Structural Safety and Reliability, Corotis et al. (eds), © 2001 Swets & Zeitinger, ISBN 90 5809 197 X

Probabilistic modeling of quasibrittle fracture and size effect

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Keywords: extreme value statistics, Weibull theory, fracture mechanics, quasibrittle materials, size effect, scaling, nonlocal damage, material length, stochastic finite element method, fractal cracks, reliability

ABSTRACT: Progress in structural design requires probabilistic modeling of quasibrittle fracture, which is typical of concrete, fiber composites, rocks, toughened ceramics, sea ice and many 'high tech' materials. The most important consequence of quasibrittle behavior is a deterministic (energetic) size effect, the theory of which evolved near the end of last century. After a review of the background, the present plenary lecture describes the recent efforts to combine the classical Weibull theory of statistical size effect due to local strength randomness with the recently developed energetic theory, and also surveys various related problems, such as the probability tail structure of the stochastic finite element methods, the random scatter in fracture testing, the role of fractal nature of cracks, the reliability provisions of design codes, and the lessons from past structural catastrophes.

1 HISTORICAL INTRODUCTION

The absence of size effect means that the nominal strength of geometrically similar structures, defined as the maximum load divided by the area of the characteristic cross section, does not depend on the structure size D. When it does, the structural failure is said to exhibit a size effect. There is no size effect in the classical continuum mechanics theories. These include elasticity with a strength limit, plasticity or any theory in which the material failure criterion is expressed in terms of the stress tensor or the strain tensor, or both, as well as fracture mechanics of bodies containing only microscopic cracks or flaws.

The size effect, which is a problem of scaling of structural response, is a very old problem. It was discussed already by Leonardo da Vinci (1500s) and Galileo Galilei (1638) (Williams 1957). The history of size effect studies can be divided in three periods.

Period 1. The period lasting until the 1980's, in which the size effect has generally been regarded as statistical, attributed to the randomness of local strength of the material, can be seen as the first period. The basic idea of the statistical size effect was qualitatively advanced by Mariotte (1686) already three and half centuries ago. But its proper mathematical formulation had to await the work of Weibull (1939, 1949, 1951, 1956), mathematically justified by the development of the extreme value statistics, and particularly the weakest link model (Fisher and Tippett 1928; also Tippett 1925, Peirce 1925, Fréchet 1927, von Mises 1936). The Weibull statistical theory was discussed and used widely [Gnedenko 1943, Gnedenko and Kolmogorov 1954, Epstein 1948, Freudenthal 1956, 1968 (and Selected Papers 1981), Weil and Daniel 1964, Bolotin 1969, Saibel 1969, Kittl and Diaz 1998, 1990, Engelund and Rackwitz 1992, Bažant and Novák 2001b, etc.]. The theory was physically justified by random distribution of microscopic flaws (Freudenthal and Gumbel 1953, 1956), and was proven to describe very well the strength of fatigue-embrittled metals and fine-grained ceramics. During the third quarter of the last century, Weibull theory reigned supreme and was applied to many other kinds of materials (Zaitsev and Wittmann 1974, Mihashi and Izumi 1977, Mihashi and Zaitsev 1981, etc.), whether or not perfectly brittle. The mechanicians in those times were not interested in the size effect, believing it belonged to the realm of statisticians (the subject was not even mentioned in Timoshenko's monumental History of the Strength of Materials, published in 1953).

Period 2. The start of the second, deterministic, period can be seen, in retrospect, in Walsh's (1972, 1976) pioneering experiments on concrete which showed discrepancies irreconcilable with the statistical theory of size effect. Stimulated by nuclear reactor research, failure theories of concrete that model the size effect as deterministic appeared in the form of the crack band model (Bažant 1976, 1982; Bažant and Oh 1983), cohesive (or fictitious) crack model (Hillerborg et al. 1976, 1983; Petersson 1961), and later the nonlocal and gradient damage models (Bažant 1984, Bažant et al. 1984, 1985, Pijaudier-Cabot and Bažant 1987, Peerlings et al. 1996, Bažant and Planas 1998). The size effects experimentally observed in concrete were described by a simple approximate size effect law deduced by energy release analysis (Bažant 1984) and later justified by asymptotic matching (Bažant 1987, 1999, 2001; Bažant and Kazemi 1990, Bažant and Chen 1987, Bažant and Planas 1998). Different size effect formulae were developed for failures with a large fracture process zone (FPZ) occurring after stable growth of large cracks or in notched specimens (Bažant 1984, Bažant and Kazemi 1991), and failures occurring at crack initiation from a smooth surface (Bažant and Li 1976a,b, Bažant 1988). It was shown that the deterministic energetic size effect is exhibited by all kinds of quasibrittle materials, i.e., the materials characterized by a large FPZ in which the material exhibits distributed microcracking with strain softening (rather than plastic yielding, as in brittle-ductile fracture of metals). Aside from concrete, the quasibrittle materials include rocks, sea ice, many 'high-tech' modern materials such as particulate and fiber composite or coarse-grained or toughened ceramics, and also cemented sands, stiff clays, paper, wood, particle board, bone, biological shells, filled elastomers, etc.

Period 3. The beginning of a third period may be seen in the recent efforts for amalgamation of the statistical and deterministic size effects in structures consisting of quasibrittle materials, which are in the focus of the present plenary lecture (e.g., Bažant and Xi 1991, Breysse 1990, Breysse and Renaudin 1996, Carmeliet 1994, Carmeliet and Hens 1994, Gutiérez 1999, Bažant and Novák 2000a,b, 2001a,b, Novák et al. 2001a,b, Frantzisconis 1998) (the uncertainties due to loads and environment are outside the scope of this lecture).

2 POWER SCALING

All the physical systems that involve no characteristic length exhibit a simple scaling, given by power laws. Let us consider geometrically similar systems, for example the beams shown in Fig. 1a, and seek to deduce the response Y (e.g., the maximum stress or the maximum deflection) as a function of the characteristic size (dimension) D of the structure; $Y = Y_0 f(D)$. (e.g., 1 ft., 1 mm). We imagine three structure sizes $1, D$ and D' (Fig. 1a). If we take size 1 as the reference size, the responses for sizes D and D' are $Y = f(D)$ and $Y' = f(D')$. However, since there is no characteristic length, we can also take size D as the reference size. Consequently, the equation

$$f(D') \,/\, f(D) \;=\; f(D' \,/\, D) \tag{1}$$

must hold (Bažant 1993, Bažant and Chen 1997; for fluid mechanics, Barenblatt 1979, Sedov 1959). This is a functional equation for the unknown scaling law $f(D)$. It has one and only one solution, namely the power law:

$$f(D) \;=\; (D/c_1)^s \tag{2}$$

where $s =$ constant and c_1 is a constant which is always implied as a unit of length measurement (e.g. 1 m, 1 inch). Note that c_1 cancels out of equation (2) when the power function (1) is substituted. Also note that when, for instance, $f(D) = \log(D/c_1)$, equation (1) is not satisfied and the unit of measurement, c_1, does not cancel out. So, the logarithmic scaling could be possible only if the system possessed a characteristic length related to c_1. (Eq. 1 is of course the fundamental reason why all the units in physics appear only in powers.)

The power scaling must apply for every physical theory in which there is no characteristic length. In solid mechanics such failure theories include elasticity with a strength limit and elasto-plasticity, as well as linear elastic fracture mechanics (LEFM), for which the FPZ is shrunken into a point.

To determine exponent s, the failure criterion of the material must be taken into account. For elasticity and plasticity, one finds that exponent $s = 0$ when response Y represents the stress or strain, for example, the maximum stress in the structure or the stress at certain homologous points (Bažant 1993). This is also true for the so-called nominal strength σ_N (or nominal stress at failure), which is a parameter of maximum load, P_{max}, and is defined as $\sigma_N = P_{max}/bD$ or P_{max}/D^2 for geometrical similarity in two of three dimensions; D = structure size (characteristic dimension).

Thus, if there is no characteristic dimension, all geometrically similar structures of different sizes must fail at the same nominal stress. By convention, this came to be known as the case of *no size effect*. In LEFM, on the other hand, $s = -1/2$, provided that similar cracks or notches are considered (this may be generally demonstrated with the help of Rice's J-integral, Bažant, F1993). If $\log \sigma_N$ is plotted versus $\log D$, the power law is a straight line (Fig. 1b). For plasticity, or elasticity with a strength limit, the exponent of the power law vanishes, i.e., the slope of this line is 0. For LEFM, the slope is $-1/2$.

3 DETERMINISTIC BACKGROUND—ENERGETIC SIZE EFFECT

A currently 'hot' subject is the quasibrittle material behavior, for which the size effect bridges two power laws pertaining to different scales. Quasibrittle materials obey on a small scale the theory of plasticity (or strength theory), characterized by material strength or yield limit σ_0, and on a large scale the LEFM, characterized by fracture energy G_f. The combination of σ_0 and G_f yields Irwin's (1958) characteristic length (material length) $\ell_0 = EG_f/\sigma_0^2$ which approximately characterizes the size of the FPZ (E = Young's elastic modulus) and separates the small and large scales. Since a characteristic length exists, a quasibrittle material cannot follow the power scaling, and so a size effect must exist.

