

*Proceedings of the Workshop on*

# **Limit Equilibrium, Plasticity and Generalized Stress-Strain In Geotechnical Engineering**

McGill University  
May 28-30, 1980

Sponsored by:  
National Science Foundation, U.S.A.  
National Science and Engineering Research Council,  
Canada  
McGill University  
Univ. of Colorado

Raymond K. Yong and Hon-Yim Ko, Workshop  
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Published by the  
American Society of Civil Engineers  
345 East 47th Street  
New York, New York 10017

*Published 1981*

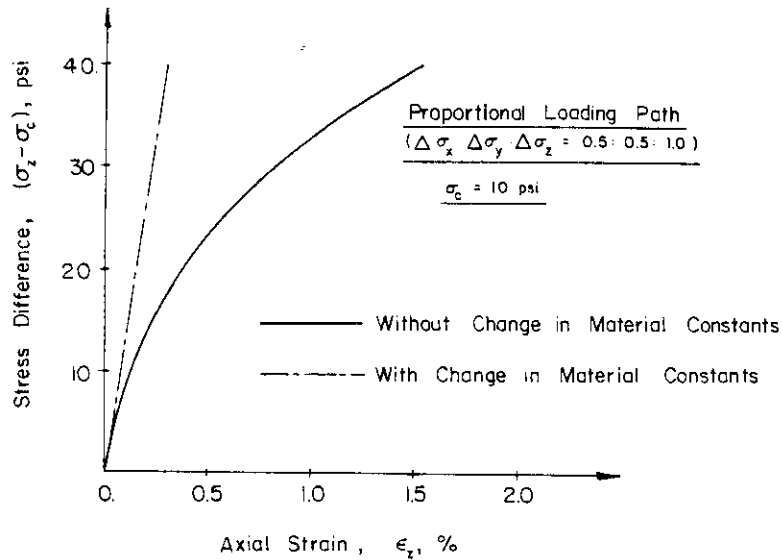


FIG. 10. Effect of Changes in Material Constants on the Model Behavior in Proportional Loading Path for Ottawa Sand

#### CRITICAL APPRAISAL OF ENDOCHRONIC THEORY FOR SOILS

by Zdeněk P. Bažant<sup>1</sup> M.ASCE, Atilla M. Ansal<sup>2</sup>, and  
 Raymond J. Krizek<sup>1</sup> M.ASCE

#### INTRODUCTION

Although endochronic (endo = internal + chronos = time) theory may not follow strictly from the basic principles of continuum mechanics and thermodynamics, this novel approach to modeling the nonlinear constitutive behavior of soils and other similar materials is quite effective and versatile. Therefore, it is worthwhile to examine closely the capabilities and limitations of this theory and to determine its relationship to other established constitutive theories with emphasis on the advantages it offers. Accordingly, the basic features of the theory will be summarized and explained, and, after calibrating the model by use of a limited set of test data, predictions of the response to certain specified types of loading will be presented and critically interpreted in a companion paper included in Section 4 of Volume 1.

#### GENERAL FORM OF THE THEORY

Endochronic theory is best regarded as a special form of viscoplasticity in which the viscosity coefficients depend on the strain rate, as proposed by Schapery (1968). Thus, the general form of the resulting constitutive equation is

$$d\tilde{\epsilon} = \tilde{C} d\tilde{\sigma} + \frac{\partial F}{\partial \tilde{\sigma}} d\tilde{\sigma} \quad (1)$$

in which  $\tilde{\epsilon}$  represents a  $(6 \times 1)$  column matrix of the components of the strain tensor,  $\tilde{\sigma}$  represents a similar column matrix of the stress components,  $\tilde{C} = \tilde{C}(\tilde{\sigma}, \tilde{\epsilon})$  is a  $(6 \times 6)$  incremental elastic stiffness matrix,  $F = F(\tilde{\sigma}, \tilde{\epsilon})$  is the loading function (Figure 1a), and  $\zeta$  is a non-decreasing independent scalar variable, termed the intrinsic time (Valanis, 1971), that depends on the

<sup>1</sup> Professor of Civil Engineering, Northwestern University, Evanston, Illinois, USA

