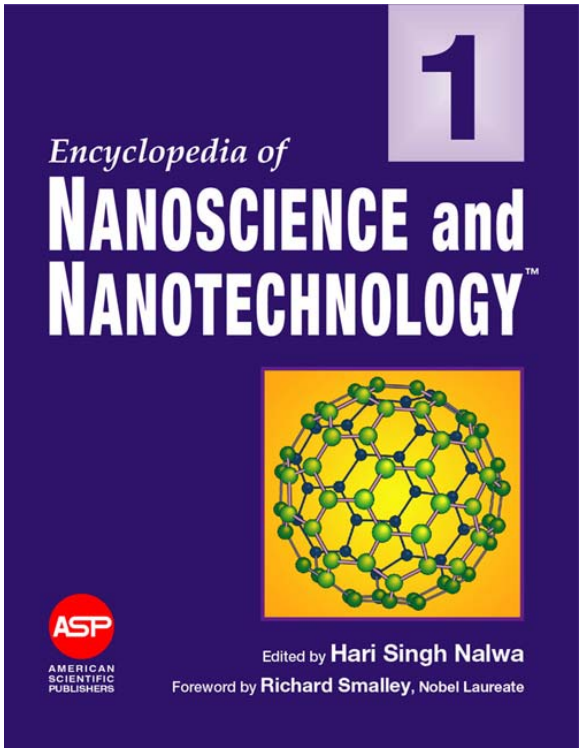

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4. THEORETICAL MODELING AND SCALING

During the 1980s and 1990s, a host of experiments on the micrometer and submicrometer scale, including microindentation [15], microtorsion [29, 30], and microbending [28], revealed a strong size effect on the yield strength and hardening of metals. Similar size effects were observed also in metal matrix composites with particle diameters in the micrometer and submicrometer scale [255, 256]. The classical plasticity theories cannot predict these size effects because they involve no material characteristic length. To explain them, several strain gradient theories were developed. The first one was a phenomenological theory by Fleck and Hutchinson [30] based on the existence of a potential. This theory was later extended and improved in several versions [31, 257, 258] while retaining the same basic structure. Another strain gradient theory which received considerable attention was the mechanism-based strain-gradient (MSG) theory [33, 35] derived under certain simplifying assumptions from the concept of geometrically necessary dislocations. Based on numerical experience, this theory was recently improved as the Taylor-base nonlocal (TNT) theory [259], and the improvement consisting in the form of strain gradient dependence of the hardening function made the theory conform to a revision proposed by Bažant [260, 261] on the basis of scaling analysis. Another noteworthy theory was the Acharya and Bassani strain gradient plasticity theory based on the idea of lattice incompatibility [262, 263], which represented a generalization of the incremental theory of plasticity. The asymptotic characters of these strain gradient theories were analyzed recently and it was found that the small-size asymptotic size effect predicted by some of the theories is excessive and unreasonable [259–261].

It might seem that the small-size asymptotic behavior of gradient plasticity is irrelevant because it is approached only at sizes below the range of validity of theory, for which the spacing of the geometrically necessary dislocations (about 10 to 100 nm) and the crystal size are not negligible, and other physical phenomena, such as surface tension, gradation of crystal size, and texture, intervene. However, knowledge of both the small-size and large-size asymptotics is very useful for developing asymptotic matching approximations for the intermediate range, for which the solutions are much harder to obtain. For the purpose of asymptotic matching, the asymptotic behavior must be physically reasonable even if attained outside the range of validity of the theory (this has been demonstrated in the modeling of cohesive fracture when the small-size plastic asymptote is often approached only for specimen sizes much smaller than the inhomogeneity size, e.g., the aggregate size in concrete [260]).

The present chapter reviews and summarizes several recent papers in which it was shown that the main theories proposed in the past, including couple stress theory, stress and rotation gradient theory, MSG, TNT, and the Acharya and Bassani theory, suffer from excessive asymptotic size effect and some exhibit an unrealistic shape of the load-deflection curve. Simple adjustments of all these theories suffice to achieve reasonable asymptotic behavior and thus to make asymptotic matching approximations feasible.

The main strain gradient theories will be briefly introduced and their asymptotic analysis presented by Bažant and Guo [264] will be outlined. After that, a simple asymptotic-matching approximation, suitable for predictions of yield limit and plastic hardening on the micrometer scale, will be presented.

4.1. Strain Gradient Theories

First we will consider the Fleck and Hutchinson phenomenological strain gradient theory [29, 30] and its successive versions. In these theories, the effect of strain gradient tensor is incorporated into the potential energy density function, in a manner similar to the classical theories of Toupin [265] and Mindlin [266] in which only linear elasticity was considered. A higher order stress tensor needs to be introduced in these theories to provide a work conjugate to the strain gradient tensor, and the boundary condition of classical solid mechanics also needs to be modified as well. The

classical J_2 deformation theory of plasticity (i.e., Hencky-type solid strain theory) is chosen as the basis of strain gradient generalization.

The Gao and Huang MSG theory [33, 35] does not use the potential energy approach (and actually, potential energy even does not exist in that theory). Rather, this theory is based on the Taylor relation between the shear strength and dislocation density. A multiscale framework is used to introduce the higher order stress tensor and to establish the virtual work balance. Numerical simulations showed that while the higher order stress tensor is affected by the material length characterizing the size of the framework cell (called the mesoscale cell), the stress and strain tensors are almost unaffected. This observation triggered a reformulation in the form of the TNT theory [259], in which the strain gradient is numerically simulated as a nonlocal variable and the higher order stress disappears. This reformulation coincided with a revision proposed by Bažant [260, 261] for entirely different reasons—namely, the observation that the presence of couple stresses, dictated by the use of a strain gradient tensor as an independent kinematic variable, causes an excessive small-size asymptotic size effect, indicating that couple stresses should be removed from the formulation.

The Acharya and Bassani strain gradient theory [262, 263] differs significantly from the previous theories. It represents a generalization of incremental plasticity rather than total strain theory. The effect of a strain gradient is considered by changing the tangential modulus in the constitutive relation, while the framework of classical plasticity theory remains.

4.1.1. Fleck and Hutchinson Theories

The first phenomenological strain-gradient theory developed by Fleck and Hutchinson [29, 30] is called the couple stress theory (denoted by CS). The subsequent modification [31] is called the stretch and rotation gradients theory (denoted by SG). Since the main idea of these two theories is the same, we will consider them jointly. To simplify the problem, only incompressible materials will be considered and the elastic part will be ignored because it is negligible compared to large plastic deformation of metals.