The analysis of distributed (smeared) cracking damage (strain softening) demonstrated (Bažant 1976, 1982, Bažant and Oh 1983) that damage localization into a crack band engenders a deterministic size effect on the postpeak deflections and energy dissipation of structures. The effect of the crack band is approximately equivalent to that of a long fracture with a sizable FPZ at the tip. Subsequently, based on an approximate energy release analysis, the following approximate size effect law was derived for structures failing after large stable crack growth (Bažant 1984):

$$\sigma_N = B\sigma_0 \left[1 + \left(\frac{D}{D_0}\right)^r\right]^{-1/2r} + \sigma_R \tag{3}$$

in which b = structure thickness in the case of 2D similarity; r, B = positive dimensionless constants, B depending on the geometry of structure (usually $r = 1$ is acceptable); D_0 = constant representing the transitional size (at which the power laws of plasticity and LEFM intersect); D_0 and B characterize the structure geometry; and σ_R = constant = residual stress (usually $\sigma_R = 0$). Eq. (3) was shown to be closely followed by the numerical results for the cohesive (fictitious) crack model (Hillerborg et al. 1976) and crack band model (Bažant 1976, Bažant and Oh 1983), as well as for the nonlocal continuum damage models. Measurements of the size effect on σ_N were shown to offer a simple way to determine the fracture characteristics of quasibrittle materials, including the fracture energy, the effective FPZ length, and the (geometry dependent) R-curve.

Quasibrittle fracture may be approximately analyzed by equivalent LEFM in which it is assumed that the tip of an equivalent LEFM crack lies ahead of the actual crack tip at distance c_f which is a material constant and represents about one half of the FPZ length. Depending on the ratio a_0/c_f where a_0 = notch length or initial (traction-free)fracture length at maximum load, two basic kinds of quasibrittle failure may be distinguished:

(i) The case where $a_0/c_f \gg 1$, in which the law (3) applies and P_{max} occurs after large stable fracture growth; and

(ii) the case where $a_0/c_f \approx 0\,(\ll 1)$, which means that P_{max} occurs at the initiation of macroscopic fracture propagation from a smooth surface.

The former kind is typical of reinforced concrete, fiber composites and sea ice (Bažant and Planas 1998, Bažant and Kazemi 1991, Bažant 1996, Walraven 1995, Walraven and Lehwalter 1994, Iguro et al. 1985, Shioya and Akiyama 1994, Okamura and Maekawa (1994), Gettu et al. 1990, Marti 1989, Bažant, Daniel and Li 1986, Bažant et al. 1999, Wisnom 1992), some unreinforced structures such as concrete gravity dams or floating ice plates in the Arctic). An example of the latter kind is the modulus of rupture (flexural strength) test, which consists in the bending of a simply supported beam of span L with a rectangular cross section of depth D and width b, subjected to concentrated load P. The maximum load is not decided by the elastically calculated stress $\sigma_1 = 3PL/2bD^2$ at the tensile face, but by the stress value $\bar{\sigma}$ roughly at distance c_f from the tensile face (which is at the middle of FPZ). Because $\bar{\sigma} = \sigma_1 - \sigma_1' c_f$ where σ_1' = stress gradient $= 2\sigma_1/D$, and also because $\bar{\sigma} = \sigma_0$ = intrinsic tensile strength of the material, the failure condition $\bar{\sigma} = \sigma_0$ yields $P/bD = \sigma_N = \sigma_0/(1 - D_b/D)$ where $D_b = (3L/D)c_f$, which is a constant because for geometrically similar beams L/D = constant. This formula for σ_N exhibits a size effect but is meaningless for $D \leq D_b$. Since the derivation is valid only for the first two terms of the asymptotic expansion in $1/D$, one may replace this formula by the following asymptotically equivalent size effect formula:

$$\sigma_N = \sigma_0 \left(1 + \frac{rD_b}{D}\right)^{1/r} \tag{4}$$

which has the same first two terms and happens to be acceptable for the whole range of D; r is any positive constant, and $r \approx 1.45$ gives the optimum fit of the existing test data (Bažant and Novák's 2000a).

To explain the mechanism of the size effect in an intuitive simple manner, consider the rectangular panel in Fig. 1d, which is initially under a uniform stress equal to σ_N. Introduction of a crack of length a with a FPZ of a certain length and width h may be approximately imagined to relieve the stress, and thus release the strain energy, from the shaded triangles on the flanks of the crack band shown in Fig. 1d. According to experimental data as well as finite element simulations, the length of the crack at maximum load may normally be assumed approximately proportional to the structure size D. The stress reduction in the triangular zones of areas $ka^2/2$ (Fig. 1d) causes (for the case $b = 1$) the energy release $U_a = 2\times(ka^2/2)\sigma_N^2/2E$. The stress drop within the crack band of width h causes further energy release $U_b = ha\sigma_N^2/E$. The total energy dissipated by the fracture is $W = aG_f$, where G_f is the fracture energy, a material property representing the energy dissipated per unit area of the fracture surface. Energy balance during static failure requires that $\partial(U_a + U_b)/\partial a = dW/da$. Setting $a = D(a/D)$ where a/D is approximately a constant if the failures for different structure sizes are geometrically similar, the solution of the last equation for σ_N yields (Bažant 1984) the approximate size effect law in (3) with $\sigma_R = 0$ (Fig. 1c).

A similar intuitive explanation can be given for compression fracture—when the band of buckling due to axial splitting cracks propagates sideway (Fig. 1f), the energy release from the triangular shaded area grows quadratically with D while the energy dissipated in the band grows linearly, when geometrically similar failures are compared. An intuitive energy explanation can also be offered for the strut-and-tie model, e.g., for the case of diagonal shear failure of a reinforced concrete beam—a softening damage band of a width that is a material property propagates across the strut (B in Fig. 1e), which releases energy from zone 5346 (R in Fig. 1e) whose area grows quadratically with beam depth D while the area of the band grows only linearly.

Rigorous derivations of the size effect law (3), which also reveal the effect of geometry, have been given by means of asymptotic analysis based equivalent LEFM (Bažant 1997b) and by means of Rice's path-independent J-integral (Bažant and Planas 1998). This law has also been verified by nonlocal finite element analysis, and by random particle (or discrete element) models of the heterogeneous microstructure of concrete. Experimental verifications have by now become abundant (e.g. Fig. 1g,h,i).

For very large sizes ($D \gg D_0$), the size effect law in (3) reduces to the power law $\sigma_N \propto D^{-1/2}$, which represents the size effect of LEFM (for geometrically similar large cracks)