<sup>2</sup> Assistant Professor of Geotechnical Engineering, Macka Civil Engineering Faculty, Istanbul Technical University, Istanbul, Turkey

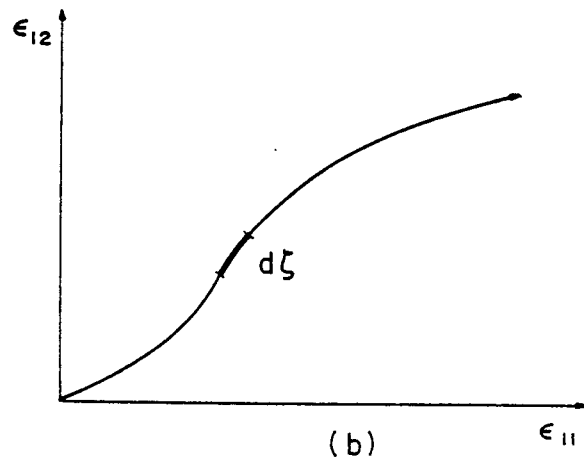
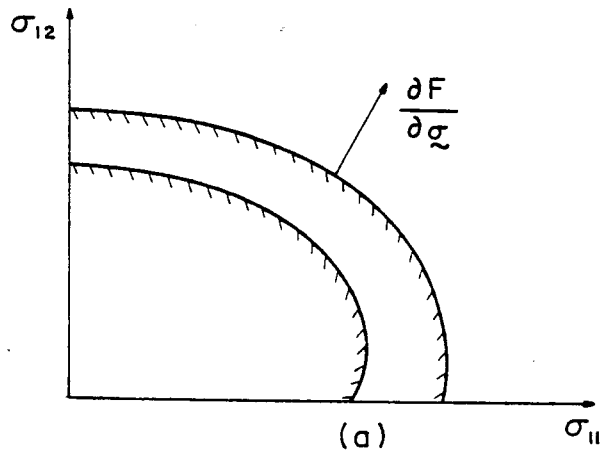


Figure 1

deformation. It is the use of this latter variable which distinguishes endochronic theory from other constitutive theories, such as incremental plasticity or hypoelasticity. The intrinsic time may be defined as the length of the path (Figure 1b) traced by successive states of the material in strain space and it may be expressed as

$$d\zeta = \sqrt{d\epsilon^T \underline{p} d\epsilon} \quad (2)$$

in which  $\underline{p}$  represents a (6 x 6) matrix of the coefficients characterizing the proper metric of the strain space and the superscript T indicates a transpose of the matrix. In applications to soils containing pore water, the stress  $\underline{\sigma}$  must be interpreted in accordance with a Biot-type two-phase medium model and may be associated with the effective stress commonly used in soil mechanics.

The forms of the metric matrix,  $\underline{p}$ , stiffness matrix,  $\underline{c}$ , and loading function,  $F$ , may be simplified considerably by evoking the conditions of initial isotropy of the material. In this case, the function  $F$  must be of the form:

$$F = F(I_1^\sigma, J_2^\sigma, J_3^\sigma, I_1^\epsilon, J_2^\epsilon, J_3^\epsilon) \quad (3)$$

in which  $I_1$ ,  $J_2$ , and  $J_3$  are the first invariant, second deviator invariant, and third deviator invariant, respectively, of either the stress tensor or the strain tensor, as indicated by superscripts  $\sigma$  and  $\epsilon$ . In all practical forms of endochronic theory applied thus far to soils and concrete, the dependence of  $F$  on the third invariants,  $J_3^\sigma$  and  $J_3^\epsilon$ , has not been considered; hence, equation (3) simplifies to

$$F = F(I_1^\sigma, J_2^\sigma, I_1^\epsilon, J_2^\epsilon) \quad (4)$$

Despite omission of the third invariants, the failure envelope obtained from the peak points of the response curves for simulated triaxial tests with proportional loading at various stress ratios manifests the form illustrated in Figure 2. Thus, a rounded triangular shape of the failure surface in the octahedral plane does not necessarily indicate an influence of the third invariants. Such a shape of the loading surface may be interpreted as an indication of the simultaneous influence of the first and second invariants of both stress and strain. Indeed, when  $F$  depends only on the first and second stress