In the classical work of Toupin [265] and Mindlin [266], and dealing only with the linear elasticity case, the strain gradient is introduced into the strain energy density W as

$$W = 1/2\lambda\varepsilon_{ii}\varepsilon_{jj} + \mu\varepsilon_{ij}\varepsilon_{ij} + a_1\eta_{ijj}\eta_{ikk} + a_2\eta_{iik}\eta_{kjj} + a_3\eta_{iik}\eta_{jjk} + a_4\eta_{ijk}\eta_{ijk} + a_5\eta_{ijk}\eta_{kij} \tag{27}$$

where λ and μ are the usual Lamé constants, $\varepsilon_{ij} = (u_{i,j} + u_{j,i})/2$ is the strain, $\eta_{ijk} = u_{k,ij}$ is the component of strain gradient tensor $\boldsymbol{\eta}$, and a_n is the additional elastic stiffness constant of the material. The sum of the first two terms on the right-hand side is the classical strain energy density function, while the other five terms are the contributions of the strain gradient tensor. Based on the strain energy density defined as (27), the Cauchy stress σ_{ij} can be defined as a work conjugate to ε_{ij} (i.e. $\sigma_{ij} = \partial W/\partial\varepsilon_{ij}$). A higher order stress tensor $\boldsymbol{\tau}$, work conjugate to the strain gradient tensor $\boldsymbol{\eta}$, needs to be defined as $\tau_{ijk} = \partial W/\partial\eta_{ijk}$. The strain energy W defined by (22) represents a linear elastic constitutive relation. There are many ways to extend it to a general nonlinear plastic material. Fleck and Hutchinson [31] chose to do it by defining a new variable, a scalar called the combined strain quantity, E, which involves both the strain tensor and the strain gradient tensor, to replace the effective strain in the J_2 theory. W is then assumed, for a general nonlinear plastic material, to be a nonlinear function of E. To define E, the strain gradient tensor $\boldsymbol{\eta}$ needs to be decomposed into a hydrostatic part $\boldsymbol{\eta}^H$ and deviatoric part $\boldsymbol{\eta}'$:

$$\eta^H_{ijk} = (\delta_{ik}\eta_{jpp} + \delta_{jk}\eta_{ipp})/4 \qquad \boldsymbol{\eta}' = \boldsymbol{\eta} - \boldsymbol{\eta}^H \tag{28}$$

Due to incompressibility, we have $\varepsilon'_{ij} = \varepsilon_{ij}$, $\eta'_{ijk} = \eta_{ijk}$. Furthermore, $\boldsymbol{\eta}'$ is decomposed into three orthogonal parts $\boldsymbol{\eta}' = \boldsymbol{\eta}'^{(1)} + \boldsymbol{\eta}'^{(2)} + \boldsymbol{\eta}'^{(3)}$ such that $\eta'^{(m)}_{ijk}\eta'^{(n)}_{ijk} = 0$ when $m \neq n$ [31]; the three invariants $\eta'^{(n)}_{ijk}\eta'^{(n)}_{ijk}$ are used to define E,

$$E = \sqrt{2\varepsilon'_{ij}\varepsilon'_{ij}/3 + \ell_1^2\eta'^{(1)}_{ijk}\eta'^{(1)}_{ijk} + \ell_2^2\eta'^{(2)}_{ijk}\eta'^{(2)}_{ijk} + \ell_3^2\eta'^{(3)}_{ijk}\eta'^{(3)}_{ijk}} \tag{29}$$

where ℓ_i are three length constants which are given different values in different version of the theory.

$$\text{For CS:} \quad \ell_1 = 0 \quad \ell_2 = \ell_{CS}/2 \quad \ell_3 = \sqrt{5/24}\ell_{CS} \tag{30}$$

$$\text{For SG:} \quad \ell_1 = \ell_{CS} \quad \ell_2 = \ell_{CS}/2 \quad \ell_3 = \sqrt{5/24}\ell_{CS} \tag{31}$$

Here ℓ_{CS} is called the material characteristic length. If the strain gradient part is ignored, scalar E becomes identical to the effective strain ε used in the classical plasticity theories.

Now the strain energy density W can be expressed as a function of E instead of ε as $W = W(E)$; thus the Cauchy stress tensor $\boldsymbol{\sigma}$ and the higher order stress tensor $\boldsymbol{\tau}$ can be expressed as

$$\sigma_{ij} = \frac{\partial W}{\partial\varepsilon_{ij}} = \frac{dW}{dE}\frac{\partial E}{\partial\varepsilon_{ij}} = \frac{2\varepsilon_{ij}}{3E}\frac{dW}{dE} \tag{32}$$

$$\begin{aligned}\tau_{ijk} &= \frac{\partial W}{\partial\eta_{ijk}} = \frac{dW}{dE}\frac{\partial E}{\partial\eta_{ijk}} \\ &= \frac{dW}{E\,dE}\left(\ell_1^2\eta'^{(1)}_{lmn}\frac{\partial\eta'^{(1)}_{lmn}}{\partial\eta_{ijk}} + \ell_2^2\eta'^{(2)}_{lmn}\frac{\partial\eta'^{(2)}_{lmn}}{\partial\eta_{ijk}} + \ell_3^2\eta'^{(3)}_{lmn}\frac{\partial\eta'^{(3)}_{lmn}}{\partial\eta_{ijk}}\right) \\ &= \frac{\ell_{CS}^2 C_{ijklmn}\eta_{lmn}dW}{E\,dE}\end{aligned} \tag{33}$$

Here C_{ijklmn} is a six-dimensional constant dimensionless tensor [264]. Since the values of ℓ_i are different in CS and SG theories, the tensor $\mathbf{C}$ will also be different in these two theories, although for each of them $\mathbf{C}$ is a constant tensor, that is, independent of ε, $\boldsymbol{\eta}$, and ℓ_{CS}. Because of the existence of higher order stress, the field equations of equilibrium must be generalized as

$$\sigma_{ik,i} - \tau_{ijk,ij} + f_k = 0 \tag{34}$$

where f_k is the body force.

4.1.2. Gao and Huang MSG Theory and TNT Theory

As the first strain gradient theory based on geometrically necessary dislocations, the MSG theory is a generalization of the incremental theory of plasticity [267]. In the MSG theory, the definition of strain gradient tensor $\eta_{ijk} = u_{k,ij}$ is the same as it is in the Fleck and Hutchinson theories, but the definition of higher order stress is different. It is defined by virtual work balance in a multiscale framework. The final constitutive relation reads [33, 260, 261, 264]

$$\sigma_{ik} = K\delta_{ik}\varepsilon_{nn} + \frac{2\sigma}{3\varepsilon}\varepsilon'_{ik} \qquad \tau_{ijk} = l_\varepsilon^2\left(\frac{K}{6}\eta^H_{ijk} + \sigma\Phi_{ijk} + \frac{\sigma_Y^2}{\sigma}\Psi_{ijk}\right) \tag{35}$$

where

$$\Phi_{ijk} = \frac{1}{\varepsilon}(\Lambda_{ijk} - \Pi_{ijk}) \qquad \Psi_{ijk} = f(\varepsilon)f'(\varepsilon)\Pi_{ijk} \tag{36}$$

$$\varepsilon = \sqrt{2\varepsilon'_{ij}\varepsilon'_{ij}/3} \qquad \eta = \sqrt{\eta'_{ijk}\eta'_{ijk}/2} \qquad \sigma = \sigma_Y\sqrt{f^2(\varepsilon) + l\eta} \tag{37}$$

and

$$\begin{aligned}\Lambda_{ijk} &= [2\eta_{ijk} + \eta_{kji} + \eta_{kij} - (\delta_{ik}\eta_{ppj} + \delta_{jk}\eta_{ppi})/4]72\\ \Pi_{ijk} &= [\varepsilon_{ik}\eta_{jmn} + \varepsilon_{jk}\eta_{imn} - (\delta_{ik}\varepsilon_{jp} + \delta_{jk}\varepsilon_{ip})\eta_{pmn}/4]\\ &\quad \times \varepsilon_{mn}/54\varepsilon^2\\ \eta^H_{ijk} &= (\delta_{ik}\eta_{jpp} + \delta_{jk}\eta_{ipp})/4 \qquad \eta'_{ijk} = \eta_{ijk} - \eta^H_{ijk}\end{aligned} \tag{38}$$