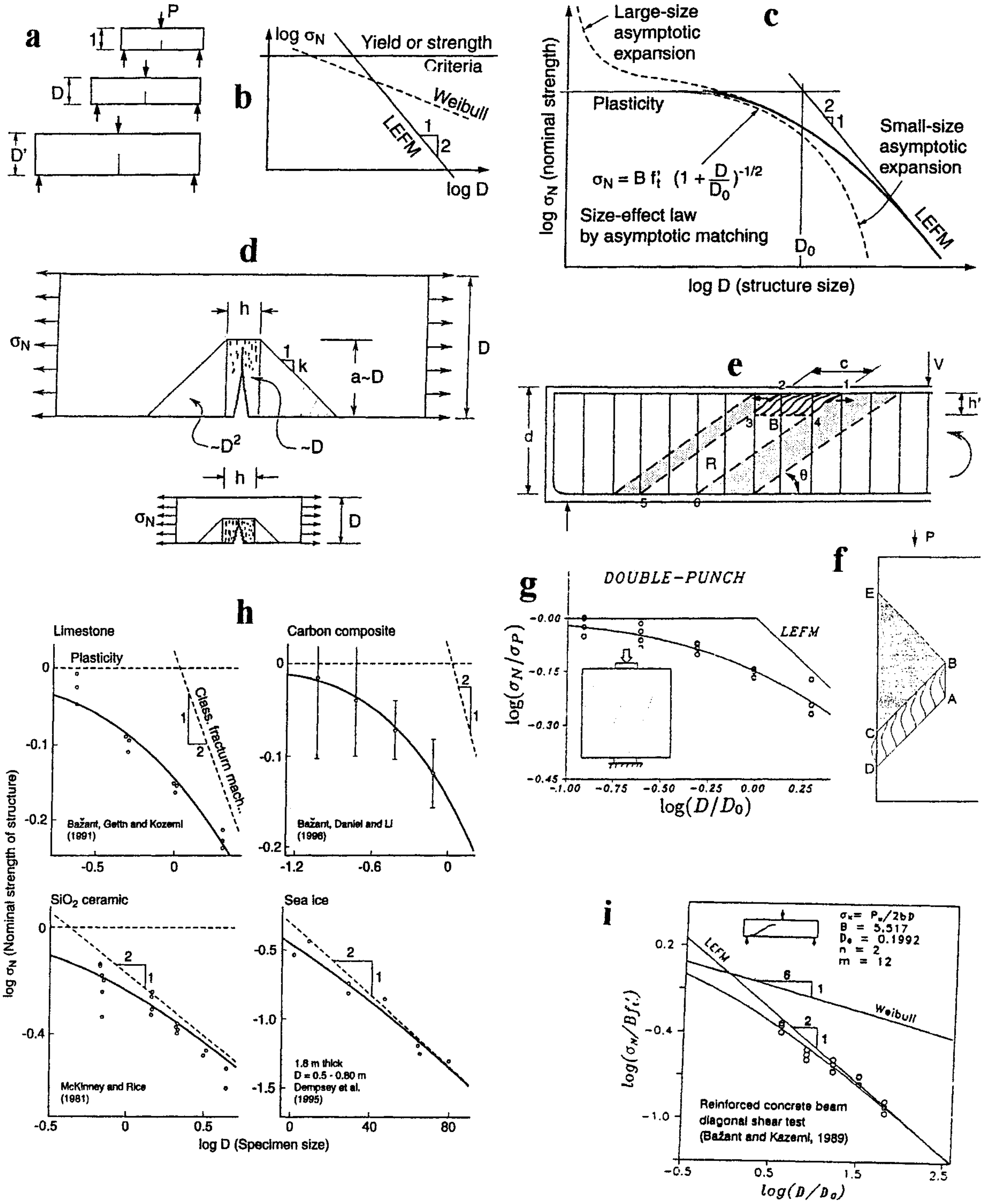

Figure 1. (a) Similar cracked structures. (b) Power scaling laws. (c) Size effect law bridging the power laws for plasticity and LEFM. (d,e,f) Zones of stress relief and energy release for tensile fracture, failure of 'compression strut' in reinforced concrete beam, and compression fracture band. (g,h,i) Size effect test data from the literature, fit by size e.ect law, for compressive punch of concrete, for notched mode I tests of limestone, carbon-epoxy composite, SiO_2 ceramic and sea ice (Dempsey's tests in Arctic Ocean with size range 0.3m–80m), and for diagonal shear or reinforced microconcrete beams.

and corresponds to the inclined asymptote of slope $-1/2$ in Fig. 1b. For very small sizes ($D \ll D_0$), this law reduces to σ_N = constant, which corresponds to the horizontal asymptote and means that there is no size effect, as in plastic limit analysis. Properly the size effect laws in Eqs. (3) and 4) should be seen as aysmptotic matching—an approximate 'interpolation' between opposite asymptotic behaviors (Barrenblatt 1979, Bender and Orszag 1978, Hinch 1991, Bažant 2001; Fig. 1c).

The distinction between the two kinds of failure is based on the stability of fracture for two kinds of geometries of the structure, loading and crack (Bažant and Cedolin 1991; Bažant and Planas 1998):

- The *positive* geometry is the case where the dimensionless energy release function $g(\alpha)$ (or the stress intensity factor K_I at constant load P) increases with a ($a = \alpha D$), i.e., $\mathrm{d}g(\alpha)/\mathrm{d}\alpha > 0$ (or $[\partial K_I/\partial a]_P > 0$). In this case, the structure, under load control, looses stability as soon as the full FPZ develops, which means that, at maximum load, the FPZ is still attached to the notch (a desired situation in testing) or to the tip of a preexisting traction-free (fatigued) crack, or to the body surface.
- The *negative* geometry is the case where $\mathrm{d}g(\alpha)/\mathrm{d}\alpha < 0$ (or $[\partial K_I/\partial a]_P < 0$). It is because of a negative geometry that a crack can grow stably at increasing load while the FPZ travels ahead. The maximum load occurs only when the structure geometry changes to positive, i.e., when $g(\alpha) = 0$ for some a.

Most types of the notched fracture specimens have a positive geometry, and so does the modulus of rupture test (flexure of an unnotched unreinforced beam) or the vertical fracture of an arch dam. The beneficial effect of reinforcement is that most reinforced concrete structures start their fracture growth with a negative geometry. So do many fiber composite structures. The 'dipping' curved fracture in a gravity dam, as well as sea ice penetration, represents also negative geometry. For unnotched structures, the size effect law (3) for large cracks can apply only if the geometry is initially negative.

4 WEIBULL THEORY AND STATISTICAL SIZE EFFECT

A three-dimensional continuous generalization of the weakest link model for the failure of a chain of links whose strengths are statistically independent random variables (Fig. 2b) leads to the cumulative distribution

$$P_f(\sigma_N) = 1 - e^{-\int_V c[\boldsymbol{\sigma}(\boldsymbol{x},\sigma_N)]\mathrm{d}V(\boldsymbol{x})}\,, \qquad c(\boldsymbol{\sigma}) = \sum_{I=1}^{3} \frac{P_1[\sigma_I(\boldsymbol{x})]}{V_r} \tag{5}$$

which represents the failure probability of the structure, provided that the structure fails as soon as a macroscopic crack initiates; σ_I = principal stresses at $\boldsymbol{x}$ just before failure ($I = 1, 2, 3$), $\boldsymbol{x}$ = coordinate vector, V = volume of structure, and $c(\boldsymbol{\sigma})$ = function giving the spatial concentration of failure probability of the material (= $V_r^{-1}\times$ failure probability of material representative volume V_r) (Freudenthal 1968); $c(\boldsymbol{\sigma})$ = concentration function (spatial density of failure probability); σ_i = principal stresses (i = 1,2,3), and $P_1(\boldsymbol{\sigma})$ = failure probability (cumulative) of the smallest possible test specimen, of volume V_r, subjected to uniaxial tensile stress σ_i. Eq. (5) is derived by noting that the survival probability , $1 - P_f$, of a chain of N links is the joint probability that all the links survive, i.e. $1-P_f = (1-P_1)^N$. For very large N, the distribution depends only on very small P_1, and so we may write $\ln(1 - P_f) = N \ln(1 - P_1) \approx -NP_1$, which immediately gives Eq. (5).

For mathematical reasons as well as physical physical ones (analysis of material flaws), the low-probability tail of $P_1(\sigma)$ must be a power law:

$$P_1(\sigma) = \left\langle \frac{\sigma - \sigma_u}{s_0} \right\rangle^m \tag{6}$$

(Weibull 1939) where m, s_0, σ_u = material constants (m = Weibull modulus, usually between 5 and 60) and the threshold σ_u is typically taken as 0 (it is normally hard to identify σ_u from test data unambiguously since very different σ_u give almost equally good fits). The weakest-link model leads (for $\sigma_u = 0$) to simple expressions for the mean of σ_N as a function of m and the coefficient of variation ω of σ_N;

$$\overline{\sigma}_N = s_0 \, \Gamma\left(1 + \frac{1}{m}\right)\left(\frac{V_r}{V}\right)^{1/m} \propto D^{-n_d/m}, \qquad \omega = \left(\frac{\Gamma(1+2m^{-1})}{\Gamma^2(1+m^{-1})} - 1\right)^{1/2} \tag{7}$$

where Γ is the gamma function, and n_d = 1, 2 or 3 for uni-, two- or three-dimensional similarity. Eq. (7) represents a power-law size effect on the mean nominal strength $\overline{\sigma}_N$ (if $\sigma_u = 0$). Since there is no size effect on ω, the expression for ω in (7) is normally used to identify m from tests. However, it is usually forgotten to check whether the ω-values for different specimen sizes are the same. This is a check on the validity of Weibull theory, which is failed by quasibrittle materials (Fig. 2f, m = 4.2, 14.0, 24.2).

In view of Eq. (7) for $\overline{\sigma}_N$, the value $\sigma_W = \sigma_N (V/V_0)^{1/m}$ for a uniformly stressed specimen can be adopted as a size-independent stress measure. Taking this viewpoint, Beremin (1983) proposed taking into account the nonuniform stress in a large crack-tip plastic zone by the so-called Weibull stress:

$$\sigma_W = \left(\sum_i \sigma_i{}^m \, V_i/V_r \right)^{1/m} \tag{8}$$

where V_i ($i = 1, 2, ...N_W$) are the elements of the plastic zone having maximum principal stress σ_{Ii}. Ruggieri and Dodds (1996) replaced the sum in (7) by an integral (see also Lei et al. 1998). Eq. (8) has been intended for the crack-tip plastic zone in metals. It seems applicable only if the crack at the moment of failure is still microscopic, that is, small compared to structural dimensions, which is not the case for quasibrittle materials.

The condition of structural failure at crack initiation is often satisfied only partially. There are many structures in which crack initiation in the high stress region (e.g., near the tensile face at midspan of a simply supported beam) gives a positive geometry, but crack initiation in the low stress region (e.g., near the neutral axis) gives a negative geometry. Thus, in the latter case, the crack can grow stably, at increasing load, which violates the conditions of validity of the classical Weibull theory. However, crack initiation at the low stress region is extremely unlikely (especially for a high m value), and so the contribution from such a region to the Weibull integral in (5) is negligible.

If the structure is not of a positive geometry (e.g., a beam with tensile reinforcement), large cracks must form before the failure can occur, which precludes Weibull type statistical analysis. Although rigorous probabilistic modeling seems prohibitively difficult, it does not matter because, for negative geometries, the size effect is predominantly energetic (deterministic). So, when the size effect is mainly statistical, the violations of statistical independence have a negligible effect, and when it is not, the question of statistical independence is irrelevant.