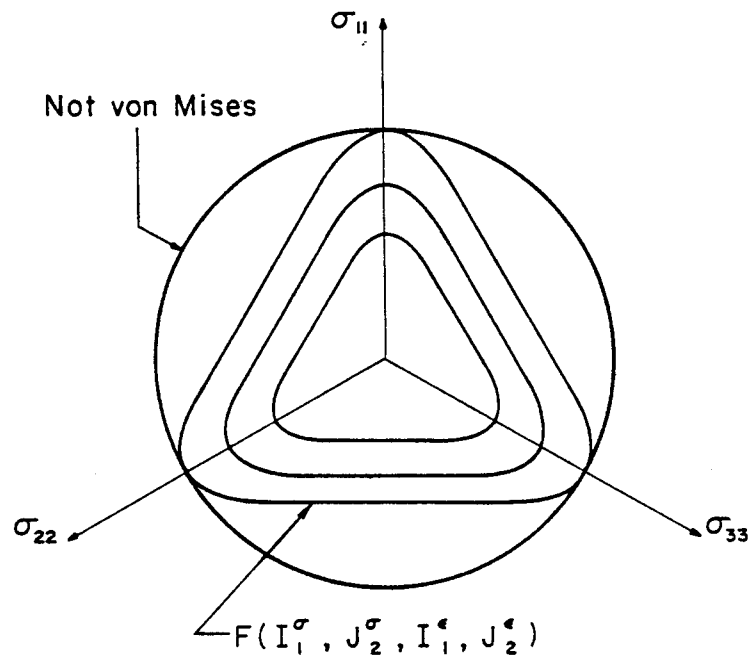


Figure 2

invariants, the projection of the failure surface in the octahedral plane is a circle, but it cannot remain a circle when  $F$  depends also on the first and second strain invariants because the strain components are generally not proportional to the stress components.

One important aspect worth noting is the fact that endochronic theory can not be brought to the incrementally linear form:

$$d\tilde{\sigma} = \tilde{D}(\tilde{\sigma}, \tilde{\epsilon}) d\tilde{\epsilon} \quad (5)$$

in which  $\tilde{D}$  is the incremental stiffness matrix for the total (elastic plus inelastic) strains. This simple fact distinguishes endochronic theory from most other constitutive theories, such as hypoelasticity, incremental hardening plasticity, plastic-fracturing theory, and total strain theory. However, if loading directions in the vicinity of a certain fixed direction are restricted, it is possible to linearize endochronic theory and obtain equation (5). In such a case a different incremental stiffness matrix,  $\tilde{D}$ , is obtained for each choice of the straining direction in whose vicinity the theory is to be linearized.

#### TREATMENT OF UNLOADING AND RELOADING

Much of the effectiveness and flexibility of endochronic theory is due to the fact that it can model the irreversibility associated with unloading without the use of any inequalities. The feature which renders this possible is the use of the square root in a quadratic form in the definition of intrinsic time (equation 2). In the case of shear straining, this expression reduces to  $|d\gamma|$  and the increment of shear stress,  $d\tau$ , may be expressed as

$$d\tau = Gd\gamma - \tau F_1 |d\gamma| \quad (6)$$

in which  $\gamma$  is the shear angle,  $G$  is the shear modulus, and  $F_1$  is a certain function of stress and strain. If loading ( $d\gamma > 0$ ) changes to unloading ( $d\gamma < 0$ ), the first (elastic) term of equation (6) changes sign while the second (inelastic) part of the stress increment retains the same sign, as indicated by the vertical downward arrows in Figure 3a. This illustrates visually that the unloading slope given by endochronic theory must be smaller than the previous loading slope. In the case of alternating loads, this property enables endochronic theory to describe hysteresis loops without any use of an unloading criterion (Figure 3b).

