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Is the cause of size effect on structural strength fractal or energetic–statistical?

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Abstract

The size effect on structural strength is an important phenomenon with a very old history. Unfortunately, despite abundant experimental evidence, this phenomenon is still not taken into account in most specifications of the design codes for concrete structures, as well as the design practices for polymer composites, rock masses and timber. The main reason appears to be a controversy between two different theories of size effect, namely the theory based on energetic–statistical scaling and the theory based on ideas from fractal geometry. This paper aims to critically analyze these two theories, examine their hypotheses and point out the limitations, in order to help code-writing committees choose a rational basis for their work. The paper begins by reviewing the theory of energetic size effect and the efforts to explain the size effect by fractal geometry. The advantages and disadvantages in modeling the structural size effect by fractals are pointed out. Certain flaws in the fractal theory of size effect are illuminated and it is shown that various aspects of this theory lack a sound physical or mathematical basis, or both. The paper ends by recommending how engineering designers and code writers should take the size effect into account.

1. Introduction

1.1. Motivation

Quasibrittle failures of concrete structures, as well as rock masses, sea ice bodies, fiber–polymer composites, ceramics and timber, exhibit a large statistical scatter. In static testing of very large structures, proper similitude is hard to achieve because of the self-weight. A statistically significant set of test results on a sufficiently large number of identical large structures cannot be obtained because of prohibitive costs. The largest concrete structures, sea ice bodies or rock formations are way beyond the range of failure testing.

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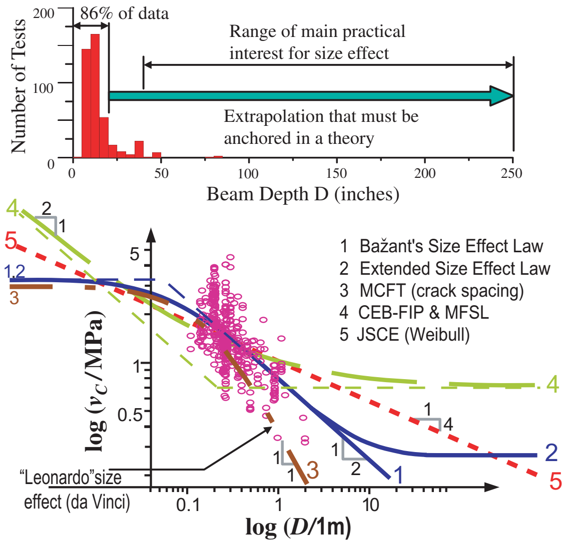

Fig. 1. Histogram (top) of ACI-445 database (bottom) for shear failures of reinforced concrete beams without stirrups (along with the plots of various proposed design formulas).

Consequently, the design of large structures must rely on extrapolation from test results on much smaller laboratory specimens (Fig. 1(top)). Having a good theory is, therefore, imperative.

Currently there exist two fundamentally different theories of size effect in quasibrittle structures: (1) the energetic–statistical theory, and (2) the fractal theory. The conflict between these two theories is a serious impediment to progress in structural design codes and practice. Despite a presidential inaugural address with the motto *ars sine scientia nihil est* ('art without science is nothing') [95], society committees will not adopt a scientific approach to size effect if they get from the literature the impression that there is no generally accepted scientific theory of this phenomenon.

For example, a few recent scaled tests of shear failure of large reinforced concrete beams [1,74], together with a few older ones [93,120], convinced most members of a code-making committee that a significant nonstatistical size effect indeed exists. But many members, aware of an apparent irreconcilable disagreement among theoreticians, feel reluctant to incorporate a scientifically based size effect formula into the building code specifications. They prefer, therefore, a purely empirical approach in which the existing small-size laboratory data are fitted by some simple intuitive formula approved by the vote of a committee, and this formula is then simply extended to large sizes.

Such an empirical approach, however, is doomed to yield an incorrect formula, for two reasons: (1) very high scatter (Fig. 1(bottom)), and (2) vast predominance of test data in the small-size range (Fig. 1(top)) which makes extrapolation to the large sizes of main concern highly uncertain [42]. Because of scatter, the existing test data can be fitted almost equally well by different formulas giving very different extrapolations to large sizes. How a purely empirical evaluation of test data can mislead is explained by Fig. 2, illustrating the kind of approach that has recently been taken by a certain society committee and has been used to claim that a code formula for the size effect on shear of reinforced concrete beams should have the form of a power law of exponent −1/3 (this would, in fact, overestimate the strength of extremely large structures by almost 100%). In Fig. 2(top) it is assumed that hypothetical test data of a limited size range, obtained for four different concretes, perfectly agree with the curves of the theoretical size effect law, with different material parameters for each concrete. Paying no attention to the theory and considering the data for the four concretes as one statistical set, this committee fits the available data by a straight line in the biloga-

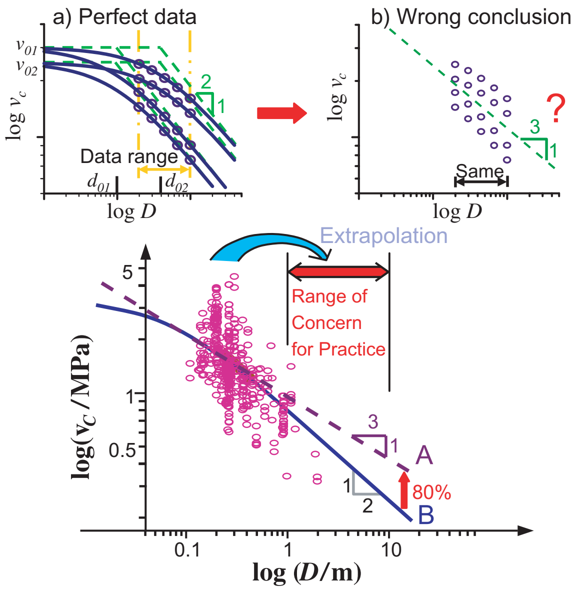

Fig. 2. Example of fallacious statistical analysis: (top left) hypothetical perfect data matching exactly the size effect law curves for four different concretes; (top right) incorrect inference made by regression of the combined data set; and (bottom) resulting error in extrapolation of database for shear of reinforced concrete beams.

rithmic plot, for which a slope of −1/3 is obtained (as shown in Fig. 2(top right)). However, such empirical extrapolation of the available test data is unsafe (causing an error of about 80% at very large structure sizes; Fig. 2(bottom)). Obviously, if the size range of available data in Fig. 2(top) happened to be different, a power law of any exponent between 0 and −1/2 could be obtained by such a fallacious approach to data analysis. To obtain a realistic extrapolation, a sound theory, justified by geometrically scaled tests of sufficient size range, performed on one and the same concrete, must be invoked for interpreting the data.

The foregoing example makes it clear that, until the theoretical disagreement is clarified, until it is demonstrated what is and what is not a sound theory, rational design code specifications guarding against the size effect will hardly be introduced by code-making committees into the concrete design codes of various countries. The purpose of this article is to provide such clarification, and thus to facilitate innovative designs and improvements of design safety and economy.

The problem of codes is particularly acute for structural engineering, and not only because large structure tests are unavailable. In that branch of engineering, many thousands of structures are designed annually. Each of them is different, and most must be designed quickly. This makes simple formulas, capturing the main trends, indispensable, despite the difficulty in developing such formulas. By contrast, for example, aircraft engineering does not need such formulas. Only a few large aircraft are designed per decade, which means that design codes are not needed, and sophisticated analyses and large-scale tests are affordable.

1.2. Size effect and its background

In the classical theories of elasticity, plasticity and continuum damage mechanics, the failure criterion is expressed in terms of stresses and strains, and no characteristic material length ℓ is present. This causes the nominal strength of geometrically similar structures, defined as $\sigma_N = P/bD$, to be independent of structure size D, i.e., there is no size effect (P = maximum load, b = structure thickness). The dependence of σ_N on D, called the size effect, is generally caused by the existence of some sort of characteristic length, ℓ.

Interest in the size effect is very old, in fact older than the mechanics of materials itself [75]. The primeval scaling idea was Galileo's [87] invention of the concepts of stress and strength, the size-independence of which correctly determines the scaling of failure loads for many materials. The first meaningful theory of size effect, understood as the deviation from Galileo's idea of constant material strength, was the statistical weakest-link theory. Although the basic concept of that theory was qualitatively proposed already in the 17th century by Mariotte [102], almost three centuries had elapsed until the correct and complete mathematical formulation of extreme value statistics (or weakest-link model) was developed by Fisher and Tippett [84], who were the first to derive what later came to be known as the Weibull distribution. Based on experiments on fatigue fracture of metals and heuristic arguments, Weibull [122] introduced this probability distribution into the theory of fatigue failure of metals and ceramics and obtained a power law for the statistical size effect.

For about half a century afterwards, Weibull's theory reigned supreme, all the experimentally observed size effects, in all the materials, being attributed to Weibull statistical theory. Serious discrepancies, however, transpired from various experiments, first conducted on concrete [121] and then on other materials, all of which are now termed quasibrittle (rocks, sea ice, fiber composites, toughened ceramics, dry snow slabs, wood, paper, etc.). These are materials that lack plasticity and are characterized by gradual softening in a fracture process zone (FPZ) that is not negligible compared to structure size D. These discrepancies led to the development of a new deterministic energetic theory [8,9,12,21–23,29,31–34,40,103], in which the size effect is explained by stress redistribution and the associated energy release due to development of large cracks or a large FPZ prior to failure. Amalgamation with the Weibull theory then led to a general energetic–probabilistic theory [19,20,35–38,44], in which the failure probability at a point of a continuum depends on a weighted average strain in a certain characteristic neighborhood of the point, rather than on the stress at that point. As far as the statistical mean of size effect in quasibrittle structures is concerned, the probabilistic generalization of the energetic theory was shown to be necessary only for failures occurring right at fracture initiation in large enough structures in which the FPZ is negligible compared to D [19,37,38]. For the variance and probability distribution of size effect, the probabilistic generalization of the energetic theory is, of course, always necessary.

2. Overview of energetic–statistical theory of size effect

2.1. Classical Weibull statistical theory

The classical statistical theory of size effect applies to structures that fail as soon as a macroscopic fracture initiates in one small material element of the structure. This theory is based on the weakest link model, the three-dimensional continuous generalization of which yields the following distribution of failure probability (Fig. 3(left)):

$$P_f(\sigma_N) = 1 - \exp\left[-\int_V c[\boldsymbol{\sigma}(\boldsymbol{x}), \sigma_N] \mathrm{d}V(\boldsymbol{x})\right] \tag{1}$$

Here $\boldsymbol{\sigma}$ = stress tensor field just before failure, $\boldsymbol{x}$ = position vector, V = spatial domain occupied by the structure, and $c(\boldsymbol{\sigma})$ = function giving the spatial concentration of failure probability of the material ($= V_r^{-1} \times$ failure probability of material representative volume V_r) (see [86,119]); $c(\boldsymbol{\sigma}) \approx \sum_i P_1(\sigma_i)/V_0$ where σ_i = principal stresses ($i = 1, 2, 3$) and $P_1(\boldsymbol{\sigma})$ = failure probability (cumulative) of the smallest possible test specimen of volume V_0 (or representative volume of the material) subjected to uniaxial tensile stress σ [122–125]

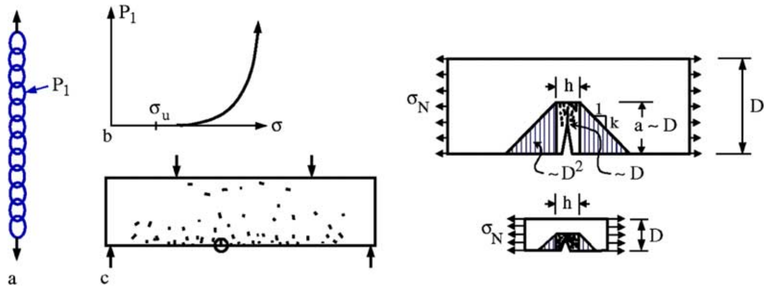

Fig. 3. (left) Chain model underlying the weakest link concept; (middle) power tail of Weibull cumulative distribution of strength of one link, fracture initiating from a microcrack at arbitrary location; and (right) intuitive explanation of energetic size effect.

$$P_1(\sigma) = \left\langle \frac{\sigma - \sigma_u}{s_0} \right\rangle^m \tag{2}$$

Here m, s_0, σ_u = material constants; m = Weibull modulus, a material parameter which characterizes the coefficient of variation of strength scatter and varies widely (between 5 and 50); s_0 = scale parameter; σ_u = strength threshold (which may usually be taken as 0), and V_0 = reference volume understood as the volume of small test specimens on which $c(\sigma)$ was measured (for recent generalization for plastic crack-tip blunting, see e.g., [46,96,114]).