In the case of quasibrittle structures, applications of the classical Weibull theory face a number of fundamental objections:

1. The fact that the size effect on $\overline{\sigma}_N$ is a power law means that the functional equation (5) is satisfied, and this implies the absence of any characteristic length.[1] But this cannot be true if the material does contain sizable inhomogeneities, as does concrete.

2. The energy release due to stress redistributions caused by a macroscopic FPZ or a stable crack growth before P_{max} gives rise to a deterministic size effect, which is ignored. Thus the Weibull theory can be valid only if the structure fails as soon as a microscopic crack becomes macroscopic.

[1] Although the length $l_r = V_r^{1/3}$ might seem to be a characteristic length, it serves merely as a unit of measurement; indeed, if V_r is changed arbitrarily to some other value V_r', it suffices to change s_0 to $s_0' = s_0(V_r'/V_r)^{1/m}$ in order to keep the value of $P_f(\sigma_N)$ the same, as may be checked by substitution into (5).

3. Every structure is mathematically equivalent to a uniaxially stressed chain or bar of a variable cross section, which means that the structural geometry and failure mechanism are ignored.

4. The size effect differences between the cases of two- and three-dimensional similarities ($n_d = 2$ or 3) are often much smaller than predicted by Weibull theory (because, for example, a crack in a beam causes failure only if it spreads across the full width of the beam).

5. Many tests of quasibrittle structures show a much stronger size effect than predicted by Weibull theory (e.g., diagonal shear failure of reinforced concrete beams; Walraven and Lehwalter 1994, Walraven 1995, Iguro et al. 1985, Shioya and Akiyama 1994, and many flexure tests of plain beams cited in Bažant and Novák 1990a,b).

6. When Weibull exponent m is identified by fitting the standard deviation of σ_N for specimens of very different sizes, very different m values are obtained. Also, the size effect data and the standard deviation data give very different m (e.g., $m = 12$ was obtained with small concrete specimens while the large-size asymptotic behavior corresponds to $m = 24$ (Bažant and Novák 1990); Fig. 2f (m varies from 4.2 to 24.2).

7. The classical theory neglects spatial correlations of material failure probabilities (which is admissible only if the structure is far larger than the autocorrelation length of the random field of local material strength).

5 WEIBULL THEORY ENHANCEMENTS AND NONLOCAL GENERALIZATION

One can discern three types of generalization of Weibull theory capturing in various ways, and to various degrees, the effect of a large FPZ and quasibrittleness.

1. Various phenomenological models for load sharing (parallel couplings), which began to appear in the mid 1900s (Daniels 1945, Grigoriu 1990). Although they can simulate some effects of a large FPZ, they are not generally predictive. Calibrating the model for one structure geometry, one cannot predict the behavior for another geometry.

2. Weibull theory adaptations to LEFM crack-tip singularity, which causes the classical Weibull integral to diverge for all realistic m values (Beremin 1983, Ritchie, Becker, Lei et al. 1998, Lin, Evans, McClintock, Phoenix 1978, etc.). For example, one excludes from the domain of Weibull integral a finite circular zone about the crack tip, in order to make the integral convergent, or only the stresses at points far enough ahead of the crack tip are considered, or the stress profile is blunted plastically, or the failure probability is averaged spatially. These approaches work well for tough metals with a large (but not very large) yielding zone at the crack tip, but are doubtful when the effective FPZ length is of the same order of magnitude as the structure size (which is typical for reinforced concrete), and are not completely general; e.g., they cannot be used for crack initiation from a smooth surface.

3. Nonlocal Weibull theory (Bažant and Xi 1991, Bažant and Novák 2000a,b). This is a general theory which has as its limit cases *both* the classical Weibull theory and the deterministic nonlocal damage mechanics developed for finite element analysis of quasibrittle materials, which means that the energetic size effect is a limiting case of this theory.

The nonlocal concept was proposed for elasticity in the 1960s (Kröner 1961, Eringen 1965, Kunin, Edelen). In the 1980s it was adopted for strain-softening continuum damage mechanics (Bažant 1984. Bažant et al. 1984, Pijaudier-Cabot and Bažant 1987), with three motivations:

1) to serve as a computational 'trick' (localization limiter) eliminating spurious mesh sensitivity and incorrect convergence of finite element simulations of damage;

2) to reflect the physical causes of nonlocality, which are:

   (a) material heterogeneity,

   (b) energy release due to microcrack formation, and

   (c) microcrack interactions; and

3) to simulate experimentally observed size effects stronger than those explicable by Weibull theory.

Causes 1 and 2 mean that microcrack formation in a heterogeneous material depends mainly on the average deformation of a representative volume of the material surrounding the microcrack, rather than on the local stress or strain at a point of a macroscopic smoothing continuum.

In keeping with the finding that, in the deterministic nonlocal theory, the spatial averaging must be applied to the inelastic part $\boldsymbol{\epsilon}''$ of the total strain $\boldsymbol{\epsilon}$ (or some of its parameters), the cumulative failure probability $P_1(\boldsymbol{\sigma})$ as a function of the local stress tensor $\boldsymbol{\sigma}$ at continuum point $\boldsymbol{x}$ is replaced in the nonlocal Weibull theory with

$$P_1 = \langle \bar{\boldsymbol{\sigma}}/s_0 \rangle^m, \quad \bar{\boldsymbol{\sigma}}(\boldsymbol{x}) = \boldsymbol{E} : [\boldsymbol{\epsilon}(\boldsymbol{x}) + \bar{\boldsymbol{\epsilon}}''(\boldsymbol{x})], \quad \bar{\boldsymbol{\epsilon}}(\boldsymbol{x}) = \int_V \alpha(\boldsymbol{s}-\boldsymbol{x})\boldsymbol{\epsilon}''(\boldsymbol{s})\mathrm{d}V(\boldsymbol{s})/\bar{\alpha}(\boldsymbol{x}) \tag{9}$$

in which $\boldsymbol{E}$ = initial elastic moduli tensor; $\alpha(\boldsymbol{x})$ = given bell-shaped weight function whose effective spread is characterized by characteristic (material) length l_0; and $\bar{\alpha}(\boldsymbol{x})$ = normalizing factor of $\alpha(\boldsymbol{x})$. The nonlocality makes the Weibull integral over a body with a sharp crack convergent for any Weibull modulus m, and it also introduces spatial correlation into the Weibull theory. Numerical calculations of bodies with large cracks or notches showed that the randomness of material strength is almost irrelevant for the size effect on the mean σ_N, except theoretically for structures extrapolated to sizes less then the inhomogeneity size in the material (Bažant and Xi 1991). So, the energetic mean size effect law (3) for the case of large cracks or large notches remains unaffected by material randomness. Intuitively, the reason is that a significant contribution to Weibull integral comes only from the FPZ, but the size of the FPZ at a crack tip is about the same regardless of the structure size. This also applies to the boundary layer of cracking, Fig. 2a, and is documented by the inelastic strain field in (Fig. 2c left, linear scale) and the field of density of contribution to the Weibull integral (right, log-scale) obtained by Bažant and Novák (2000a) in nonlocal beam flexure analysis.

However, the size effect law for failures at crack initiation from a smooth surface (Eqs. 3 and 4 in the paper by Bažant and Novák, 2001, in this volume) is affected by randomness for the case of very large sizes, the effect becoming important for plain concrete beams or plates of thickness ≥1m, and major for 10 m (arch dams); Bažant and Novák 2000a,b; Fig. 2d,e). The standard deviation of σ_N becomes size dependent. Furthermore, for very large plain concrete beams, the material randomness becomes more important when the maximum bending moment acts within a longer segment of the beam (e.g., 4-point versus 3-point loading of a beam). The asymptotic limits of the nonlocal Weibull theory are the deterministic energetic size effect for $D \to 0$ and the Weibull statistical size effect for $D \to \infty$. Their asymptotic matching approximation leads to the following approximate formula for the mean size effect(Bažant 2001, Fig. g,d,e):

$$\sigma_N = \sigma_0 \left[\left(\frac{D_b}{\eta D_b + D} \right)^{rn/m} + \frac{rD_b}{\eta D_b + D} \right]^{1/r} \qquad (rn/m < 1) \tag{10}$$

where η and r are empirical constants. The special case for $\eta = 0$ was shown to fit the bulk of the existing test data on the modulus of rupture and closely agree with numerical predictions of the nonlocal Weibull theory over the size range 1:1000 (Bažant and Novák 2000b). Aside from the two aforementioned asymptotic limits, the formula also satisfies a third asymptotic—namely that the deterministic size effect on the modulus of rupture must be recovered for $m \to \infty$.

During the quarter-century since the work of Zech and Wittmann in 1977, the value $m = 12$ has been generally accepted for the Weibull modulus of concrete. However, the studies of size effect in quasibrittle materials with the nonlocal Weibull theory revealed that this is merely an apparent value characterizing the standard deviation of σ_N for small laboratory specimens. When the specimen size increases, the m-value corresponding to the standard deviation increases as well (Fig. 2f).