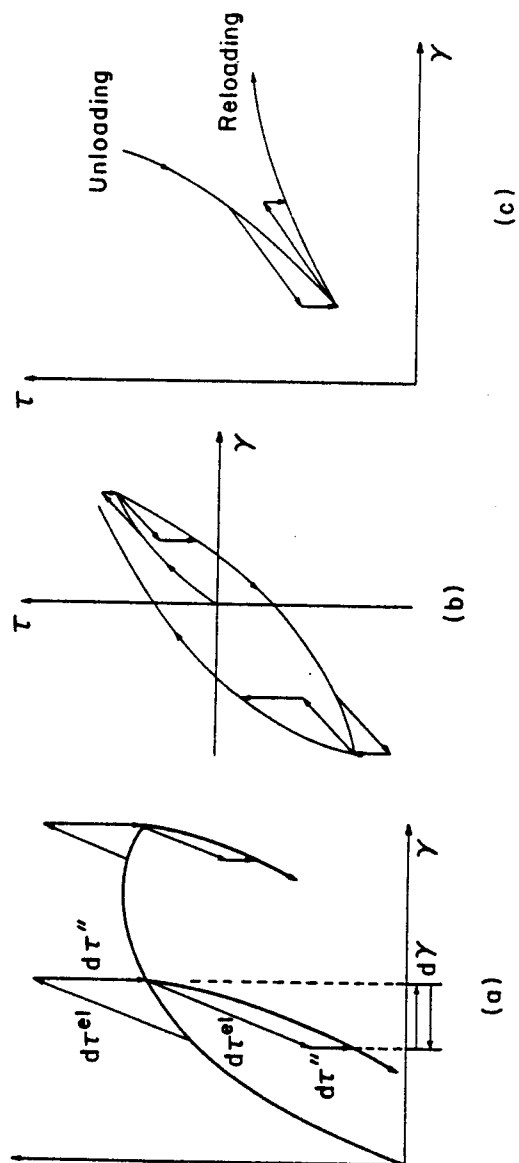


Figure 3

The foregoing attractive property, however, works to the disadvantage of endochronic theory when a small unloading is followed by reloading. In this case, one obtains a reloading slope that is smaller than the previous unloading slope and the response does not form a closed loop (Figure 3c). Although this is not strictly unacceptable in all situations, a reloading slope that is steeper than the unloading slope should be obtained in most cases (for example, pure deviatoric straining). The only way to achieve such behavior with endochronic theory is to avoid relying on the intrinsic time alone to model the irreversibility phenomenon and to introduce an unloading criterion.

One method to obtain closed hysteresis loops for small unload-reload cycles and assure fulfillment of Drucker's postulate was described by Bažant (1978) and applied by Bažant, Krizek, and Shieh (1980). This method consists of two relatively simple corrections. First, the intrinsic time increment,  $d\tau$ , must be replaced by  $c d\tau$ , in which  $c$  is a correction coefficient which is taken as unity for virgin loading and less than unity for unloading and reloading. The fulfillment of a certain condition on the values of  $c$  for unloading, reloading, and virgin loading is also necessary to satisfy Drucker's stability postulate (Bažant, 1978). Furthermore, a three-way loading-unloading-reloading criterion is needed (Figure 4a). Unloading is characterized by  $dW < 0$ , in which  $W$  is some loading function (for example, the work stored in the material), and virgin loading, as well as reloading, are characterized by  $dW > 0$ . The distinction between the latter two conditions may be made on the basis of the maximum energy,  $W_0$ , stored in the material up to the current time. If  $W_0$  is larger than the current  $W$ , we have reloading, and, if  $W_0$  is equal to  $W$ , we have virgin loading. The second correction which must be introduced in endochronic theory is a certain particular form of kinematic hardening, called jump-kinematic hardening (Bažant, 1978) which involves moving the center of the loading surface to the last extreme stress point whenever loading changes to unloading or unloading to reloading. The points to which the center of the loading surface is "jumped" are indicated in the stress-strain diagram in Figure 4b, and in the stress-space plots of the subsequent loading surfaces in Figures 4c, 4d, and 4e. With the use of kinematic hardening it is impossible for the current state point to move inward from the loading surface; only outward movement from the current loading surface can be obtained.

