Reprinted from First International Conference on
Structural Mechanics in Reactor Technology,
Berlin, September 1971, organized by BAM, Berlin
and Commission of the European Communities, Brussels,
Vol. 3, Paper H2/3

NUMERICALLY STABLE ALGORITHM WITH INCREASING TIME STEPS FOR INTEGRAL-TYPE AGING CREEP

Z.P. BAŽANT,

Department of Civil Engineering, The Technological Institute, Northwestern University, Evanston, Illinois, U.S.A.

ABSTRACT

The creep law of concrete is assumed to obey the principle of superposition and is expressed by hereditary integrals with non-convolution type kernels, which are expanded in a degenerate form. For each material point, a set of certain additional hidden variables, fully characterizing the previous stress history, is defined and it is found that in a step-by-step integration in time these variables can be computed from simple recurrent formulas. This makes storage of the past stress history unnecessary and circumvents thus the main obstacle for creep analysis of large structures. Unlike the step-forward integration with a rate-type creep law, the method is found to be stable for any size of the time step. This enables the time steps in relaxation-type problems to be gradually increased and the asymptotic values of stresses to be closely approached. Results of a simple numerical example corroborate this conclusion.

1. INTRODUCTION

In stress analysis of prestressed concrete pressure vessels it is necessary to pay careful attention to creep of concrete. At the present level of knowledge it is appropriate to assume that creep of concrete obeys the principle of superposition and may be formulated in terms of hereditary integrals with given memory kernels. However, this formulation of stress-strain law presents considerable difficulties when a large structure is to be analyzed because the complete history of stress for every element of the structure must be stored and long sums in each time step of analysis must be evaluated. This requires much computer time and over-taxes the capacity of the central memory of computers presently available so that storage on tapes is necessary. Such difficulties have been experienced, for example, by Cederberg and David [1]. The intent of the present paper is to show a step-by-step algorithm which allows these problems to be overcome.

2. CREEP LAW

If attention is restricted to working stress levels and strain reversals are excluded, the creep law of concrete can be approximately considered as linear and isotropic, and may be expressed as follows (Cf., e.g., [2]):

$$\epsilon(t) - \epsilon^{o}(t) = \frac{1}{3}\int_0^t J^V(t,t')\, d\sigma(t') \tag{1a}$$

(Stieltjes integrals)

$$e_{ij}(t) = \frac{1}{2}\int_0^t J^D(t,t')\, ds_{ij}(t') \tag{1b}$$

where s_{ij} and e_{ij} = components of the stress and strain deviators in cartesian coordinates x_i referred to by subscripts i and j; σ and ϵ = volumetric stress and strain components; $\epsilon^{o}(t)$ = shrinkage plus temperature dilatation; t = time from casting of concrete; $J^V(t,t')$ and $J^D(t,t')$ = memory kernels = volumetric and deviatoric creep functions (or compliances) representing strain ϵ or e_{ij} in time t caused by a constant unit stress σ or s_{ij} applied in time t'(unit step loading); $J^V(t,t) = 1/K(t)$ and $J^D(t,t) = 1/G(t)$ where K(t) = bulk modulus and G(t) = shear modulus. The volumetric and deviatoric creep compliances are related to the uniaxial creep compliance J(t,t') as follows

$$J^D(t,t') = 2(1+\nu)J(t,t') \;, \quad J^V(t,t') = 6(\tfrac{1}{2} - \nu)J(t,t') \tag{2}$$

where ν = Poisson's ratio, which may approximately be considered as independent of t and t', and equal about 0.18. Note that $J(t,t) = 1/E(t)$ where E(t) = Young's modulus in time t. A simple expression for function J(t,t') can be found, e.g., in reference [1].

Utilizing creep law (1), one should be aware, however, that eq. (1) is correct only for conditions of constant water content and temperature. Otherwise the memory kernels depend on the history of water content and temperature [3]. But quantitative knowledge of these effects is still incomplete.

