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NUMERICAL ANALYSIS OF CREEP OF REINFORCED PLATES

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Approximating the hereditary integrals (generally of non-convolution type) by finite sums, the integral-type creep problem is converted to a sequence of elasticity problems with initial strains. In this manner a highly accurate, fourth-order method of time integration is set up and applied to an orthotropic layered plate, or a plate reinforced by an orthotropic system of bars or fibres. Applying a well-known method for elastic problems with initial strains, it is shown how the inelastic strains in a layered plate can be replaced by an equivalent lateral distributed load. The method was verified by means of a numerical example of a rectangular plate. For the special case of a degenerate memory function, a modification, reducing substantially the requirements for computer storage and time, is derived.

Symbols

d_b = thickness of isotropic layer b of plate
k_1, k_2 = curvature parameters given by (9)
q = distributed lateral load of plate, in direction of z
q^1 = fictitious q equivalent to inelastic strains
t = time
w = deflection of plate, in direction of z
w_{xy} = $\partial^2 w/\partial x \partial y$
w^1 = deflections due to q^1
x, y = rectangular coordinates in the plane of plate
z = lateral coordinte throughout the thickness of plate
$A_{1\alpha}$, $B_{1\alpha}$ = component functions of a degenerate memory function, (20)
E_b, E_{1b} = Young's modulus of layer b, and modulus given by (10)
D_b = cylindrical stiffness of layer b (2)
D_x, D_y, D_{xy} = total bending and torsional stiffness of the layered plate (2)
$D_{x\alpha}$, $D_{y\alpha}$, $D_{xy\alpha}$ = bending and torsional stiffnesses of the orthotropic layer (reinforcement) (2)
G_b = shear modulus of isotropic layer b
I_b = $d_b^3/12$
L_{1b}, L_{2b} = memory functions of isotropic layer b, corresponding to E_1 and G_b (11, 12)
M_{xb}, M_{yb}, M_{xyb} = bending moments and torsional moment in isotropic layer b
$M_{1b} = M_{2b} + M_{yb}$, $M_{2b} = M_{xb} - M_{yb}$
M_{1b}^0, M_{2b}^0, M_{xyb}^0 = fictitious prestress moments equivalent to inelastic strains (16)
M_{1b}^1, M_{2b}^1, M_{xyb}^1 = values of M_{xb}, M_{yb}, M_{xyb} due to q^1
$S_{1\alpha}$, $S_{2\alpha}$, S_{xyb} = given by (21), (23)
ε_x, ε_y, ε_{xy} = normal and shear components of strain tensor in layer b
$\varepsilon_1 = \varepsilon_x + \varepsilon_y$, $\varepsilon_2 = \varepsilon_x - \varepsilon_y$
ε_1^0, ε_2^0, ε_{xy}^0 = inelastic strains in the sense of ε_1, ε_2, ε_{xy} (13, 14)
ν_b = Poisson ratio for layer b
σ_x, σ_y, τ_{xy} = normal stresses and shear stress in layer b

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, $\sigma_y = \sigma_x - \sigma_y$
= prestresses equivalent to inelastic strains, in the sense of σ_1, σ_2, τ_{xy} (15)
= time as integration variable
, b . . . for layer a or b
(r) . . . for time $t_{(r)}$
0. . . for quantities due to inelastic strains

1. Introduction

forced concrete plates or layered plates represent practically impor- :tures whose creep can usually be solved only numerically. For roblems at small strains the numerical method is, in general, well- d is based on Theorem 1 given in the Appendix. This theorem was for volumetric inelastic strains already in 1838 by DUHAMEL [6] and widely utilized in thermoelasticity. For deviatoric strains and an naterial, this theorem was first deduced by REISSNER in 1931 [6]. nt approaches it was later independently derived by ESCHELBY [6] n anisotropic material) by BAŽANT [1, 2, 4]. For the finite element n equivalent technique of solution of the effect of initial (inelastic) been developed separately [7]. First application to a two-dimension- ı in creep of homogeneous plates (non-linear creep of the rate type) made by LIN [5] and by BAŽANT [1, 2].
his paper* a method of application of Theorem 1 to a reinforced plate will be presented. In addition, a highly accurate algorithm ution of memory-type creep problems will be shown and verified by a numerical example.

2. Basic relationships for elastic layered plates

r to stepping into the proper subject of creep, it is necessary to some well-known relationships for elastic plates [7]. Assuming ormals to the middle surface of plate remain straight and perpendic- ctions w of a plate must satisfy the equation:

$$D_x \frac{\partial^4 w}{\partial x^4} + 2D_{xy} \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 w}{\partial y^4} = q \tag{1}$$

y = rectangular coordinates; $q = q(x, y)$ = distributed load; D_x, rigidity constants. Consider that the plate consists of two layers f which layer b is isotropic and layer a is othotropic. Then

s paper is based on author's Internal Research Report No. 68/2, "Approximate Linear and Nonlinear Creep Problems. Initial Strain Method", Department of ering, University of Toronto, December 1968.

