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Size effect on buckling strength of eccentrically compressed column with fixed or propagating transverse crack

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Abstract The strength and size effect of a slender eccentrically compressed column with a transverse pre-existing traction-free edge crack or notch is analyzed. Rice and Levy's spring model is applied to simulate the effect of a crack or notch. An approximate, though accurate, formula is proposed for the buckling strength of the column of variable size. Depending on the eccentricity, the crack at maximum load can be fully opened, partially opened or closed. The size effects in these three situations are shown to be different. The exponent of the power-law for the large-size asymptotic behavior can be $-1/2$ or $-1/4$, depending on the relative eccentricity of the compression load. Whether the maximum load occurs at initiation of fracture growth, or only after a certain stable crack extension, is found to depend not only on the column geometry but also on its size. This means that the definition of positive or negative structural geometry (as a geometry for which the energy release rate at constant load increases or decreases with the crack length) cannot be extended to stability problems or geometrically nonlinear behavior. Comparison is made with a previous simplified solution by Okamura and coworkers. The analytical results show good agreement with the available experimental data.

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1 Introduction

The interaction between stability and fracture has been investigated for several decades (e.g., Bažant and Cedolin 1991, chapter 12; Bažant and Ohtsubo 1977; Bažant et al. 1979), but except for the work of Okamura et al. (1969) the basic problem of stability of a slender column with propagating crack received little attention, and the corresponding size effect apparently remains unexplored. The present paper will address this kind of problem—the size effect in a pre-cracked or notched column under eccentric compression. The nonlinear spring model of Rice and Levy (1972) will be applied to simulate the behavior of a pre-existing traction-free crack or notch in a beam or column.

2 Crack in column modeled as a rotational spring

We consider a single-edge notched hinged column under eccentric compression (Fig. 1a). The width, D, of the column is chosen as the characteristic size of the structure. The column is treated as two-dimensional, in either plain stress or plain strain. The former assumption is accurate when $b \ll D$, and the latter when $b \gg D$. For the sake of simplicity, we set the thickness $b = 1$ (Fig. 1a) and the load, P, is considered to be the load per unit thickness (with the physical dimension of N/m). The solution for plane strain is obtained from the solution for plane stress when Young's modulus E and Poisson's ratio ν are replaced by $E' = E/(1-\nu^2)$ and $\nu' = \nu/(1-\nu)$. The structure and notch geometries are defined by the following dimensionless parameters:

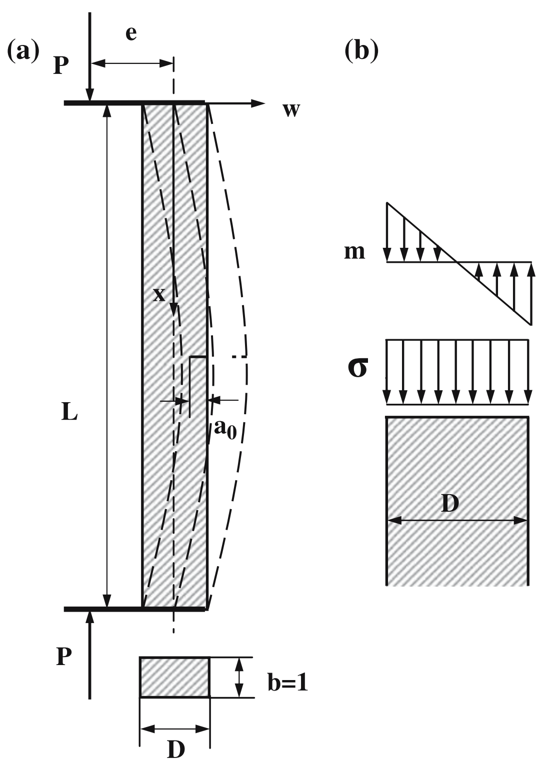

Fig. 1 (**a**) Single-edge notched column with hinged ends under eccentric compression; (**b**) decomposition of centric load and moment, and the corresponding compressive stress σ and nominal bending stress m

$$\lambda = \frac{L}{D}, \quad \alpha_0 = \frac{a_0}{D}, \quad \xi = \frac{e}{D} \tag{1}$$

where L = height of the column; λ = its slenderness; a_0 = initial notch length; α_0 = relative initial notch length; e = eccentricity of the compression load; and ξ = relative eccentricity. The length of opened crack or notch is denoted as a and its relative length as $\alpha = a/D$.

When $a_0 = 0$ (i.e., when there is no pre-existing crack or notch), the second-order beam theory for a slender column leads to the equilibrium equation

$$M(x) = EIw''(x) = -P\,[w(x) + e] \tag{2}$$

where $I = D^3/12$ and $w(x)$ = deflection of the column as a function of length coordinate x. The relative deflection is defined as $\bar{w} = w/D$. Because of symmetry, only one half of the column needs to be considered ($0 \le x \le L/2$), and the boundary conditions for this half read

$$w(0) = 0, \quad w'(L/2) = 0 \tag{3}$$

The solution is well-known:

$$w(x) = e\left[\tan\frac{kL}{2}\sin kx + \cos kx - 1\right] \tag{4}$$

where $k = \sqrt{P/EI}$. The maximum deflection occurs at the middle cross section and is

$$w_{\max} = w(L/2) = \mathrm{e}\left(\sec\frac{kL}{2} - 1\right),$$
$$\text{or} \quad \bar{w}_{\max} = \frac{w_{\max}}{D} = \xi\left(\sec\frac{kL}{2} - 1\right) \tag{5}$$

where $\bar{w}_{\max}$ is maximum relative deflection.

If there is a crack or notch ($a_0 > 0$), its effect on the overall deflections can be modeled by a softening rotational spring (Rice and Levy 1972), and then the second-order beam theory can still be used. The middle cross section of column transmits centric compression load P and bending moment $M = -P[e + w(L/2)]$. The corresponding nominal compression stress σ and nominal bending stress m can be defined (Fig. 1b, as if the crack or notch were fully closed):

$$\sigma = \frac{P}{D}, \quad m = \frac{-6M}{D^2} = \frac{6P(w_{\max} + e)}{D^2} \tag{6}$$

The nominal stress σ_N can be defined as the average compression stress $\sigma = P/D$ in the cross section of unit width, while the nominal strength is the σ_N value at maximum load. From linear elastic fracture mechanics (LEFM), the mode I stress-intensity factor K as a function of the relative notch length α is known and can be written as

$$K = K(\alpha) = \left[mg_b(\alpha) - \sigma g_t(\alpha)\right]\sqrt{D} \tag{7}$$

where $g_b(\alpha)$, $g_t(\alpha)$ are functions giving the stress-intensity factor for unit m or σ (Gross and Srawley 1965; Fig. 2):

$$\begin{aligned} g_b(\alpha) &= \sqrt{\alpha}\Big(1.99 - 0.41\alpha + 18.70\alpha^2 - 38.48\alpha^3 \\ &\qquad + 53.85\alpha^4\Big) \\ g_t(\alpha) &= \sqrt{\alpha}\Big(1.99 - 2.47\alpha + 12.97\alpha^2 - 23.17\alpha^3 \\ &\qquad + 24.80\alpha^4\Big) \end{aligned} \tag{8}$$

Here the calculation of M must of course include the effect of deflection (geometrical nonlinearity).