For the numerical solution of creep problems it is convenient to approximate the memory kernels $J^V(t,t')$ and $J^D(t,t')$ by the so-called degenerate kernels, which may be written, taking J^V as an example, in the form [4,5] :

$$J^V(t,t') = \frac{1}{K(t')} + \sum_{n=1}^{m} \frac{1}{K_n(t')}\left(1 - e^{-(t-t')/\tau_n}\right) \tag{3}$$

where τ_n are constant retardation times and $K_n(t')$ are coefficients dependent only on t'. Expression (3) represents an expansion of function $J^V(t,t')$ for constant t' into the so-called Prony series [6-8]. (This series, being a series of real exponentials, represents the real counterpart of Fourier series.) It is known that any creep curve can be approximated by Prony series with any desired accuracy. For this purpose the values $\tau_1, \ldots \tau_m$ may be chosen but the accuracy of approximation depends on the choice made. For a good accuracy the values $\tau_1, \ldots \tau_m$ should be evenly distributed in the log τ_n-scale and the closer they are to each other (or the greater is n), the better accuracy is achieved. For all practical purposes it is sufficient to choose the τ_n-values by decades, i.e. $\tau_n = 10^{n-1}\tau_1$ (n=1,...m); τ_1 should roughly coincide with the point where the creep curve plotted in the log (t-t')-scale begins to rise, and τ_m with the longest elapsed time considered in the problem or the point where the creep curve in log (t-t')-scale levels off, whichever is smaller. For a given unit creep curve, i.e. function $J^V(t,t')$ for a fixed t', the K_n-values can be determined by the method of least squares or (more simply) by the collocation method [6-8]. Then, choosing various values t', the dependence of K_n upon t' may be assessed.

3. NUMERICAL ALGORITHMS USED IN THE PAST

For step-by-step numerical integration the time period under consideration is subdivided by discrete times t_o, $t_1, \dots t_N$ into N time steps $\Delta t = t_r - t_{r-1}$. In problems involving a steady loading the time step Δt may be increased with r. The first time step Δt need be chosen about $0.1\tau_1$, but the subsequent time steps Δt may be gradually increased up to about $\Delta t \approx 0.2\tau_m$. Such increasing time steps allow approach to the terminal values of stress in an acceptable number of steps (N=100, e.g.).

If the hereditary integrals in eqs. (1) are replaced by finite sums (using, for instance the trapezoidal rule for the chosen subdivision), the increments $\Delta\sigma_r$, Δs_{ij_r} and $\Delta\epsilon_r$, Δe_{ij_r} in each time step are found to be related by a linear elastic stress-strain law with initial strains $\Delta\epsilon''_r$ and $\Delta e''_{ij_r}$, which are expressed as a linear function of all previous values of σ and s_{ij}, i.e. the values σ_o, $\sigma_1, \dots \sigma_r$, $s_{ij_o}, \dots s_{ij_r}$. Details of the procedure can be found, e.g., in references [5,4,9,10]. As has already been mentioned in the introduction, the fact that it is necessary to store all previous stress values $\sigma_o, \sigma_1, \dots \sigma_r$ for all elements of the structure and evaluate long sums from these values makes the method unsuitable for large structures.

The storage and time requirements can be substantially reduced by using the degenerate form of the memory kernel, eq. (3). One such method was proposed by Selna [4], and for nonlinear creep by Bazant [5]. In this method, the hereditary integral in eq. (1a), for instance, is expressed in the form:

$$\int_o^t J^V(t,t')d\sigma(t') = S_o(t) - \sum_{n=1}^{m} e^{-t/\tau_n} S_n(t) \tag{4}$$

where
$$S_o(t) = \frac{1}{3}\int_o^t \left(\frac{1}{K(t')} + \sum_{n=1}^{m} \frac{1}{K_n(t')}\right) d\sigma(t') \tag{5a}$$

$$S_n(t) = \frac{1}{3}\int_o^t \frac{1}{K_n(t')} e^{t'/\tau_n} d\sigma(t') \ , \ n=1,2,\dots m \tag{5b}$$

The determination of the increment of hereditary integral (4) from time t_{r-1} to time t_r requires only the values $S_o(t_{r-1})$ and $S_n(t_{r-1})$ of the auxiliary variables S_o, S_n to be known. The values $S_o(t_r)$ and $S_n(t_r)$ can be determined solely on the basis of the immediately preceding values $S_o(t_{r-1})$ and $S_n(t_{r-1})$ and increment $\Delta\sigma$ in the r-th step Δt, so that no previous history of σ need be stored. One needs, of course, to store the current values of S_o, $S_1, \dots S_n$ in each step (for each element of the structure) but their number is generally much smaller (usually $m \le 5$) than the number of steps (typically N=100). Nevertheless, this method fails when the spectrum of retardation times τ_n is wide, as it is for concrete. Namely, when $t' = 10^4\tau_1$, e.g., the value of e^{t'/τ_n} in eq. (5b) equals e^{10000}, which results in an overflow.