where K is the elastic bulk modulus. Equation (37) defines the new hardening rule of the material in which σ_Y is the yield stress; ε and σ are the effective strain and stress; η is the effective strain gradient, which is proportional to the density of geometrically stored dislocations; $\sigma_Y f(\varepsilon)$ represents the classical plastic hardening function; l is the material intrinsic length (similar to parameter ℓ_{CS} used in the Fleck and Hutchinson theories [29–31]); $\varepsilon'_{ij} = \varepsilon_{ij} - \varepsilon_{nn}/3$ is the deviatoric strain; η^H_{ijk} is the volumetric part of strain gradient tensor; and l_ε is the size of the so-called "mesoscale" cell which is expressed by Gao et al. [33] as

$$l_\varepsilon = \beta(G/\sigma_Y)b \tag{39}$$

Here G is the shear modulus, b is the Burgers vector, and β is an empirical factor whose value is suggested to be between 1 to 10 [33]. The equilibrium equations are the same as (29). It is also interesting to consider a more general hardening relation,

$$\sigma = \sigma_Y[f^q(\varepsilon) + (l\eta)^p]^{1/q} \tag{40}$$

where p and q are positive exponents; and MSG theory corresponds to the case $p = 1$, $q = 2$.

When the MSG theory is used in numerical simulations, the results show that when the value of l_ε is changed, the stress and strain do not change much, although the higher order stress does. This means that the existence of the higher order stress offers no advantage [35] (aside from the fact that they make the asymptotic scaling problematic [261, 264]). Upon noticing this fact, the MSG theory has been replaced by the TNT theory, in which the higher order stress tensor is removed.

In the TNT theory, the strain gradient is not an independent variable but a nonlocal variable defined by numerical integration. The gradient term $\varepsilon_{ij,k}$ can be numerically approximated in a nonlocal form as [259]

$$\varepsilon_{ij,k} = \int_{V_{\text{cell}}}[\varepsilon_{ij}(\mathbf{x}+\xi) - \varepsilon_{ij}(\mathbf{x})]\xi_m\, dV\left(\int_{V_{\text{cell}}}\xi_k\xi_m\, dV\right)^{-1} \tag{41}$$

in which V_{cell} is a sufficiently small representative cell surrounding point described by $\mathbf{x}$. To simplify the integration, V_{cell} can be chosen as a cube centered at $\mathbf{x}$, and then the strain gradient η_{ijk} can be expressed as

$$\eta_{ijk} = \frac{1}{I_\varepsilon}\int_{V_{\text{cell}}}[\varepsilon_{ik}\xi_j + \varepsilon_{jk}\xi_i - \varepsilon_{ij}\xi_k]\, dV \quad \text{with}$$
$$I_\varepsilon = \int_{V_{\text{cell}}}\xi_1^2\, dV = \frac{1}{12}l_\varepsilon^5 \tag{42}$$

where l_ε is the size of the cube. Furthermore, one may introduce the volumetric part $\boldsymbol{\eta}^H$ and the deviatoric part $\boldsymbol{\eta}'$ of tensor $\boldsymbol{\eta}$, and the effective strain gradient invariant $\eta = \sqrt{\eta'_{ijk}\eta'_{ijk}/2}$, which is identical to that defined in the MSG theory. Because the strain gradient tensor does not function in (40) as an independent kinematic variable, we need not define the corresponding work-conjugate higher order stress tensor. For $p = 1$, $q = 2$, the constitutive relation is [259]

$$\sigma_{ik} = K\delta_{ik}\varepsilon_{nn} + \frac{2\sigma}{3\varepsilon}\varepsilon'_{ik}$$

where

$$\sigma = \sigma_Y\sqrt{f^2(\varepsilon) + l\eta} \tag{43}$$

Since the new higher order stress is absent, the equilibrium equation of the TNT theory is the same as in the classical theory (i.e., $\sigma_{ij,i} + f_j = 0$).

4.1.3. Acharya and Bassani Theory

The Acharya and Bassani strain gradient theory is a generalization of the classical incremental plasticity theory, in which the strain gradient is assumed to affect only the instantaneous modulus. The strain gradient is considered to be a measure of lattice incompatibility and is introduced only through the second-order tensor as [262, 263]

$$\alpha_{ij} = e_{jkl}\varepsilon^p_{il,k} \tag{44}$$

where e_{jkl} is the alternating symbol and ε^p is the plastic strain. Introducing the invariant:

$$\alpha = \sqrt{2\alpha_{ij}\alpha_{ji}} \tag{45}$$

Acharya and Bassani modified the classical J_2 flow theory as follows [262, 263]:

$$\tau = \sqrt{\sigma'_{ij}\sigma'_{ij}/2} \qquad \dot\tau = \dot\tau_{cr} = h(\gamma^p, \alpha)\dot\gamma^p \tag{46}$$

$$\dot\varepsilon^p_{ij} = (\dot\gamma^p/2\tau)\sigma'_{ij} \qquad \dot\sigma_{ij} = C_{ijkl}(\dot\varepsilon_{kl} - \dot\varepsilon^p_{kl}) \qquad \gamma^p = \sqrt{2\varepsilon^p_{ij}\varepsilon^p_{ij}/3} \tag{47}$$

Here the instantaneous hardening modulus h depends not only on plastic strain invariant γ^p but also on plastic strain gradient invariant α. An example of this function is [262, 263]

$$h(\gamma^p, \alpha) = h_0\left(1 + \frac{\gamma^p}{\gamma_0}\right)^{N-1}\left[1 + \frac{l^2(\alpha/\gamma_0)^2}{1 + c(\gamma^p/\gamma_0)^2}\right]^{1/2} \quad (48)$$

where l is the material intrinsic length, and h_0, γ_0, c, and N are positive material constants.

There exist other strain gradient theories, but generally they are similar to one of the theories introduced here. For example, the Chen and Wang [268, 269] strain gradient theory is similar to the Fleck and Hutchinson theories.

4.2. Asymptotic Analysis of Strain Gradient Theories

For the purpose of scaling analysis, we need to consider geometrically similar structures of different sizes. This means that the structures are also similarly loaded. It is obvious that the strain gradient theories must reduce to the classical plasticity theory when the structure size is very large. To discuss the asymptotic cases, it is necessary to introduce dimensionless variables. Diverse sets of such variables could be chosen but only one is easy to interpret,

$$\begin{gathered} \bar{x}_i = x_i/D \qquad \bar{u}_i = u_i/D \qquad \bar{\varepsilon}_{ij} = \varepsilon_{ij} \\ \bar{\eta}_{ijk} = \eta_{ijk}D \qquad \bar{f}_k = f_k D/\sigma_N \end{gathered} \quad (49)$$

where D is the characteristic length of the structure, and σ_N is the nominal strength. For geometrically similar structures the strain distribution may often be assumed to be the same, and then $\bar{x}_i$, $\bar{u}_i$, $\bar{\varepsilon}_{ij}$, and $\bar{\eta}_{ijk}$ will be size independent; that is, they will be the same for structures of different sizes. Consequently, the asymptotic behavior of the strain gradient tensor must be $\eta_{ijk} \propto 1/D$.