The fact that the Weibull modulus of the classical (local) theory is not a constant is one reason why this theory does not apply. However, in the context of the nonlocal Weibull theory and by data fitting with the statistical energetic formula, the large-size asymptotic value of Weibull modulus of a quasibrittle material is in fact found to be a constant—about $m = 24$ for concrete (Bažant and Novák 2000a,b); Fig. 2f. This, of course, could have been expected for physical reasons: When the structure size is far larger (say, 100× larger) than the FPZ size, a quasibrittle material becomes a perfectly brittle material, following LEFM (thus, e.g., the global fracture of a concrete dam must follow LEFM). Conversely, a perfectly brittle material such as a fine-grained ceramic must doubtless become quasibrittle when the structure is small enough, as for instance in MEMS (micro-electro-mechanical systems).

6 NEED FOR IMPROVING THE STOCHASTIC FINITE ELEMENT METHOD (SFEM)

The SFEM has become a powerful tool for calculating the statistics of deflections and stresses of arbitrary structures (e.g., Schuëller 1997a,b, Kleiber and Hien 1992, Ghanem and Spanos 1991, Liu et al. 1987, Deodatis and Shinozuka 1991, Shinozuka and Deodatis 1988, Takada 1990). Compared to SFEM, the nonlocal Weibull theory has two limitations:

- It does not yield the statistics of stiffness, deflections and stresses during the loading process.
- The failure probability should be related to the probability that the first eigenvalue λ_1 of the tangential stiffness matrix of the structure, K_t, becomes nonpositive, which is not the case for the Weibull theory, whether local or nonlocal.

However, the nonlocal Weibull theory offers three significant advantages over SFEMs:

1. It is simpler, since an autocorrelated random field is not needed (a certain kind of spatial correlation is implied by the characteristic length of the nonlocal averaging operator).
2. In contrast to autocorrelation length in SFEMs, the nonlocal characteristic length ℓ has a clear physical meaning and can be easily evaluated from the size effect tests using simple LEFM-based formulae (ℓ = LEFM shape factor times the size D_0 obtained as the intersection of the asymptotes of an optimally matched size effect law).
3. In the limit of infinite size, the nonlocal Weibull theory reduces to the classical (local) Weibull theory, while the same cannot be said about SFEMs (in their contemporary form).

The last point is of a rather fundamental nature. It relates to the far-out tail of the probability distribution of the tangential stiffness. In this regard, the following physical argument should be noted:

When the structure size is scaled up to infinity ($D \to \infty$), the FPZ becomes infinitely small compared to the structure size D (i.e., a point in the dimensionless coordinates $\boldsymbol{\xi} = \boldsymbol{x}/D$). In that case, failure (of a structure of positive geometry) must occur at fracture initiation. Therefore, the classical (local) Weibull theory must apply, and the failure then depends *only on the far-off tail* of the local strength distribution. Thus, extrapolation to very large sizes is a way to *identify the far-off tail* of the local strength distribution (it can be seen as a physical substitute for the importance sampling).

For the existing SFEMs, however, it has not been demonstrated that they would converge to Weibull theory and reproduce the Weibull size effect as $D \to \infty$, neither analytically nor computationally. Whether this requirement is satisfied by the existing SFEMs is doubtful.

This has some implications for the structure of the tail of the probability distribution of the stiffness coefficients, deflections and stresses. Normally, the Gaussian or log-normal distributions for the material stiffness characteristics are assumed in SFEM. Since the probability distribution of the structural tangent stiffness matrix is essentially a weighted sum of the elemental distributions (i.e., the distributions of the stiffness characteristics of a small representative volume of the material), the distribution of the structural stiffness coefficients may be expected to be Gaussian, with the exception of

(1) the far-out tail of probability distribution, and

(2) the states of damage localization which may (though need not) occur just before reaching the peak load and are characterized by the fact that the stiffness of one or several finite elements in a small failing zone totally dominates the structural stiffness.

The connection between the structural stiffness and failure rests on stability analysis (e.g. Bažant and Cedolin 1991, ch. 4, 10 and 13). During the loading process, the maximum load (which represents a failure state under the conditions of load control), is reached when the tangential stiffness matrix of the structure, $\boldsymbol{K}_t$, ceases being positive definite, i.e., when the first eigenvalue λ_1 of $\boldsymbol{K}_t$ ceases being positive. Therefore,

$$\textsf{Failure probability}(u) \;=\; \mathrm{Prob}\,(\,\lambda_1(u) \le 0\,) \tag{11}$$

where it is indicated that the failure probability and the first eigenvalue may be regarded as functions of the displacement u (since, in order to achieve computational stability, u rather than P needs to be controlled during loading because a smoothly evolving matrix $\boldsymbol{K}_t$ is singular at maximum load). (For the sake of simplicity, we omit one further condition needed to distinguish the maximum load state from a bifurcation state, which can be stable or unstable and is also characterized by a loss of positive definiteness of $\boldsymbol{K}_t$.)

Based on the knowledge of the possible limiting forms of extreme value distributions, it appears reasonable to impose on the SFEM the following condition (which is strictly required only for a very large chain structure but may in practice be convenient to apply systematically for any structure of any size):

$$\textsf{Tail of}[\,\mathrm{Prob}\,(\,\lambda_1(u) \le 0\,)] \;=\; F_W[P(u)] \tag{12}$$

Here $F_W(P)$ is the cumulative Weibull distribution function (Weibull 1939), which has a power-law tail with a threshold (the Weibull distribution would better be called the Fisher-Tippett-Weibull distribution because, in mathematics, it was derived by Fisher and Tippett already in 1928); F_W is here properly considered as an implicit function of the controlled displacement u because the tangential stiffness changes its sign at maximum load. Condition (12) ensues by excluding all the other possibilities, which are as follows.

If we consider a population of N statistically independent random variables X_i ($i = 1, 2, ...N$) with arbitrary but identical statistical distributions $\mathrm{Prob}(X_i \le x) = P_1(x)$, henceforth called the elemental distribution ($x = \sigma/s_0$ = scaled stress, X_i = scaled random strength), we have for the distribution of $Y_N = \min_{i=1}^N X_i$ for very large N the general expression:

$$P_N(y) = 1 - e^{-NP_1(y)} \quad \text{where} \quad P_N(y) = \mathrm{Prob}(\,\min_{i=1}^N X_i \;\le\; y\,) \tag{13}$$

where $P_N(y) = P_f$ = failure probability of structure, provided that the failure of one element causes the whole structure to fail. As Fisher and Tippett (1928) proved, there exist three and