Alternatively, a reduction in the storage requirements can be achieved by replacing the integral-type creep law (1) with a rate-type creep law. If $J^V(t,t')$ is given in the degenerate form (3), the volumetric creep law (1a) may be converted into a system of n first order differential equations between σ, ϵ and further hidden state variables. For numerical integration of the creep problem, these differential equations may be approximated by finite difference equations in time. However, when the spectrum of retardation times is wide, this method fails too. Namely, it can be shown that for numerical stability the time step Δt may not exceed the value $2\tau_1$. Thus, if the time period for which the solution is desired equals $10^4\tau_1$, which is a typical value, one would have to use over 10^4 time steps in order to reach the terminal values of stress. The running time on the computer thus becomes unacceptably long, which offsets the advantage of reduced storage requirements.

4. PROPOSED NUMERICAL ALGORITHM

Introducing the degenerate memory kernel (3), eq. (1a) may be written as follows:

$$\epsilon(t) = \frac{1}{3}\int_0^t \Big[K^{-1} + \sum_n K_n^{-1}\Big]_{t'} d\sigma(t') - \sum_{n=1}^{m} \epsilon_n^*(t) + \epsilon^o(t) \tag{6}$$

where the quantities,

$$\epsilon_n^*(t) = \frac{1}{3} e^{-t/\tau_n} \int_0^t e^{t'/\tau_n} K_n^{-1}(t') \frac{d\sigma(t')}{dt'} dt' \ , \quad n=1,2,\ldots m \tag{7}$$

represent hidden material variables which characterize the effect of the past history. Because the term e^{t'/τ_m} can become extremely large, as has already been mentioned, a good numerical approximation of the integral in eq. (7) in regard to the exponential term is imperative. Assuming that $d\sigma(t)/dt$ and $E_n(t)$ have constant values within each time step $< t_{s-1}, t_s >$, $s=1,2,\ldots r$, and change discontinuously in discrete times $t_1, t_2, \ldots t_r$, eq. (7) may be brought to the form:

$$\epsilon^*_{n_r} = \frac{1}{3} e^{-t_r/\tau_n} \sum_{s=1}^{r} \Big[K_n^{-1} \frac{d\sigma}{dt}\Big]_{s-\frac{1}{2}} \int_{t_{s-1}}^{t_s} e^{t'/\tau_n} dt' \tag{7a}$$

which can be integrated (exactly) as follows:

$$\epsilon^*_{n_r} = \frac{1}{3} e^{-t_r/\tau_n} \sum_{s=1}^{r} e^{t_s/\tau_n} \lambda_{n_s} \Delta\sigma_s / K_{n_{s-\frac{1}{2}}} \tag{8}$$

where $$\lambda_{n_s} = \Big(1 - e^{-\Delta t_s/\tau_n}\Big)\tau_n/\Delta t_s \tag{9}$$

Subscript r refers to time t_r, e.g. $\epsilon^*_{n_r} = \epsilon^*_n(t_r)$, or to the increment during the r-th step, e.g. $\Delta\sigma_s = \sigma_s - \sigma_{s-1} = \sigma(t_s) - \sigma(t_{s-1})$. Subscript $s-\frac{1}{2}$ refers to the average value in the s-th step Δt, e.g. $K_{n_{s-\frac{1}{2}}} = \frac{1}{2}\big(K_{n_{s-1}} + K_{n_s}\big)$.