4.2.1. Asymptotic Analysis of the Fleck and Hutchinson Theories

Scaling and Size Effect The Fleck and Hutchinson strain gradient theory can be used to generalize various particular forms of classical constitutive relations for plasticity. A stress–strain relation in the form of a general power law relation may be chosen as an example, in which the strain energy density is [29–31]

$$W = \frac{n}{n+1}\sigma_0 E_0\left(\frac{E}{E_0}\right)^{(n+1)/n} \quad (50)$$

where σ_0, E_0, and n are positive material constants. For hardening materials, $n \geq 1$; typically $n \approx 2$–5 for normal metals. According to (32) and (33), the constitutive relation then reads

$$\sigma_{ik} = \frac{2}{3}\sigma_0\left(\frac{1}{E_0}\right)^{1/n} E^{(1-n)/n}\varepsilon_{ik} \quad (51)$$

$$\tau_{ijk} = \sigma_0\left(\frac{1}{E_0}\right)^{1/n} \ell_{\mathrm{CS}}^2 E^{(1-n)/n} C_{ijklmn}\eta_{lmn} \quad (52)$$

It is now useful to define dimensionless variables:

$$\begin{gathered} \bar{\tau}_{ijk} = \tau_{ijk}/(E_0\ell_{\mathrm{CS}}) \qquad \bar{\sigma}_{ij} = \sigma_{ij}/\sigma_0 \qquad \overline{E} = E \\ \bar{\eta}^{(l)}_{ijk} = \eta'^{(l)}_{ijk} D \qquad (l = 1, 2, 3) \end{gathered} \quad (53)$$

Then the constitutive relation can be expressed as

$$\bar{\sigma}_{ik} = \frac{2}{3}\left(\frac{1}{E_0}\right)^{1/n} \overline{E}^{(1-n)/n}\bar{\varepsilon}_{ik} \quad (54)$$

$$\bar{\tau}_{ijk} = \left(\frac{1}{E_0}\right)^{1/n} \frac{\ell_{\mathrm{CS}}}{D}\overline{E}^{(1-n)/n} C_{ijklmn}\bar{\eta}_{lmn} \quad (55)$$

The equilibrium equation (29) can be rewritten as

$$\partial_i\bar{\sigma}_{ik} - \frac{\ell_{\mathrm{CS}}}{D}\partial_i\partial_j\bar{\tau}_{ijk} + \frac{\sigma_N}{\sigma_0}\bar{f}_k = 0 \quad (56)$$

where $\partial_i = \partial/\partial\bar{x}_i =$ derivatives with respect to the dimensionless coordinates. Substituting (49) and (55) into (56), one obtains the dimensionless field equation of equilibrium in the form

$$\begin{aligned} &\frac{2}{3}\left(\frac{1}{E_0}\right)^{1/n}\partial_i\left(\overline{E}^{(1-n)/n}\bar{\varepsilon}_{ik}\right) - \left(\frac{\ell_{\mathrm{CS}}}{D}\right)^2\left(\frac{1}{E_0}\right)^{1/n} \\ &\times \partial_i\partial_j\left(C_{ijklmp}\overline{E}^{(1-n)/n}\bar{\eta}_{lmp}\right) = -\frac{\sigma_N}{\sigma_0}\bar{f}_k \end{aligned} \quad (57)$$

Following Bažant [260, 261] and Bažant and Guo [264], we may simplify the analysis by replacing the surface fractions with body forces applied in a very thin boundary layer, the thickness of which tends to zero. This ensures that all the boundary conditions are homogeneous. When the structure is sufficiently large, $\ell_{\mathrm{CS}}/D \to 0$, $\bar{\tau}_{ijl}$ vanish, according to (55), and the equilibrium equations reduce to the classical equilibrium equations, as required. The combined strain quantity, E, reduces to the classical effective strain because the strain gradient part can be ignored compared to the strain part. Then the strain energy density function takes the normal form as a function of the strain only.

As proposed by Bažant and Guo [264], it is interesting to look at the opposite asymptotic character of the theory when the structure size tends to zero, $\ell_{\mathrm{CS}}/D \to \infty$. At first, the dimensionless combined strain quantity can be rewritten as

$$\begin{aligned} \overline{E} &= \sqrt{2\varepsilon'_{ij}\varepsilon'_{ij}/3 + \left(\ell_1^2\bar{\eta}'^{(1)}_{ijk}\bar{\eta}'^{(1)}_{ijk} + \ell_2^2\bar{\eta}'^{(2)}_{ijk}\bar{\eta}'^{(2)}_{ijk} + \ell_3^2\bar{\eta}'^{(3)}_{ijk}\bar{\eta}'^{(3)}_{ijk}\right)/D^2} \\ &\propto D^{-1} \qquad \text{for } \ell_{\mathrm{CS}}/D \to \infty \end{aligned} \quad (58)$$

If one defines a size-independent dimensionless variable

$$\overline{H} = \sqrt{\ell_1^2\bar{\eta}'^{(1)}_{ijk}\bar{\eta}'^{(1)}_{ijk} + \ell_2^2\bar{\eta}'^{(2)}_{ijk}\bar{\eta}'^{(2)}_{ijk} + \ell_3^2\bar{\eta}'^{(3)}_{ijk}\bar{\eta}'^{(3)}_{ijk}}\Big/\ell_{\mathrm{CS}} \quad (59)$$

the asymptotic behavior is seen to be

$$\overline{E} \approx \frac{\ell_{\mathrm{CS}}}{D}\overline{H} \qquad \text{for } \ell_{\mathrm{CS}}/D \to \infty \quad (60)$$

Substituting (60) into (57), the asymptotic form of the equilibrium equation reads

$$\frac{2}{3}\left(\frac{\ell_{CS}}{D}\right)^{(1-n)/n}\left(\frac{1}{E_0}\right)^{1/n}\partial_i\left(\overline{H}^{(1-n)/n}\bar{\varepsilon}_{ik}\right)-\left(\frac{\ell_{CS}}{D}\right)^{(1+n)/n}\times\left(\frac{1}{E_0}\right)^{1/n}\partial_i\partial_j\left(C_{ijklmp}\overline{H}^{(1-n)/n}\bar{\eta}_{lmp}\right)=-\frac{\sigma_N}{\sigma_0}\bar{f}_k \tag{61}$$

After multiplying this equation by $(D/\ell_{CS})^{(n+1)/n}$ and taking the limit of the left-hand side for $\ell_{CS}/D \to \infty$, one gets the following asymptotic form of the equilibrium equations:

$$\partial_i\partial_j\left(C_{ijklmp}\overline{H}^{(1-n)/n}\bar{\eta}_{lmp}\right)=\chi\bar{f}_k \quad \text{with } \chi=\overline{E}_0^{1/n}\frac{\sigma_N}{E_0}\left(\frac{D}{\ell_{CS}}\right)^{(n+1)/n} \tag{62}$$

Because the left-hand side of the foregoing equation, as well as the dimensionless body force $\bar{f}_k$, is independent of size D and because the boundary conditions are homogeneous and thus size independent, the parameter χ must be size independent. Thus, upon solving σ_N from (62), one finds that the small-size asymptotic scaling law is

$$\sigma_N=\sigma_0\chi\overline{E}_0^{-1/n}\left(\frac{\ell_{CS}}{D}\right)^{(n+1)/n} \tag{63}$$

or

$$\sigma_N \propto D^{-(n+1)/n} \tag{64}$$

For plastic hardening materials, we have $1 < (n+1)/n \leq 2$. Although the result (64) applies only to the special case of strain energy density function given by (50), the analytical technique used here is general. It is even suitable to the strain energy density function defined directly in terms of strain and strain gradients, rather than as the combined strain quantity. For example, if the strain energy density function is defined as (27) for the case of linear elasticity, a similar analysis can be made and it is found that the size effect law for very small sizes reads [264]

$$\sigma_N \propto D^{-2} \tag{65}$$

This also shows that (64) is quite general because (65) can be regarded as a special case of (64) in which the strain hardening exponent $n = 1$.