Distribution (1) with (2) generally implies the mean and the coefficient of variation of the nominal strength to be

$$\bar{\sigma}_N \propto \Gamma(1 + m^{-1})(D_0/D)^{n_d/m}, \quad \omega = [\Gamma(1 + 2m^{-1})/\Gamma^2(1 + m^{-1}) - 1]^{1/2} \tag{3}$$

in which n_d = number of dimensions in which the structure is scaled ($n_d = 1, 2, 3$). Because, according to Weibull theory, ω is independent of D, the measurement of ω is normally used as the easiest way to identify m from tests (but unfortunately it is usually forgotten to check whether the same ω is obtained from size effect tests and seatter test specimens of various sizes, which is a condition of validity of Weibull theory; e.g. [127,136]).

As far as quasibrittle structures are concerned, application of the classical Weibull theory faces a number of objections:

(1) The fact that the Weibull size effect is a power law

$$\sigma_N \propto D^{-n_d/m} \tag{4}$$

implies the absence of any characteristic length, which cannot be correct if the material exhibits an FPZ of a finite, non-negligible, size [2,11].
(2) The energy release due to stress redistribution caused by macroscopic FPZ or stable crack growth before P_{max} gives rise to a deterministic size effect which is ignored.
(3) Every structure is mathematically equivalent to a uniaxially stressed bar (or chain, Fig. 3).
(4) The tests of concrete, ice, fiber composites, rocks and ceramics show a much stronger size effect than explicable by Weibull theory.
(5) The m-values inferred from scatter are not independent of structure size D and do not match the m-values determined from size effect tests [15].
(6) Spatial correlations of material failure probabilities caused by nonlocal behavior are not taken into account.
(7) The predicted difference in size effect between two- and three-dimensional similarity ($n_d = 2$ or 3) in flexure tests is excessive.

Consequently, in the case of concrete structures, the mean size effect of the classical Weibull theory appears applicable only to the failure of extremely thick plain (unreinforced) structures, e.g., to the flexural fracture of arch dams.

2.2. Deterministic energetic theory of quasibrittle size effect

In quasibrittle materials, the failure is characterized by both a critical energy per unit area (or the fracture energy, G_f, of dimension Pa m) and a critical stress (or the tensile strength, f'_t, of dimension Pa). From dimensional analysis, it is clear that a material with such properties must possess a characteristic length, $\ell_0 = EG_f/f'^2_t$ (E = Young's elastic modulus); ℓ_0 approximately characterizes the length of FPZ, as proposed by Irwin [94] (and for concrete promulgated by Hillerborg et al. [90,91]). Linear elastic fracture mechanics applies asymptotically when $\ell_0/D \to 0$, and in that limit case f'_t becomes irrelevant. The cohesive (or fictitious) crack model (originated by Barenblatt [3,4]) is needed when ℓ_0/D is neither negligible nor very large. In the asymptotic limit $D/\ell_0 \to 0$, G_f becomes irrelevant and the cohesive crack model leads to an elastic body with a perfectly plastic crack of yield strength f'_t.

The general approximate size effect laws for quasibrittle failures described by the cohesive crack model or nonlocal damage models can be most generally derived by asymptotic matching—a procedure in which the large-size asymptotic behavior described by LEFM is expanded into an asymptotic series in terms of powers of ℓ_0/D, the small-size asymptotic behavior described by plasticity is expanded into an asymptotic series in terms of powers of D/ℓ_0, and both series are then approximately matched to obtain a size effect law approximately applicable over the entire size range [21,28]. For structures with pre-existing notches or large pre-existing fatigue-weakened cracks, this leads to the size effect law [9]

$$\sigma_N = Bf'_t\left(1+\frac{D}{D_0}\right)^{1/2}, \quad Bf'_t = \sqrt{\frac{EG_f}{g'(\alpha_0)c_f}}, \quad D_0 = c_f\frac{g'(\alpha_0)}{g(\alpha_0)} \tag{5}$$

Here B is a dimensionless geometry-dependent parameter, which depends on the geometry of the structure and of the crack, and can be calculated from the cohesive crack model [41]; P = applied load; K_I = stress intensity factor; $\alpha_0 = a_0/D$; a_0 = initial crack length; D_0 = geometry-dependent parameter called the transitional size; and $\check{c}_f$ = half-length of FPZ proportional to ℓ_0; the primes denote derivatives; and $g(\alpha_0) = K_I^2(\alpha_0)b^2D/P^2$ = dimensionless LEFM energy release function for the given geometry ($g(\alpha_0) = k^2(\alpha_0)$ where $k(\alpha_0)$ = dimensionless stress intensity factor [40]). The first formula in (5) has also been derived in several other ways—analytically, from the cohesive crack model [21, Eq. (9.40),115]; by a combination of dimensional analysis with asymptotic matching [43]; and by asymptotic expansions of equivalent LEFM [16]; and was also verified by simulations with nonlocal finite elements, discrete elements (lattice or particle models), and crack band model (see also [110]).

For structures failing at crack initiation, right after the FPZ has formed, the asymptotic matching leads to a different size effect law [10,18,21,24,31,32]:

$$\sigma_N = \sigma_\infty\left(1+\frac{rD_b}{\eta D_b + D}\right)^{1/r}, \quad \sigma_\infty = \sqrt{\frac{EG_f}{c_f g'(0)}}, \quad D_b = \left\langle \frac{-g''(0)c_f}{4g'(0)} \right\rangle \tag{6}$$

where r, η = empirical positive constants (of the order of 1); $\langle \ldots \rangle$ denotes the positive part of the argument; B = dimensionless geometry-dependent parameter; and D_b = constant that can be regarded as the thickness of the boundary layer of cracking in beam flexure.

There exists also a third case of size effect. This is the case of stable growth of a large crack in which the cohesive stresses have not been reduced to zero by previous fatigue; see [21]. But the plots of the corresponding formula and of (5) are rather similar and hardly distinguishable in comparison with experiments.

Let us now give a simple intuitive argument leading to equation (5). Consider the rectangular panel in Fig. 3(right), which is initially under a uniform stress equal to σ_N. Introduction of a crack of length a with an FPZ of a certain length and width h may be approximately imagined to relieve the stress, and thus release the strain energy, from the triangular zones on the flanks of the crack band shown in Fig. 3(right). The slope k of the effective boundary of the stress relief zone need not be known; what is important is that, for geometrically similar panels, k is independent of the size D. It is also assumed that the situations at failure are approximately geometrically similar, i.e., $a/D \approx$ constant (which is true in many, but not all situations, and must be justified experimentally or computationally). The stress reduction in the shaded triangular zones, of areas $ka^2/2$ (Fig. 3(right)), causes the strain energy release $U_a = 2(ka^2/2)\sigma_N^2/2E$ (for the case $b = 1$). The stress drop within the crack band of width h causes further strain energy release $U_b = ha\sigma_N^2/E$. The total energy dissipated by fracture is $W = aG_f$, where G_f is the fracture energy, a material property representing the energy dissipated per unit area of the fracture surface. Energy balance during static failure requires that $\partial(U_a + U_b)/\partial a = \mathrm{d}W/\mathrm{d}a$. Setting $a = D(a/D)$ (where a/D is approximately a constant if the failures for different structure sizes are geometrically similar), the solution of the last equation for σ_N yields the approximate size effect law in (5) (see [9] and Fig. 4(top left)).

Likewise, one can give a simple intuitive argument [31] leading to equation (6). For the sake of illustration, let us consider the modulus of rupture test, i.e., the flexural failure of a simply supported beam of span L with a rectangular cross-section of depth D and width b, subjected to a concentrated load P at midspan (Fig. 5(top left)). Due to material heterogeneity, causing the quasibrittle behavior, the maximum

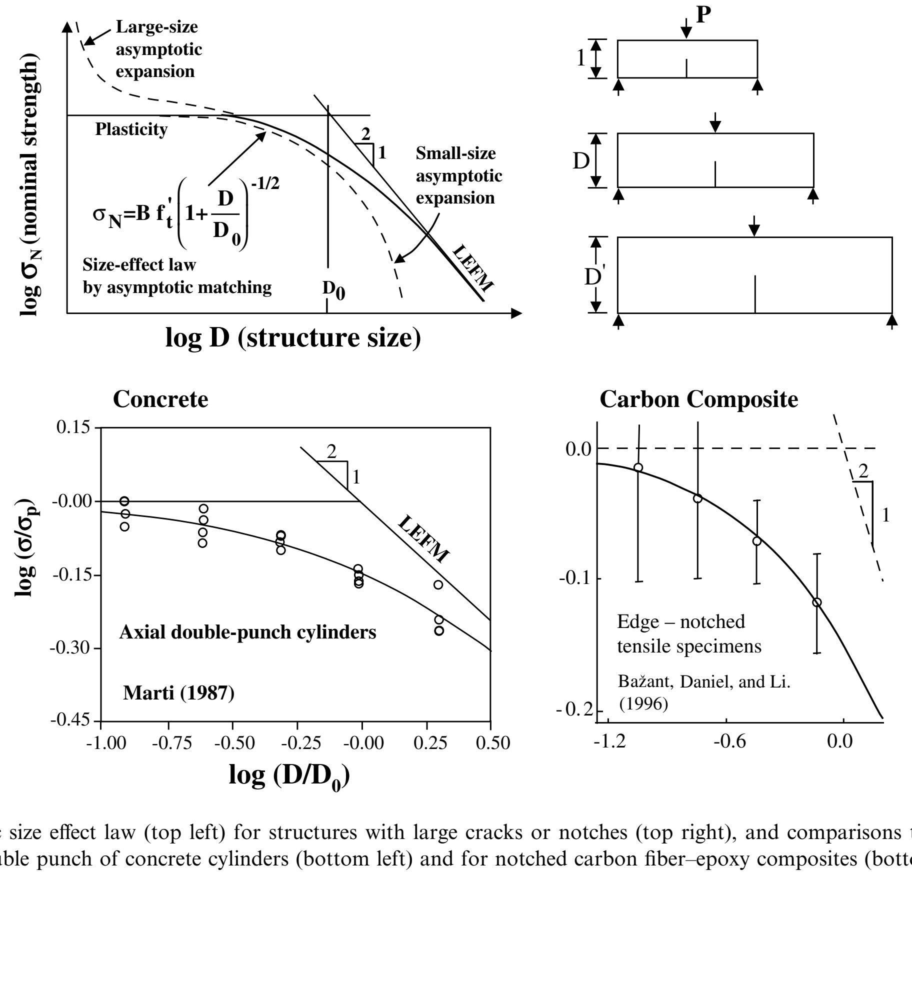

Fig. 4. Energetic size effect law (top left) for structures with large cracks or notches (top right), and comparisons to tests data for compression double punch of concrete cylinders (bottom left) and for notched carbon fiber–epoxy composites (bottom right).