Subtracting from eq. (8) the analogous expression for $\epsilon^*_{n_{r-1}}$, the following recurrent formula may be acquired:

$$\epsilon^*_{n_r} = \frac{1}{3} \lambda_{n_r} \Delta\sigma_r / K_{n_{r-\frac{1}{2}}} + \epsilon^*_{n_{r-1}} e^{-\Delta t_r/\tau_n}, \quad n=1,\ldots m \tag{10}$$

Hence, the values of $\epsilon^*_{n_s}$ for $s < r-1$ need not be stored.

Eq. (6) may be approximated by the following incremental finite difference equation:

$$\Delta\epsilon_r = \frac{1}{3}\Big[K^{-1} + \sum_n K_n^{-1}\Big]_{r-\frac{1}{2}} \Delta\sigma_r - \sum_{n=1}^{m} \Delta\epsilon^*_{n_r} + \Delta\epsilon^o_r \tag{11}$$

Expressing the increment $\Delta\epsilon^*_{n_r} = \epsilon^*_{n_r} - \epsilon^*_{n_{r-1}}$ with the help of eq. (10), eq. (11) takes the form

$$\Delta\epsilon_r = \frac{\Delta\sigma_r}{3K''_r} + \Delta\epsilon''_r \tag{12}$$

where
$$K''_r = \left[K^{-1}_{r-\frac{1}{2}} + \sum_{n=1}^{m}\left(1 - \lambda_{n_r}\right)K^{-1}_{n_{r-\frac{1}{2}}}\right]^{-1} \tag{13}$$

$$\Delta\epsilon''_r = \sum_{n=1}^{m}\left(1 - e^{-\Delta t_r/\tau_n}\right)\epsilon^*_{n_{r-1}} + \Delta\epsilon^o_r \tag{14}$$

The deviatoric stress-strain law (1b) may be subjected to analogous transformations which finally yield

$$\Delta e_{ij_r} = \frac{\Delta s_{ij_r}}{2G''_r} + \Delta e''_{ij_r} \tag{15}$$

where
$$G''_r = \left[G^{-1}_{r-\frac{1}{2}} + \sum_{n=1}^{m}\left(1 - \lambda_{n_r}\right)G^{-1}_{n_{r-\frac{1}{2}}}\right]^{-1} \tag{16}$$

$$\Delta e''_{ij_r} = \sum_{n=1}^{m}\left(1 - e^{-\Delta t_r/\tau_n}\right)e^*_{ij_{n_{r-1}}} \tag{17}$$

$$e^*_{ij_{n_r}} = \tfrac{1}{2}\lambda_{n_r}\Delta s_{ij_r}/G_{n_{r-\frac{1}{2}}} + e^*_{ij_{n_{r-1}}}e^{-\Delta t_r/\tau_n} \tag{18}$$

Supposing that in a given creep problem the stresses have already been calculated up to the time t_{r-1}, the values K''_r, G''_r, $\Delta\epsilon''_r$ and $\Delta e''_{ij_r}$ may be determined from eqs. (13), (16), (14), (17). Equations (12), (15) may then be regarded as a fictitious linear elastic stress-strain law, in which $\Delta\epsilon''_r$ and $\Delta e''_{ij_r}$ = pseudo-inelastic (or pseudo-initial) strains, and E''_r and G''_r = pseudo-instantaneous elastic moduli. The problem of determination of $\Delta\sigma_r$ and $\Delta\epsilon_r$ is thus formally reduced to a linear elasticity problem with initial strains. Computer solution of this problem will usually be carried out by the finite element method. When the elasticity problem is solved, the new values of hidden variables ϵ^*_n and $e^*_{ij_n}$ at time t_r are computed from eqs. (10) and (18) and the analysis of the next time step Δt_{r+1} may be begun.

The creep problem is thus converted to a series of linear elasticity problems.

As has already been noted, the proposed method does not require any history to be stored. Unlike in the method based on eqs. (6) and (7), there is no reason for the hidden variables to cause overflow (because always $0 < \lambda_{n_r} < 1$). Experience with some simple problems has further shown that the method is stable for any size of step Δt, in contrast with the step-forward integration based on the rate-type counterpart of the creep law.

It is noted that the method of approximation of integral (7) bears some marks of analogy with a method developed and tested by Taylor, Pister and Goudreau [11] for thermorheologically simple non-aging materials, such as polymers, characterized by a degenerate relaxation function or a generalized Maxwell model.