$$D_x = D_b + D_{x_a}, \quad D_{xy} = D_b + D_{xy_a}, \quad D_y = D_b + D_{y_a}$$
$$D_b = I_b E_b/(1-\nu_b^2), \quad I_b = d_b^3/12 \tag{2}$$

where E_b, ν_b = Young's modulus and Poisson ratio for the isotropic layer b; d_b = thickness of layer b; D_{x_a}, D_{y_a}, D_{xy_a} = bending and torsional rigidities of the orthotropic layer a; subscripts a or b refer to the layers a or b. Actually the plate may consist even of more layers if these layers have the same elastic properties; then the constants D_b or D_{x_a}, D_{y_a}, D_{xy_a} must express the sum of bending and torsional rigidities (with respect to the same middle surface) of all the layers with the same properties. A system of reinforcing bars (or fibres) in directions x and y may be viewed as a special case of orthotropic layer a, such that $D_{xy_a} \approx 0$; D_{x_a} and D_{y_a} = total bending rigidities of all reinforcement in directions x and y. For the sake of simplicity it is assumed that every layer is symmetrical with respect to the middle plane of plate; this implies, e.g., that the reinforcing bars are distributed symmetrically. Later also the following relationships will be needed:

$$\varepsilon_1 = -k_1 z, \qquad \varepsilon_2 = -k_2 z, \qquad \varepsilon_{xy} = -w_{xy} z/2 \tag{3}$$

$$\sigma_1 = M_{1b} z/I_b, \quad \sigma_2 = M_{2b} z/I_b, \quad \tau_{xy} = M_{xyb} z/I_b \tag{4}$$

$$M_{1b} = -E_{1b} I_b k_1, \; M_{2b} = 2G_b I_b k_2, \quad M_{xyb} = -2G_b I_b w_{xy} \tag{5}$$

where

$$\varepsilon_1 = \varepsilon_x + \varepsilon_y, \quad \varepsilon_2 = \varepsilon_x - \varepsilon_y \tag{6}$$

$$\sigma_1 = \sigma_x + \sigma_y = E_{1b}\varepsilon_1, \quad \sigma_2 = \sigma_x - \sigma_y = 2G_b \varepsilon_2, \quad \tau_{xy} = 2G_b \varepsilon_{xy} \tag{7}$$

$$M_{1b} = M_{xb} + M_{yb}, \quad M_{2b} = M_{xb} - M_{yb} \tag{8}$$

$$k_1 = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}, \quad k_2 = \frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{\partial y^2}, \quad w_{xy} = \frac{\partial^2 w}{\partial x \partial y} \tag{9}$$

$$E_{1b} = \frac{E_b}{1-\nu_b}, \quad G_b = \frac{E_b}{2(1+\nu_b)} \tag{10}$$

Here z = normal coordinate; σ_x, σ_y, τ_{xy} = normal stresses and shear stress in layer b ($\sigma_z = 0$); ε_x, ε_y, ε_{xy} = normal strains and shear strain ($\varepsilon_{xz} = \varepsilon_{yz} = 0$, ε_z is generally nonzero); M_{xb}, M_{yb} = bending moments in layer b, M_{xyb} = torsional moment, G_b = shear modulus.

3. Numerical integration of the creep problem

It will be assumed that the isotropic layer b exhibits linear creep and the layer a does not creep at all.

he case of plane stress, assumed to exist in the plate, the linear creep tegral-type, satisfying the conditions of isotropy, may be written in ing form:

$$\varepsilon_1(t) = \frac{\sigma_1(t)}{E_{1b}(t)} + \int_{t_0}^{t} \frac{\sigma_1(\tau)}{E_{1b}(\tau)} L_{1b}(t,\tau)\, d\tau = E_{1b}^{-1} \sigma_1(t)$$
$$\varepsilon_2(t) = \frac{\sigma_2(t)}{2\,G_b(t)} + \int_{t_0}^{t} \frac{\sigma_2(\tau)}{2\,G_b(\tau)} L_{2b}(t,\tau)\, d\tau = E_{2b}^{-1} \sigma_2(t) \tag{11}$$
$$\varepsilon_{xy}(t) = \frac{\tau_{xy}(t)}{2\,G_b(t)} + \int_{t_0}^{t} \frac{\tau_{xy}(\tau)}{2\,G_b(\tau)} L_{2b}(t,\tau)\, d\tau = G_b^{-1} \tau_{xy}(t)$$

and τ = time, or age of concrete; E_{1b}^{-1} and G_b^{-1} are creep operators, ling to the elastic constants E_{1b}^{-1} and G_b^{-1}; E_{1b}^{-1} and G_b^{-1} have the form Volterra's integral operators whose kernels L_{1b}, L_{2b} (the memory are of non-convolution type because of aging of concrete. The elastic $_b$, E_{2b} are in general also functions of age of concrete. In the numerical the following forms have been considered:

$$) = -\frac{\partial}{\partial\tau}\left[\left(0{,}6 + \frac{100}{\tau}\right) \frac{t-\tau}{t-\tau+60}\right], \quad \frac{L_{1b}(t,\tau)}{E_{1b}} = 0.44 \frac{L_{2b}(t,\tau)}{G_b} \tag{12}$$

y be adopted as a reasonable approximation for concrete in steady ental conditions; t and τ is given in eq. (12) in days. The time variation l G_b has been neglected in the numerical example.

the numerical solution the given time interval (t_0, t_1) may be subdivid- crete times $t_{(0)}, t_{(1)}, \ldots t_{(n)}$ in n equal subintervals Δt. The hereditary or $t = t_{(r)}$ in eq. (11) may be approximated by the finite sums

$$\varepsilon^0_{1(r)} = \sum_{s=0}^{r} c^r_{(s)} L_{1b}(t_{(r)}, t_{(s)})\, \sigma_{1(s)}/E_{1b(s)}$$
$$\varepsilon^0_{2(r)} = \sum_{s=0}^{r} c^r_{(s)} L_{2b}(t_{(r)}, t_{(s)})\, \sigma_{2(s)}/(2G_{b(s)}) \tag{13}$$
$$\varepsilon^0_{xy(r)} = \sum_{s=0}^{r} c^r_{(s)} L_{2b}(t_{(r)}, t_{(s)})\, \tau_{xy(s)}/(2\,G_{b(s)})$$

are constants and subscript (r) pertains to time $t_{(r)}$, e.g. $\sigma_{1(s)} =$ en the creep law (11) takes the form

$$\varepsilon_{1(r)} = \sigma_{1(r)}/E_{1b(r)} + \varepsilon^0_{1(r)}, \quad \varepsilon_{2(r)} = \sigma_{2(r)}/(2G_{b(r)}) + \varepsilon^0_{2(r)},$$
$$\varepsilon_{xy(r)} = \tau_{xy(r)}/(2G_{b(r)} + \varepsilon^0_{xy(r)} \tag{14}$$