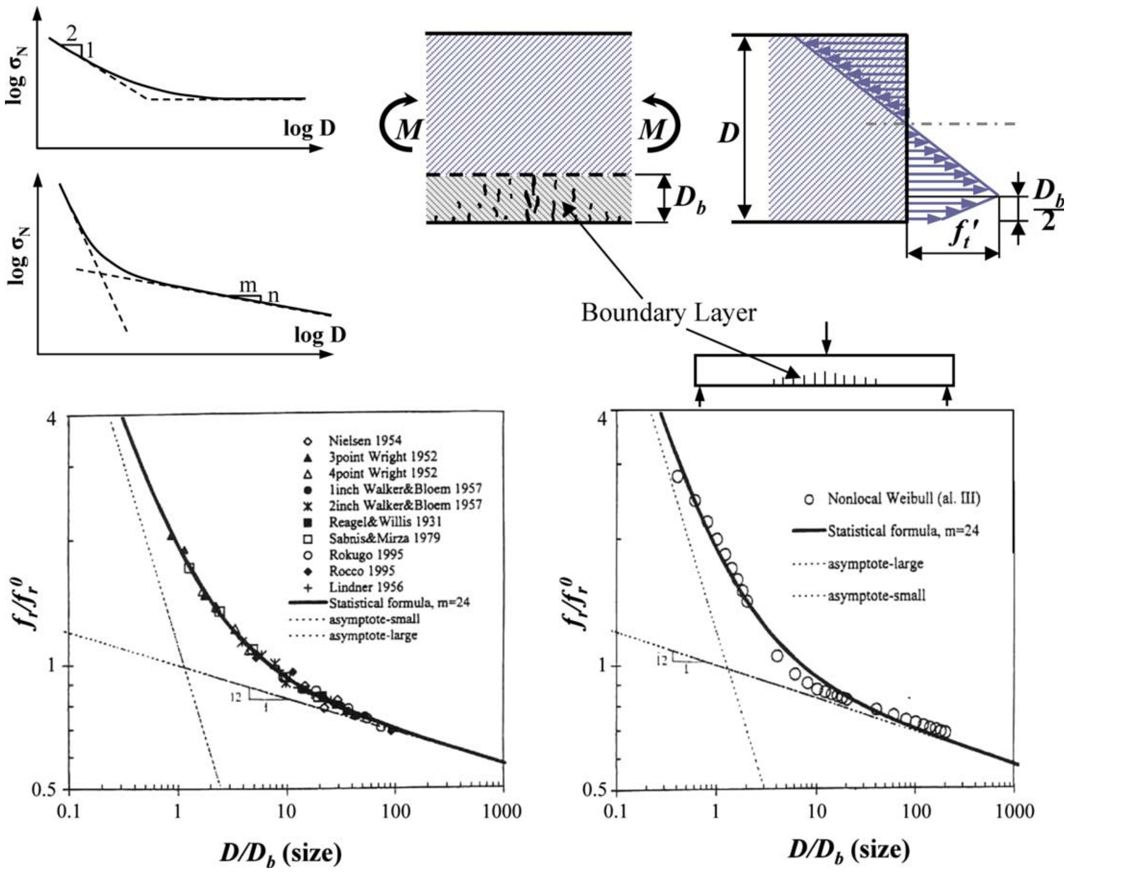

Fig. 5. (top left) Curves of energetic and energetic–statistical size effects for failures at crack initiation; (top right) explanation of energetic size effect due to stress redistribution at macrocrack initiation from a finite-size microcracking zone; (bottom) energetic–statistical size effect compared to tests of modulus of rupture of concrete (left) and to numerical simulations with nonlocal energetic–statistical theory (right) [35,36].

load is not decided by the stress $\sigma_1 = 3PL/2bD^2$ at the tensile face, but by the average stress value within a boundary layer of microcracking having thickness c_f that is a material property (equal roughly to one to three maximum inhomogeneity sizes). This value may be approximated by the stress $\bar{\sigma}$ roughly at distance $c_f/2$ from the tensile face (which is at the middle of FPZ). Because $\bar{\sigma} = \sigma_1 - \sigma_1' c_f/2$ where $\sigma_1' =$ stress gradient $= 2\sigma_1/D$, and also because $\bar{\sigma} = \sigma_0 =$ intrinsic tensile strength of the material, the failure condition $\bar{\sigma} = \sigma_0$ yields $P/bD = \sigma_N = \sigma_0/(1 - D_b/D)$ where $D_b = (3L/2D)c_f$, which is a constant because for geometrically similar beams $L/D =$ constant. This expression, however, is unacceptable for $D/D_b \to 0$. But since the foregoing derivation is valid only for small enough c_f/D, one may replace this formula with (6), which is asymptotically equivalent for $D/D_b \to 0$ (Fig. 5(top middle)). This formula happens to also satisfy the asymptotic conditions of plastic failure for $D/D_b \to 0$, and thus is acceptable for the whole range of D (r is any positive constant). The values $r = 1$ or 2 have been used for concrete, while $r \approx 1.44$ gives the optimum fit of the concrete test data from the literature [18].

Derivations of (5) and (6), applicable to arbitrary structure geometry, have been given in terms of asymptotic analysis based on Rice's path-independent J-integral [18,21], and also on the basis of equivalent LEFM [15].

2.3. Combined statistical–energetic size effect in quasibrittle materials

The Weibull statistical theory and the energetic theory (which arose from the fracture propagation concept of Griffith) have been combined in the nonlocal Weibull theory [35,36,44], which has both aforementioned theories as its asymptotic limits. The deterministic energetic size effect is obtained for not too large structure sizes, and the Weibull statistical size effect is obtained as the asymptotic limit for very large structures ($D \to \infty$), provided that the failure (or instability) occurs at macrocrack initiation and that the structure geometry is positive. [1]

The nonlocal concept was proposed, within the context of elasticity, in the 1960s [77,78,92], and has later been extended to hardening plasticity [79]. In the 1980s, it was adapted to strain-softening continuum damage mechanics and strain-softening plasticity [25,109], with three motivations:

(1) to serve as a computational 'trick' (localization limiter) eliminating spurious mesh sensitivity and incorrect convergence of finite element simulations of damage;
(2) to reflect the physical causes of nonlocality, which are three:
    (a) material heterogeneity,
    (b) energy release due to microcrack formation, and
    (c) microcrack interactions; and
(3) to simulate the experimentally observed size effects that are stronger than those explicable by Weibull theory. Because of material heterogeneity, the macroscopic continuum stress at a material point must depend mainly on the average deformation of a representative volume of the material surrounding that point, rather than on the local stress or strain at that point.

In the deterministic nonlocal theory for strain-softening damage or plasticity, the spatial averaging must be applied only to the inelastic part ϵ'' of the total strain ϵ (or some of its parameters), rather than to the total strain itself [109]. Accordingly, the cumulative failure probability $P_1(\boldsymbol{\sigma})$, considered in the classical Weibull theory as a function of the local stress tensor $\boldsymbol{\sigma}$ at material point $\boldsymbol{x}$, must be replaced by a function of a nonlocal variable [24,35,36,39,44]. The nonlocal stress is not acceptable as a nonlocal variable because it decreases when the average strain increases. A suitable nonlocal variable is the nonlocal strain or, more precisely, the nonlocal inelastic part of strain. The material failure probability is thus defined in the nonlocal Weibull theory as

$$P_1 = \langle \bar{\boldsymbol{\sigma}}/s_0 \rangle^m \tag{7}$$

where

$$\bar{\boldsymbol{\sigma}}(\boldsymbol{x}) = \boldsymbol{E} : [\boldsymbol{\epsilon}(\boldsymbol{x}) - \bar{\boldsymbol{\epsilon}}''(\boldsymbol{x})], \quad \bar{\boldsymbol{\epsilon}}''(\boldsymbol{x}) = \frac{1}{\bar{\alpha}(\boldsymbol{x})} \int_V \alpha(\boldsymbol{s} - \boldsymbol{x}) \boldsymbol{\epsilon}''(\boldsymbol{s}) \mathrm{d}V(\boldsymbol{s}) \tag{8}$$

in which $\boldsymbol{\epsilon}''$ = inelastic part of strain tensor; E = initial elastic moduli tensor; $\bar{\alpha}(\boldsymbol{x})$ = normalizing factor of $\alpha(\boldsymbol{s} - \boldsymbol{x})$; and $\alpha(\boldsymbol{s} - \boldsymbol{x})$ = a bell-shaped nonlocal weight function whose effective spread is characterized by

[1] A 'positive' geometry is a geometry of structure (including geometry of crack and loading) for which the stress intensity factor increases as the crack extends at constant load (negative geometry—the opposite). Structures of positive geometry fail, under controlled load (or dead load), as soon as the process zone forms. Nearly all notched fracture specimens are of positive geometry. The exceptions are a large panel with a small crack loaded by concentrated loads on the crack faces, the reverse-tapered double-cantilever specimen, and the Charpy notched specimen; so is the flexural failure of unreinforced beams. Many reinforced concrete structures have an initially negative geometry, and so do many structures with a compression zone in front of fracture tip, e.g., a gravity dam with a dipping crack produced by over flow.

characteristic (material) length ℓ_0, which for example governs the minimum possible spacing of parallel cracks and is in general different from (and much smaller than) ℓ_0 characterizing the FPZ length.

The nonlocality makes the Weibull integral over a body with crack tip singularity converge for any value of Weibull modulus m, and it also introduces into the Weibull theory spatial statistical correlation. Numerical calculations of bodies with large cracks or notches showed that the randomness of material strength is almost irrelevant for the size effect on the mean σ_{N} [44]. Therefore, the energetic mean size effect law (5) for the case of large fatigued cracks or large notches remains almost unaffected by material randomness. Intuitively, the reasons are that: (1) a significant contribution to Weibull integral comes only from the FPZ, the size of which remains constant if the structure size is increased, and (2) mechanics almost dictates the crack path, so that the FPZ cannot sample locations of different local strength.

One case in which the statistical size effect on the mean strength $\bar{\sigma}_{\mathrm{N}}$ is important is the failure at crack initiation in structures of positive geometry that are much larger than the inhomogeneity size. This is the case of bending of very thick plain concrete beams or plates (exemplified by the flexural failure of an arch dam, typically about 10 m thick) [35,36]. In this case, Eq. (6) needs a correction for large D [22]. Based on asymptotic matching arguments, this equation needs to be generalized as [19] (Fig. 5(top right and bottom))

$$\bar{\sigma}_{\mathrm{N}} = \sigma_0 \left[\left(\frac{D_{\mathrm{b}}}{\eta D_{\mathrm{b}} + D} \right)^{rn_{\mathrm{d}}/m} + \frac{rD_{\mathrm{b}}}{\eta D_{\mathrm{b}} + D} \right]^{1/r} \tag{9}$$

where $rn_{\mathrm{d}}/m < 1$; η, $r =$ empirical constants. This formula, which gives the statistical mean of size effect, asymptotically approaches the Weibull size effect as $D \to \infty$. It was shown to fit quite well the bulk of the existing test data on the modulus of rupture of concrete and to closely agree with numerical predictions of the nonlocal Weibull theory over the size range 1:1000 [36,39]. Parameter η can be taken as zero for the fitting of the existing test data, however, a nonzero value is needed for the purposes of asymptotic matching, in order to match the asymptotic behavior of the cohesive crack model (or crack band model) for $D \to 0$ (normally approached only for sizes less than the size of material inhomogeneities). Aside from the

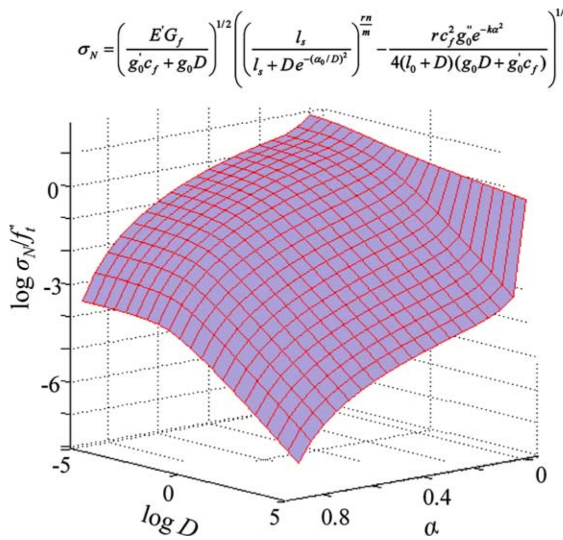

Fig. 6. Universal (mean, energetic–statistical) size effect law describing the transition between failures at crack initiation and failures for large notches or cracks formed in a stable manner before failure [43] ($\alpha = a_0/D =$ relative notch or crack length; ℓ_s; ℓ_0; $r =$ constants, $g_0 = g(\alpha_0)$, $g_0' = g'(\alpha_0)$) (the upper rim of the surface shown, $\alpha = 0$, may be compared to MFSL).

two aforementioned asymptotic limits, the formula also satisfies, as a third asymptotic condition, the requirement that the deterministic formula (6) must be recovered for $m \to \infty$. The statistical distribution of the size effect has also been deduced; it represents a transition from the Gaussian distribution for $D \to 0$ to the Weibull distribution for $D \to \infty$ [24,39].