To check whether or not the discrete approximation of the stress-strain law may cause any numerical instability, it is possible to restrict attention to the solution of strains corresponding to a general prescribed (bounded) stress history. Equation (10) may then be regarded as a linear finite difference equation for the discrete variable $\epsilon^*_{n_r}$. The equation is nonhomogeneous because of the presence of a term with prescribed values of $\Delta\sigma_r$. The solution of a homogeneous equation may be sought in the form $\epsilon^*_{n_r} = a^r$ whose substitution in eq. (10) yields the characteristic equation:

$$a = e^{-\Delta t_r/\tau_n} \tag{19}$$

Obviously $0 < a < 1$. Hence, the solutions of the homogeneous part of eq. (10) are of an exponentially decaying character (and also do not exhibit any oscillations). Examination of eqs. (14) and (12) shows that ϵ_r must be of the same character as $\epsilon^*_{n_r}$. Thus the numerical error cannot be amplified in the subsequent steps and the method is stable. The same is true for the nonhomogeneous equation, provided that σ_r has also an exponentially decaying character.

It should be mentioned that if the forward difference approximation of the rate-type creep law associated with the degenerate creep function (3) is subjected to a similar analysis, solutions of the type a^r with $a < -1$ are found when $\Delta t > 2\tau_1$ = double the shortest retardation time.

It is interesting to examine the range of values which the pseudo-instantaneous modulus K''_r, eq. (13), can take. From eq. (9) it is seen that $\lambda_{n_r} \to 1$ for $\Delta t_r/\tau_n \to 0$, and $\lambda_{n_r} \to 0$ for $\Delta t_r/\tau_n \to \infty$ For $\Delta t_r \approx 10\tau_n$ approximately $\lambda_{n_r} \approx 0$, so that

$$1/K''_r \approx 1/K_{r-\frac{1}{2}} + \sum_{p=1}^{n} 1/K_{p_{r-\frac{1}{2}}}$$

The deformation due to all exponential components of creep function (3) whose retardation times are substantially shorter than step Δt_r is thus taken approximately as instantaneous. This agrees with what may be expected on the basis of intuitive judgment.

4. SIMPLE NUMERICAL EXAMPLE

To demonstrate the numerical stability and efficiency of the proposed method, the problem of uniaxial stress relaxation at constant strain has been programmed and solved. The uniaxial creep function has been considered in the form

$$J(t,t') = E_1^{-1}\left[1 + 2.35(1.25t'^{-0.118})f(t-t')\right](0.85 + 4/t')^{\frac{1}{2}} \tag{20}$$

where

$$f(t-t') = 0.236\left(1 - e^{-(t-t')/5}\right) + 0.420\left(1 - e^{-(t-t')/50}\right) + 0.180\left(1 - e^{-(t-t')/500}\right) + 0.125\left(1 - e^{-(t-t')/5000}\right) \tag{20a}$$

t and t' being in days; $E_1 = 5\times10^6$ psi. Expression (20a) has been determined as an approximation to the function $(t-t')^{0.6}\left[10 + (t-t')^{0.6}\right]^{-1}$, which has been recently recommended by ACI Committee 209 [12]. The dependence upon t' in eq. (20) was also taken from this recommendation. The constant strain ϵ of magnitude 10^{-6} was considered to be introduced at the time t_o = 35 days. The time steps Δt were gradually increased with time in such a manner that in $\ln(t-t_o)$-scale the time steps would appear as constant. The first time step after strain introduction was taken as 0.1 day, and the terminal time as 29 031 days after load introduction. The solution of stress was computed for various numbers N of steps within this time interval, the first step being always the same. The stress values obtained (in psi) are summarized in the table below:

Time elapsed (in days)	2.321	53.881	1250.7	29031
N=13	4.1434	2.3223	1.7410	1.5320
N=25	4.1458	2.3368	1.7506	1.5411
N=49	4.1464	2.3417	1.7531	1.5438
N=97	4.1465	2.3430	1.7537	1.5443
N=193	4.1466	2.3434	1.7539	1.5445

From this example it is seen that the convergence is excellent and the method is very accurate, since even the very crude subdivision with 13 steps yielded results which were sufficiently accurate from the practical point of view. It has also been checked that the results are identical with those obtained by the direct method in which the hereditary integrals are evaluated as sums over the whole preceding history of stress.