Assume that the stresses have already been calculated up to the time $t_{(1-r)}$ and that the values of the stresses $\sigma_{1(r)}, \sigma_{2(r)}, \sigma_{xy(r)}$ have been estimated. The most simple estimate is $\sigma_{1(r)} \approx \sigma_{1(r-}$ obtained by extrapolation, e.g. by the formula $\sigma_{1(r)} = \sigma_{1(r-1)} + 3\sigma_{1(r-2)}\, \sigma_{1(r-3)}$ whose error is of order Δt^4. Then the values $\varepsilon^0_{1(t)}, \varepsilon^0_{2(r)}, \varepsilon^0_{xy(r)}$ may be computed, using eqs (13), and represent thus known quantities in eqs (14). Therefore eqs (14) may be formally regarded as a fictitious elastic stress-strain law with prescribed initial (inelastic) strains. Solving the elasticity problem with these initial strains, and given applied loads and prescribed displacements in time $t_{(r)}$, new values for the stresses $\sigma_{1(r)}, \sigma_{2(r)}, \sigma_{xy(r)}$ are obtained.

The basic feature of the numerical algorithm outlined is that the time integration of a creep problem is converted to a sequence of elasticity problems with prescribed initial strain. Each of these problems may be converted to a problem without initial strains according to Theorem 1 in the Appendix. How this may be implemented will be explained now.

4. Effect of inelastic strains in layered elastic plates

Because of the linearity of creep law, the distributions of $\varepsilon^0_{1(r)}, \varepsilon^0_{2(r)}, \varepsilon^0_{x\ (r)}$ or $\sigma^0_{1(r)}, \sigma^0_{2(r)}, \sigma^0_{xy(r)}$ across the viscoelastic layer must be linear. Denoting the resultants of $\sigma^0_{x(r)}, \sigma^0_{y(r)}, \tau^0_{xy(r)}$ over the viscoelastic layer by $M^0_{xb(r)}, M^0_{yb(r)}, M^0_{xyb(r)}$ and putting $M^0_{1b(r)} = M^0_{xb(r)} + M^0_{yb(x)}$, $M^0_{2b(x)} = M^0_{xb(x)} - M^0_{yb(x)}$, the following holds true:

$$\sigma^0_{1(r)} = E_{1b(r)}\ \varepsilon^0_{1(r)} = M^0_{1b(r)}\, z/I_b$$
$$\sigma^0_{2(r)} = 2\,G_{b(r)}\, \varepsilon^0_{2(r)} = M^0_{2b(r)}\, z/I_b \tag{15}$$
$$\tau^0_{xy(r)} = 2\,G_{b(r)}\, \varepsilon^0_{xy(r)} = M^0_{xyb(r)}\, z/I_b$$

Expressing $\sigma_{1(s)}, \ldots$ from eq. (4), substituting into (14) and taking into account eqs (15), it follows that:

$$M^0_{1(r)} = E_{1b(r)} \sum_{s=0}^{r} c^r_{(s)} L_{1b}(t_{(r)}, t_{(s)})\, M_{1b(s)}/E_{1b(s)}$$
$$M^0_{2b(r)} = 2G_{b(r)} \sum_{s=0}^{r} c^r_{(s)} L_{2b}(t_{(r)}, t_{(s)})\, M_{2b(s)}/(2G_{b(s)}) \tag{16}$$
$$M^0_{xyb(r)} = 2G_{b(r)} \sum_{s=0}^{r} c^r_{(s)} L_{2b}(t_{(r)}, t_{(s)}\, M_{xyb(s)}/(2G_{b(s)})\,.$$

The loading state designated by F^1 in Theorem 1 is represented by a distributed load $q^1_{(r)}$ which is in equilibrium with the prestresses $\sigma^0_{1(r)}, \sigma^0_{2(r)}, \tau^0_{xy(r)}$. Then also in equilibrium with $M^0_{1b(r)}$, $M^0_{2b(r)}$, $M^0_{xyb(r)}$ and is determined according e differential equation of equilibrium of plate. Thus

$$q^1_{(r)} = -\frac{1}{2}\frac{\partial^2}{\partial x^2}(M^0_{1b(r)} + M^0_{2b(r)}) - \\ -\frac{1}{2}\frac{\partial^2}{\partial y^2}(M^0_{1b(r)} - M^0_{2b(r)}) - 2\frac{\partial^2 M^0_{xyb(r)}}{\partial x\,\partial y}\,. \tag{17}$$

rally, at the boundaries additional loads may be required to balance , etc. However, at the simply supported edge no additional moment disted along the edge is needed. (This would not be true if shrinkage or therlilatation were considered.) The vertical loads and the torques, and for a ɔed edge also the moments balancing $M^0_{1b(r)}\ldots$ are generally non-zero, ırse, but have no effect on the plate. Therefore, with $q^1_{(r)}$ the loading state fully described.