A more difficult fracture scaling problem arises in the transitional situation in which there is a notch or initial crack not much larger than the material inhomogeneities, e.g., the aggregate size or the thickness of a layer in a laminate. Information on this transition is computationally best generated by random particle-lattice models for concrete microstructure (e.g., [26]), but can also be obtained by nonlocal Weibull model [39]. Asymptotic matching approach to this problem has led to a universal mean size effect law presented in [43] and pictured in Fig. 6 (for the deterministic case, see [32]). The formula of this law, shown in Fig. 6, satisfies six asymptotic conditions: (1) for a large a_0 (initial crack or notch length) and $D \to \infty$—convergence to LEFM; (2) for large a_0 and $D \to 0$—to a plastic limit (with no size effect) required by cohesive crack model, crack band model or nonlocal models; (3) for $a_0 = 0$ (failure at crack initiation, and $D \to \infty$—to pure Weibull scaling; (4) for $a_0 = 0$ and $D \to 0$—again to the aforementioned plastic limit; (5) for $m \to \infty$—to deterministic theory; and finally (6) for $l_0 \ll D \ll \eta\ell_0$ (i.e., between the nonlocal characteristic length ℓ_0 and FPZ length $\eta\ell_0$, as explained in [24])—convergence to an intermediate asymptotic behavior (in the sense rigorously mathematically discussed in [6]).

3. Fractal theories of size effect on nominal strength of structures

3.1. Fractal characterization of crack randomness, disorder and roughness

The fact that the surface roughness of cracks in many materials as well as the distributions of microcracks are physical fractals, i.e., can be described over a certain limited range by fractal concepts, is not in doubt (see [57,99,108,111,116,117,129] and references therein). To date, many attempts have been made to relate the fractal dimension (or roughness) of a crack to material properties such as the toughness, and some experimental studies have been conducted to find some universality in the fractal dimension (or roughness exponent, see [55]). However, it seems that there is no simple universal relationship between fractal dimension and fracture toughness or fractal energy, and no relationship between fractal dimension and material characteristic length (length and width of the fracture process zone). Looking at the literature on the fractal properties of fracture, one can see that most studies are experimential.

An important point to note is that simply knowing the fractal dimension of a fracture surface or a distribution of microcracks does not help in understanding the mechanics of failure. There exist only a few theoretical studies on the mechanical consequences of fractality of cracks (see [12,13,15,21,27,49–53,130–135]). Unfortunately, there have been conceptual mistakes in some recent papers. One can also find some unnecessarily sophisticated and unjustified generalizations of classical results that contribute nothing but complexity and are practically unusable.

Several researchers have tried to use fractals in the modeling of structural size effect. Among these, one can mention Weiss [126], Borodich [52,54], Gol'dshtein and Mosolov [89] and Bažant [15]. Gol'dshtein and Mosolov [89] used fractals and multifractals for the purpose of size effect, although with limited results. Self-affine crack roughness [47,73] was considered by Weiss [126] and Morel and coworkers [104–106] to model the size effect on different scales using the transition property of self-affine curves. Borodich [52] used self-similar fractals in the modeling of multiple fracture. What particularly calls for a critical examination in this paper, which is focused on the size effect law needed for design, are the formulations of Carpinteri and co-workers [58,70]). It will be seen, however, that these formulations, while thought-provoking, lack a solid basis and have various fundamental flaws.

3.2. The so-called 'multifractal' scaling law (MFSL)

A possible role of fractality in the size effects was discussed in 1990 in relation to sea ice [45]. The partly fractal nature of crack surfaces and of the distribution of microcracks in concrete was considered in 1994 as the physical origin of the size effects observed on quasibrittle structures, and the so-called 'multifractal' scaling law (MFSL), which can be used for concrete structures failing at fracture initiation from a smooth surface, was proposed [58]. This law reads

$$\sigma_N = \sqrt{A_1 + (A_2/D)} \tag{10}$$

where A_1, A_2 = constants; see also [59–62,64–67]. In an attempt for mathematical derivation (critically discussed in Appendix A), it was argued [58] that (i) the fractal nature of crack surface or microcrack distribution, or both, requires the scaling $\sigma_N \propto \sqrt{D}$, while (ii) the disappearance of fractality at very large-scales implies the vanishing of size effect, i.e. σ_N = constant. In view of these two assumed opposite asymptotic properties, the MFSL formula (10) was then proposed as a way to achieve a smooth transition.

What Carpinteri calls 'multifractals' is different from the concept of multifractals in mathematics and should properly be called 'scale-dependent fractals' (see [72]) (a multifractal is a measure, supported on some geometrical set; see e.g. Feder [82] for details).

The MFSL does not capture, in contrast to formula (9), the transition to Weibull size effect for very large sizes. But that the MFSL allows a good fit of the test data on the modulus of rupture of concrete (i.e., the flexural strength of unreinforced beams) on the laboratory scale is undeniable. However, a formula identical to MFSL has been derived from fracture mechanics [12,18,21,31] based on a reasonable hypothesis already discussed—namely that there exists a boundary layer of cracking that has a fixed thickness and causes stress redistribution with energy release before the maximum load is reached. In fact, the MFSL, Eq. (10), is found to be identical to the special case of formula (6), as well as its probabilistic generalization (9), if one sets [18]

$$r = 2, \quad \eta = 0, \quad m \to \infty \tag{11}$$

$$A_1 = EG_f/c_f g'(0), \quad A_2 = -EG_f g''(0)/2c_f[g'(0)]^3 \tag{12}$$

The last two equations relate the MFSL parameters to $g(\alpha)$, the dimensionless energy release function of LEFM, and thus they automatically provide the geometry dependence of A_1 and A_2, if it is accepted that the explanation of (10) is energetic rather than 'multifractal'.

3.3. Comparisons and problems of MFSL

Two kinds of comparisons of MFSL need to be discussed: one meaningless and one meaningful. First the meaningless. The proponents of MFSL, since 1995 until the present (see e.g. [63], or Fig. 2 in [70], or Figs. 10–13 in [71]), have never compared MFSL in their works to (6). Rather, they have habitually compared it to (5). But this equation, which is the size effect law for structures with large notches or with large cracks at maximum load, has never been claimed to apply for structures failing at the initiation of a macroscopic crack, the sole case for which the MFSL can be applied. Comparisons of these two formulas, intended for different situations, only sow confusion and make no sense at all.

Now the meaningful comparison, which is a comparison of (10) with (6). This comparison is crucial and must be clarified before the size effect can be introduced into practical design.

Although the fact that many features of disorder in the microstructure of concrete and other quasibrittle materials can be characterized in terms of fractal concepts is not questioned, a number of objections must be raised against the proposition that the size effects on the nominal strength of structures, and the MFSL (Eq. (10)) in particular, follow from the hypothesis of fractal microstructure. They are as follows:

1. The stress redistribution and energy release phenomena associated with large cracks or notches, or large FPZ, or both, must cause a size effect. They are by now well proven from many angles, experimentally as well as theoretically. Therefore, if fractality of the microstructure should be regarded as a physical source of size effect on the structural level, it cannot be introduced as a replacement of the stress redistribution and energy phenomena. It could come only as some additional feature, a refinement. For example, it is certainly possible that microstructure fractality could help to explain the statistical scatter about the mean energetic size effect or some properties of the fracture energy, the strength limit and the softening stress-separation law of a cohesive crack.
2. In contrast to the energetic concept, Eq. (12), the fractal concept of structural size effect does not provide information on the dependence of the size effect parameters on the geometry of the structure. It cannot provide it unless the fractal boundary value problem (which is not even well-defined) is solved. This fact alone suffices to render the MFSL almost useless for practical applications.
3. The 'MFSL' was based on a series of hypotheses but does not follow from these hypotheses by a valid mathematical procedure (see Appendix A). In particular, the argument based on dimensional analysis [118] is mathematically inconsistent.
4. For very small sizes D, the MFSL (Eq. (10)) gives the scaling law $\sigma_N \propto D^{-1/2}$. The asymptotic small-size power scaling law of exponent $-1/2$ (which corresponds to the value $r = 2$ in (6)) agrees reasonably well with the data for concrete. In view of the universality of this exponent in the random walk and percolation models, this is no surprise. However, the value $-1/2$ is an unproven conjecture which does not follow from the fractal hypothesis (Appendix A). Moreover, it has not been explained why the scaling law exponent should be independent of the fractal dimension D_f that characterizes the cracking morphology. Should not the exponent be different for very different fractal dimensions D_f, for example when D_f deviates from the Euclidean dimension by 40% or by 2%? In the extreme, one may consider the difference of the fractal dimension from the Euclidean dimension to be almost vanishing, for instance 0.001%, and in that case one must expect the scaling exponent to be very close to 0 because a sudden jump from $-1/2$ to 0 as $D_f \to 0$ would obviously be irrational. This argument demonstrates that the exponent of the power scaling law cannot be independent of the fractal dimension if the fractal hypothesis is adopted (see [133] for similar considerations and recovery of classical results as a limit case of their fractal counterpart).
5. The lack of fractality on a large-scale is assumed to imply the absence of size effect. In comparison with the small-size value $-1/2$, this implies that the size effect exponent would depend on the fractal dimension. However, as already pointed out, this conflicts with the fact that, for very small sizes, the exponent is implicitly considered as independent of that dimension.
6. If the hypothesis of fractal origin of the structural size effect were justified, and if the argument for fractal-based scaling were valid, then an n-fold increase of beam width would have to cause the same size effect on σ_N as an n-fold increase of beam depth. But this is not the case. The beam width has almost no effect on the modulus of rupture. This empirical conflict alone suffices to reject the hypothesis of fractal origin of structural size effect.
7. In microscopic observations, the fractality of fracture surfaces and microcrack distributions is observed for up to only about 1.5 orders of magnitude of refinement. Such a range of fractality is much narrower than the range important for size effect laws. Besides, within such a narrow range, a rough crack regarded as a fractal can be described about equally well by statistical models for surface roughness. Normally, the fractality needs to be experimentally observed through about six orders of magnitude for the fractal scaling to be considered a very good model.
8. The mathematical arguments advanced in support of the fractal concept invoked [68,69] the renormalization group transformation [5,6]. This transformation relates the power scaling law for one size range to a different power scaling law for an adjacent size range, giving the intersection of these two power laws (which appear as straight lines of different slopes in a bi-logarithmic plot). For quasibrittle materials, however, the salient question is not only the intersection but also the gradual transition between these

two power laws in the size coordinate. This transition occupies several orders of magnitude (about 3) of structure size D and represents the size range of practical interest. The fractal approach as formulated so far [68,69] says nothing about this transition, while the energetic approach based on stress redistribution does.

The fractal arguments have also been extended to provide an alternative explanation for the dependence of fracture energy of quasibrittle material on structure size, and for the apparent variation of fracture energy (evaluated according to LEFM) during the initial crack growth, called the R-curve (resistance curve) [68]. However, both phenomena have already been explained by nonfractal concepts—by the initial growth of FPZ and by nonlocal continuum damage models simulating this growth [40]). It is not denied that fractal concepts could play some role. However, this role would have to be a refinement of the energetic (or stress redistribution) mechanism, rather than its replacement (the same as in point 1 above).

Questions also arise with regard to the recent attempts at general fractal-based continuum mechanics in which the stress, strain, energy density and mass density have fractal dimensions [68,69]. The difficulty of developing such a radical theory is apparent from the fact that the concept of fractal stress has so far been enunciated only in the uniaxial setting, i.e., merely as a force per unit measure of a lacunar cross-section rather than a multidimensional tensor. The same uniaxial limitation also applies to the concept of fractal strain. In more than one dimension, it is necessary to define the surface on which a force (the fractal stress) is supposed to act. However, a fractal surface is nowhere differentiable, and so the normal to such a surface is not defined. Thus the generalizations made in [68,69] appears to be unjustified and useless, achieving nothing but artificial complexity (see Section 4.4).