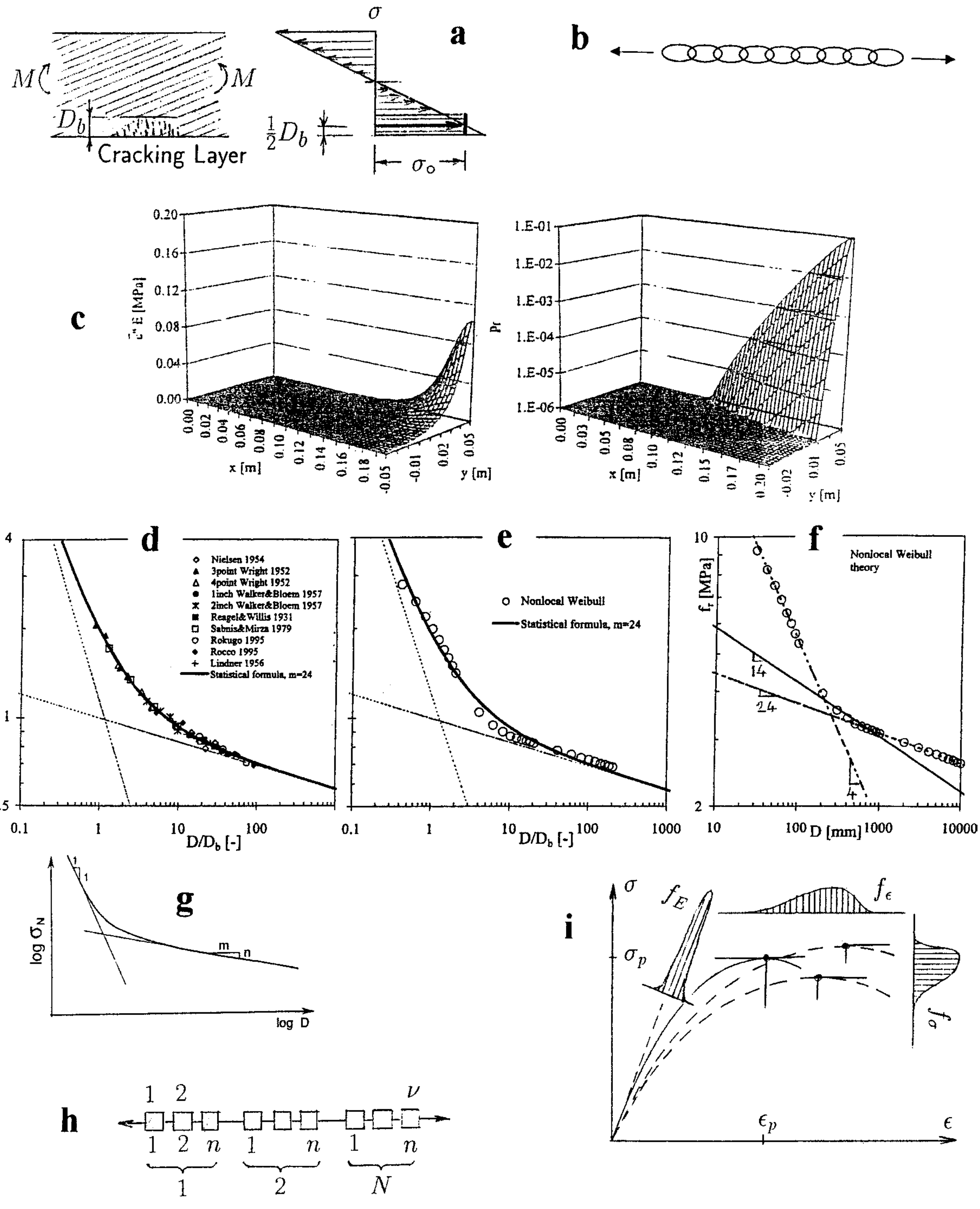

Figure 2. (a) Approximate effect of stress redistritution due to cracking in boundary layer. (b) Chain structure. (c) Field of inelastic strain in beam flexure (linear scale) and corresponding density field of contributions to failure probability (log-scale). (d,e) Relative flexural strength versus relative size for 10 test series from the literature, and results of nonlocal Weibull calculations, fit by energetic-statistical formula for crack initiation. (f) Differences in apparent Weibull moduli m in classical definition corresponding to nonlocal Weibull calculations in different size ranges. (g) Energetic- statistical size effect for failures at crack initiation. (h) Chain subdivided into segments. (i) Stress-strain formula with three random parameters.

only three asymptotic forms (or limiting forms for $N \to \infty$) of the extreme value distribution $P_N(y)$:

$$\text{1) Weibull distribution:} \qquad P_N(y) = 1 - e^{-y^m} \tag{14}$$

$$\text{2) Fisher-Tippett-Gumbel distribution:} \qquad P_N(y) = 1 - e^{-e^y} \tag{15}$$

$$\text{3) Fréchet distribution:} \qquad P_N(y) = 1 - e^{|y|^{-m}} \tag{16}$$

(Case 2 is usually called the Gumbel distribution, but Fisher and Tippett derived it much earlier and Gumbel gave them credit for it.) Case 1 is obtained if the elemental distribution $P_1(y)$ has a power-law tail with a finite threshold, which is normally taken as 0 (the simplest case is the rectangular probability density function, for which $m = 1$). Case 2 is obtained if $P_1(y)$ has an exponentially decaying tail, and case 3 if $P_1(y)$ has an infinite tail with an inverse power law (such as $|\sigma|^{-m}$) (see also Bouchaud and Potters 2000).

The basis of Fisher and Tippett's (1928) ingenious proof can be dissected into three arguments:

(1) The extreme of a sample of $\nu = Nn$ independent identical random variables x (the strengths of the individual links of a chain) can be regarded as the extreme of the set of N extremes of the subsets of n variables (segments of the chain, with n links in each, Fig. 2h). This recursive property is the key idea.

(2) The asymptotic forms of the distributions of the extremes of samples of sizes n and Nn must be the same if an asymptotic form exists (both $n \to \infty$ and $N \to \infty$).

(3) When the size of the samples is increased from n to Nn, the mean and standard deviation of the distribution of the extremes must, in general, change. Thus, an asymptotic distribution of the extremes, as a limit for $N \to \infty$, cannot exist. What may exist is an asymptotic *form* (or shape) of the extreme value distribution. The forms of the distributions of samples of sizes N and n are identical if the former can be transformed into the latter by a linear transformation of variables, $\sigma' = a_N \sigma + b_N$ where a_N and b_N are functions of N ($N \sim$ structure size) (note that this transformation is equivalent to changing the mean and standard deviation of the distribution, but not any higher moments). Thus (if we prefer the terminology of the weakest link model), the argument of joint probability of survival of all N segments of the chain yields for the asymptotic form of the cumulative distribution of the survival probability $F(\sigma) = 1 - P_f = 1 - P_N$ of a very long chain the functional equation:

$$F^N(\sigma) = F(a_N \sigma + b_N) \tag{17}$$

In this manner, Fisher and Tippett reduce the problem to finding the solution $F(\sigma)$ of this equation. They prove that there exist three and only three forms of solution—those in (14)–(16). By substituting these forms into functional equation (17), one can check that indeed this equation is satisfied, and the substitutions give the dependence of a_N and b_N on N, which in turn represents the dependence of the mean and the standard deviation of each asymptotic distribution on N ($N \sim$ structure size).

For structural strength, the infinite negative tails of P_N distribution appear unrealistic, for physical as well as conceptual reasons. Consequently, the Fréchet distribution and the Fisher-Tippett-Gumbel distribution must be excluded. So, for the tail distribution, there is no other rational possibility but the Weibull distribution. This lends support to the condition (12). Besides, the Weibull distribution provides overall the best match of the experimental evidence, although most of it is on fatigue fracture. The other two distributions do not fit the test data on structural strength.

There are of course techniques in SFEM, e.g., the importance sampling, to calculate failure loads of extremely small probability. Unfortunately, though, the fact that the failure load calculations with the existing SFEMs do not give probability distributions with a Weibull power-law tail, and that they do not lead to the Weibull power-law size effect when the structure size is scaled up to infinity, implies that calculations of the loads of a very small failure probability, such as 10^{-7}, cannot be realistic.

In design codes, the safety factors relate the mean failure load prediction (roughly the same as the median, or failure probability 0.5) to the failure load with a desired extremely low probability, typically about 10^{-7}. This is illustrated by the upper arc in Fig. 3k, spanning almost 7 orders of magnitude. Since the existing experimental validations of SFEMs have been confined mainly to the standard deviation, the current SFEMs, with their exponential tails, might be realistic for calculating only loads of failure probability no less than about 10^{-2}. So, an empirical safety factor is needed to relate this load to the failure load of the desired probability such as 10^{-7}. This is illustrated by the lower arc in the figure, spanning 5 orders of magnitude. Comparing the lengths of the two arcs, we are sobered with the impression that not too much has yet been gained through the development of SFEMs, as far as structural safety is concerned.

Although a rigorous mathematical formulation based on condition (12) seems in general quite difficult, a semi-intuitive justification may be seen in the following observations:

1. For a chain structure whose elements have a smooth stress-strain curve through the peak stress region (Fig. 2i), λ_1 at states near the maximum load is totally dominated by K_t of one element—the element with the lowest strength (lowest peak of the stress-strain curve), and

2. the same property must approximately hold for non-chain structures of positive geometry, or else the structure would not be failing at crack initiation.

There may be various ways to meet condition (12) and, in view of the foregoing observations, the following proposition represents one:

Proposition I: The random material properties should be defined so that, at least on approach to the strength limit, the distribution of the slope $K_t = \mathrm{d}\sigma/\mathrm{d}\epsilon$ of the random tensile stress-strain curve as a function of the strength limit (peak of the curve) would be a Weibull distribution or, more generally, a distribution with a power-law tail and a non-negative threshold (Fig. 2i).

Even if the Weibull distribution is adopted for the material tangent stiffness at any load, the deflections and stresses at loads not close to the failure load would have Gaussian distributions (except the far-out tails), by virtue of the central limit theorem. To control the variance of the Gaussian distribution, Weibull parameters s_0 and m could be varied as a function of the current stress, strain or damage in the material element.

Consider now the simple example of a chain of elements (links) $i = 1, 2, ...N$. Although the stress-strain curve of the elements is generally a Markovian process, one may restrict consideration to a family of curves with random parameters. For instance, as the simplest expression, consider the curves of the type introduced in Bažant and Chern (1985):

$$\sigma = E\,\epsilon\, e^{-\epsilon^n/n\epsilon_p^n}, \qquad 1/n = \ln(E\epsilon_p/\sigma_p) \tag{18}$$

where E = Young's modulus and $n > 0$. One can check that σ_p = peak stress (maximum stress, strength of element) and ϵ_p = strain at peak stress (Fig. 2i). In general, σ_p, ϵ_p and E may be taken as independent random variables, characterized by the cumulative distributions

$$\mathrm{Prob}(\sigma_p < y) = F_\sigma(y), \qquad \mathrm{Prob}(\epsilon_p < z) = F_\epsilon(z), \qquad \mathrm{Prob}(E < \eta) = F_E(\eta) \tag{19}$$

According to the foregoing arguments, the distribution $F_\sigma(y)$ must have a Weibull (power-law) tail. The distributions $F_\epsilon(z)$ and $F_E(\eta)$ can be arbitrary but must be such that always $n > 0$, which requires that $\sigma_p < E\epsilon_p$ for any realization. This inequality is easy to meet if both $F_\epsilon(z)$ and $F_E(\eta)$ are bounded from below (e.g., if they are both taken as Weibull), while the distribution $F_\sigma(y)$ must be bounded from above, in addition to being bounded from below. Thus, $F_\sigma(y)$ must rise from a power-law tail but, strictly speaking, cannot be the Weibull distribution through the whole range, although in practice it can be Weibull if the probability of exceeding $E\epsilon_p$ is negligible, say 10^{-9} (this is easy to satisfy, especially for high m, because the Weibull probability density distribution has on the right a rapidly decaying exponential tail; Fig. 3a, right). Intuitively selecting the distributions of ϵ_p and σ_p, one could not expect to satisfy these requirements. A trivial satisfactory case, albeit not very realistic, occurs when ϵ_p and E are deterministic, in which case all the random stress-strain curves are affine.