Introduction of a crack or notch will cause additional displacement and rotation of one end of the column relative to the other. Since the effect of axial displacement on the critical load is negligible (e.g., Bažant and Cedolin 1991), only the additional rotation needs to be considered. Thus the cracked or notched cross section can be modeled as a rotational spring (Rice and Levy 1972). If the notch is fully opened, the additional rotation (in plane stress) is

$$\theta = \frac{12}{E}\left[C_b(\alpha)m - C_t(\alpha)\sigma\right] \tag{9}$$

where the dimensionless compliance coefficients $C_b(\alpha)$ and $C_t(\alpha)$ are computed from $g_b(\alpha), g_t(\alpha)$ as (Rice and Levy 1972):

$$\begin{aligned} C_b(\alpha) = \alpha^2 \Big(&1.98 - 3.28\alpha + 14.43\alpha^2 - 31.26\alpha^3 \\ &+ 63.56\alpha^4 - 103.36\alpha^5 + 147.52\alpha^6 \\ &- 127.69\alpha^7 + 61.50\alpha^8\Big) \end{aligned} \tag{10}$$

$$\begin{aligned} C_t(\alpha) = \alpha^2 \Big(&1.98 - 1.91\alpha + 16.01\alpha^2 - 34.84\alpha^3 \\ &+ 83.93\alpha^4 - 153.65\alpha^5 + 256.72\alpha^6 \\ &- 244.67\alpha^7 + 133.55\alpha^8\Big) \end{aligned} \tag{11}$$

The curves of these two functions are shown in Fig. 2 (Okamura et al. 1969, also used a rotational spring model, but ignored the effect of compression stress σ, i.e., assumed that $C_t(\alpha) = 0$).

The spring model is based on the assumption that the notch is fully opened. If the relative eccentricity ξ is very small, however, m can be smaller than σ, and the pre-existing crack or notch will then be fully closed at the beginning (see Fig. 3a). As the load P increases, m becomes larger than σ because of the increase of $w_{\max}$, and the notch will become partially opened (see Fig. 3b). The notch will be fully opened only when $w_{\max}$ or ξ is large enough (Fig. 3c), and it can propagate in a stable manner if P keeps increasing (Fig. 3d). In this paper, we will first assume the notch to be fully opened and calculate the deflection curve $w(x)$. Then we will discuss the criteria to judge the state of the crack or notch in detail.

When the cracked or notched cross section is modeled as a rotational spring, the equilibrium equation of the column (2) remains valid, and the well-known general solution is

$$w(x) = A\sin kx + B\cos kx - e \tag{12}$$

Constants A and B can be determined from the boundary conditions. Because of introducing the rotational spring, the boundary conditions are

$$w(0) = 0, \quad w'(L/2) = \frac{\theta}{2} = \frac{6}{E}\left[C_b(\alpha)m - C_t(\alpha)\sigma\right] \tag{13}$$

Here m is related to $w_{\max} = w(L/2)$ as given by (6). Substituting (6) and (12) into (13), one can obtain:

$$A = \frac{ek\sin\frac{kL}{2} + \frac{36C_b(\alpha)e\sigma}{ED}\cos\frac{kL}{2} - \frac{6C_t(\alpha)\sigma}{E}}{k\cos\frac{kL}{2} - \frac{36C_b(\alpha)\sigma}{ED}\sin\frac{kL}{2}}, \quad B = e \tag{14}$$

The maximum deflection of the column is

$$w_{\max} = \frac{ek\sin\frac{kL}{2} + \frac{36C_b(\alpha)e\sigma}{ED}\cos\frac{kL}{2} - \frac{6C_t(\alpha)\sigma}{E}}{k\cos\frac{kL}{2} - \frac{36C_b(\alpha)\sigma}{ED}\sin\frac{kL}{2}} \times \sin\frac{kL}{2} + e\left(\cos\frac{kL}{2} - 1\right) \tag{15}$$

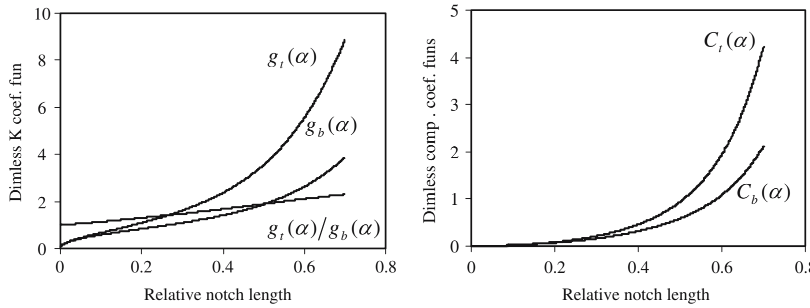

Fig. 2 (Left) Functions giving the dimensionless stress-intensity factor coefficients $g_b(\alpha)$ and $g_t(\alpha)$ defined in (8), and their ratio $g_t(\alpha)/g_b(\alpha)$ (α = relative notch length); (Right) functions giving dimensionless compliance coefficients $C_b(\alpha)$ and $C_t(\alpha)$ defined in (11)

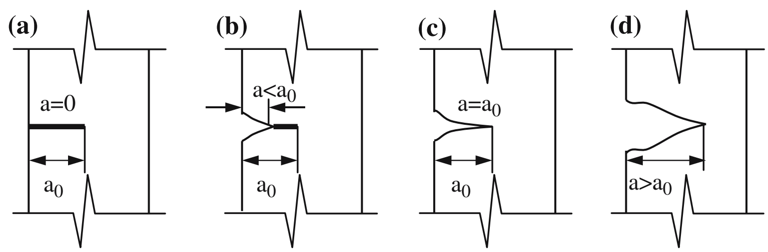

Fig. 3 Status of pre-existing crack or notch: (**a**) fully closed; (**b**) partially opened; (**c**) fully opened; (**d**) propagating

The critical nominal stress σ_{N1} can be computed by

$$k \cos \frac{kL}{2} - \frac{36C_b(\alpha)\sigma}{ED} \sin \frac{kL}{2} = 0 \tag{16}$$

If the notch is fully opened, A, as well as $w_{\max}$, will be infinite when the nominal stress σ approaches σ_{N1}. Thus we know that $P_1 = \sigma_{N1}D$ = the buckling load of the notched column. So σ_{N1} can be called the "nominal buckling strength" of the notched column. If there is no notch, the buckling load is the well-known Euler load $P_0 = \pi^2 EI/L^2$, and the corresponding nominal stress σ_{N0} is:

$$\sigma_{N0} = \frac{\pi^2 EI/L^2}{D} = \frac{\pi^2}{12\lambda^2} E \tag{17}$$

where σ_{N0} may be called the "initial nominal buckling strength". The ratio σ_{N1}/σ_{N0} indicates the ultimate load capacity of the notched column, provided that the notch is fully opened and will not propagate.