It may be noted that a similar controversy about the applicability of fractal models has developed in turbulence [21], a field that is still far from fully understood despite a century of research. Dimotakis and Catrakis [72,76] recently explain that the problem of turbulence is too complex to be modeled by a fractal of constant fractal dimension and show that a fractal with scale-dependent variable fractal dimension would have to be used.

3.4. Crack characterization by self-affine roughness and simulation of R-curve

While a fractal curve is obtained by self-similar disturbances relative to the *local* direction of the curve (i.e., by repeatedly scaling down each multisegment section and substituting it for a single segment on the lower-scale), a self-affine curve is obtained by imposing lateral disturbances transverse to the direction of the global fracture direction, which are self-similar only in the transverse direction (Fig. 7(top left)).

Morel and coworkers [104–106] studied the consequences of self-affine crack roughness with a focus on mildly heterogeneous materials in which the source of crack roughness appears to be the interaction of the crack front with microcracks. To simulate their experimental observations of crack surfaces in wood (pine and spruce), they described the fluctuations of the asperity height Δh along the crack curve as a transition between two power laws.

Specifically, Morel et al. considered a generalization of the Family-Vicsek law [81]—an 'anomalous' scaling law according to which $\Delta h(l,y) \approx A l^{\zeta_{\rm loc}} \xi(y)^{\zeta-\zeta_{\rm loc}}$ for $l \ll \xi$ (large-scale of observation) and $\Delta h(l,y) \approx A\xi(y)^{\zeta}$ for $l \gg \xi$ (small-scale of observation) where $l =$ length of observation window (or ruler length), $\xi(y) = By^{1/z}$, $\zeta_{\rm loc} =$ local roughness exponent, $\zeta =$ global roughness exponent, $y =$ distance from notch tip, and A, B, $z =$ constants (z called the dynamic exponent).

In [105], Morel et al. present an energetic analysis of the size effect on the nominal strength $\sigma_{\rm N}$, analogous to that in [15]. The result is similar to that in Fig. 7(bottom left) from [15], except that they assumed not one power law but a transition between two power laws for crack roughness, which causes that the size effect curve found (Fig. 3 in [105]) is the same as a transition in Fig. 7 from the nonfractal curve for small sizes (which eliminates the rising portion of the curve) to the fractal curve for large sizes, with an asymptotic

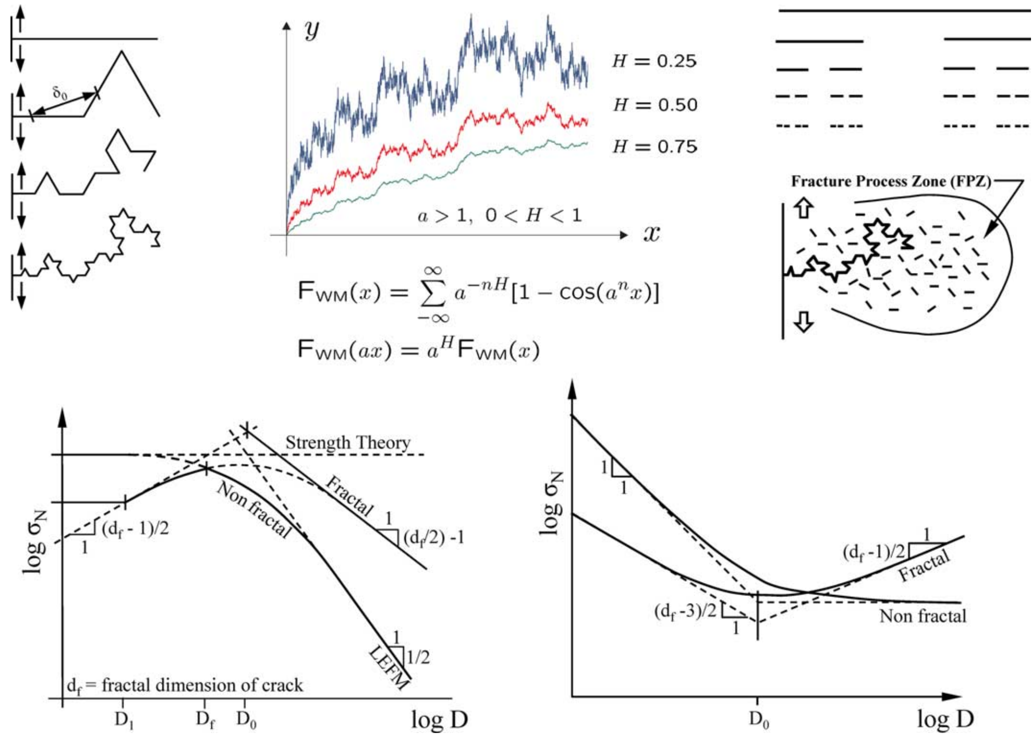

Fig. 7. (top left) Fractal curve; (top below) curve with self-affine roughness; (top right) one-dimensional lacunar fractal set at progressive refinements; (top below) cracking zone surrounding fractal crack tip; (bottom) comparisons of fractal and nonfractal size effects for structures with (left) and without (right) notches or large cracks at maximum load.

slope significantly less than $-1/2$. The transition from one size effect curve to another is caused by the fact that the 'anomalous' scaling law possesses a lower bound on the scale of crack roughness.

As already mentioned, such a milder asymptotic curve would disagree with the experimental evidence for concrete, sea ice, some rocks and fiber composites. In [105], a good agreement is nevertheless found with tests of geometrically similar notched wood specimens. However, the size range of these tests has been limited and it seems that a size effect curve as in Fig. 4(top right), with a long transition to the terminal slope of $-1/2$, could also match these test results (the asymptotic slope $-1/2$ being approached only beyond the size range of these tests). The reason that an asymptotic slope milder than $-1/2$ is obtained in [105] is that the assumed large-scale power law for crack roughness is considered to have no upper bound. Whether this is true deserves further study, especially since generally the ranges of fractal regimes are not unbounded.

Although the *R*-curves (e.g., [40]) are not the focus of this study, it may be mentioned that the growth of roughness of a self-affine crack leads to *R*-curve behavior. This is explained by noting that if the nominal (projected) crack length is increased by Δa, its measure increases in proportion to $(\Delta a)^{1/H-1}$, and thus the critical energy release rate increases as

$$G_R^{\mathrm{f}} \sim 2\gamma_{\mathrm{f}}(\Delta a)^{1/H-1} \tag{13}$$

where γ_{f} is a specific surface energy per unit of a fractal measure. This means that G_R^{f} starts from $2\gamma_{\mathrm{f}}$ and increases with the projected length. The self-affine roughness model of Morel et al. has led to a good agreement with *R*-curve measurements on wood [104–106]. However, such *R*-curves can be simulated more simply by the cohesive crack model, crack band model and nonlocal models, which better describe the fact that the fracture process zone is finite, while the self-affine and fractal models for the R-curve presume the fracture behavior to be describable point-wise. It is also strange that the power law exponents for the R-curves for pine and spruce are found in these studies to be rather different (0.42 and 0.73), which suggests lack of universality.

4. Some fundamental problems of structural size effect based on material fractality

4.1. The concept of fractal crack

Before proceeding further, it will be helpful to review some basic ideas of fractality. When Mandelbrot introduced fractals [98,100], he concentrated on similarities between mathematical and physical (or empirical, natural) fractals. However, the modern papers on the subject study these two kinds of fractals separately [97] or at least they emphasize the difference between them [50]. The former fractals are observed in nature on a bounded region of size scale, while the latter kind gives mathematical models of these observations in the ideal limit case in which the size range tends to infinity. For example, if the length of a rough curve is measured by a ruler of a certain length δ_0, representing the resolution (Fig. 7(left)), the measured length a_δ will evidently depend on δ_0. If the curve is a physical fractal over a certain limited range, then this is described by the equation

$$a_\delta = \delta_0 (a/\delta_0)^{d_\mathrm{f}} \tag{14}$$

where a is the projected (or smooth, Euclidean) crack length, and exponent $d_\mathrm{f} > 1$. This exponent is usually called the fractal dimension (or the physical fractal dimension). The relation (14) can be modeled by mathematical fractals. In this case, the resolution $\delta_0 \to 0$ while $a_\delta \to \infty$. Fractal dimensions of mathematical fractals can be defined in various ways. However, all of them are based on covering a fractal set by objects such as cubes, squares or line segments, the size of which is at least δ_0, considering the sums of δ_0^s over the covered domain for various values of exponent s, and calculating a fractal measure of the set. Generally, the measure is either 0 or ∞. The value $s = d_\mathrm{f}$ at which the measure jumps from 0 to ∞ is called the mathematical (in particular, Hausdorff or Hausdorff–Besikovitch) fractal dimension [80,98]. Note that only for $s = d_\mathrm{f}$ the fractal measure of a set may have a nonzero bounded value [80] (it was suggested to call such sets as d_f-measurable [51]).

For a mathematical fractal model of a crack, Borodich noted [48,49,107] that the total energy dissipation $\mathscr{W}_\mathrm{f}$ would be infinite if a finite amount of energy G_f were assumed to be dissipated per unit crack length because the crack length $a_\delta \to \infty$ (he even called this as the 'fractal cracking paradox' [49,50]). To overcome this conceptual difficulty, he refined Barenblatt's idea [5,7] to refer physical quantities to units of fractal measure m_{d_f} and introduced a new concept of specific energy-absorbing capacity of a fractal crack $\beta(d_\mathrm{f})$, which may also be called the fractal fracture energy, having a dimension that is not J/m^2 but J/m$^{d_\mathrm{f}+1}$. Note that originally the Barenblatt–Borodich idea was applied only to scalar objects such as the mass [5] or energy [48]. For example, if $\hat{G}_{d_\mathrm{f}}$ is the fracture energy related to $m_{d_\mathrm{f}}(a)$ then $\hat{G}_{d_\mathrm{f}} m_{d_\mathrm{f}}(a)$ is the fracture energy of a fractal crack of nominal length a. Using this notation one can calculate the fracture energy of a fractal crack of nominal length ka ($k =$ constant). Because $m_{d_\mathrm{f}}(ka) = k^{d_\mathrm{f}} m_{d_\mathrm{f}}(a)$, the energy is found to increase k^{d_f} times. Note, however, that the idea cannot be applied to vectorial quantities, introducing terms such as the force per unit of a fractal measure, because, mathematically, no normal to a fractal surface exists.

For a physical fractal crack, one can introduce the energy G_f with the standard dimension J/m^2 and attribute it to the lower limit of validity of (14) [48]. For both mathematical and physical fractal cracks, one may then write

$$\mathscr{W}_\mathrm{f}/b = G_\mathrm{fl} a^{d_\mathrm{f}} \tag{15}$$

in which b = width of body; $\mathscr{W}_\mathrm{f}$ = total energy dissipation; and G_fl represents either the fractal fracture energy $\hat{G}_{d_\mathrm{f}}$ or the fracture energy G_f for the smallest δ_0 in (14). Based on this fractal concept of fracture energy, quasibrittle fracture propagation of a fractal crack has been analyzed [15] under the assumption that standard continuum mechanics and energy balance conditions are applicable on the scale of the structure as well as the FPZ. Some recent studies [68,69] developed the Barenblatt–Borodich idea to refer various physical quantities to a unit of the fractal measure m_{d_f} of the object. In these studies, however, this idea was applied to vectorial quantities such as the force, stress and strain, which is not acceptable.