The deflections $w^1_{(r)}$ due to the load $q^1_{(r)}$ can be solved from eq. (1) with ppropriate boundary conditions. The corresponding internal forces may ımputed from eqs (9), (5), (8). Finally, according to Theorem 1,

$$w_{(r)} = w^1_{(r)} + w^l_{(r)} \\ M_{1b(r)} = M^1_{1b(r)} - M^0_{1b(r)} + M^l_{1b(r)}, \ldots \tag{18}$$

: $w^l_{(r)}$, $M^l_{1b(r)}$, ... is the elastic solution, due to the given applied loads in $t_{(r)}$, alone.

5. Algorithm of numerical integration

The algorithm of time integration as outlined after eq. (14) can be made efficient by essentially the same refinements as those used in solving a n of integral equations [6]. Thus, when the final values of $M_{1b(r)}$, $M_{2b(r)}$ $I_{xyb(r)}$ for the *r-th* step have been found, their accuracy may be improved ration, recalculating the values of $M^0_{1b(r)}$, $M^0_{2b(r)}$ and $M^0_{xyb(r)}$ (eqs 16) and :ing the solution of the elastic plate.

For the evaluation of the hereditary integrals according to eqs (16), efficients $c^r_{(s)}$ cannot be selected according to the Simpson's rule because ımber of subintervals between t_0 and $t_{(r)}$ is alternately even and odd. A nuıl integration formula without this limitation and with the same order or (Δt^4) is:

$$\int_{t_{(0)}}^{t_{(r)}} f(t)\,dt = \frac{\Delta t}{24}\,[9(f_{(0)} + f_{(r)} + 19(f_{(1)} + f_{(r-1)}) - 5(f_{(2)} + f_{(r-2)}) + \\ + f_{(3)} + f_{(r-3)} + \sum^{r-2} (-f_{(s-1)} + 13 f_{(s)} + 13 f_{(s+1)} - f_{(s+2)})]. \tag{19}$$

It is valid for $r \geq 3$. For $r = 2$ the Simpson's rule may be applied. Obviously, a special procedure is needed for the evaluation of the hereditary integrals over the first step Δt $(r = 1)$ if the use of the trapezoidal rule, whose error is of order Δt^2, is to be avoided. A suitable procedure is the successive approximations, a technique known to be very efficient for systems of integral equations. To maintain the same order of error (Δt^4), the successive approximations must be applied (at least) for the first three steps. First some estimate (0-th approximation) of the values $M_{1b(1)}$, $M_{1b(2)}$, $M_{1b(3)}$, $M_{2b(1)}\ldots M_{xyb(3)}$ must be made, e.g. putting $M_{1b(3)} = M_{1b(2)} = M_{1b(1)} = M_{1b(0)}$ etc. Then the values for the middle of the first subinterval Δt, denoted by $M_{1b(.5)}, \ldots$, may be determined according to the fourth order interpolation formula which reads: $M_{1b(.5)} = (5M_{1b(0)} + 15M_{1b(1)} - 5M_{1b(2)} + M_{1b(3)})/16$. The values $M^0_{1b(1)}, \ldots M^0_{xyb(3)}$ must be calculated, using solely the values of $M_{1b(s)}, \ldots$ from the preceding approximation. In the first step Δt, the values $M_{1b(.5)}, \ldots$ enable the use of the Simpson's rule. Finally, the elasticity problems with initial strains may be solved for each of the first three steps Δt, obtaining the improved values of $M_{1b(r)}, \ldots$ for the next approximation.

The algorithm of time integration just described is represented in Fig. 1.

It is necessary to note that this higher-order integration method may be utilized only in the time intervals, in which all the applied loads evolve as sufficiently smooth functions of time [6], i.e. as continuous functions with continuous first three derivatives. Otherwise it makes sense to use only the simple algorithm, involving no extrapolation (i.e. starting with the values $\sigma_{1b(r)} = \sigma_{1b(r-1)}$ or $\sigma_{1b(r)} = 0$) and no successive approximations in the first three steps, and applying the trapezoidal (or rectangular) formula for the numerical evaluation of hereditary integrals. If there are not too many discontinuities in the time variation of loads and their derivatives, one can, of course, solve every time interval between two discontinuities independently by the fourth order method as described.

Numerical example

As a test example the solution of a rectangular plate was programmed. The edges $x = 0$ and $x = a$ were considered as fixed, and the edges $y = 0$ and $y = b$ as simply supported. A constant, uniform load q applied in time t_0 was considered. The numerical data were: $a = b = 400$ cm, $D_b = 4 \times 10^7$ kp/cm² (kp = force kilogram), $Dx_a = D_y$, $Dy_a = D_b/4$, $Dxy_a = 0$, $\nu_b = 0{,}15$, $t_0 = 60$ days, $t_1 = 180$ days. For the solution of the elasticity problem, the finite difference method with a square grid of mesh size $\Delta x = \Delta y = a/16$ was adopted; the functions M_{1b}, $M^0_{1b}, \ldots$ were represented by the arrays of their nodal values, and the partial derivatives in eqs (17), (1), (9) were replaced by the finite difference expressions. The elastic analysis of the plate was thus reduced to a system of algebraic equations. The results of analysis (on IBM 7094) according to the flow chart in Fig. 1 are given in Table I for different numbers of subdivision, n, of the time interval (t_0, t_1). Some of the results are graphically represented in Figs 2a, b, c. It is noteworthy that the time changes in the distribution of the relative (not the absolute) values of stresses in the reinforcement are very small (Fig. 2c) while the shifts in the