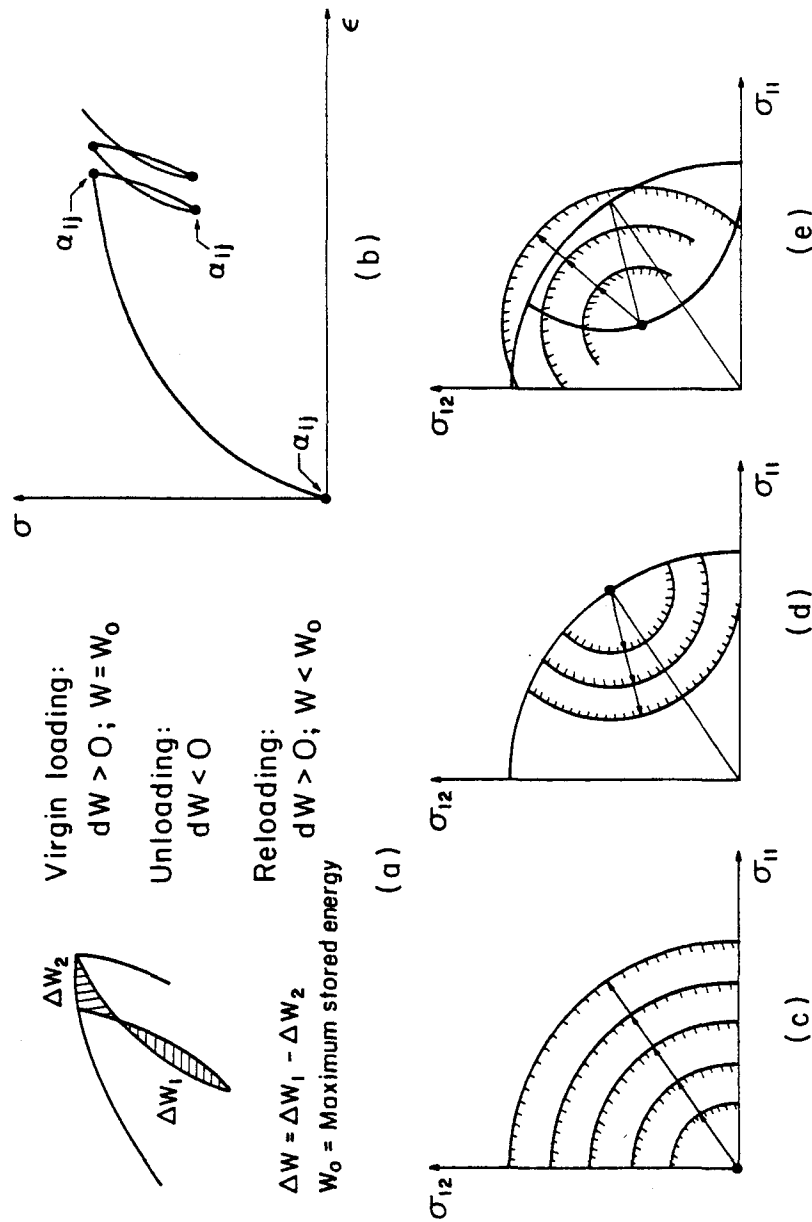


Figure 4

#### ESSENTIAL DIFFERENCES FROM OTHER THEORIES

The most significant difference between various constitutive models, such as plastic, endochronic, and total strain (deformation) models, is the behavior for so-called loading to the side (that is, loading which is parallel to the loading surface and approximately normal to the proportional loading path in stress space). The stiffness of the inelastic response for loading to the side (and generally the stiffness of the response for loading of any direction) may be characterized and instructively visualized by means of the inelastic stiffness locus (Bažant, 1978). This locus is defined as the locus of all strain increment vectors,  $d\epsilon_{ij}$  (that is, a nine-dimensional vector formed of the components of  $d\epsilon_{ij}$ ), that give the same magnitude  $\|d\epsilon_{ij}''\|$  of the inelastic strain increments,  $d\epsilon_{ij}''$ . The distance of a point on the locus for any loading direction is proportional to the stiffness modulus for inelastic strain in that direction.