4.2. Could crack surface fractality or self-affine roughness be the cause of size effect?

Let us now summarize the analysis in [15], which showed that the answer to the above question is negative. A crack was considered as a fractal curve in a plane (Fig. 7(left)). Eqs. (14) and (15) were taken as the point of departure. To take into account finite (projected) length $2c_\mathrm{f}$ of the FPZ, the approximation by equivalent LEFM was used. In this approximation, the tip of the equivalent LEFM (sharp) crack is assumed to lie in the middle of the FPZ. For the case of specimens with notches or structures failing only after the development of large traction-free (fatigued) cracks, the matching of the large-size and small-size asymptotic expansions was shown to yield, instead of Eq. (5), the following approximate fractal size effect law:

$$\sigma_\mathrm{N} = \sigma_\mathrm{N}^0 D^{(D_\mathrm{f}-1)/2}\left(1+\frac{D}{D_0}\right)^{-1/2} \tag{16}$$

For the limit case $D \gg D_0$, corresponding to the fractal generalization of LEFM, this yields the large-size asymptotic behavior

$$\sigma_\mathrm{N} \propto D^{(D_\mathrm{f}/2)-1} \quad \text{where} \quad -1/2 \leqslant (D_\mathrm{f}/2) - 1 \leqslant 2 \tag{17}$$

because $1 \leqslant D_\mathrm{f} \leqslant 2$. Obviously, the condition that the limit case of LEFM, $\sigma_\mathrm{N} \propto D^{-1/2}$, must be recovered for $D_\mathrm{f} \to 1$ is satisfied.

For failure at crack initiation, the asymptotic analysis furnished, instead of Eq. (6), the result:

$$\sigma_\mathrm{N} = \sigma_\mathrm{N}^\infty D^{(D_\mathrm{f}-1)/2}\left(1+\frac{D_\mathrm{b}}{D}\right) \tag{18}$$

The large-size asymptotic behavior is

$$\sigma_\mathrm{N} \propto D^{(D_\mathrm{f}-1)/2} \quad \text{where} \quad 0 \leqslant (D_\mathrm{f}-1)/2 \leqslant 1/2 \tag{19}$$

The plots of (16) and (18) are shown in Fig. 7(bottom) in comparison with the nonfractal laws (5) and (6) representing the limit case for $D_\mathrm{f} \to 1$.

By judging the consequences for size effect, one may decide whether or not the hypothesis of fractal fracture energy is realistic [15,21]. Fig. 7(bottom) reveals that the fractal case disagrees with the available experimental evidence. For the case of structures failing only after large crack growth, the rising portion of the plot has never been seen in experiments, and there are many data showing that the asymptotic size effect is equal or very close to a power law of exponent −1/2 ([21,40] e.g.), rather than an exponent of much smaller magnitude predicted from the fractal hypothesis. This is clear by looking at Fig. 7(bottom). For the case of structures failing at crack initiation, the kind of plots seen in this figure, with a rising size effect curve for large sizes, is also never observed. Thus it is inevitable to conclude that the hypothesis of a size effect caused by crack fractality, with a fractal fracture energy, is contradicted by test data and thus untenable.

The existence of limited fractal characteristics of fracture surfaces in various materials is of course not disputed, but crack surface fractality cannot cause a size effect in the mean. As for describing the statistical scatter of size effect about the mean, fractals might nevertheless be helpful.

What is the physical reason for the unrealistic consequences of the fractal hypothesis? Doubtless it is the fact that the crack front is not sharp but is surrounded by a large fracture process zone involving microcracks and frictional slips (in Fig. 7(right)). The development of a wide zone of extensive microcracking has been evidenced by locating sources of acoustic emissions, as well as by thermographic and holographic measurements. Besides, the length difference between the partially fractal crack curve and its smooth projection cannot account for the huge difference in energy dissipation because fractality of crack surface is limited to only about 1.5 orders of magnitude, whereas the fracture energy G_{f} of quasibrittle materials such as concrete is known to be several orders of magnitude larger than the surface energy γ of the solid from which the structure is made. Consequently, far more energy is dissipated in the volume of the FPZ than on the surface of the final crack, which is created by coalescence of some of the microcracks in the FPZ. An additional reason for this discrepancy is that, in concrete, frictional slips in the fracture process zone account for more than 50% of energy dissipation, as corroborated by the fact that only a part of crack opening is recovered upon unloading [14].

As an alternative approach, mode I fracture in a solid with fractal crack has recently been studied by Yavari et al. [133] in terms of a fractal stress intensity factor $K_{\mathrm{I}}^{\mathrm{f}}$. This study, which has also been extended to self-affine cracks, has been limited to a generalization of LEFM, i.e., quasibrittle materials have not been considered. The asymptotic near-tip field of stress tensor σ_{ij} is in this study assumed to be written in a separated form similar to the expression in LEFM, but with different exponents;

$$\sigma_{ij}(r,\theta) = K_{\mathrm{I}}^{\mathrm{f}} r^{-\alpha} f_{ij}(\theta) \tag{20}$$

where (see [128,135])

$$\alpha = 1 - D_{\mathrm{f}}/2, \quad 1 \leqslant D_{\mathrm{f}} \leqslant 2 \quad [K_{\mathrm{I}}^{\mathrm{f}}] = FL^{-1-D_{\mathrm{f}}/2} \quad \text{self-similar cracks} \tag{21}$$

$$\alpha = \begin{cases} 1 - 1/2H, & \frac{1}{2} \leqslant H < 1 \\ 0, & 0 < H \leqslant \frac{1}{2} \end{cases} \quad [K_{\mathrm{I}}^{\mathrm{f}}] = FL^{-1-1/2H} \quad \text{self-affine cracks} \tag{22}$$

Here F, L = quantities of force and length dimension; H is the roughness exponent of self-affine crack, and the physical dimension of $K_{\mathrm{I}}^{\mathrm{f}}$ depends on H or the fractal dimension D_{f}. The stress tensor, of course, is well defined only farther away from the crack, which is a measure-zero set in the plane. It is considered that the fractal fracture toughness $K_{\mathrm{Ic}}^{\mathrm{f}}$ (critical fractal stress intensity factor) is a function of fractal dimension D_{f} or roughness exponent H of the crack. Similar to classical LEFM, it is assumed that a fractal crack propagates if $K_{\mathrm{I}}^{\mathrm{f}} = K_{\mathrm{Ic}}^{\mathrm{f}}$. Yavari [135] and Yavari et al. [133] further restrict consideration to a mode I crack of length $2a$ so small that there exists a remote stress field σ not disturbed by the crack (a situation typical of steel or fine-grained ceramics, but not relevant to quasibrittle materials). Based on dimensional arguments and in analogy to LEFM, they show that

$$K_{\mathrm{I}}^{\mathrm{f}} = \chi(D_{\mathrm{f}})\sigma\sqrt{\pi a^{2-D_{\mathrm{f}}}} \tag{23}$$

where a is the projected length of the fractal crack, σ is the remote stress, and χ is a coefficient depending on D_{f}. Considering an infinite body in which the only dimension is the crack length, they set $D = a$ and $\sigma = \sigma_{\mathrm{N}}$. Inversion of (23) then yields the scaling law:

$$\sigma = \sigma_{\mathrm{N}} = \frac{K_{\mathrm{I}}^{\mathrm{f}}}{\chi(D_{\mathrm{f}})\sqrt{\pi D^{2-D_{\mathrm{f}}}}} \propto D^{D_{\mathrm{f}}/2-1} \tag{24}$$

This scaling relation is identical to the special limiting case of Bažant's [15] analysis of quasibrittle fractal fracture, Eq. (17). This analysis rests on two, quite plausible, hypotheses—that the first law of thermodynamics (energy balance) must apply globally, and that the solutions to the fractal and nonfractal boundary value problems are the same except near the fractal crack surface. In view of our inability to solve boundary value problems with fractal boundaries, this equivalence is important to retroactively justify the hypotheses that the fractal stress intensity factor can be defined as in (20) and that its relationship to a remote uniform stress field σ can be written in the form of (23).

4.3. Could lacunar fractality be the cause of size effect?

In view of the difficulties with crack surface fractality, as just described, it was proposed to deal with another type of fractality—the lacunar (or rarefying) fractality of the solid caused by an array of microcracks (Fig. 7(top right)) or microvoids. First, we need to summarize the critical analysis of this hypothesis given in [15].

From distance, one can see in the material only large cracks. But, looking closer, one can discern that each crack is discontinuous and consists of shorter mini-cracks, with mini-gaps between them. Looking still closer, one can discern that each mini-crack is also discontinuous and again consists of shorter microcracks with microgaps between them, etc. Refinement ad infinitum produces a fractal set, with a fractal dimension D_{f} that is less than the Euclidean dimension of the space (Fig. 7 top right). It seems that the microcrack systems in concrete might be described by this type of fractality, but only for about one order of magnitude of crack size, which is hardly enough to justify a fractal treatment.

More recently, it has been argued that lacunar fractality could be the cause of size effect on the structural level [61,62,66,68,69]. For a small-scale, the fractal dimension d_{f} of the arrays of microcracks is considered to be distinctly less than 1, and for a large-scale equal to 1. It is supposed that, for the failure of a small structure, the small-scale matters, and for the failure of a large structure, the large-scale matters. Hence, as it was argued, there should be a transition from a power scaling law corresponding to small-scale fractality to another power scaling law corresponding to large-scale fractality, the latter having size exponent 0 for the nominal strength, i.e., no size effect. Thus, as it was proposed, the size effect in a plot of $\log \sigma_{\mathrm{N}}$ versus $\log D$ would be given by a transitional curve between an inclined asymptote and a horizontal asymptote. The inclined asymptote was considered to have the slope $-1/2$ (Fig. 7(bottom)), which leads to the MFSL (Eq. (10)).

However, even though for some of the existing test data on the modulus of rupture the slope $-1/2$ is not unreasonable, the mathematical argument that was used to arrive at this slope is incorrect (Appendix A). Besides, there exist, for specimens of sizes as small as possible for the given aggregate size, other test data that indicate that the initial slope can be much less than $-1/2$. As already pointed out, the formula of MFSL is a special case for $r = 2$ of formula (6) derived strictly from fracture mechanics, without any recourse to fractals; however, the optimal fit of the available test data is obtained for $r = 1.44$ [18,36].

On closer scrutiny, the hypothesis of lacunar fractality appears to lead to the classical Weibull statistical theory. If the failure is assumed to be controlled by lacunar fractality of the solid, rather than large cracks, it obviously implies that the failure occurs at crack initiation. In that case, the mathematical formulation must be akin to Weibull theory [15]. Labeling the aforementioned small and large-scales of observations of the lacunar material by superscripts A and B, the Weibull distributions of the strength of a small material element at small and large-scales may be written as

$$\varphi[\sigma(\boldsymbol{x}); D_{\mathrm{f}}^{A}] = \left\langle \frac{\sigma_{\mathrm{N}} S(\xi) c_{\mathrm{f}}^{1-D_{\mathrm{f}}^{A}} - \hat{\sigma}_{u}^{A}}{\hat{\sigma}_{0}^{A}} \right\rangle^{m} \tag{25}$$

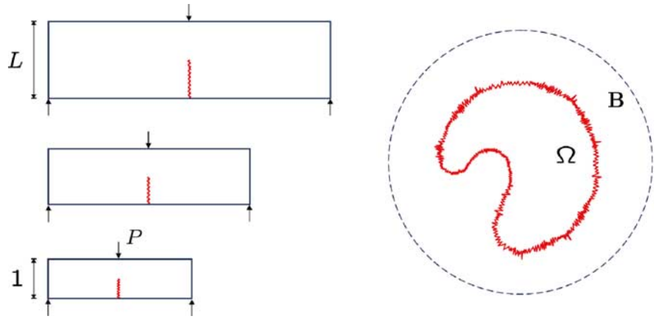

Fig. 8. (left) Fractal-cracked structures of different sizes; (right) a bounded domain Ω (imagined as a finite disk B) with fractal boundary in the plane (Ω has finite area and infinite perimeter).

$$\varphi[\sigma(\boldsymbol{x}); D_{\rm f}^B] = \left\langle \frac{\sigma_{\rm N} S(\xi) c_{\rm f}^{1-D_{\rm f}^B} - \hat{\sigma}_u^B}{\hat{\sigma}_0^B} \right\rangle^m \tag{26}$$

in which the stress in the small material element of random strength has been written as $\sigma = \sigma_{\rm N} S(\boldsymbol{\xi})$; function S must be the same for all sizes of geometrically similar structures; and $\boldsymbol{\xi} = \boldsymbol{x}/D$ = dimensionless coordinate for the nonfractal (nonlacunar) case on the global structural level.