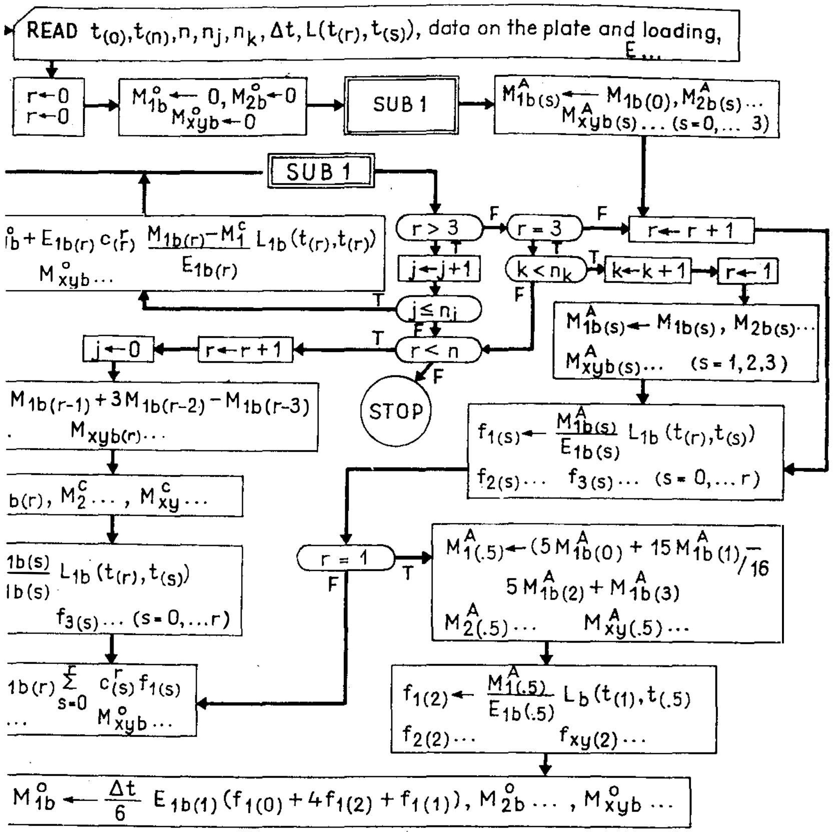


chart for the integration of creep problem, valid for $n < 3$. (Subscript i for the e is not written; n = subdivision of time; n_j, n_k = number of iterations or approxi- = true, F = false.) *SUB1*. For an elastic plate with the actual instantaneous $E_{2b(r)}$, $G_{b(r)}$ in time $t_{(r)}$ calculate w, $M_{1b(r)}$, $M_{2b(r)}$, $M_{xyb(r)}$ due to initial strains by M^0_{1b}, M^0_{2b}, M^0_{xyb} [(20), (1), (2), (10), (6), (21)]. Then add the values of w, M_{1b}, M_{xyb} due to applied loads in time $_{(r)}$. Write w, M_{1b}, M_{2b}, M_{xyb}, $t_{(r)}$

Table I

Numerical results

$x = a/2$, $y = b/2$, w	$x = 3a/16$, $y = b/2$, w	$x = a/2$, $y = b/2$			
		M_{xb}	M_{yb}	M_{xa}	M_{ya}
10.8487	8.124	−231.96	−172.90	−409.32	−48.767
10.7866	8.0775	−231.08	−168.41	−406.97	−48.475
10.7857	8.0768	−231.01	−168.18	−406.94	−48.472
10.7862	8.0772	−230.98	−168.13	−406.96	−48.475

The computer results indicated that more than two iterations per step bring hardly any improvement in accuracy.

It should be noted that the problem discussed is practically relevant only for the stresses due to lateral loads in symmetrically prestressed concrete plates. This analysis cannot be applied to non-prestressed concrete plates because the phenomenon of cracking has not been accounted for. Another area of application are the plates of fibre-reinforced plastics and laminated plates.