It can be shown (Bažant, 1978) for incremental plasticity that the inelastic stiffness locus is a straight line parallel to the tangent to the loading surface (Figure 5a). The fact that this locus must be a straight line is also evident from the linearity property of incremental plasticity. Since the loading direction parallel to the loading surface intersects the elastic stiffness locus at infinity, the response to such loading is obtained as perfectly elastic. Recently, however, it has become widely accepted that the actual response resulting from loading to the side is not or should not be perfectly elastic, but softer than elastic. For this reason, various theories which introduce inelastic strain due to loading to the side are being developed. This trend is reflected in the vertex-hardening models for the plasticity of metals and other materials. In general, the vertex concept does not involve a fixed vertex (corner) at a predetermined place on the loading surface, but a vertex which is always superimposed on the current stress point of the loading surface and moves jointly with this point. Conceptually, the most simple and effective vertex model appears to be that of Rudnicki and Rice (1975) for which the inelastic stiffness locus is obtained as the smooth curve shown in Figure 5b.

For endochronic theory it can be shown that the inelastic stiffness locus is either a circle centered around the current stress point on the loading

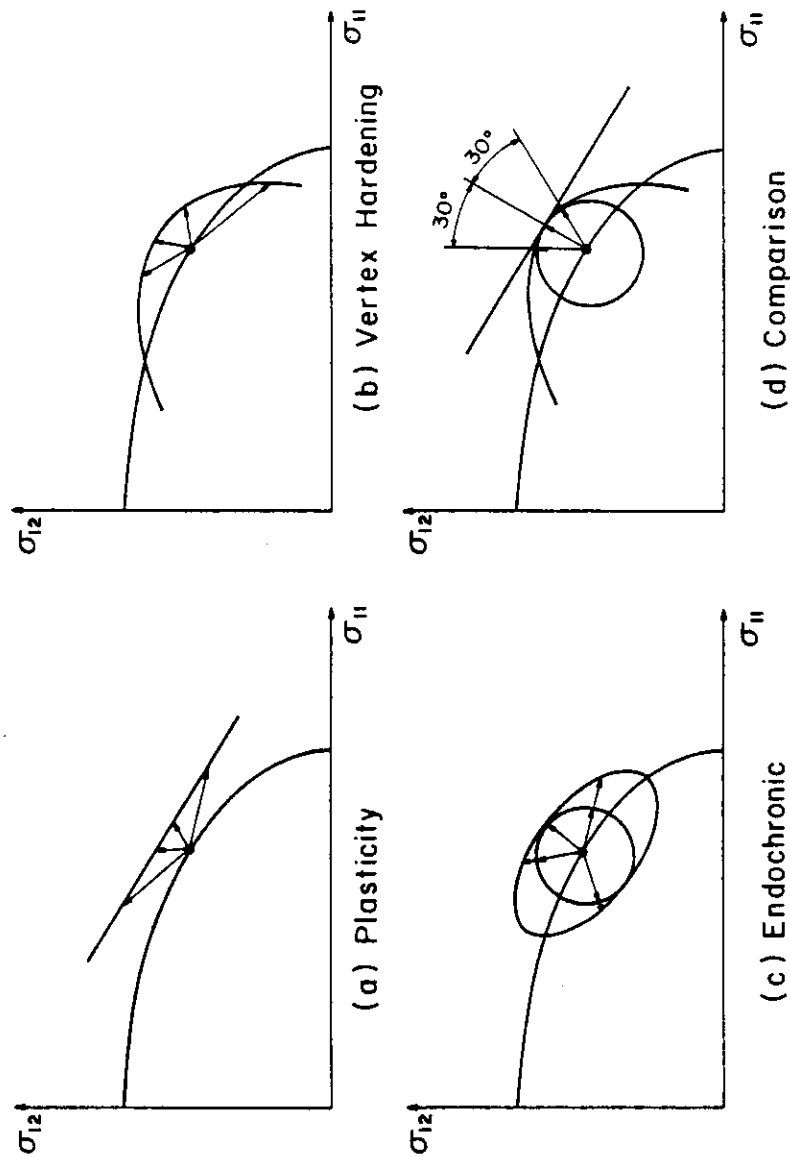


Figure 5

surface or an ellipse (Figure 5c). Since for this shape of the inelastic stiffness locus the tip of the vector  $de_{ij}$  for loading to the side or for unloading is at a finite distance from the loading surface, the response for these loading directions is inelastic. Accordingly, given that one accepts the widespread opinion that the response for loading tangent to the loading surface is not perfectly elastic, endochronic theory appears to be more realistic. It should also be noted that the curved form of the inelastic stiffness locus for endochronic theory is similar to that for vertex-hardening plasticity, as well as for the total strain (deformation) theory. In view of the curved form of the inelastic stiffness locus, endochronic theory represents a development in the same direction as the introduction of vertex-hardening models in plasticity.