To compute σ_{N1}, (16) can be rewritten as

$$\cot\left(\lambda\sqrt{\frac{3\sigma}{E}}\right) = 6C_b(\alpha)\sqrt{\frac{3\sigma}{E}} \tag{18}$$

where $k = \sqrt{P/EI} = 2\sqrt{3\sigma/E}/D$. Thus we see that σ_{N1} depends on λ and $C_b(\alpha)$, but is independent of ξ and $C_t(\alpha)$. Letting $z = \lambda\sqrt{3\sigma/E}$ and $\kappa = 6C_b(\alpha)/\lambda$, we can simplify the equation as $\cot z = \kappa z$. Using the approximation formula derived in Appendix A, we can represent the nominal buckling strength of notched column, σ_{N1}, as

$$\sigma_{N1} = \frac{E/3\lambda^2}{(2/\pi)^2 + 6C_b(\alpha)/\lambda} \tag{19}$$

One can also use a better but more complex formula

$$\sigma_{N1} = \frac{E}{3\lambda^2}\left[\left(\frac{2}{\pi}\right)^2 - \left(1 - \frac{8}{\pi^2}\right)\frac{6C_b(\alpha)/\lambda}{1 + 15.6C_b(\alpha)/\lambda} + \frac{6C_b(\alpha)}{\lambda}\right]^{-1} \tag{20}$$

where $C_b(\alpha)$ can be computed from (11). The value σ_{N1} predicted by (19) is a little smaller (not more than by 2.5% less) than the actual value, which means it is a safe and good estimate. The error of (20) is smaller than 0.1%. The ratio σ_{N1}/σ_{N0} is shown in Fig. 4 for some typical values of λ.

3 Load–deflection response

Because the boundary condition $w(0) = 0$ is always valid, regardless of the state of the crack or notch, we know that the constant B in the general

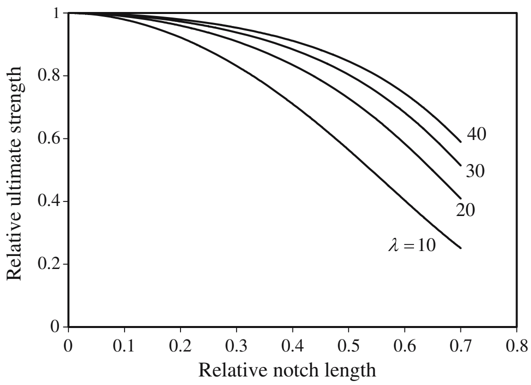

Fig. 4 The relative ultimate strength σ_{N1}/σ_{N0} (λ = slenderness of column)

solution (12) will always be e, and only the constant A will be affected by the situation of the notch. If we know $w_{\max}$, A can be computed and $w(x)$ can then be determined. So the plot of σ_N versus $\bar{w}_{\max} + \xi$ can be used to represent the load–deflection response.

If the notch is fully opened from the beginning, (12), (14) and (15) already define the load–deflection response of the notched column ($\alpha = \alpha_0$ here). But the notch is not always fully opened at the beginning. From Eq. 7 for the stress-intensity factor K, one knows the notch will be fully opened if and only if $K(\alpha_0) \geq 0$ (see also Okamura et al. 1969). Therefore

$$\bar{w}_{\max} + \xi \geq \frac{g_t(\alpha_0)}{6 g_b(\alpha_0)} = \frac{1}{6} F(\alpha_0) \tag{21}$$

where $F(\alpha) = g_t(\alpha)/g_b(\alpha)$; the curve of $F(\alpha)$ is shown in Fig. 2. Because, at the beginning, $w_{\max} = 0$, the criterion for the notch to be fully opened from the beginning is

$$\xi \geq F(\alpha_0)/6 = \xi_2 \tag{22}$$

Thus we can determine the load–deflection curve as (15) if (22) is satisfied. If $m \leq \sigma$, the combined stress at the notch mouth is compressive or zero, and then the notch will be fully closed (Okamura et al. 1969). So the criterion for notch to be fully closed is

$$\bar{w}_{\max} + \xi \leq 1/6 \tag{23}$$

The condition for the notch to be fully closed at the beginning is

$$\xi \leq 1/6 = \xi_1 \tag{24}$$

Thus, if $\xi \leq 1/6$, the notch will be fully closed at the beginning. In this situation, the load–deflection response is characterized by (5) until the value of $\bar{w}_{\max} + \xi$ reaches 1/6. When $\bar{w}_{\max} + \xi = 1/6$, the corresponding nominal stress σ'_{N1} can be obtained from (5):

$$\sigma'_{N1} = \frac{\arccos^2 6\xi}{3\lambda^2} E \tag{25}$$

If $1/6 < \bar{w}_{\max} + \xi < F(\alpha_0)/6$, the notch will be partially opened. In this case, the current relative notch length $\alpha = \chi\alpha_0$ $(0 < \chi < 1)$ can be calculated from $K(\alpha) = 0$, or

$$\bar{w}_{\max} + \xi = F(\alpha)/6 = F(\chi\alpha_0)/6 \tag{26}$$

This is consistent with (21) and (23) because $F(0) = 1$ and $F(\alpha)$ is an increasing function of α, as shown in Fig. 2. So, at the beginning, the notch will be partially opened if and only if

$$1/6 < \xi < F(\alpha_0)/6 \tag{27}$$

When the notch is partially opened, the load–deflection response can be determined by the following procedure. If, for given $\bar{w}_{\max}$, we have $1/6 < \bar{w}_{\max} + \xi < F(\alpha_0)/6$, the current relative notch length α can be computed from (26). Then the nominal stress σ_N can be obtained from (15) because it is the only unknown variable there. When $\bar{w}_{\max} + \xi$ reaches $F(\alpha_0)/6$, the crack is fully opened ($\alpha = \alpha_0$) and the corresponding nominal stress σ'_{N2} can be obtained by solving the following equation:

$$\xi + \sqrt{\frac{3\sigma}{E}} \left[C_b(\alpha_0)F(\alpha_0) - C_t(\alpha_0)\right] \sin\left(\lambda\sqrt{\frac{3\sigma}{E}}\right) - \frac{F(\alpha_0)}{6} \cos\left(\lambda\sqrt{\frac{3\sigma}{E}}\right) = 0 \tag{28}$$

Obviously, σ'_{N2} is a smooth function of ξ, as well as λ and α_0. The value of σ'_{N2} is 0 if $\xi \geq F(\alpha_0)/6$.