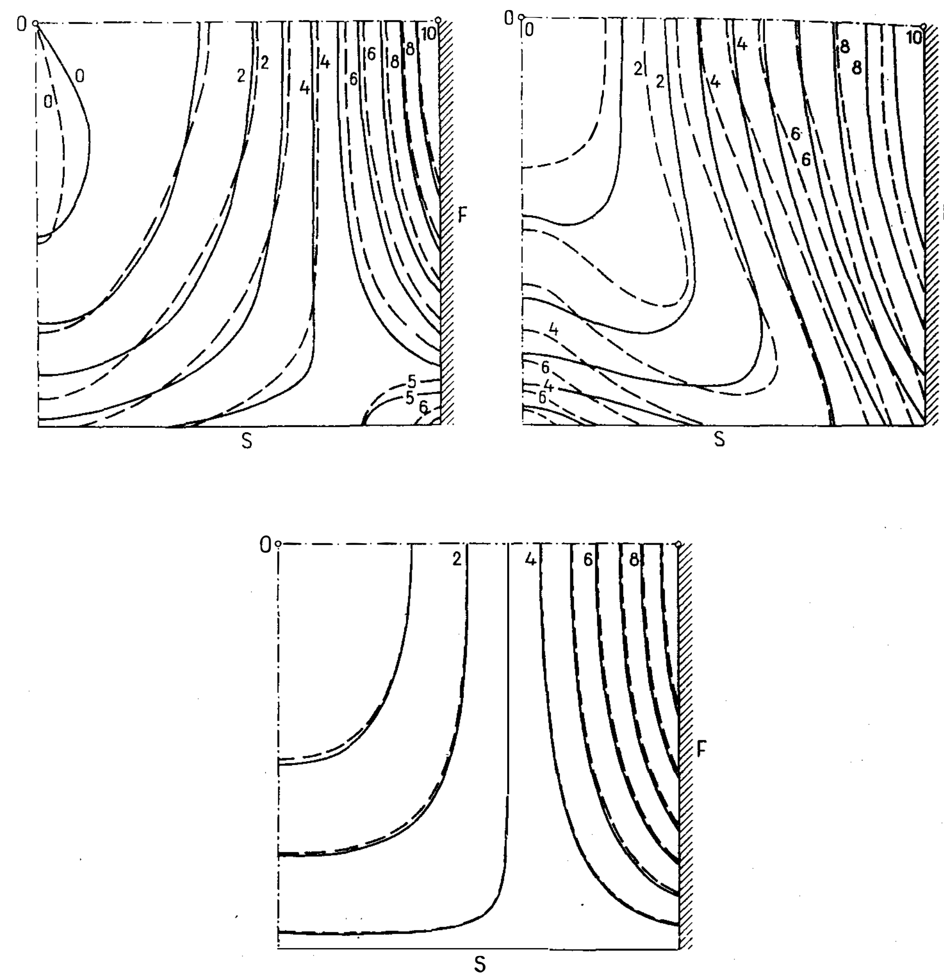


Fig. 2. Lines of equal relative values in the right lower quadrant of plate. (Dashed lines pertain to the initial state at time t_0, continuous lines to time t_1; S denotes the simply supported edge, F the fixed edge and O the center of plate.) a) Maximum bending moments in the isotropic layer b. (In time t_0, O corresponds to the stress value − 192,8, 10 to 629, 3; in time t_1, O corresponds to −168,2, 10 to 416,8). b) Minimum bending moments in the isotropic layer b. (In time t_0, O corresponds to − 312,8, 10 to 94,4, in time t_1, O corresponds to − 231,0, 10 to 166,3.) c) Bending moments in the reinforcement (layer a) in x − direction. (In time t_0, O corresponds to − 290,4, 10 to 629,3; in time t_1, O corresponds to − 406,9, 10 to 875,2.)

```
AABBBCCDDEE-.   ,'PQRST-.,'UVWWXY-.  ,'ABCDE-.  ,'PQST-.  ,U
AABBBCCDDEE-..  ,'PQRST-.  ''UVWXY-.  ,'ABCDE-.  ,'PQST-.  ,'
AABBBCCDDEE-..  ,'PQRST-.  ,'UVWXY-.  ,'ABCDE-.  ,'PQST-.  ,'
AABBBCCDDEE-..  ,'PPRST-.  ,'UVWXY-.  ,'ABCDE-.  ,'PQST-.  ,'
AABBBCCDDEE-..  ,''PQST-.  ,'UVWXY-.  ,'ABCDE-.  ,'PQST-.  ,'
ABBBBCCDDEF-..  ,''PQRST-.  ,'UWXY-.  ,'ABCDE-.  ,'PQRT-.  ,'
BBBBCCCDDEE-..  ,''PQRST-.  ,'UVWXY-.  ,'ABCE-.  ,'PQRST.  ,'
BBBBCCCDDEE-..  ,,'PQRST-.  ,'UVWXY-.  ,'ABCDE.  ,'PQRST-  ,'
BBBBCCDDDEE-..  ,,'PQRST--.  ,'VWXY-.  ,'ABCDE-.,'PQRST-.  '
BBBCCCDDDEE-..  ,,'PQQRST-.  ,'UVWXY-  ,'ABCDE-.  ,PQRST-.  .
BBCCCCDDEEE-..   ,''PQRST-.  ,'UVWXY-.  ,'ACDE-.  ,'PRST-.  ,
CCCCCDDDEEE--.   ,''PQRSTT-.  .'UVWX-.  ,'ABCD-.  ,'PQRS-.  ,
CCCCDDDDEEE--.   ,,'PQRRST-.  ,'UVWXY-.,'ABCDE-  ,'PQRST-.,
CCCDDDDEEEE--.   ,,'PPQRSTT-.  ,UVWXY-.  ,'BCDE-.  ,PQRST-.
DDDDDDEEEEE--..   ,''PQRRST-.  ,'UVWXY.  ,'ABCE-.  ,'PQST-.
DDDDDEEEEEE--...   ,,'PQQRST-.  ,'UVWXY-.  ,ABCDE-  ,'PQRST-.
DDDEEEEEEE---...   ,,'PPQRSST-.  ,'UVWX-.  ,'ABDE-.  .PQRST-.
EEEEEEEE----...   ,,''PQRRST-.  ,'UVWXY-.  'ABCD-.  ,'PQRST.
EEEEEE-------...    ,''PQQRRST-.  ,'UVWX-.  ,'ACDE-.,'PQRST-
-------------...    ,''PPQRRST-.  ,,UVWXY-.  'ABCD-.  ,'PQRS-
 -----------...     ,,'PPQQRST--.  ,'UVWX-.  .'ACDE-.,'PQRST
 .............      ,,''PQQRRST-.   ,UVWXY-.  'ABCD-.  .'PQRS
 ............      ,,,''PPQQRST--.  ,'UVWXY.  ,'ACDE-.  ,PQRS
       ...         ,,,''PPQQRSST-.   ,UVWXY-.  'ABCD-.  ,'PQR
 ,,,           ,,,,,''PPPQQRSST--.  ,'UVWXY-  ,'ABDE-.  ,'PQ
 ',,,,,,,,,,,,,,,,'''PPPQQRRSTT-.  ,'UUVWX-.  ,'BCDE-.  ,'P
P''''''',,,,,,,'''''PPPQQQRRSTT--.  ,'UVWXY-.  'ABCDE-.  'P
PPPPP''''''''''''''PPPQQQRRSSTT-.  ,,'UVWXY.  ,'ABCE-.  ,'
RQQQQQQPPPPPPPPPPPPPQQQQRRSSTT--.   ,'UVVXY-.  ,'ABCE-.  ,
SSSRRRRRQQQQQQQQQQQQQRRRRRSSTT--..  ,''UVWXY-.  ,'ABDE-.
TTTTSSSSSRRRRRRRRRRRRRRRRRSSTTT--..  ,,'UUVWXY-.  ,'ACDE-.
..---TTTTSSSSSSSSSSSSSSSSSTTTT---.   ,,'UVWXYY-.  ,'BCDE-
,    ....----TTTTTTTTTTTTTTTT----..   ,,''UVWXY--.  ,'ABDE
U''',,,     .....-----------------....   ,,''UVVWXY--.  ,'ABC
WVVVUUU'''',,,,        ..........      ,,,''UUVWWXYY-.  ,'AB
YYXXWWWVVVUUUU''',,,,,,,            ,,,,,'''UVVWWXXY-.  ,'A
..--YYYXXXWWWVVVUUUU'''',,,,,,,,,,,,,,'''UUVVVWXXY-.  ,'
```