7 CRACK FRACTALITY AS A CAUSE OF SIZE EFFECT?

The random scatter observed in fracture experiments is doubtless largely controlled by the random disorder on the microstructural level. Aside from various statistical descriptions, much of this disorder can be described by fractal concepts. The crack surface irregularity in concrete and other quasibrittle materials can be characterized as a self-affine invasive fractal surface with a fractal dimension $d_f > 1$ (Fig. 3f), and the microcrack distributions as a lacunar fractal set (or Cantor set) with $D_f < 1$ (Fig. 3g).

Carpinteri (1994a,b) proposed an intriguing idea—that the size effect on σ_N might be caused by crack fractality. But they offered merely intuitive geometric arguments. While providing one viable alternative way to describe the effect of random disorder in the microstructure on the material fracture characteristics, their idea, however, appears invalid in terms of a direct influence on the structural size effect.

The idea was examined by mechanical analysis in Bažant (1997) under the hypothesis that standard (nonfractal) continuum mechanics and the laws of continuum mechanics apply on the global scale of structure. If the crack surface has a fractal dimension, the fracture energy must have a fractal dimension, too, as proposed by Mosolov and Borodich. The use of this concept in an energy-based fracture analysis showed that the predictions do not agree with reality. Patently unrealistic size effects were shown to result from the hypothesis of crack surface fractality for large cracks as well as crack initiation (Fig. 3h,i) and, based on a recursive ad infinitum argument similar to Fisher-Tippett's, a physically meaningful form of the hypothesis of lacunar fractality of the microcrack distribution was shown to be mathematically equivalent to the classical Weibull size effect theory.

8 RANDOM SCATTER IN FRACTURE TESTING

Statistical analysis of quasibrittle fracture requires at least the values of standard deviation of the fracture energy, G_f, and the effective fracture process zone length, c_f. It is appealing to take advantage in this regard of the enormous statistical basis (238 tests) that exists in the literature. The existing test data pertain basically to two types of testing method:

(I) The size effect method (SEM) (RILEM 1990), with two other methods which test in the maximum load range and give similar results [Jenq and Shah's (1985) method, similar to Wells (1961) and Cottrell's (1963) method for metals, and Nallathambi and Karihaloo's (1986) method], and

(II) the work-of-fracture method based on the area under the measured load-deflection curve of a notched specimen, as proposed by Hillerborg for his fictitious crack model (Hillerborg et al. 1976, Bažant and Planas 1998).

The differences among the test results involve not only random scatter but also large systematic differences between the concretes tested in different laboratories. To allow statistical comparisons, the deterministic trends must be extracted first, at least approximately. By fitting the results of 238 test series compiled from the literature, the following approximate, admittedly very crude, formulae for the deterministic (mean) trend were obtained by least-square fitting (Bažant and Becq-Giraudon 2001):

$$G_f = \alpha_0 \left(f_c'/0.051\right)^{0.46} [1 + (d_a/11.27)]^{0.22} (w/c)^{-0.30}, \qquad G_F \approx 2.5 G_f \tag{20}$$

$$\ln c_f = \gamma_0 \left(f_c'/0.022\right)^{-0.019} [1 + (d_a/15.05)]^{0.72} (w/c)^{0.2} \qquad (\omega_{c_f} = 47.6\%) \tag{21}$$

where G_f fracture energy of concrete in N/m obtained by the size effect method and other type I methods, f'_c = compression strength in MPa, d_a = maximum aggregate size in mm, w/c = water-cement ratio in the mix, c_f effective fracture process zone length in mm, and parameter α_0 distinguishes river and crushed aggregates. Knowing also the scatter of the elastic modulus E, one can further estimate the coefficients of variation of δ_{CMOD} and K_c.

Fig. 3d,e shows the plots of measured versus predicted values of fracture energy, for the size effect method (and related methods) of type I, and for Hillerborg's work-of-fracture method. The coefficients of variation of the vertical deviations from the straight line of slope 1 in Fig. 3d,e are

$$\omega_{Gf} = 17.8\ \% \quad (77 \text{ tests}), \qquad \omega_{GF} = 29.9\ \% = 1.66\ \omega_{Gf} \quad (161 \text{ tests}) \tag{22}$$

where subscripts G_f and G_F refer to the size effect method (with the related peak load methods) and the work-of-fracture method. Fig. 3c shows the histogram of 77 test data for the size effect method and related methods, plotted on the Weibull probability paper (the plots for the Gaussian and log-normal distributions deviate from a straight line significantly more, Bažant and Becq-Giraudon 2001).

Based on the researches of Planas and Elices (Bažant and Planas 1998) and on further mathematical justification in Bažant (2001a), the common fitting of the type I and type II data was based on the assumption that the ratio of the corresponding fracture energies is $G_F/G_f \approx 2.5$ (G_F corresponds to the area under the entire stress-displacement curve of cohesive crack, while G_f corresponds to the area under the initial tangent of that curve, Fig. 3j; the G_F values in Fig. 3e are converted to the G_f values according to this ratio, to facilitate comparison, with no effect on ω_{GF}).

The large discrepancy between the coefficients of variation in Eq. (22) (Fig. 3d,e) reveals that the size effect method gives a much smaller random scatter than the work-of-fracture method. This important advantage should be noted in the current debates of a standardized test. Further note that calculations of the maximum loads of structures require only the knowledge of the initial portion (slope) of the softening stress-displacement curve of the cohesive crack model (Fig. 3k). The tail of this curve is needed to calculate the post-peak softening behavior of structures. Thus it makes little sense to conduct the work-of-fracture tests and then use the result for calculating the maximum load, the most frequent objective in practice. By adjusting the cohesive stress-displacement curve of the assumed shape (Fig. 3k) to the measured G_F, one in fact imparts the high coefficient of variation also on the initial slope of that curve, and the result is that the fracture energy that matters for the maximum load now has an unnecessarily high coefficient of variation, 29.9% instead of 17.8%.

It might be objected that the statistical comparison in Fig. 3d,e is contaminated by errors in the formula used to capture the mean effects of the differences in concrete composition. True, but there is no reason for such errors to favor one type of test. To obtain more reliable comparisons, all the 238 data sets should ideally be obtained for one and the same concrete, under the same laboratory conditions. Not only would then the comparison of testing methods be shielded from criticism, but a significantly smaller coefficient of variation could be expected. However, inferring the standard deviations for other concretes would then become more uncertain.

The physical reason for G_f having a much smaller scatter than G_F is that the tail of the softening stress-displacement curve exhibits much more random scatter than its initial portion, which is a property well known from testing.

The high value of ω_{cf} means that a distribution with a threshold, e.g. the Weibull or log-normal distribution, must be assumed for c_f. Note that the results of structural analysis as well as testing are much less sensitive to c_f than G_f, and only the order of magnitude of c_f is important.

9 REINTERPRETATION OF PAST STRUCTURAL DISASTERS

Because of very large safety factors as well as a size effect hidden in the safety factor for dead load, a single inadequacy of the current design methods and codes can hardly bring