Small-Size Asymptotic Load–Deflection Response For some special cases (e.g., the pure torsion of a long thin wire or the bending of a slender beam), the symmetry conditions require displacement distribution to remain similar during the loading process. For such cases, the dimensionless displacement $\bar{u}_k$ can be related to a single parameter, w, and characterized by displacement profile $\hat{u}_k$ as $\bar{u}_k = w\hat{u}_k$ [260, 261, 264]. Since $\hat{u}_k$ is dimensionless, it must be independent of the size D. Displacements $\bar{u}_k$ evolve during the proportional loading process while the distribution profile remains constant. Thus the parameter w can be considered as the displacement norm, $\|\bar{u}_k\|$. It follows that the strain, strain gradient, and combined strain quantity are all proportional to w. Therefore, $\overline{E}$ can be similarly represented as $\overline{E} = w\widehat{E}$, where $\widehat{E}$ is a size independent profile function, and w can be regarded as the deflection magnitude. Substituting this relation into the dimensionless constitutive relation (54) and (55), one can easily get

$$\bar{\sigma}_{ik}=w^{1/n}\hat{\sigma}_{ik} \qquad \bar{\tau}_{ijk}=w^{l/n}\hat{\tau}_{ijk} \tag{66}$$

where $\hat{\sigma}_{ik}$ and $\hat{\tau}_{ijk}$ are both size independent profile functions. Substituting these relations into dimensionless equilibrium equation (56), one finds that the load-deflection curve must have the form

$$\bar{f}_k \propto w^{1/n} \tag{67}$$

This relation is similar to the traditional strain–stress relation derived from the strain energy density function (50). The reason the load deflection curve begins with a vertical tangent is that the initial elastic response is assumed to be negligible.

Example One important example is the microtorsion of a thin wire, for which a strong size effect was demonstrated [29–31] and described by strain gradient theories. The strain energy density function W is defined as $W = \sigma E^{N+1}/(N+1)$. Compared with (50), one finds that $N = 1/n$. The radius of the wire, D, is chosen as the characteristic size of the structure. The deformation is characterized by the twist angle per unit length, κ. The nominal stress can be defined as $\sigma_N = T/D^3$, where T is the torque. For different radii of the wire, we compare the σ_N values corresponding to the same dimensionless twist $\bar{\kappa} = \kappa D$. The nominal stress can be expressed according to the CS theory as follows:

$$\sigma_N=\frac{T}{D^3}=\frac{6}{N+3}\sigma_0\bar{\kappa}^N\left\{\left[\frac{1}{3}+\left(\frac{\ell_{CS}}{D}\right)^2\right]^{(N+3)/2}-\left(\frac{\ell_{CS}}{D}\right)^{N+3}\right\} \tag{68}$$

When $\ell_{CS}/D \to \infty$, one has

$$\left[\frac{1}{3}+\left(\frac{\ell_{CS}}{D}\right)^2\right]^{(N+3)/2}-\left(\frac{\ell_{CS}}{D}\right)^{N+3}\approx\frac{N+3}{2}\left(\frac{D}{3\ell_{CS}}\right)^2\left(\frac{\ell_{CS}}{D}\right)^{N+3} \tag{69}$$

from which

$$\sigma_N \propto D^{-N-1}=D^{-(n+1)/n} \tag{70}$$

For the load-deflection response, we now obtain the following relation between the load T and the deformation $\bar{\kappa}$:

$$T \propto \kappa^N=\kappa^{1/n} \tag{71}$$

4.2.2. Asymptotic Analysis of the Gao and Huang MSG Theory and TNT Theory

Scaling and Size Effect The dimensionless variables defined in (49) also need to be used here, and further dimensionless variables need to be defined as follows:

$$\begin{gathered}\bar{\varepsilon}=\varepsilon \qquad \bar{\eta}=\eta D \qquad \bar{\sigma}_{ij}=\sigma_{ij}/\sigma_Y \\ \bar{\tau}_{ijk}=\tau_{ijk}/(\sigma_Y l) \qquad \bar{\sigma}=\sigma/\sigma_Y\end{gathered} \tag{72}$$

$$\begin{gathered}\bar{\eta}^H_{ijk}=\eta^H_{ijk}D \qquad \overline{\Lambda}_{ijk}=\Lambda_{ijk}D \qquad \overline{\Pi}_{ijk}=\Pi_{ijk}D \\ \overline{\Phi}_{ijk}=\Phi_{ijk}D \qquad \overline{\Psi}_{ijk}=\Psi_{ijk}D\end{gathered} \tag{73}$$

These definitions are meaningful because η^H_{ijk}, Λ_{ijk}, and Π_{ijk} are all homogeneous functions of degree 1 of tensors η_{ijk} and ε_{ij}. It is not difficult to obtain a dimensionless version of the constitutive law of the MSG theory

$$\begin{aligned}\bar{\sigma}_{ik}&=\frac{K}{\sigma_Y}\delta_{ik}\bar{\varepsilon}_{nn}+\frac{2\bar{\sigma}}{3\bar{\varepsilon}}\bar{\varepsilon}'_{ik} \\ \bar{\tau}_{ijk}&=\frac{l_\varepsilon^2}{lD}\left(\frac{K}{6\sigma_Y}\bar{\eta}^H_{ijk}+\bar{\sigma}\overline{\Phi}_{ijk}+\frac{1}{\bar{\sigma}}\overline{\Psi}_{ijk}\right)\end{aligned} \tag{74}$$

where l and l_ε are two characteristic material lengths. The corresponding dimensionless equilibrium equation reads

$$\partial_i\bar{\sigma}_{ik}-\frac{l}{D}\partial_i\partial_j\bar{\tau}_{ijk}+\frac{\sigma_N}{\sigma_Y}\bar{f}_k=0 \tag{75}$$

Same as before, the boundary conditions can again be considered as homogeneous and the applied loads replaced by body forces $\bar{f}_k$ applied within a thin surface layer. The asymptotic behavior for a very large structure is simple. When $l/D\to 0$, we also have $l_\varepsilon/D\to 0$. Thus $\bar{\tau}_{ijk}$ tends to zero, according to Eq. (69), and all the equations reduce to the standard equations of classical plasticity theory, which means that there is no size effect, as required by classical plasticity.

