможно получить численное решение, применяя например методы конечных разностей или конечных элементов. Поэтому проблема сводится к нахождению λ для наименьшей реальной части, удовлетворяющей уравнению $\text{Det}(A_{ij}) = 0$, где A_{ij} — большая матрица, коэффициенты которой зависят линейно от $\mu = \lambda(\lambda + 1)$. Вообще λ и A_{ij} — комплексными. Решения можно получить или приведением проблемы собственного значения стандартной матрицы относительно μ , или последовательными преобразованиями к системам неоднородных линейных уравнений. Из исследований компьютером подтверждена практическая осуществимость метода, а также возможность получения высокоточных результатов. Представлены решения для трещин и надрезов, ограничивающихся у плоскости или конической поверхности, и для трещин, ограничивающихся косно у полупространственной поверхности. Во этих случаях λ есть реальное, а особенность всегда более слабое ($\lambda > p$), чем значение ее на особенной линии, возможно и исчезание ее ($\lambda > 1$). Кроме того, проанализированы упругие напряжения под клин-образном, жестким, скользящим штампом, или во угле края трещины, и гармонические функции трехгранных пирамидальных надрезов. Здесь установлено что $\lambda < p$. Получено и простое аналитическое решение для одного класса частных случаев, примененное для проверки некоторых численных результатов.