For the fractal (lacunar) case, this is generalized as $\sigma = \sigma_{\rm N} S(\boldsymbol{\xi}) c_{\rm f}^{1-D_{\rm f}}$ because the stress of the material element, in the case of lacunar structure of the solid, must be considered to have a nonstandard, fractal dimension. Obviously, the Weibull constants $\hat{\sigma}_0$ and $\hat{\sigma}_u$ must now be considered to have fractal dimensions as well, but Weibull modulus m must not. An equation of the type of Eqs. (25) or (26) was written by Carpinteri, et al., however, further analysis consisted of geometric and intuitive arguments. We will now sketch a recently published mechanical analysis [17].

In Weibull theory of failure at the initiation of macroscopic fracture, every structure is equivalent to a long bar of variable cross-section [44] (Fig. 9). The lacunarity argument means that a small structure is considered subdivided into small material elements, in which a low fractal exponent $D_{\rm f} < 1$ is what matters (Fig. 9(a)), while a large structure is subdivided into proportionately larger material elements with $D_{\rm f} = 1$ (Fig. 9(c)). However, a direct comparison of these small and large material elements is not objective, since structures made of the *same* material must be compared. The large material elements of the large structure (Fig. 9(c)) must be divisible into the small elements considered for the small structure (Fig. 9(a)), which may be identified with the representative volume of the material for which the material properties are defined. If the large elements were not divisible into the small ones, it would imply that the material of the small structure is not the same. So we must consider that the large material elements can be subdivided into the small material elements, as shown in Fig. 9(b). Accordingly, we may now calculate the failure probability of the large structure on the basis of the refined subdivision into the small elements, as shown in Fig. 9(b). We note that the failure probability $P_{\rm f}$ of the large structure subdivided into large elements ΔV_{Bj} $(j = 1, 2, \cdots N)$, and the failure probability $P_{{\rm f}j}^B$ of the large element ΔV_{Bj} of the large structure subdivided into small elements ΔV_{Aij}, must satisfy the following relations based on the weakest link model underlying Weibull theory (based on the fact that the failure of one element implies failure of the whole structure):

$$-\ln(1 - P_{\rm f}) = \sum_j \varphi(\sigma_{\rm N} S_j^B; D_{\rm f}^B) \Delta V_{Bj}/V_r \tag{27}$$

$$-\ln(1 - P_{{\rm f}j}^B) = \sum_j \varphi(\sigma_{\rm N} S_{Aij}; D_{\rm f}^A) \Delta V_{Aij}/V_r \tag{28}$$

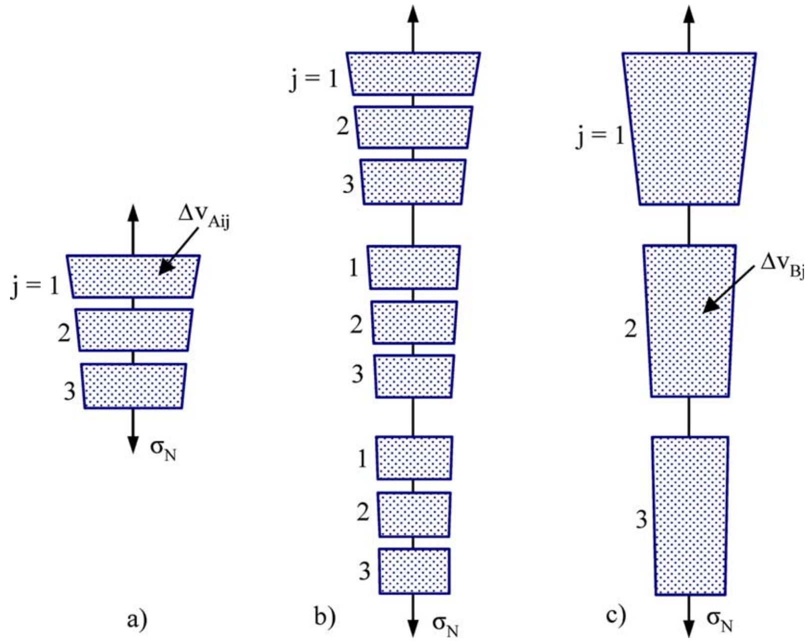

Fig. 9. Bar with lacunar fractality subdivided into segments.

Now, since we may subdivide each element B of the large structure into the small elements A if the material is the same, we have the recursive condition:

$$-\ln(1-P_{\mathrm{f}}) = -\sum_j \ln(1-P_{\mathrm{f}j}^B) = \sum_j \sum_i \varphi(\sigma_{\mathrm{N}} S_{ij}^A; D_{\mathrm{f}}^A)\Delta v_{Aij}/V_r \tag{29}$$

Now, upon equating this to (27), we see that, in order to meet the requirement of the objective existence of the same material, the Weibull characteristics on scales A and B must be different, and precisely such that

$$\varphi(\sigma_{\mathrm{N}} S_j^B; D_{\mathrm{f}}^B) = (\Delta V_j^B)^{-1} \sum_i \varphi(\sigma_{\mathrm{N}} S_{ij}^A; D_{\mathrm{f}}^A)\Delta V_{ij}^A \tag{30}$$

From Eqs. (29) and (30) it follows that, for structures made of the same material failing at crack initiation (i.e., following the weakest link model), the consideration of different scales cannot yield different scaling laws. The same power law size effect must ensue from the hypothesis of lacunar fractality of the material, regardless of the scale considered. So the lacunarity argument leads to Weibull theory and offers nothing new in terms of the size effect on structural strength, although it might offer something new for the understanding of failure mechanism on the microlevel.

In summary, the scaling law of the nominal strength of a structure failing at the initiation of fracture in a lacunar fractal solid must be identical to the scaling of the classical Weibull theory. The only difference is that the values of Weibull parameters may depend on the lacunar fractality. This aspect could be quantified if the values of these parameters could be predicted by some sort of fractal micromechanics.

Before closing this subject, it should be noted that the concept of lacunar fractality should be used more carefully. According to Mandelbrot [101], who discussed the idea of lacunarity in his treatise [98], the concept of fractal lacunarity appears dubious from the mathematical viewpoint. Noting that the lacunarity of fractals comes into play only when fractals have the same fractal dimension but different distributions of holes (lacunes), Mandelbrot argued that a measure of lacunarity could be the multiplicative pre-factors of fractal scaling laws, and that the fractal dimension itself cannot describe lacunarity. As explained by Mandelbrot [101], a single fractal dimension cannot properly describe a fractal set with holes. Describing lacunarity requires more than just a number. There are many examples of fractals with the same fractal dimensions but with very different distributions of holes. This fact should be taken into account by any lacunar fractal model because media with different distribution of holes (or defects, microcracks) have, in general, different mechanical behaviors.

4.4. Problems of continuum mechanics for bodies with fractal surfaces of discontinuity

To make the fractal approach to size effect more fundamental, it has recently been attempted to develop 'continuum mechanics of fractal media' (CMFM). However, it has been unclear what is CMFM. In continuum mechanics, each representative volume element is treated as a mathematical point. This implies the assumption that the size of the representative volume element is very small compared to the smallest dimensions of the structure. The structures can have a very complicated microstructure, but these details are not seen in the homogenized continuum. Consider three cases in which one might be tempted to introduce CMFM.

4.4.1. Continuum with a fractal microstructure

An example may be a fractal distribution of twins. Its description, however, does not require any generalization of continuum mechanics. Although the macroproperties might be affected by fractal characteristics, the homogenized continuum is simply a continuum and the standard continuum mechanics is applicable.

4.4.2. Continuum with a fractal distribution of microcracks, voids or other defects

Some researchers have in this case been tempted to use the so-called lacunar fractals for such a medium [61,62,66,68,69]. A lacunar fractal in $\boldsymbol{R}^n$ is a spongy-shaped subset with fractal dimension strictly less than n. Such a set may be thought of as a topological ball with infinitely many holes in it. A natural concept consistent with the axioms of the classical continuum mechanics is to consider the distributed holes or defects as a special microstructure. Its homogenization leads to the usual continuum. Some authors [69] nevertheless proposed to define a 'fractal stress' as a density of force per unit of a fractal measure. However, to accept such a fractal concept would require overcoming two problems. First, as already explained, it would be necessary to define clearly what is meant by a fractal continuum. Second, the definition of a fractal density of force would have to be generalized to more than one dimension.We will come back to this at the end of this section.

To be more precise, consider now a closed fractal curve in $\boldsymbol{R}^2$ (Fig. 8(right)). According to Jordan's theorem, this curve partitions the plane into two subsets. If the interior region, denoted as Ω, is bounded, then it must have a finite area, i.e.,

$$\mathrm{vol}(\Omega) < \infty \tag{31}$$

This means that it is meaningful to speak about a fractal surface in the framework of continuum mechanics. Having a body $\mathscr{B}$, we could consider a sub-body $\mathscr{P}$ with a fractal boundary $\partial\mathscr{P}$. This sub-body (Fig. 8(right)) has no unit normal vector on $\partial\mathscr{P}$ but has a finite volume and hence a finite mass. If a fractal set $\Omega \subset \boldsymbol{R}^3$ has fractal dimension $D_{\mathrm{f}} < 3$, then (by definition of fractal dimension) $\mathrm{vol}(\Omega) = 0$. Every (time-independent) problem in continuum mechanics is studied in $\boldsymbol{R}^3$, i.e., the continuum body $\mathscr{B}$ is embedded in $\boldsymbol{R}^3$. In any continuum mechanics problem, a finite body must certainly be considered to have a finite and nonzero mass (because the notion of a 'body' with zero mass would be physically meaningless). It is argued that, in a continuum body with defects, any cross-section as a subset of $\boldsymbol{R}^2$ is fractal, with some fractal dimension $d_{\mathrm{f}} < 2$ associated with microcracks and other defects. But this means that the continuum $\mathscr{B}$ has dimension $D_{\mathrm{f}} < 3$ and hence has volume zero. It follows that the body has a zero mass unless the mass density is assumed to be fractal as well. Obviously, the mass density would change with any evolution of D_{f} (as assumed in the proposed derivation of MFSL, Appendix A). However, this would be absurd. On the scale of a macroscopic homogenized continuum, there are no fractal holes. Knowing the distribution of microcracks and cavities, the microstructure can in principle be homogenized as a continuum with effective properties depending on the distribution of defects. This again demonstrates that CMFM for case 2 is a meaningless artifice.

4.4.3. Continuum with fractal surfaces of discontinuity (fractal cracks)

There is experimental evidence that fracture surfaces are fractals within some limited range of scales. This is true even for materials with relatively mild inhomogeneity, for which the source of crack fractality is believed to be the interaction of a dynamically propagating crack front with material inhomogeneities (see [56,112,113]). The first question that comes to mind is whether it is acceptable to have a fractal crack, a fractal interface, a fractal surface of discontinuity, etc., in a continuum. In the writers' opinion, the answer to this question should be yes. Existence of a fractal surface of discontinuity (say, in the deformation gradient) does not contradict any accepted fundamental concept. However, the classical Hadamard jump condition does not apply here, because a unit normal is not defined on a fractal surface. All that needs to be known is that the displacement field is continuous everywhere along the fractal surface.

Fractal roughness of cracks, however, creates enormous mathematical problems. For many years, fracture mechanicians have had great success working with smooth cracks. However, assuming a non-smooth crack makes the elasticity problem virtually intractable.

It may be helpful to classify physical quantities as *primary* and *secondary*. The primary quantities are those quantities that can be measured directly. Examples are the force, mass, energy, etc. No matter what the geometry of a continuum is, the force has always the same meaning and the same physical dimension because it can be measured directly. Similarly, the mass is a primary quantity and, in any continuum mechanics formulation, must have the same physical dimension. The secondary quantities are those defined through the primary quantities. For example, the surface traction is a secondary quantity defined in terms of force and area. Secondary quantities can be geometry dependent and could have nonstandard physical dimensions. For example, the stress intensity factor and energy release rate are secondary quantities and can have nonstandard dimensions for a fractal crack.