So, different ξ values will lead to different load–deflection response. Figure 5 shows various load–deflection curves for a notched column ($\lambda = 30$,

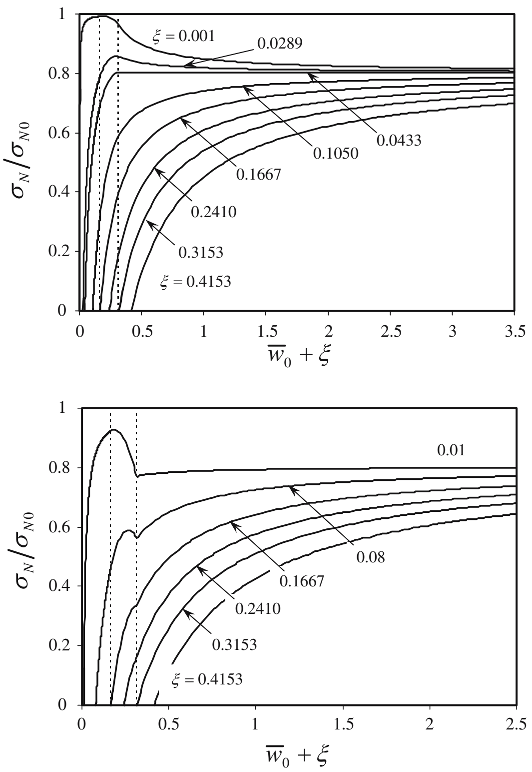

Fig. 5 (Top) Various load–deflection curves for a notched column ($\lambda = 30, \alpha_0 = 0.5$), in which $\xi_c = 0.0433$ and $\sigma_{N1} = 0.8037$. The dotted lines are $\bar{w}_0 + \xi = 1/6$ and $F(\alpha_0)/6$ (=0.3153). (Bottom) Various load–deflection curves for a notched column using Okamura et al.'s model ($\lambda = 30, \alpha_0 = 0.5$). The dotted lines are $\bar{w}_0 = 1/6$ and $F(\alpha_0)/6$ (=0.3153)

$\alpha_0 = 0.5$). Two cases may be distinguished: (i) If ξ is very small, and if the crack or notch is fully closed (that is, $\bar{w}_{\max} + \xi \leq 1/6$), the nominal stress σ can be very close to the initial nominal buckling strength σ_{N0}, while after the crack or notch partially or fully opens, the nominal stress decreases and eventually approaches σ_{N1}. (ii) On the other hand, if ξ is large enough, the nominal stress will always be smaller than σ_{N1}.

There is a critical case between these two cases. When $\xi = \xi_c$, the nominal stress σ_N reaches σ_{N1} at the point $\bar{w}_{\max} + \xi = F(\alpha_0)/6$, and then keeps unchanged while the deflection can increase arbitrarily. At this critical case, constant A in (14) should be indeterminate, that is,

$$ek \sin \frac{kL}{2} + \frac{36C_b(\alpha_0)e\sigma}{ED} \cos \frac{kL}{2} - \frac{6C_t(\alpha_0)\sigma}{E} = 0 \tag{29}$$

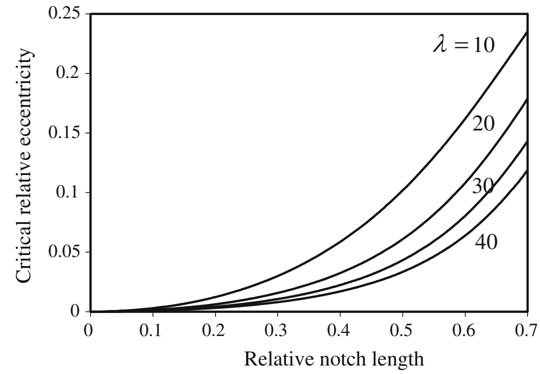

Fig. 6 Critical relative eccentricity ξ_c

at $\sigma = \sigma_{N1}$. Since σ_{N1} satisfies (16), the critical relative eccentricity ξ_c is

$$\xi_c = C_t(\alpha_0) \left[36C_b^2(\alpha_0) + E/3\sigma_{N1}\right]^{-1/2} \tag{30}$$

Figure 6 shows ξ_c as a function of α for typical λ.

4 Fracture and size effect analysis

Up to now, all the nominal stresses we defined (e.g., $\sigma_{N0}, \sigma_{N1}, \sigma'_{N1}$ and σ'_{N2}) have been size-independent. This is to be expected because stability analysis as such introduces no characteristic length and thus no transitional size effect. No such size effect can arise unless some material characteristic length is present.

If LEFM is applied to this problem, the notch will propagate when the mode I fracture toughness, K_c, is reached. If $\xi < \xi_c$, the maximum nominal stress σ_N is achieved before the notch is fully opened, and then σ_N will decrease and eventually approach σ_{N1}. So, in this case, we can use σ_{N1} as the column strength and need not consider propagation. This also holds when $\xi = \xi_c$. So we need to consider only the situation where $\xi > \xi_c$.

Let the nominal stress at the beginning of crack propagation be defined as the "nominal fracture strength", σ_{N2}. σ_{N2} can be obtained by solving $K(\alpha) = [g_b(\alpha)m - g_t(\alpha)\sigma]\sqrt{D} = K_c$, that is,

$$K = \left[6(\bar{w}_{\max} + \xi)g_b(\alpha) - g_t(\alpha)\right]\sigma\sqrt{D} = K_c \tag{31}$$

Here $\alpha = \alpha_0$. When $K(\alpha_0) > 0$, the notch is fully opened, and the maximum deflection may then be obtained from (15). Equation 31 can be rewritten as

$$\xi + \sqrt{\frac{3\sigma}{E}}\left\{C_b(\alpha)\left[\frac{K_c}{\sigma\sqrt{D}g_b(\alpha)} + F(\alpha_0)\right] - C_t(\alpha_0)\right\}\sin\left(\lambda\sqrt{\frac{3\sigma}{E}}\right) - \left[\frac{K_c}{6\sigma\sqrt{D}g_b(\alpha)} + \frac{F(\alpha_0)}{6}\right]\cos\left(\lambda\sqrt{\frac{3\sigma}{E}}\right) = 0 \quad (32)$$

which is similar to (28). From (31), we observe that σ_{N2} depends on D. The larger the size D, the smaller the fracture strength σ_{N2}. This is because K_c, together with material strength, implies the existence of a material characteristic length.

First consider two asymptotic cases, $D \to 0$ and $D \to \infty$. Then we can derive the size effect property and construct an asymptotic matching approximate formula for σ_{N2}.