The opposite asymptotic character for sufficiently small structures ($l/D\to\infty$ and $l_\varepsilon/D\to\infty$) is more interesting. The general hardening rule (40) can be rewritten with dimensionless variables as

$$\bar{\sigma}=[f^q(\varepsilon)+(l\bar{\eta}/D)^p]^{1/q} \tag{76}$$

Thus, we have $\bar{\sigma}\approx(l\bar{\eta}/D)^{p/q}$ when $l/D\to\infty$. Substituting (74) into (75), we can express the equilibrium equation as follows:

$$\begin{aligned}&\partial_i\left[\frac{K}{\sigma_Y}\delta_{ik}\bar{\varepsilon}_{nn}+\frac{2}{3\bar{\varepsilon}}\left(\frac{l\bar{\eta}}{D}\right)^{p/q}\bar{\varepsilon}'_{ik}\right]-\left(\frac{l_\varepsilon}{D}\right)^2 \\ &\quad\times\partial_i\partial_j\left[\frac{K}{6\sigma_Y}\bar{\eta}^H_{ijk}+\left(\frac{l\bar{\eta}}{D}\right)^{p/q}\overline{\Phi}_{ijk}+\left(\frac{D}{l\bar{\eta}}\right)^{p/q}\overline{\Psi}_{ijk}\right] \\ &\quad=-\frac{\sigma_N}{\sigma_Y}\bar{f}_k\end{aligned} \tag{77}$$

When $l/D\to\infty$, the five terms on the left-hand side of the foregoing equation are, in sequence, of the order of

$$O(1)\quad O(D^{-p/q})\quad O(D^{-2})\quad O(D^{-2-p/q})\quad O(D^{2+p/q}) \tag{78}$$

When $D\to 0$, the fourth term is generally the dominant one, and so we get the asymptotic form of the equilibrium equation,

$$\partial_i\partial_j(\bar{\eta}^{p/q}\overline{\Phi}_{ijk})=\chi_1\bar{f}_k \tag{79}$$

with

$$\chi_1=\left(\frac{l}{l_\varepsilon}\right)^2\frac{\sigma_N}{\sigma_Y}\left(\frac{D}{l}\right)^{2+p/q} \tag{80}$$

Since D is not present in the left-hand side of (80) and the boundary conditions are also homogeneous, the parameter χ must be independent of D. Thus, the general small-size asymptotic scaling law of MSG theory reads [260, 261, 264]

$$\sigma_N=\sigma_Y\chi_1\left(\frac{l_\varepsilon}{l}\right)^2\left(\frac{l}{D}\right)^{2+p/q}$$

and for $\frac{p}{q}=\frac{1}{2}$

$$\sigma_N\propto D^{-5/2} \tag{81}$$

This asymptotic size effect is very strong [260, 261, 264]. It is much stronger than the normal linear elastic fracture mechanics size effect, which is $\sigma_N\propto D^{-1/2}$, or the typical Weibull size effect, which is around $\sigma_N\propto D^{-0.1}$.

There are also some special cases. For example, in the case of microbending, $\overline{\Phi}_{ijk}=0$ for all i,j,k, which makes the fourth term on the left-hand side of (77) vanish; in the case of incompressible material, $\eta^H_{ijk}=0$, which makes the third term on the left-hand side of (77) zero. So the general size effect law will change to $\sigma_N\propto D^{-2}$ for microbending of a compressible material, and to $\sigma_N\propto D^{-2+p/q}$ for microbending of an incompressible material (in detail, see [264]).

The size effect $D^{-5/2}$, as well as D^{-2}, is enormous and unrealistic. This is a consequence of the last three terms on the left-hand side of (77), which represent contributions from the couple stresses. A detailed analysis showed that the couple stresses are not necessary to fit the test results and to ensure the virtual work balance [264]. Based on this analysis, Bažant [260, 261] and Bažant and Guo [264] proposed a modified version of the MSG theory in which the couple stresses are made to vanish. This led to a theory identical to the TNT theory [260, 261, 264], which was proposed on the basis of numerical experience with varying the "mesoscale cell size" l_ε. Let us now analyze the asymptotic size effect of this theory. The dimensionless variables defined for the MSG theory may again be used for TNT theory. The dimensionless constitutive relation of the TNT theory reads

$$\bar{\sigma}_{ik}=\frac{K}{\sigma_Y}\delta_{ik}\bar{\varepsilon}_{nn}+\frac{2\bar{\sigma}}{3\bar{\varepsilon}}\bar{\varepsilon}'_{ik} \tag{82}$$

and the differential equation of equilibrium in terms of the dimensionless variables takes the form

$$\partial_i\bar{\sigma}_{ik}+\frac{\sigma_N}{\sigma_Y}\bar{f}_k=0 \tag{83}$$

For large enough sizes, $D/l\to\infty$, the asymptotic behavior will be identical to the classical theory of plasticity, which implies no size effect. For very small sizes, $D/l\to 0$, we have

$$\bar{\sigma}=[f^q(\varepsilon)+(l\bar{\eta}/D)^p]^{1/q}\approx(l\bar{\eta}/D)^{p/q} \tag{84}$$

and the equilibrium equation can be rewritten as follows:

$$\partial_i\left[\frac{K}{\sigma_Y}\delta_{ik}\bar{\varepsilon}_{nn}+\frac{2}{3\bar{\varepsilon}}\left(\frac{l\bar{\eta}}{D}\right)^{p/q}\bar{\varepsilon}'_{ik}\right]=-\frac{\sigma_N}{\sigma_Y}\bar{f}_k \tag{85}$$

Obviously, the second term on the lefthand side dominates when $D/l \to 0$, and so the asymptotic form of the equilibrium equation is

$$\partial_i\left(\bar{\eta}^{p/q}\frac{\bar{\varepsilon}'_{ijk}}{\bar{\varepsilon}}\right)=\chi\bar{f}_k \qquad \text{with } \chi=-\frac{3}{2}\frac{\sigma_N}{\sigma_Y}\left(\frac{D}{l}\right)^{p/q} \tag{86}$$

Same as before, χ is sizeindependent, and consequently the small-size asymptotic scaling law for the TNT theory is

$$\sigma_N=-\frac{2\sigma_Y}{3}\chi\left(\frac{l}{D}\right)^{p/q}$$

and for $\frac{p}{q}=\frac{1}{2}$

$$\sigma_N \propto D^{-1/2} \tag{87}$$

Four possible cases of small-size asymptotic scaling for the MSG theory and the TNT theory are shown in Figure 45.

Small-Size Asymptotic Load-Deflection Response The characteristic features of the small-size asymptotic load-deflection curves will now be determined. The MSG theory will be analyzed first, and the TNT theory can be treated as a special limiting case of the MSG theory. Again we consider only the special cases where the relative displacement profile does not change during the loading process. Same as before, the displacement can be characterized by parameter w as $\bar{u}_k = w\hat{u}_k$, where $\hat{u}_k$ is the displacement profile, which is not only independent of D but also invariable during the proportional loading process. Similarly, the strain and strain gradient can be expressed as $\bar{\varepsilon}_{ij} = \varepsilon_{ij} = w\hat{\varepsilon}_{ij}$, $\bar{\varepsilon} = \varepsilon = w\hat{\varepsilon}$, $\bar{\eta}_{ijk} = w\hat{\eta}_{ijk}$, $\bar{\eta} = w\hat{\eta}$. Since variables Λ_{ijk} and Π_{ijk} are homogeneous functions of degree 1 of both $\boldsymbol{\eta}$ and $\boldsymbol{\varepsilon}$, their corresponding dimensionless variables can also be expressed as products of w and a dimensionless profile function (i.e., $\overline{\Lambda}_{ijk} = w\widehat{\Lambda}_{ijk}$ and $\overline{\Pi}_{ijk} = w\widehat{\Pi}_{ijk}$). Because of the factor $1/\varepsilon$ in the definition of Φ_{ijk} [see Eq. (36)], we have $\overline{\Phi}_{ijk} = \widehat{\Phi}_{ijk}$, which means that $\overline{\Phi}_{ijk}$ is independent of w. When we consider the beginning of the load-deflection diagram at $D \to 0$, we also have $w \to 0$, which is a limit not discussed during previous sizeeffect analysis. When the effect of w is considered, the five terms on the left-hand side of (77) are proportional, in sequence, to the functions as follows:

$$\begin{array}{ccccc} w & (w/D)^{p/q} & w/D^2 & w^{p/q}/D^{2+p/q} & \\ & w^{1-p/q} & w^{1-p/q}/D^{2-p/q} & & \end{array} \tag{88}$$