The first step in formulating a continuum theory of bodies with fractal surfaces of discontinuity is to define traction or a similar quantity on a fractal surface. Suppose that a fractal surface $\mathscr{S}$ separates a body $\mathscr{B}$ into two sub-bodies $\mathscr{B}_1$ and $\mathscr{B}_2$. These sub-bodies exert forces on each other across $\mathscr{S}$. The force is a primary quantity and always exists no matter how rough and irregular $\mathscr{S}$ is. Note that, for points in $\mathscr{B} \setminus \mathscr{S}$, the usual stress tensor is defined and we just need to worry about points that lie on $\mathscr{S}$ (a measure-zero set).

One may be tempted to define a fractal traction as

$$\mathbf{t}_{\mathrm{f}}(\mathbf{x}, t) = \lim_{\Delta m_D \to 0} \frac{\Delta \mathbf{F}}{\Delta m_D}$$

where Δm_D is a fractal measure. This and similar definitions, however, are mathematically inconsistent because they ignore the fact that a single fractal dimension is not enough to describe a fractal surface. The above definition does not recover the classical definition in the limit $D_{\mathrm{f}} \to 2$ (see [135]). This shows that further geometrical information about $\mathscr{S}$ is necessary, to be able to define a reasonable 'fractal traction'. All that we can say is that traction has the dimension of force per unit of a fractal measure. Similarly, the driving force (or configurational force) on a fractal interface must have the dimension of energy per unit of a fractal measure. Thus it now becomes clear that 'continuum mechanics of fractal media' in the sense of [69] is not meaningful.

In [69], some of the concepts of classical continuum mechanics are nevertheless generalized to 'fractal media'. This generalization, however, tends to be misleading because it goes too far in making several unjustified assumptions which lead to some incorrect conclusions.

A 'fractal stress' in [69] is defined as force per unit of a fractal measure in the case of a solid that is a fractal with fractal dimension less than three (a 'spongy' solid). While this 'fractal stress' is defined only for a uniaxial tension specimen, the same concept is used in a three-dimensional solid without any justification. However, the 'fractal stress tensor' is not even defined in [69]. Defining such a density of force might be acceptable in one dimension, but not in three dimensions (see [133,135] for more details). Eq. (5) of [69] defines a fractal traction and it is then implicitly assumed that it exists at any 'singular point'. One could consider this traction to be some measure supported on the fractal solid (i.e., a measure being zero everywhere else), but again it is not made clear what is meant by a continuum.

In Section 2.3 of [69], a 'fractal strain' is defined, but again only in one dimension. The fact that the strain, no matter how defined, must be some measure of the local deformation of the solid is ignored. Later in [69], the three-dimensional 'fractal strain' is defined in a peculiar way which will be discussed shortly. A very special deformation for defining the one-dimensional 'fractal strain' is considered—namely the deformation of a bar with a fractal distribution of localized deformations.

In view of postulating the 'fractal stress' and 'fractal strain', Section 4 of [69] then introduces a constitutive equation with fractional derivatives, without offering any sound physical or mathematical grounds. Section 5 all of a sudden defines a 'fractal strain' field as a fractional gradient of the displacement field. It is not clear at all why the strain is defined in this strange way and why such a quantity is supposed to describe the local deformation of a solid.

Section 5.2 mentions a stress vector, however, without defining it for a three-dimensional solid. Thus the 'equilibrium equations' in Eq. (22) are meaningless, and it is not even clear why these should be regarded as equilibrium equations. To determine the equilibrium equations for any system, the natural approach would be to begin with the integral form of the balance of linear momentum and of angular momentum, and then try to adapt it to fractality. Instead, [69] merely looks at the classical equilibrium equations and, without considering the physical meaning of a fractional divergence, replaces the divergence operator by a fractional divergence. Last but not least, the fractal 'principle of virtual work', as used in [69], is simply a replacement of the classical terms of the principle of virtual work by some physically meaningless quantities that have fractional dimensions.

In short, it is not justified to generalize the classical continuum mechanics to 'fractal media' by simply replacing the classical differential operators by some fractional operators and stress and strain tensors with some ill-defined 'fractal' stress and strain tensors.

5. Conclusions

1. The so-called 'multifractal scaling law' (MFSL) is identical to a special case of an energetic–statistical scaling law for failures at crack initiation, which has been derived from fracture mechanics taking into account the finiteness of the thickness of the boundary layer of cracking.
2. Comparing and contrasting the MFSL to Bažant's original size effect law [9] (as seen in the papers by the proponents of MFSL, e.g. [63,70]), makes no sense and is misleading. The former is applicable only for failures occurring at macrocrack initiation (from a zone of microcracking), while the latter, derived from fracture mechanics, can be applied only for structures with a notch or a large stress-free (fatigued) crack formed prior to the maximum load. These two types of failure necessarily follow different size effect laws.

3. The existing mathematical derivation of MFSL from fractal concepts includes problematic steps which invalidate it. The MFSL does not mathematically follow from the fractal hypotheses made by its proponents (Appendix A).
4. The proposition that a structural size effect is caused by lacunar fractality of the material, particularly in the fracture process zone, leads to Weibull statistical theory of size effect. This theory is applicable only to unnotched structures that have a positive geometry (i.e., fail at macrocrack initiation) and are far larger than the fracture process zone as well as the material inhomogeneities (a situation that does not arise in the quasibrittle range of structural response).
5. The recent proposal [69] for introducing fractal stresses, fractal strains, fractal thermodynamic potentials, etc., is not mathematically consistent and practically usable, and it brings about unnecessary complexity (as explained in Section 4.4).
6. The fractal theories proposed thus far cannot predict the dependence of size effect law parameters on the structure geometry. On the other hand, the energetic and statistical–energetic theories are able to predict this dependence.
7. The fractal concepts could, at most, serve only as a refinement, but *not a replacement*, of the size effect caused by energy release due to stress redistribution engendered by a large crack or a large fracture process zone. The energetic and statistical sources of size effect are undeniable.
8. The theory of size effect is now ripe for implementation in design codes and engineering practice. The arguments summarized in this paper show that the correct approach is the energetic–statistical theory of size effect.

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Appendix A. Problematic steps in mathematical derivation of MFSL

For a mathematical derivation of MFSL, its proponents have generally been citing reference [58], even in the latest works. MFSL was formulated as a smooth transition between the small-size asymptotic size effect, which was taken as $\sigma_N \propto D^{-1/2}$, and the large-size asymptotic behavior considered as free of size effect, i.e. $\sigma_N = \text{constant}$. The problem is with the claim that the fractal concept mathematically leads to the small-size asymptotic scaling law $\sigma_N \propto D^{-1/2}$, in other words, to the slope −1/2 of the left-side asymptote of MFSL in a plot of log σ_N versus log D. It has apparently passed unnoticed that the derivation of this scaling law has many serious problems, which will now be summarized.

1. Eqs. (1)–(8) of [58] are used to claim that the nominal (ultimate) strength of a body containing fractal defects of fractal dimension $2 - d_\sigma$ must obey the size effect law $\sigma_u = \sigma_u^* b^{-d\sigma}$ where $\sigma_u^* = \text{constant}$, $b = D/D_1 =$ dimensionless size ratio of geometrically similar bodies ($D_1 =$ reference size), and $d_\sigma =$ dimensional decrement from the Euclidean dimension of space. The proposed derivation [58] rests on the simplifying assumption that the stress field is a uniformly distributed uniaxial tensile stress.

Further it rests on the implied hypothesis (obviously questionable) that any stress redistribution and energy release that may occur prior to the maximum load can be ignored (in other words, the energetic size effect is ruled out a priori). The exponent −1/2 attributed to the small-size asymptotic scaling law is supposed to be solely a consequence of a peculiar situation called the 'extreme disorder'. The microstructure is considered to be replete with small planar defects having orientations characterized by spherical coordinates (ϕ, θ) of their normals. The probability density distribution, $p(a)$, of defects of all sizes a is characterized in [58] by Eq. (21a) or (34), which may be, for clarity, rewritten as

$$[p(k^\beta a_{\max})\Delta a]n\Delta\Omega = 1, \quad n = \rho(kb)^3, \quad \Delta\Omega = \frac{\Delta\phi}{2\pi}\frac{\Delta\theta}{2\pi} \tag{A.1}$$

where a = defect size, $a_{\max}$ = maximum defect size in the body, $\Delta\Omega$ = element of the normalized surface area Ω of a sphere whose points characterize the orientation of defect normal, ρ = ratio of the number n of defects in a certain volume to that volume = spatial density of defects (with the dimension of length^{-3}); k denotes the dimensionless size ratio, apparently the same quantity that was in an earlier part of [58] denoted differently (namely as b); and β = exponent which reduces to 1 for the nonfractal case (Eq. (21a) of [58]). For this equation to be dimensionally correct, it must further be assumed that b doubtless was intended, from this point on, to represent not the dimensionless size but the actual characteristic size of structure (with the dimension of length). Furthermore, two corrections are required in this equation: (1) the expression given for $\Delta\Omega$ must be replaced with an element of solid angle, $\Delta\Omega = \sin\theta\Delta\phi\Delta\theta/4\pi$; and (2) integrations over ϕ and θ on the left-hand side of (A.1) need to be carried out because defects $a_{\max}$ can have any of all possible orientations, not just one arbitrary orientation (ϕ, θ). But even after these corrections, one must note that the defects of *maximum* size cannot have the same probability distribution of a as the ensemble of all defects (as considered in Eqs. (22)–(32) of [58]) but could have only one of the three possible extreme value distributions (Fréchet [83], Gumbel [84,85] or Weibull), of which only the Weibull distribution would be realistic here because a non-negative threshold on a must exist.

2. Aside from the relation $\sigma_u = u_u^* b^{-d_\sigma}$, the subsequent argument tacitly implies further three hypotheses which are questionable. The first implied hypothesis is self-similarity of the distributions of defect sizes in geometrically similar bodies of different sizes (otherwise it would be illegitimate to use Eqs. (21b)–(30) and (35) of [58] to obtain an inverse power distribution for $p(a)$). As a second implied hypothesis, the maximum size of defects is simply assumed to scale up with the body size b, and to do so as a power function (Eq. (37) of [58]), which is written as $a_{\max}(k) = k^\beta a_{\max}(1)$ where $\beta = 3/(N+1)$ and is then used to infer that $\sigma_u \propto a_{\max}^{-\lambda}$ and $\sigma_u \propto b^{-\lambda}$. Here N = constant such that $1 \leqslant N < \infty$, and k (instead of b) stands for the dimensionless size ratio. As a third implied hypothesis, the maximum defect size $a_{\max}$ is treated as nonrandom when the scaling is considered, although in reality it should more properly be considered as randomly distributed according to Weibull distribution. These three implied hypotheses seem to be rather arbitrary and unreasonably restrictive, apparently made for convenience of the argument (they seem particularly dubious when the body size exceeds the size of a fully developed FPZ).
3. There is a logical gap in passing from Eqs. (29) to Eq. (31) in [58]. The reason for the sudden appearance of exponent N, allegedly 'measuring in some way the degree of disorder', is unclear. So is the sudden incorporation of exponent ζ to account for some unspecified 'secondary effects' (presumably including the stress redistribution with energy release?).
4. Why the condition called the 'maximum disorder' (for which N is supposed to tend to 2, as argued below Eqs. (39) and (40) in [58]), rather than just some degree of disorder, should occur for a vanishing structure size? This alleged property is unproven, even under the aforementioned assumptions.

In consequence of the aforementioned problems with the derivation, the property that the left-size asymptote of the MFSL in a bi-logarithmic plot should have the slope −1/2 must be considered as unproven by the fractal argument in [58]. It does not logically ensue from fractal concepts. If only the fractal viewpoint is considered, this property is merely an empirical assumption, which happens to yield an acceptable fit of the test data on the modulus of rupture. On the other hand, from the viewpoint of fracture mechanics, this property, representing a special case of (6), is of course reasonable.

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