When $D \to 0$, we know from (31) that $\bar{w}_{\max}$ must be infinitely large because $\sigma < \sigma_{N1}$. This means that

$$\sigma_{N2} \to \sigma_{N1} \quad \text{when } D \to 0 \quad (33)$$

When $D \to \infty$, however, the asymptotic behavior is more complex. There are three cases: (i) $\xi < F(\alpha_0)/6$, (ii) $\xi > F(\alpha_0)/6$, and (iii) $\xi = F(\alpha_0)/6$.

If $\xi < F(\alpha_0)/6$, we know that $\bar{w}_{\max} + \xi$ reaches $F(\alpha_0)/6$ at $\sigma = \sigma'_{N2} > 0$ and $K = 0$ at this point. When $D \to \infty$, from (31), we know that $6(\bar{w}_{\max} + \xi)g_b(\alpha) - g_t(\alpha) \to 0$ because $\sigma > \sigma'_{N2}$. This means that $\sigma_{N2} \to \sigma'_{N2}$ when $D \to \infty$. Denote $\sigma_{N2} = \sigma'_{N2} + \Delta\sigma$ ($\Delta\sigma > 0$), and assume that $\mathrm{d}\bar{w}_{\max}/\mathrm{d}\sigma = H$ at $\sigma = \sigma'_{N2}$ (from Eq. 15, or from the load–deflection curve, it follows that $H > 0$). Then, at $\sigma = \sigma_{N2}$, we have $6(\bar{w}_{\max} + \xi)g_b(\alpha) - g_t(\alpha) \approx 6Hg_b(\alpha)\Delta\sigma$ and

$$6Hg_b(\alpha)\Delta\sigma\left[\sigma'_{N2} + \Delta\sigma\right]\sqrt{D} = K_c \quad (34)$$

It is not difficult to find that the foregoing equation holds when $\xi = F(\alpha_0)/6$ (but $\sigma'_{N2} = 0$ in this case). Based on (34), we have

$$\sigma_{N2} - \sigma'_{N2} = \Delta\sigma = \frac{\sqrt{\left[6Hg_b(\alpha)\sigma'_{N2}\right]^2 D + 24K_cHg_b(\alpha)\sqrt{D}} - 6Hg_b(\alpha)\sqrt{D}\sigma'_N}{12Hg_b(\alpha)\sqrt{D}} \quad (35)$$

If $\xi < F(\alpha_0)/6$ (then $\sigma'_{N2} > 0$), the asymptotic size effect can be derived as

$$\sigma_{N2} - \sigma'_{N2} = \Delta\sigma \approx \frac{K_c}{6Hg_b(\alpha)\sigma'_{N2}\sqrt{D}} \propto D^{-1/2}, \quad \text{when } D \to \infty \quad (36)$$

where $\alpha = \alpha_0$. When $\xi = F(\alpha_0)/6$, we have $\sigma'_{N2} = 0$, and the corresponding asymptotic size effect yields

$$\sigma_{N2} = \Delta\sigma \approx \sqrt{\frac{K_c}{6Hg_b(\alpha)}}D^{-1/4} \propto D^{-1/4}, \quad \text{when } D \to \infty \quad (37)$$

which also ensues naturally from

$$6Hg_b(\alpha)\,(\Delta\sigma)^2\sqrt{D} = K_c \quad (38)$$

Since σ'_{N2} is a smooth function of ξ if the structural geometry is fixed (that is, if λ and α_0 are given constants), we know that $\Delta\sigma$ is a smooth function of ξ and D. Now it is interesting to find that a mathematically smooth function can have discontinuous asymptotic behavior: the exponent of the asymptotic power-law size effect jumps from $-1/2$ to $-1/4$ at $\xi = F(\alpha_0)/6$.

If $\xi > F(\alpha_0)/6$, we still have $\sigma'_{N2} = 0$. When $D \to \infty$, we know that $\sigma_{N2} \to 0$ in (31) because the first term on the left-hand side of (31), i.e. $6(\bar{w}_{\max} + \xi)g_b(\alpha_0) - g_t(\alpha_0) \geq [6\xi - F(\alpha_0)]g_b(\alpha_0)$. Similarly, we may denote $\sigma_{N2} = \Delta\sigma$, and assume $\mathrm{d}\bar{w}_{\max}/\mathrm{d}\sigma = H$ at $\sigma = 0$ (H is also positive here). Then, at $\sigma = 0$, one has $\Delta\bar{w}_{\max} \approx H\Delta\sigma$ and

$$\left[6H\Delta\sigma + 6\xi - F(\alpha)\right]g_b(\alpha)\Delta\sigma\sqrt{D} = K_c \quad (39)$$

This is also true for $\xi = F(\alpha)/6$. Hence $\Delta\sigma$ can be computed as

$$\sigma_{N2} = \Delta\sigma = \frac{\sqrt{\left[6\xi - F(\alpha)\right]^2 g_b^2(\alpha)D + 24K_cHg_b(\alpha)\sqrt{D}} - \left[6\xi - F(\alpha)\right]g_b(\alpha)\sqrt{D}}{12Hg_b(\alpha)\sqrt{D}} \quad (40)$$

When $\xi > F(\alpha)/6$, the asymptotic size effect is

$$\sigma_{N2} = \Delta\sigma \approx \frac{K_c}{[6\xi - F(\alpha)]g_b(\alpha)\sqrt{D}} \propto D^{-1/2},$$
$$\text{when } D \to \infty \quad (41)$$

where $\alpha = \alpha_0$. If $\xi = F(\alpha)/6$, it is not surprising to find that the asymptotic behavior remains characterized by (37). Once again, $\Delta\sigma$ is a smooth function of ξ and D (provided that λ and α_0 are constants for given geometry), but the asymptotic behavior is not smooth: the exponent jumps from $-1/4$ to $-1/2$ when ξ increases from $F(\alpha)/6$.

When $\xi < F(\alpha_0)/6$ or $\xi > F(\alpha_0)/6$, the exponent of the asymptotic size effect is $-1/2$, which is similar to the normal size effect caused by LEFM (Bažant and Planas, 1998; Bažant, 2004, 2005). When $\xi = F(\alpha_0)/6$, the exponent is different. This is caused by the second-order theory. In classical LEFM without second-order geometric effects, the stress-intensity factor K can be represented as

$$K = f(\alpha)\sigma\sqrt{D} \quad (42)$$

where $f(\alpha)$ is a function of α only. In the second-order theory, though, Eq. 42 does not hold anymore. It must be revised as

$$K = f(\alpha, \sigma)\sigma\sqrt{D} \quad (43)$$

where $f(\alpha, \sigma)$ is a function of both α and σ. This is verified by (34), (38) and (39). As shown in (7), the nominal moment m is involved in the stress-intensity factor K. However, m includes some second-order term of $\Delta\sigma$, because m = distance times σ, while the distance in second-order theory depends on $\Delta\sigma$. When $\xi < F(\alpha_0)/6$ or $\xi > F(\alpha_0)/6$, although there is a second-order term of $\Delta\sigma$ in K, the first-order term of $\Delta\sigma$ will dominate for $D \to \infty$ because $\Delta\sigma \to 0$. Hence the asymptotic size effect is still similar to the classical LEFM, in which there is only the first-term of $\Delta\sigma$ in K. When $\xi = F(\alpha_0)/6$, however, the first-order term of $\Delta\sigma$ vanishes and only the second-order term $\Delta\sigma$ (38) is left. So the exponent in the asymptotic size effect naturally drops from $-1/2$ to $-1/4$.