In the case of MSG theory with $p = 1$, $q = 2$, these functions are

$$w \qquad \sqrt{w/D} \qquad w/D^2 \qquad w^{1/2}/D^{5/2} \qquad w^{1/2}/D^{3/2} \tag{89}$$

and (77) can then be expressed as

$$-\frac{\sigma_N}{\sigma_Y}\bar{f}_k = a_1 w + a_2 w^{p/q}D^{-p/q} + a_3 wD^{-2} + a_4 w^{p/q}D^{-2-p/q} + a_5 w^{1-p/q}D^{-2+p/q} \tag{90}$$

where parameters a_i are constants independent of D and w. Since the force $\bar{f}_k$ should decrease when w decreases, one knows that $1 - p/q > 0$, which implies $p < q$. As we discussed before, if only $D \to 0$ is considered (or, in other words, $D \ll w$), the dominant term is $a_4 w^{p/q}D^{-2-p/q}$, which means that

$$\bar{f}_k \propto w^{p/q} \qquad (\text{for } w \gg D) \tag{91}$$

For the MSG theory, this gives

$$\bar{f}_k \propto w^{1/2} \qquad (\text{for } w \gg D) \tag{92}$$

We need to consider another asymptotic case in which $w \ll D$ (e.g., at the beginning of the load-deflection diagram). The dominant term in this case is either $a_4 w^{p/q}D^{-2-p/q}$ or $a_5 w^{1-p/q}D^{-2+p/q}$, depending on the value of p/q. The asymptotic load-deflection behavior is

$$\bar{f}_k \propto w^r \qquad r = \min\{p/q, 1-p/q\} \qquad (\text{for } w \ll D) \tag{93}$$

For MSG theory, the dominant term is $a_4 w^{1/2}D^{-5/2}$, and so the load $\bar{f}_k$ initially increases in proportion to $w^{1/2}$. Thus one has the asymptotic load-deflection relation for MSG theory as

$$\bar{f}_k \propto w^{1/2} \qquad \text{for all } w \tag{94}$$

As discussed in the preceding section, some terms in (77) may vanish in some special cases, and a similar analysis can be applied to these special cases. For example, for microbending of an incompressible material, the third and fourth terms in (77) vanish, and as a result (92) changes as follows:

$$-\frac{\sigma_N}{\sigma_Y}\bar{f}_k = a_1 w + a_2\sqrt{w/D} + a_5 w^{1/2}/D^{3/2} \tag{95}$$

The asymptotic load-deflection curve is simple because the last term on the right-hand side of (95) dominates when

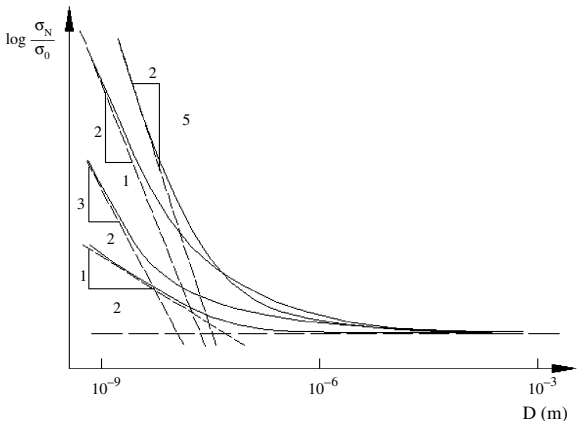

Figure 45. Four possible small-size asymptotic scaling curves for the MSG theory and the TNT theory.

D is small, regardless of the ratio of w/D, and so the asymptotic load-deflection relation for this special case is again

$$\bar{f}_k \propto w^{1/2} \qquad \text{for all } w \tag{96}$$

This means the vanishing of some terms in flexure problems will not change the asymptotic load-deflection behavior. The TNT theory can be treated as a special case of the MSG theory. If the last three terms on the right-hand of the MSG equilibrium equation (77) vanish, the equation becomes identical to equilibrium equation (85) of the TNT theory. So (85) can be expressed as

$$-\frac{\sigma_N}{\sigma_Y}\bar{f}_k = a_1 w + a_2\sqrt{w/D} \tag{97}$$

For small enough D and w, the dominant term will be the second term on the right-hand side of (97), and so we have

$$\bar{f}_k \propto w^{1/2} \qquad \text{for all } w \qquad \text{(TNT theory)} \tag{98}$$

It should be noted that the elastic part of the response has been neglected, which is why the load-deflection curves in (94), (96), and (98) begin with a vertical tangent.

Example The experiment of microtorsion of a thin wire can also be analyzed by the MSG theory [35] or TNT theory. Equation (35) in [35] can be transformed to the dimensionless formula

$$\sigma_N = \frac{T}{D^3} = \sigma_Y \frac{2\pi\bar{\kappa}}{3}\int_0^1 \left\{ \frac{\bar{\sigma}}{\bar{\varepsilon}}\left(\rho^2 + \frac{l_\varepsilon^2}{12D^2}\right) + \frac{l_\varepsilon^2 f(\bar{\varepsilon}) f'(\bar{\varepsilon})}{12D^2\bar{\sigma}} \right\} \rho\, d\rho \tag{99}$$

where σ_N is the nominal stress, T is the torque, D is the radius of wire (which is also chosen as the characteristic length of the structure), σ_Y is the yield stress of macroscale metal, and $\bar{\kappa} = \bar{\eta} = \kappa D$ is the dimensionless specific angle of twist, where κ = actual specific angle of twist (i.e., the rotation angle per unit length of wire). Substituting $\bar{\sigma} = \sqrt{f^2(\bar{\varepsilon}) + l\bar{\eta}/D} \approx \sqrt{l\bar{\kappa}/D}$ into this formula, we find that, for $D \to 0$, the dominant part is obtained by integration of the second term, which leads to the following small-size asymptotic form:

$$\sigma_N = \sigma_Y\left(\frac{\pi}{18} l_\varepsilon^2 l^{1/2}\int_0^1 \frac{\rho}{\hat{\varepsilon}} d\rho\right)\bar{\kappa}^{1/2} D^{-5/2} \tag{100}$$

This result verifies the conclusions (81) and (89), where $\bar{\kappa}$ is considered as a measure of deflection, analogous to w in (89). The asymptotic load-deflection behavior of the TNT theory can be obtained similarly.