It has been mentioned that endochronic theory can not be expressed in an incrementally linear form (equation 5), even if unloading is excluded. In this respect, endochronic theory differs from most existing nonlinear constitutive theories, including incremental plasticity, hypoelasticity, and the total strain (deformation) theory. Nevertheless, it is possible to linearize endochronic theory in the vicinity of a chosen fixed loading direction for the given stress state under consideration, and this brings the theory to the form of equation (5). This is achieved by replacing the increment of intrinsic time,  $d\zeta$ , given by equation (2), by the linearized expression

$$d\zeta = \frac{\epsilon^T p d\epsilon}{|\epsilon|} \quad (7)$$

This linearization transforms endochronic theory to a stress-strain relation of essentially the same form as in incremental plasticity. Geometrically, this linearization corresponds to the replacement of the circular or curved inelastic stiffness locus by a tangent straight line. The response of this linearized theory is then very close to the response indicated by endochronic theory for all straining directions that deviate by less than  $30^\circ$  from the chosen stress direction about which the linearization is made (Figure 5d). Within this fan of directions it is obviously impossible to find much difference between the various theories which lead to a different form of the inelastic stiffness locus.

The foregoing consideration shows that, in order to obtain experimental information to prove or refute the validity of endochronic theory or any other

theory, it is necessary to conduct experiments in which the loading path forms a sharp corner with a sudden and large change in the loading direction (that is, so-called loading to the side). For loading paths which do not involve such sharp changes in the loading direction and remain close to a proportional loading path, the test data (with experimental scatter) can probably be represented equally well by various theories and it will be virtually impossible to evaluate the advantages of any particular theory. However, highly nonproportional loading paths with sudden changes in the loading direction are often typical of final failure modes; for example, the stress condition in the soil mass directly under a footing prior to failure consists essentially of a vertical stress, whereas at the moment of failure the strain consists chiefly of shear on the plane of the previous vertical stress. Significant differences between various theories are also obtained in unloading and cyclic loading.

Another property for which various theories manifest significant differences are the so-called cross effects, such as the effects of the shear strain increment,  $d\gamma_{12}$ , on the normal stress increment,  $d\sigma_{11}$ , the shear stress increment,  $d\tau_{12}$ , on the normal strain increment,  $d\epsilon_{11}$ , or one shear strain increment,  $d\gamma_{12}$ , on the shear stress increment,  $d\tau_{23}$ , on another plane. In the linear or linearized form of the incremental constitutive equation (equation 5), this is manifested by non-zero values of the stiffness matrix coefficients corresponding to these components (the upper right corner and lower left corner of the stiffness matrix). In endochronic theory, as well as certain other formulations, these cross effects are rather significant and their presence is required to model various salient constitutive properties, such as densification or dilatancy due to monotonic and cyclic shear strain. This phenomenon leads to non-coaxiality of stress and strain and precludes the use of an orthotropic form of the incremental stiffness matrix (in particular, an isotropic form of this matrix). Thus, the differences between endochronic theory and the so-called incremental orthotropic or isotropic models are rather significant, although these differences can be clearly discerned only for certain loading paths.

#### ADVANTAGES OF ENDOCHRONIC THEORY

From a practical point of view, the following useful features of endochronic theory may be summarized:

1. The theory is rather convenient and effective for representing unloading irreversibility, and this makes it particularly useful for cyclic loading.

Since the intrinsic time increases with the number of loading cycles, this variable is suitable as a measure of the softening or hardening produced in the soil by cyclic loading. This fact, which was first exploited by Valanis (1971), enables endochronic theory to represent rather simply the stiffening and contraction exhibited by hysteresis loops from one cycle to the next.