Since here $\sigma'_{N2} = 0$, we can also write (41) as $\sigma_{N2} - \sigma'_{N2} \propto D^{-1/2}$. Based on the asymptotic scaling properties obtained, we can construct approximate asymptotic matching formulas. When $\xi < F(\alpha_0)/6$ or $\xi > F(\alpha_0)/6$, the following formula gives a good fit of σ_{N2}:

$$\sigma_{N2} = \sigma'_{N2} + \left(\sigma_{N1} - \sigma'_{N2}\right)\left[1 + \left(\frac{D}{D_0}\right)^r\right]^{-\frac{1}{2r}} \quad (44)$$

where parameter r governs the abruptness or smoothness of the size effect transition between the asymptotic power laws. The value of D_0 can be obtained from the large-size asymptote. When $\xi < F(\alpha_0)/6$, it follows from Eqs. 36 and 44 that

$$D_0 = \left[\frac{K_c}{6Hg_b(\alpha)\sigma'_{N2}(\sigma_{N1} - \sigma'_{N2})}\right]^2 \quad (45)$$

One example is shown in Fig. 7 (in which $D_0 = 3.548 \times 10^{-5}, r = 0.45$). If $\xi > F(\alpha_0)/6$, based on (41) and (44) we have

$$D_0 = \left\{\frac{K_c}{[6\xi - F(\alpha)]g_b(\alpha)(\sigma_{N1} - \sigma'_{N2})}\right\}^2 \quad (46)$$

The fit is shown in Fig. 7 (in which $D_0 = 2.218 \times 10^{-4}, r = 0.31$). Because the asymptotic scaling property at $\xi = F(\alpha_0)/6$ is different from other cases, a separate approximate formula is needed, and it may be written as:

$$\sigma_{N2} = \sigma_{N1}\left[1 + \left(\frac{D}{D_0}\right)^r\right]^{-\frac{1}{4r}} \quad (47)$$

Similarly as before, D_0 can be determined from the asymptote (37):

$$D_0 = \left[\frac{K_c}{6Hg_b(\alpha)\sigma_{N1}^2}\right]^2 \quad (48)$$

Figure 7 shows the result of fitting formula (47) (in which $D_0 = 1.585 \times 10^{-5}, r = 0.36$).

The start of notch propagation does not necessarily imply that a peak load is reached. For structures with negative geometry, the load can increase even after the crack begins to propagate. To determine the type of fracture geometry, we need to analyze the load–deflection response during crack propagation. When the notch propagates, the stress-intensity factor $K = K_c$ according to

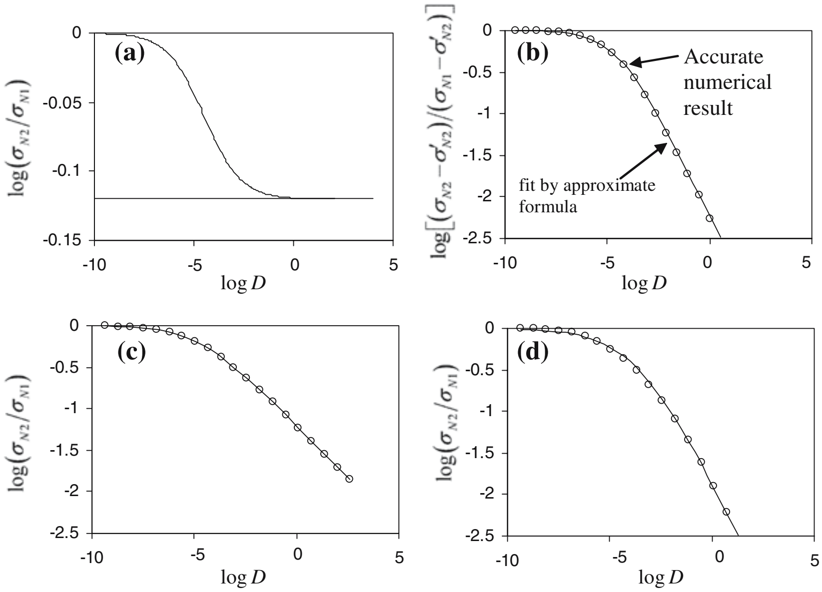

Fig. 7 (**a**) Size effect curve of the nominal fracture strength σ_{N2} for $\xi < F(\alpha_0)/6$,$\sigma_{N2} \to \sigma'_{N2}$ when $D \to \infty$, here $\lambda = 30, \alpha_0 = 0.5, \xi = 0.1$; (**b**) fit the previous case by approximate formula (44), here $D_0 = 3.548 \times 10^{-5}$ m, $r = 0.45$; (**c**) size effect curve of the nominal fracture strength σ_{N2} and fit by approximate formula (47) for $\xi = F(\alpha_0)/6$,$\sigma_{N2} \to 0$ when $D \to \infty$, here $\lambda = 30, \alpha_0 = 0.5, \xi = 0.3153, D_0 = 1.585 \times 10^{-5}$ m, $r = 0.36$. The slope of this curve at $D \to \infty$ is different from other curves; (**d**) size effect curve of the nominal fracture strength σ_{N2} and fit by approximate formula (44) for $\xi > F(\alpha_0)/6$, $\sigma_{N2} \to 0$ when $D \to \infty$, here $\lambda = 30, \alpha_0 = 0.5, \xi = 0.4, D_0 = 2.218 \times 10^{-4}$ m, $r = 0.31$

LEFM must be constant. Thus the relation between σ_N and α is still governed by (31). So the load–deflection response can be obtained by increasing α gradually. At the initial state, $\alpha = \alpha_0$. First the relative crack length is increased to $\alpha = \alpha_0 + \Delta\alpha$, where $\Delta\alpha$ is very small. σ_N can be computed by solving (32), and $\bar{w}_{\max}$ can then be obtained from (15).

One example is shown in Fig. 8. Note that the nominal stress and the deflection are not necessarily increasing while the notch propagates. When it does, we have

$$\mathrm{d}K = (\partial K/\partial\alpha)\mathrm{d}\alpha + (\partial K/\partial\sigma)\mathrm{d}\sigma = 0 \tag{49}$$

To judge whether the geometry is negative or positive, we need to compute $\partial K/\partial\alpha$. Generally we have $\partial K/\partial\sigma > 0$ (we will see that even this assumption is not always true in the case of a stability problem). If $\partial K/\partial\alpha > 0$, one can get $\mathrm{d}\sigma < 0$ from (49), that is, the nominal stress decreases, which means positive geometry. When $\partial K/\partial\alpha < 0$, we have $\mathrm{d}\sigma > 0$, which means negative geometry.