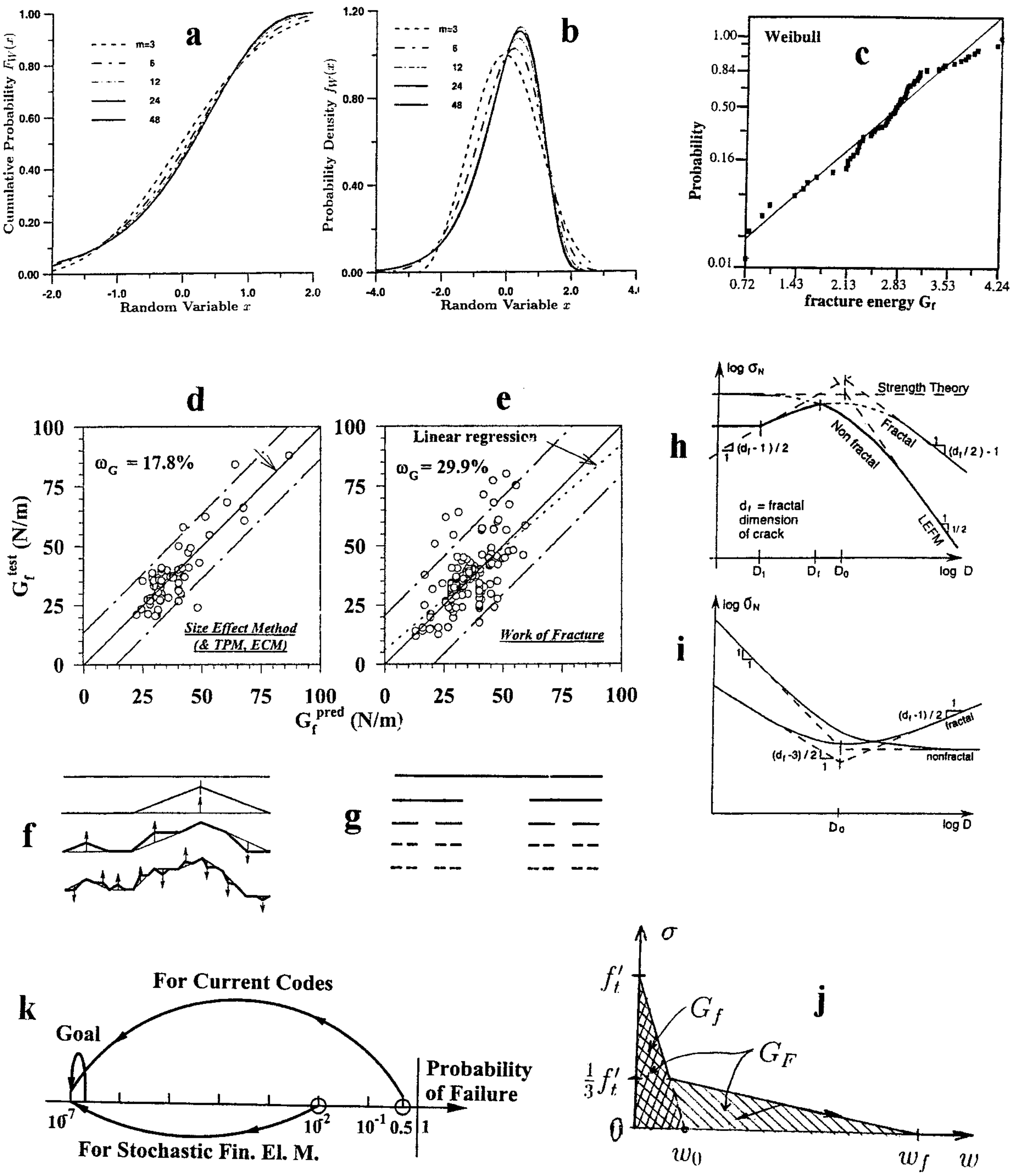

Figure 3. (a,b) Cumulative Weibull distribution and its density, for various Weibull moduli m. (c) 77 size effect test data on fracture energy from the literature, plotted on Weibull probability paper. (d,e) Plots of measured versus predicted values of size effect fracture energy (77 test series) and Hillerborg fracture energy (161 test series). (f,g) Invasive affine fractality of crack surface and lacunar fractality of microcracks. (h,i) Unreasonable size effects for large cracks and crack initiation ensuing from the hypothesis of fractal size effect. (j) Softening stress-crack separation curve of cohesive crack model of concrete. (k) Safety factors relating failure probability to calculations.

down a structure. Thus it is not surprising that in most structural failures, several causes are combined. Practically all the famous catastrophes of structures have in the past been plausibly explained without revoking the size effect. However, it now transpires that many of these explanations have been incomplete. In the light of the latest research, the fracture size effect should, for example, be added as a significant contributing factor (Bažant and Novák 2000b) to the explanations of the following catastrophes:

1) While the direct cause of the tragic failure of the Malpasset arch dam in French Maritime Alps in 1959 was an excessive movement of the rock abutment, it now transpires that the maximum tolerable movement must have been only about 45% of the value deduced at that time from the standard strength tests of concrete.

2) The same, but with a reduction to 40%, applies to the Saint-Francis Dam (failed in 1928).

3) Whereas the direct cause of the failure of the Schoharie Bridge on New York Thruway in 1987 was river bed scouring in a flood, the nominal bending strength of the concrete foundation plinth that broke was only about 54% of the value deduced from the standard tensile strength tests.

Based on today's knowledge of size effect for large fractures, nominal strength reductions ranging from 30% to 50% compared to code-based design must have also occurred in the following catastrophes:

4) The sinking of the Sleipner oil platform in Norway in 1991, caused by shear fracture of a tri-cell (an incorrect placement of reinforcement and an error in finite element analysis due to incorrect meshing were the originally cited causes).

5) The columns of Hanshin viaduct, Kobe (failed in 1995 earthquake).

6) The columns of Cypress Viaduct, Oakland (failed in 1989 earthquake).

7) The bridge columns in Los Angeles earthquake (1994).

Insufficient confining reinforcement was, of course, the primary cause of the last three failures, as originally cited in the reports on these disasters.

10 SIZE EFFECT HIDDEN IN EXCESSIVE DEAD LOAD FACTOR IN DESIGN CODES

The dead load factors currently used in concrete design codes have recently been criticized by structural engineering statisticians as unjustifiably large. Proposals for reducing these factors drastically have been made. However, such a reduction would be dangerous of if the fracture size effect were not simultaneously incorporated into the code provisions (Bažant and Frangopol 2001).

The larger the structure, the higher is the percentage of the own weight contribution D_1 to the ultimate load U. So, if the load factor for the own weight is excessive, structures of large size are overdesigned from the viewpoint of strength theory or plastic limit state design—the theory underlying the current building codes. However, such an overdesign helps to counteract the neglect of size effect in the current codes, which is inherent to plastic limit analysis concepts (Bažant 2001a). Doubtless it is the reason why the number of structural collapses in which the size effect was a contributing factor has not been much larger than we have seen so far.

Denote by $\hat{L}$ and $\hat{D}$ the internal forces caused by the live load and the dead load, and by U the internal force caused by ultimate loads, i.e., the loads magnified by the load factors. Using the load factors currently prescribed by the building code (ACI Standard 318, 1999), one has $U = 1.4\ \hat{D} + 1.7\ \hat{L}$

Take it now for granted that the dead load factor 1.4 is excessive and that a realistic value, justified by statistics of dead load, should be μ_D. Then the ratio of the required ultimate design value of the internal force to the realistic ultimate value, named the overdesign ratio (Bažant 2001, Bažant and Frangopol 2001) is

$$R = \frac{U_{\text{design}}}{U_{\text{real}}} = \frac{1.4\ \hat{D} + 1.7\ \hat{L}}{\mu_D\ \hat{D} + 1.7\ \hat{L}} \tag{23}$$

Consideration will now be limited to dead loads caused by the own weight of structures, which for example dominate the design of large span bridges. For a bridge of very large span, the dead load may represent 90% of the total load, and the live load 10%. In that case, the overdesign ratio is $R = (1.4 \times 0.9 + 1.7 \times 0.1)/(\mu_D \times 0.9 + 1.7 \times 0.1)$. For the small scale tests which were used to calibrate the present code specifications, the own weight may be assumed to represent less that 2% of the total load. In that case, the overdesign ratio is $R_0 = (0.4 \times 0.02 + 1.7 \times 0.98)/(\mu_D \times 0.02 + 1.7 \times 0.98)$. It seems reasonable to assume that the own weight of a very large structure cannot be underestimated by more than 5%. This means that $\mu_D = 1.05$. So, $R \approx 1.28$, $R_0 \approx 1.00$. It follows that, compared to the reduced scale laboratory tests used to calibrate the code, a structure of a very large span is *overdesigned*, according to the current theory, by about **28**% (Bažant 2001). Such overdesign compensates for a size effect in the ratio of about 1.28. This is approximately the size effect for very large spans that is unintentionally hidden in the current code specifications.

Further note that a hidden size effect also exists in various indices proposed for reliability-based codes. This is due to the fact that the reliability implied in the code increases with the contribution of the dead load to the overall gravity load effect (in detail, see Bažant and Frangopol 2001).

11 PATH TO HAPPIER FUTURE

Hopefully, the present lecture has demonstrated that, if the future should bring a significant improvement in reliability of failure predictions of quasibrittle structures, particularly the concrete structures, fiber composites, sea ice and rock masses, the probabilists and fracture mechanicians must collaborate. In civil engineering, the dead load safety factor cannot be reduced without incorporating the size effect into the concrete design codes, which pertains to the concrete code specifications for all the failure types termed ‘brittle’. Conversely, it makes little sense to introduce size effects into the code without improving its probabilistic structure. Likewise, the results of sophisticated finite element analyses that correctly simulate the mean (deterministic) energetic size effect get absurdly devalued when the current, less than rational, safety factors are applied. To reach a happier future, synergy will be imperative.

12 ACKNOWLEDGMENT

Partial financial support under Grant ONR-N00014-91-J-1109 from Office of Naval Research (monitored by Dr. Yapa D.S. Rajapakse) is gratefully acknowledged. Thanks are due to Drahomír Novák, Associate Professor at the Institute of Structural Mechanics, Technical University Brno, for very valuable discussions, and to Martin Z. Bazant, assistant professor of mathematics at M.I.T., for pointing out to the writer certain implications of the cardinal results of Fisher and Tippett.

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