5. CONCLUSION

The method presented herein allows a substantial reduction of storage and machine time requirements in creep problems for linear hereditary time-variable materials.

ACKNOWLEDGMENT

The present work constitutes a part of a project on "Inelastic deformations of concrete due to mass transport" sponsored by the National Science Foundation under Grant No. GK-26030.

REFERENCES

[1] CEDERBERG, H., DAVID, M., "Computation of creep effects in prestressed concrete pressure vessels using dynamic relaxation," Nuclear Engineering & Design, Vol. 9, 1969, 439-448

[2] BAŽANT, Z. P., "Creep of concrete and structural analysis" (in Czech), SNTL(State Publ. of Techn. Lit.), Prague, 1966; see also Acta Technica (Prague), 1966, pp. 82-109.

[3] BAŽANT, Z. P., "Thermodynamic theory of concrete deformation with explanation of drying creep," Proc., ACI Symp. "Designing for Effects of Creep,Shrinkage and Temperature," 1971; see also Materials & Structures (RILEM), No. 13, 1970, pp. 3-36, and Nuclear Eng. & Design, Vol. 14, 1970, 308-318.

[4] SELNA, L. G., "A concrete creep, shrinkage and cracking law for frame structures," Amer. Concrete Institute Journal, Proc. Vol. 66, Oct. 1969, 847-848, with Supplement No. 66-71.

[5] BAŽANT, Z. P., "Numerical solution of non-linear creep problems with application to plates," Intern. J. of Solids and Structures, Vol. 7, Jan. 1971, 83-97.

[6] WILLIAMS, M. L., "Structural analysis of viscoelastic materials," AIAA Journal, May 1964, 785-808.

[7] COST, T. M., "Approximate Laplace transform inversions in viscoelastic stress analysis," AIAA Journal, Vol. 2, Dec. 1964, 2157-2166.

[8] SCHAPERY, R. A., "Approximate methods of transform inversion for viscoelastic stress analysis," Proc., 4th U. S. Nat. Congress of Appl. Mechanics, Vol. 2, 1075-1085, held in Berkeley, Cal., June 1962, publ. by ASME.

[9] BAŽANT, Z. P., "Numerical analysis of creep of an indeterminate composite beam," Trans. ASME,J. of Appl. Mechanics, Dec. 1970, 1161-1164.

[10] BAŽANT, Z. P., "Linear creep problems solved by a succession of generalized thermoelasticity problems," Acta Technica ČSAV (Prague), 1967, No. 5, 581-594.

[11] TAYLOR, R. L., PISTER, K. S., GOUDREAU, G. L., "Thermoelastic analysis of viscoelastic solids," Report No. 68-7, Structural Engrg. Lab., Univ. of California, Berkeley, June 1968.

[12] Amer. Concrete Institute Committee 209, Subcomittee II (Chairman D. E. Branson), Report on "Prediction of creep, shrinkage and temperature effects in concrete structures, Symp. on "Designing for effects of creep shrinkage and temperature," held in New York, Apr. 1970, publ. by ACI in 1971.

DISCUSSION

Q

G. L. ENGLAND, U. K.

The main feature of interest of the creep analysis reported is clearly the economy of computer time which shows a superposition method of analysis to be used without incurring great expense, at least for concrete maintained at uniform temperature. Could the author please indicate whether he believes that the method can be extended to include the analysis of non-uniformly heated concrete structures, where the creep behaviour becomes non-homogeneous, without increasing the effort and cost of solution so substantially that other methods of analysis become more attractive than the method described. Are any numerical examples available for structures heated non-uniformly ?

A

Z. P. BAŽANT, U. S. A.

Extension to variable temperature is in my opinion possible but I have not tried to do so because in this case I believe the rate-type formulation of the creep law to be more suitable, as has been described in my other paper at this Conference. For this formulation an analogous algorithm can be developed, which is shown at the end of the unabbreviated text of my other paper to be published in Nuclear Engineering and Design, (1972) Vol. 20, No. 2.