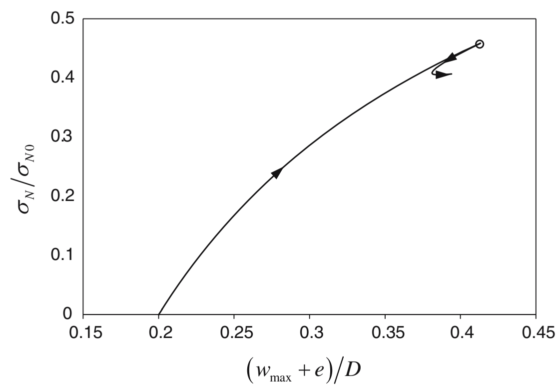

Fig. 8 Load–deflection curve of column taking into account crack propagation, in which the nominal stress and deflection are not necessarily increasing during the propagation ($\lambda = 30, \alpha_0 = 0.1, \xi = 0.2, D_0 = 0.001$ m, $r = 0.31, E = 3.5 \times 10^{10}$ N/m^2, $K_c = 3.65 \times 10^5$ N/m$^{-3/2}$

From (31), we have

$$\frac{\partial K}{\partial\alpha} = \left[6\frac{\partial\bar{w}_{\max}}{\partial\alpha}g_b(\alpha) + \bar{w}_{\max}g'_b(\alpha) - g'_t(\alpha)\right]\sigma\sqrt{D} \tag{50}$$

Because the value of $\partial\bar{w}_{\max}/\partial\alpha$ depends on σ, while σ depends on the size D, we find that the sign of $\partial K/\partial\alpha$ will depend on size D. From (43), we know that the sign of $\partial K/\partial\alpha = \partial f(\alpha,\sigma)/\partial\alpha$ depends on σ, and eventually it depends on D because of the dependence of σ on D. The physical meaning is also clear—in LEFM, the deformation profile, or the current geometry, is assumed to be unchanged when the load (or the nominal stress) increases, and this is why $f(\alpha)$ is independent of σ. But in the second-order theory, required for stability problems, the type of fracture geometry depends on the level of nominal stress. For geometrically similar structures of different sizes, the type of fracture geometry remains the same only at beginning of loading. When the notch starts to propagate, the level of nominal stress is different for structures of different sizes, and obviously the current type of fracture geometry is also different.

So, whether the fracture geometry is positive or negative, depends in stability problems on the structure size.

Although $\partial K/\partial\alpha$ based on (31) or $\mathrm{d}\sigma/\mathrm{d}\alpha$ based on (32) are both too complex to be written analytically, we can easily determine their sign in the asymptotic cases.

When $D \to 0$, we have $\sigma_{N2} \approx \sigma_{N1}$. Because $\mathrm{d}\sigma_{N1}/\mathrm{d}\alpha < 0$, the geometry must be positive for sufficiently small structure sizes. This observation is independent of ξ. However, when $D \to \infty$, the sign of geometry depends on ξ. Because, in Eqs. 36, 41 and 37, the variables H and σ'_{N2} both depend on α,

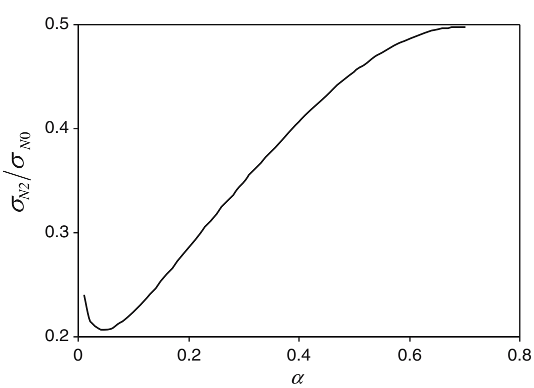

Fig. 9 Relation between nominal fracture strength σ_{N2} and relative length α of pre-existing crack or notch ($\lambda = 30, \xi = 0.15, D = 1\,\mathrm{m}, E = 3.5 \times 10^{10}\,\mathrm{N/m^2}, K_c = 3.65 \times 10^5\,\mathrm{N/m^{-3/2}}$)

we do not have a simple conclusion for this asymptotic case. Actually, the load–deflection curve during notch propagation can be very complex. For example, when the geometry parameters are $\lambda = 30, \xi = 0.15, D = 1\mathrm{m}$, the material parameters are $E = 3.5 \times 10^{10}\,\mathrm{N/m^2}, K_c = 3.65 \times 10^5\,\mathrm{N/m^{-3/2}}$, and the corresponding plot of σ_{N2} versus α is shown in Fig. 9. If the initial relative crack length is very small (e.g., <0.03), the geometry is positive. After the nominal fracture strength is reached and the notch starts to propagate, the nominal stress will first decrease but then increase after the relative notch length exceeds 0.05.

5 Comparisons with Okamura et al.'s model and experiments

Okamura et al. (1969) also applied the rotational spring to model the notched column under eccentric compression, and used Eq. 7 to compute the stress-intensity factor. However, they ignored the effect of σ on the rotation, which means that $C_t(\alpha) = 0$ in their model. One may check that, if $C_t(\alpha)$ is set to 0, all of the present analysis is still valid. Thus Eq. 15 (but with $C_t(\alpha)$=0) may be used to determine the load–deflection response of Okamura et al.'s model if the notch is fully opened. Since σ_{N1} is independent of $C_t(\alpha)$, the nominal buckling stress for our model and Okamura et al.'s model is the same.

If the notch is fully closed, then, of course, the load–deflection response can be described by (5). When the notch is partially opened, the same procedure can be followed to determine the load–deflection curve. Then it is found from Eq. 29 that $\xi_c = 0$ in Okamura et al.'s model. What does that mean? Figure 5 shows various load–deflection curves of Okamura et al.'s model. The figure makes it clear that the nominal stress cannot be larger than the nominal buckling stress σ_{N1} when the notch is fully opened. Even in the case of very small ξ, the nominal stress will decrease quickly when the crack partially opens, and it will always be smaller than σ_{N1} when the notch is fully opened. Figure 5 shows that when the notch is fully opened, the difference between these two models is very small (normally less than 2%). This is reasonable to expect because the larger the $\bar{w}_{\max} + \xi$, the smaller