ıle showing how the lines in Fig. 2b for $t = t_1$ were simulated on printer. (The interval he maximum value and the minimum value was subdivided in 50 equal subintervals, the following series of 50 characters was assigned ABCDE-, 'PQRST-., 'UVWXY-., 'ABCDE-., 'PQRST-.,')

6. Simplification for a degenerate memory function

the kernels in (11) have the form

$$L_{1b}(t, \tau)/E_{1b} = \sum_{\alpha} A_{1\alpha}(\tau) B_{1\alpha}(t), \quad L_{2b}(t, \tau)/G_b = \sum_{\alpha} A_{2\alpha}(\tau) B_{2\alpha}(t) \tag{20}$$

ı $A_{1\alpha}$, $B_{1\alpha}$, ... are given functions of one variable, the integral-type v may be transformed to a rate-type creep law, in form of a differential

equation of order n_α [3]. In such a case the requirements for computer storage are known to be much lower. The same result can be, however, achieved without abandoning the integral-type creep law and the numerical algorithm described. Indeed, substituting (20) into (11), it may be found that (13) may be replaced by the following equations

$$\varepsilon^0_{1(r)} = \sum_{\alpha} B_{1\alpha(r)} S_{1\alpha(r)}, \quad \varepsilon^0_{2(r)} = \sum_{\alpha} B_{2\alpha(r)} S_{2\alpha(r)}, \quad \varepsilon^0_{xy(r)} = \sum_{\alpha} B_{2\alpha(r)} S_{xy\alpha(r)} \tag{21}$$

where

$$S_{1\alpha(r)} = \int_0^{t_r} A_{1\alpha}(\tau)\, \sigma_1(\tau)\, d\tau, \quad S_{2\alpha(r)} = \int_0^{t_r} A_{2\alpha}(\tau)\, \sigma_2(\tau)\, d\tau,$$
$$S_{xy\alpha(r)} = \int_0^{t_r} A_{2\alpha}(\tau)\, \tau_{xy}(\tau)\, d\tau. \tag{22}$$

Then, replacing these integrals with finite sums according to formula (19), it is possible to obtain the recurrent equations which follow

$$S_{1\alpha(r)} = S_{1\alpha(r-1)} + \frac{\Delta t}{24}(8 f_{(r)} + 23 f_{(r-1)} - 11 f_{(r-2)} +$$
$$+ 5 f_{(r-3)} - f_{(r-4)}), \quad S_{2\alpha(r)} = \ldots, \quad S_{xy\alpha(r)} = \ldots \tag{23}$$

where $f_{(r)} = A_{1\alpha(r)} G_{1(r)}$. They are applicable for $r > 3$ and the starting values are:

$$S_{1\alpha(3)} = \frac{3\Delta t}{8}(f_{(0)} + 3 f_{(1)} + 3 f_{(2)} + f_{(\lambda)}), \ldots \tag{24}$$

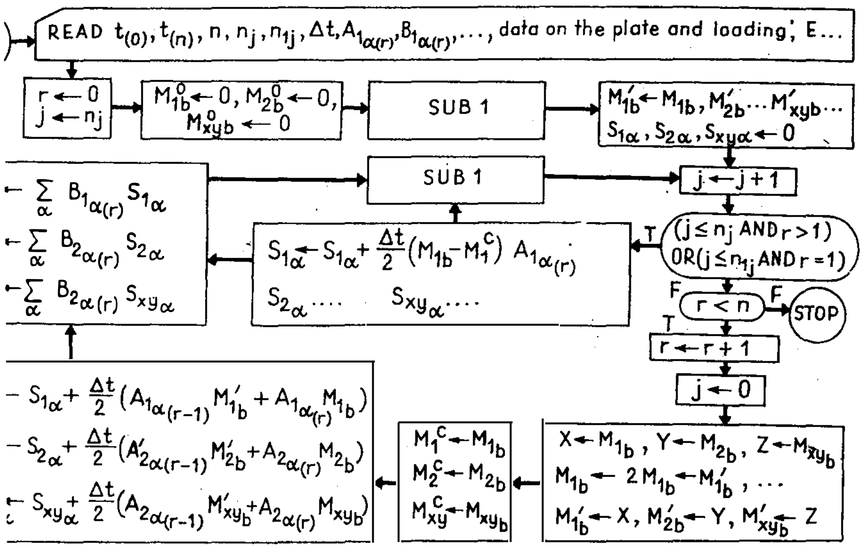
It us seen that the storage in computer of the entire history of stresses σ_1, σ_2, σ_{xy} is not needed; only the history over the last four steps Δt nedds to be stored. It may be verified that the amount of storage required is the same as for the corresponding rate-type creep law and the numerical method of the fourth order. The amount of computer time for evaluation of ε^0_1, ε^0_2, ε^0_{xy} becomes also substantially reduced. The general form of the flow chart in Fig. 1 remains unchanged.