2. Compared to classical plasticity, the fact that the intrinsic time is independent of the loading surface and its evolution lends the theory considerable flexibility and enables it to represent diverse phenomena more easily. Furthermore, certain of the characteristic material functions, such as the hardening function, softening function, and dilatancy function, have a relatively simple, albeit intuitive, physical interpretation, and this provides endochronic theory with an advantage in helping to understand the behavior of the model.
3. From the fundamental theoretical point of view, the main difference between endochronic theory and other theories is obtained for loading to the side. In this case endochronic theory always exhibits a softer response for loading than incrementally linear theories, such as classical incremental plasticity. Since a softer response is obviously safer with regard to failure predictions (particularly because failure often occurs with a stress path of this type), endochronic theory will yield conservative predictions. At the same time, a softer or inelastic response for loading to the side is probably more correct, since all microstructural models for inelastic behavior of materials point to the lack of purely elastic response for this type of loading.
4. Endochronic theory is particularly effective for modeling cross effects, such as dilatancy due to shear, and cross hardening (for example, hardening of the volumetric response due to accumulated shear).
5. Finally, endochronic theory provides a relatively simple way to introduce strain-rate effects.

There is, however, one major disadvantage to endochronic theory. The constitutive equation can not be integrated explicitly to give the response curves for various basic types of tests; this complicates data fitting and requires the use of trial-and-error iterative approaches, possibly combined with optimization computer techniques. One noteworthy exception is the case of cyclic pure shear straining, for which endochronic theory yields relatively simple explicit



expressions that can be fitted to test data quite easily (Cuellar et al, 1977). As a consequence of these difficulties in the data fitting procedure, it has thus far been necessary to use a relatively large number of material parameters to represent the hardening, softening, and dilatancy functions, and this has made it impossible to attach a clear physical significance to each numerical parameter. Hopefully, further work will lead to improvements in this regard.

#### CONCLUSION

Endochronic theory provides a very flexible and effective approach for modeling the nonlinear behavior of soils. It is able to handle a wide range of phenomena (such as strain rate effects, dilatancy due to shear, hardening and softening, etc) and seems to predict the proper behavior for cases involving highly non-proportional loading with a sharp corner on the stress path.

#### ACKNOWLEDGMENT

Financial support by the U.S. National Science Foundation under Grant No ENG-7807777 is gratefully acknowledged.

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## PLASTICITY MODELS FOR SOILS

Theory and Calibration

by

E. Mizuno<sup>1</sup> and W. F. Chen<sup>2</sup>, M. ASCE

### 1. Introduction

The mechanical behavior of soil and rock is complicated and they can not be modelled accurately as a continuum. At present, however, the concept of continuum mechanics has been used extensively in the mathematical modelling of these materials. These include the applications of linear elastic models, nonlinear elastic models, and elastic-plastic models to geotechnical engineering problems. Although the models such as hyper-elastic or hypoelastic can represent the phenomena such as dilatancy and hardening or softening of soil behavior, the effect of plastic strain induced during loading can not be predicted within the framework of an incremental Hooke's law with variable moduli which are functions of the stress and/or strain levels.

Current research in soil constitutive modelling is moving toward the development of three-dimensional stress-strain relations based on the principles of plasticity as well as elasticity.

Herein, three types of soil models are described. The first type was used for prediction before the workshop was held, thus without the benefit of the test results. The second and third types are subsequently developed and used after the workshop.

- (i) Nonlinear elasticity material model with the Mohr-Coulomb or the Drucker-Prager surface as failure criterion.
- (ii) Mohr-Coulomb type of elastic-plastic material model with two different sizes of elliptical hardening cap which are defined respectively on the tensile meridian plane ( $\theta = 0^\circ$ ) and the compressive meridian plane ( $\theta = 60^\circ$ ). (Cap Model I)

<sup>1</sup>Research Assistant, School of Civil Engineering, Purdue University, West Lafayette, IN 47907

<sup>2</sup>Professor of Structural Engineering, School of Civil Engineering, Purdue University, West Lafayette, IN 47907