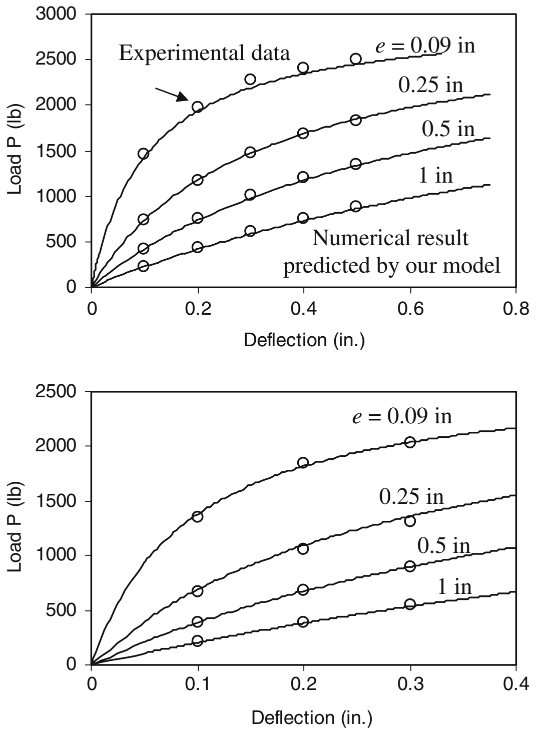

Fig. 10 Comparison with experiments ($\lambda = 26.5, D = 0.5$ in, $E = 10.5 \times 10^6$ psi): (Top) load–deflection curve for a column with relative length $\alpha = 0.18$ of pre-existing crack or notch; (Bottom) load–deflection curve for a column with $\alpha = 0.39$

the effect of σ on the rotation. When $\bar{w}_{\max} + \xi < 1/6$, the notch is fully closed and the load–deflection responses for these two models are the same. So the difference between these two models is only significant when the notch is partially opened and ξ is very small.

Okamura et al. (1969) compared their model with the experimental results of Liebowitz et al. (1967). For the same load, the deflection predicted by Okamura et al.'s model is always larger than the experimental result. This is a consequence of the fact that they ignored the effect of σ on rotation. Because stress σ is always compressive, it tends to close the notch and reduce the rotation, which will reduce the deflection. So the deflection predicted by the present model will be smaller than that computed from Okamura et al.'s model. That is, our model is more accurate, though the differences between these two models are quite small.

The results of experiments are compared to the present model in Fig. 10, in which $\alpha = 0.18$ and 0.39. The other parameters are $\lambda = 26.5, D = 0.5$ in, $E = 10.5 \times 10^6$ psi.

6 Conclusions

1. A nonpropagating pre-existing transverse edge crack or notch reduces the nominal buckling strength, σ_{N1}, of a column under eccentric compression. This is described by two approximate formulas based on asymptotic matching. One formula is simple and accurate enough for engineering purposes (with an error less than 2.5%). The other is less simple but more accurate (with an error less than 0.1%).
2. The greater the slenderness of the column, the smaller the strength reduction due to pre-existing transverse crack or notch.
3. Depending on the load eccentricity and the lateral deflection, the pre-existing crack or notch at maximum load can be fully opened, fully closed, or partially opened. For different relative eccentricities of load, the size effect is of different types. The power-law exponent of the large-size asymptotic size effect is $-1/2$ in general cases, but it jumps to $-1/4$ when the relative eccentricity equals a certain particular value.
4. Depending on the column size, the maximum load can occur at the beginning of crack propagation or after a certain stable crack growth. This demonstrates that, in stability problems, the distinction between positive or negative fracture geometries is not size independent. Whether the crack grows at decreasing or increasing load depends not only on structure geometry (or shape) but also on the structure size.
5. The results of analysis are in satisfactory agreement with the existing experimental results.

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Appendix A: Approximation Formula of Equation $\cot z = \kappa z$

For the equation $\cot z = \kappa z$ ($\kappa \geq 0$ here), assume the solution is $z = f(\kappa)$. The numerical result of

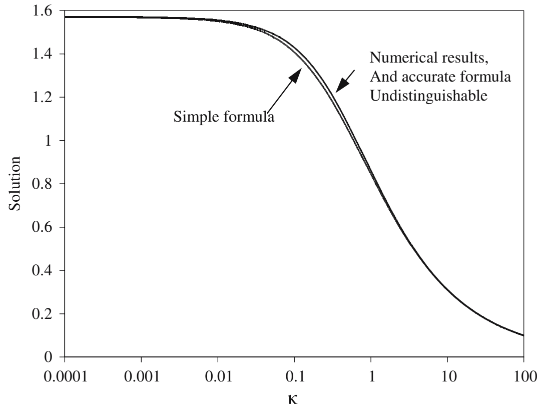

Fig. 11 Solution of equation $\cot z = \kappa z$: numerical results and approximation formulae

$f(\kappa)$ is shown as Fig. 11, our objective is to find an approximation formula for $f(\kappa)$.

When $\kappa \to \infty$, $z \to 0$, we have

$$\kappa z = \cot z = \frac{1}{\tan z} \approx \frac{1}{z}, \quad \text{when } \kappa \to \infty \tag{51}$$

and the approximate solution is

$$f(\kappa) \approx \frac{1}{\sqrt{\kappa}}, \text{ when } \kappa \to \infty \tag{52}$$

When $\kappa \to 0$, $z \to \pi/2$, we have

$$\kappa z = \cot z = \tan\left(\frac{\pi}{2} - z\right) \approx \frac{\pi}{2} - z, \text{ when } \kappa \to 0 \tag{53}$$

and the approximate solution is

$$f(\kappa) \approx \frac{\pi/2}{1+\kappa} \approx \frac{\pi}{2}(1-\kappa), \quad \text{when } \kappa \to 0 \tag{54}$$

Because $\kappa \to 0$, this can further be simplified to

$$f(\kappa) \approx \pi/2, \quad \text{when } \kappa \to 0 \tag{55}$$

A simple approximation satisfying (52) and (55) is

$$f_1(\kappa) = \left[\left(\frac{2}{\pi}\right)^2 + \kappa\right]^{-1/2} \tag{56}$$

and we have

$$0.975 f(\kappa) \le f_1(\kappa) \le f(\kappa), \quad \text{for any } \kappa \ge 0 \tag{57}$$

This means that the fit is quite close (Fig. 11).

When $\kappa \to 0$, Eq. 56 implies that $f_1(\kappa) \approx \pi/2 - \pi^3\kappa/16$. This does not satisfy the first order asymptote (54). A better approximation formula can be constructed by making it satisfy (54) (Fig. 11):

$$f_2(\kappa) = \left[\left(\frac{2}{\pi}\right)^2 - \left(1 - \frac{8}{\pi^2}\right)\frac{\kappa}{1+\eta\kappa} + \kappa\right]^{-1/2} \tag{58}$$

The value of η can be obtained by data fitting. When $\eta = 2.6$, the error is within 0.1%:

$$0.999 f(\kappa) \le f_2(\kappa) \le 1.0002 f(\kappa) \tag{59}$$

This also confirms that, as expected, an asymptotic matching approximation is better than an non-matching one.

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