In the case of degenerate memory functions, the storage requirements are in fact proportional to the desired order of error. If one is contented with the trapezoidal rule for the evaluation of the hereditary integrals in (11), then

$$S_{1\alpha(r)} = S_{1\alpha(r-1)} + \frac{\Delta t}{2}(A_{1\alpha(r-1)}\, \sigma_{1(r-1)} + A_{1\alpha(r)}\, \sigma_{1\alpha(r)}, \quad S_{2\alpha(r)} = \ldots, \quad S_{xy\alpha(r)} = \ldots \tag{25}$$

so that besides the current values of stresses only the current values and the last preceding values of $S_{1\alpha}$, $S_{2\alpha}$, $S_{xy\alpha}$ ($\alpha = 1, \ldots n_\alpha$) need to be stored. The

.art for this case is represented in Fig. 3 in which the notation $M_{1b} =$ $t_{(r)}), \ldots\ M'_{1b} = M_{1b}(t_{(r-1)}) \ldots$ is used.



low chart for the simplified integration of creep problem in the case of degenerate 'unction. (Second order method with the trapezoidal rule for the evaluation of hereditary integrals; SUB1 is the same as in Fig. 1)

is worth noting that the Arutyunian's and Maslov's formula [2, 3]

$$L_{1b}(t, \tau)\, E_b^{-1}(\tau) = -\frac{\partial}{\partial \tau}\left[E_b^{-1}(\tau) + \sum_{\alpha=1}^{n_\alpha} \varphi_\alpha(\tau)(1 - e^{-\alpha(t-\tau)})\right] \tag{26}$$

tly utilized for concrete, is a special case of the degenerate memory ı (20).

7. Conclusions

The effect of inelastic (creep) strains, linearly distributed over the ıs of a layer of layered plate, can be replaced by applied loads according (15)—(17).

Creep of a layered (or reinforced) plate can be solved as a sequence of y problems with inelastic strains. A highly accurate algorithm of time ion is given in Fig. 1; it is applicable when the loads evolve as suffi- ɜmooth functions of time.

It is advantageous to introduce a creep law involving degenerate functions. Then the entire history of stresses need not be stored and ıntial saving in computer storage and time is possible (Fig. 3).

4. If the plate has a simple shape, the finite difference method in space coordinates is suitable.

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Appendix

Theorem 1. — Let the constitutive equation for small strains be

$$\sigma = C\,(\epsilon - \epsilon^0) \tag{A1}$$

where σ = stress tensor, ϵ = strain tensor, ϵ^0 = initial strain tensor, C = fourth rank tensor of elastic moduli; C and ϵ^0 may depend on space coordinates. Introduce the prestress tensor

$$\sigma^0 = C\,\epsilon^0 \tag{A2}$$

and define F^1 as the state of volume and surface loads which equilibrate σ^0. Then in a given body under zero applied loads the stresses are $\sigma^1 - \sigma^0$, the (linearized) strains are ϵ^1 and the (small) displacements are u^1 where σ^1, ϵ^1, u^1 is the solution of the same body (with the given prescribed displacements) for loads F^1 and zero initial strains.

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Numerische Berechnung des Kriechens armierter Platten. Die Annäherung der Kriechintegrale (im allgemeinen von Nicht-Konvolution) durch endliche Summen verändert das Problem des Kriechens vom Integraltyp zu einer Folge von elastischen Problemen mit Anfangsformänderungen. Auf diese Weise wurde eine sehr genaue Methode der Zeitintegration von vierter Ordnung ausgearbeitet und auf eine orthotropisch geschichtete, oder mit einem orthotropischen System von Stahleinlagen oder Fasern armierte Platte angewendet. Durch den Gebrauch einer wohlbekannten Methode zur Lösung der Elastizitätsprobleme mit Anfangsverformungen wird demonstriert, wie die elastischen Verformungen in einer schichtigen Platte durch ein gleichwertiges System von stetig verteilten Lasten ersetzt werden können. Die Methode wurde mit dem numerischen Beispiel einer rechteckigen Platte nachgewiesen. Für den speziellen Fall eines entartenen Kriechkernels wird eine abgeänderte Methode abgeleitet, die die Erfordernisse an Speicherung und Zeit für die Rechenanlage wesentlich reduziert.

Числовой анализ ползучести железобетонных плит *(З. П. Базант)*. Приближение дегенерированных интегралов неспирального типа с помощью конечных количеств преобразует явление ползучести интегрального типа в ряд последующих друг за другом задач по упругости с начальной деформацией. Таким образом удалось разработать очень точный метод временного интегрирования четвертой степени и примеиить его для некоторой многослойной ортотропной пластины или для некоторой ортотропной пластины, армированной металлической или волокнистой системой. Применяя хорошо известный метод, учитывающий начальные деформации, для задач по упругости, видно, что каким образом можно заменить в некоторой многослойной пластине неупругие деформации эквивалентной распределяющейся системой сил, действующей в боковом направлении. Доказательство правильности метода производится на числовом примере прямоугольной четырехугольной пластины. Для специального случая некоторой дегенерированной мемориальной функции приведен вывод такого модифицированного метода, который значительно сокращает требования по накоплению и времени на вычислительной машине.