Another special case is the application of the MSG theory to the microbending of incompressible metals [35]. Equation (29) in [35] can be transformed to the dimensionless version

$$\sigma_N = \frac{M}{D^2} = 2\sigma_Y \int_0^{1/2}\left[\frac{2}{\sqrt{3}}\bar{\sigma}\rho + \frac{\bar{\kappa} l_\varepsilon^2 f(\bar{\varepsilon}) f'(\bar{\varepsilon})}{9D^2\bar{\sigma}}\right] d\rho \tag{101}$$

where D is the beam depth (the characteristic dimension of the structure), M is the bending moment, $\bar{\kappa} = \bar{\eta} = \kappa D$ is the dimensionless bending curvature, and κ is the actual bending curvature. When $D \to 0$, the small-size asymptotic form is

$$\sigma_N = \sigma_Y\left(\frac{2}{9} l_\varepsilon^2 l^{-1/2}\int_0^{1/2} f(\bar{\varepsilon}) f'(\bar{\varepsilon}) d\rho\right)\bar{\kappa}^{1/2} D^{-3/2} \tag{102}$$

This verifies for this special case the asymptotic behavior $\sigma_N \propto D^{-2+p/q}$, as well as (96). Letting $l_\varepsilon = 0$, one finds that this asymptotic character also applies to the TNT theory.

4.2.3. Asymptotic Analysis of Acharya and Bassani's Theory

Let us now give a simple analysis of the asymptotic behavior of the Acharya and Bassani strain gradient theory [263]. We define dimensionless variables $\bar{u}_i = u_i/D$, $\bar{\varepsilon}_{ij,k} = \varepsilon_{ij,k} D$, and that $\bar{\alpha}_{ij} = \alpha_{ij} D$. When $D \to 0$, the asymptotic behavior of the plastic hardening modulus ($\bar{\gamma}^p = \gamma^p$) defined by (48) is found to be

$$h(\gamma^p, \alpha) = h_0\left(1 + \frac{\gamma^p}{\gamma_0}\right)^{N-1}\left[1 + c(\gamma^p/\gamma_0)^2\right]^{-1/2}\frac{\bar{\alpha}}{\gamma_0}\frac{l}{D} \propto D^{-1} \tag{103}$$

This shows that, at the same strain level, the plastic hardening modulus (slope of load deflection curve) increases as D^{-1} when $D \to 0$. If the elastic part is neglected for very large plastic strain, the nominal stress must also scale asymptotically as D^{-1}. This asymptotic size effect is again quite strong (but not as strong as in the MSG theory). This size effect can be reduced by modifying the hardening function $h(\gamma^p, \alpha)$. For example, if the hardening modulus is redefined as

$$h(\gamma^p, \alpha) = h_0\left(1 + \frac{\gamma^p}{\gamma_0}\right)^{N-1}\left[1 + \frac{l\alpha/\gamma_0}{1 + c(\gamma^p/\gamma_0)}\right]^{1/2} \tag{104}$$

then the asymptotic scaling becomes more reasonable:

$$h(\gamma^p, \alpha) \propto D^{-1/2} \quad \text{when } D \to 0 \tag{105}$$

4.3. Asymptotic-Matching Approximation Formula

After determining the asymptotic behaviors of strain gradient plasticity theories, one can obtain an asymptotic matching approximation for a smooth transition of the nominal strength in the intermediate size range. In [260, 261, 264], a smooth transition between the case of no size effect for $D \to \infty$ and the case of power law size effect $\sigma_N \propto D^{-s}$ for $D \to 0$ ($s > 0$) has been described by the simple asymptotic-matching approximation

$$\sigma_N = \sigma_0\left[1 + \left(\frac{D_0}{D}\right)^{2s/r}\right]^{r/2} \tag{106}$$

where r is a constant to be determined by data fitting, while parameters σ_0 and D_0 can be determined by either the asymptotic size effect formula or data fitting. This formula was shown to fit the results for microtorsion and microbending (Figs. 46 and 47).

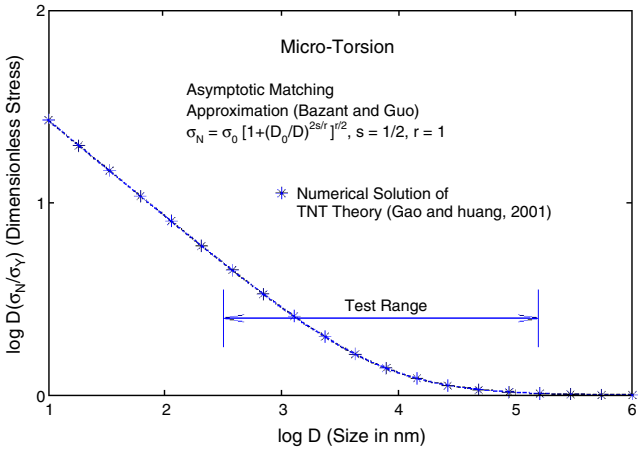

Figure 46. Asymptotic-matching approximation for microtorsion.

4.4. Concluding Remarks on Strain Gradient Theories

In many applications of interest (e.g., microelectronics and MEMS), characteristic dimensions are in excess of 100 nm–1 μm. Modeling the mechanics of such systems at the atomistic level is beyond present computational capabilities. Therefore, extension of continuum theories to account for size scales is of high relevance. Here we have discussed many of the existing theories. At the same time, their range of applicability was examined through small-size asymptotics.

Even though the small-size asymptotic behavior is obtained only below the size range of applicability of the theory (>100 nm), it is useful to pay attention to it. Several main theories show unreasonable small-size asymptotic behavior, which impairs the representation of experimentally observed behavior in the practical size range and spoils asymptotic matching approximations. Simple adjustments of the theories suffer to obtain reasonable asymptotics and make asymptotic matching approximation meaningful.

Finally, an analogy with quasibrittle materials such as concrete, rocks, sea ice, and fiber composite may be mentioned [270]. For them, too, the small size as well as large size asymptotic behaviors are attained only outside the range of validity of the theoretical models (cohesive crack model, crack band model, nonlocal damage models), that is, for specimen sizes much smaller than the inhomogeneity size or much larger than the largest constructable structures. Yet the knowledge of two-sided asymptotics has been shown to be very helpful to achieving good asymptotic matching approximations for the intermediate practical range.

It is also important to emphasize that strain gradient theories cannot explain the size scale effects observed in fcc metals in the absence of strain gradients [36, 37]. Clearly, new continuum theories are needed to be able to predict these size effects.

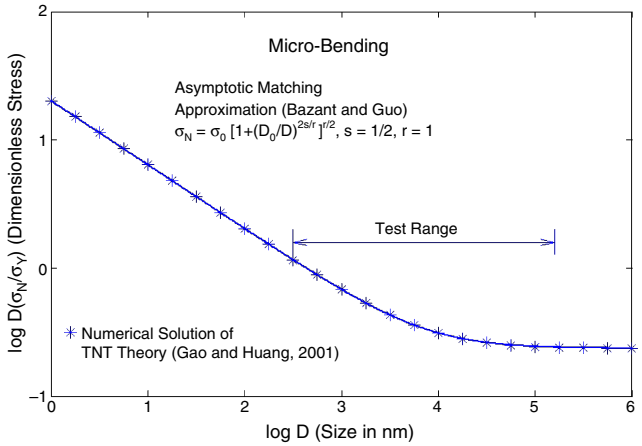

Figure 47. Asymptotic-matching approximation for microbending.

GLOSSARY

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