Asymptotic Matching Analysis of Scaling of Structural Failure Due to Softening Hinges. I: Theory

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Abstract: Propagation of a crack from the tensile face of a beam causes postpeak softening, i.e., the bending moment decreases at increasing rotation of the hinge. Examples are unreinforced concrete beams and plates, foundations plinths, retaining walls, tunnel linings, or arch dams. Softening in an inelastic hinge is also caused by compression crushing of concrete. This happens in reinforced concrete beams that are prestressed, overreinforced, retrofitted by laminates, or subjected to a large enough axial compressive force, which is typical of columns as well as frames or arches with a large enough horizontal thrust. Hinge softening may also be caused by plastic buckling of flanges in deep thin-wall steel beams. An inevitable consequence of inelastic hinge softening in statically indeterminate structures requiring more than one inelastic hinge to fail is a size effect. Although finite element solutions are possible, general analytical formulas for the size effect in such structures do not exist, because of the complexity of response. The idea of this two-part paper is to exploit the technique of asymptotic matching in order to derive approximate formulas for the entire size range. Exact analytical solutions of the nominal strength of structure are derived for the large-size asymptotic case, for which the hinges soften one by one rather than simultaneously, and for the small-size asymptotic case, for which the classical plastic limit analysis applies. Matching these asymptotic solutions by a smooth formula then yields simple, yet general, size effect laws for the peaks and troughs of the load-deflection diagram through the entire size range. The size effect found is very different from the classical size effect in quasibrittle structures failing due to a single dominant crack. The theory is developed in the present Part I, and analysis of its implications is relegated to Part II.

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Introduction

The load capacity of beams, frames, and plates is generally calculated according to plastic limit analysis. This classical approach, however, runs into serious problems if the material in the inelastic hinge fails by fracture or by cracking damage. In that case, the moment-rotation diagram of the inelastic hinge does not reach a horizontal yield plateau but exhibits postpeak softening.

By now it is widely accepted that, as a matter of principle, fracture or damage with postpeak softening always gives rise to a (nonstatistical) size effect on the nominal strength of quasibrittle structure (Bažant 1976, 1984, 1999, 2001a,b, 2002b; Bažant and Chen 1997; Bažant and Planas 1998). Although the size effect on the postpeak ductility of beam structures with softening hinges has been analyzed from the stability viewpoint (Bažant 1976; Bažant and Cedolin 1991; Chap. 13), the general laws of the size effect on load capacity and the postpeak load–deflection diagram of structures failing by softening hinges still remain unsolved. This two-part study [whose summary was presented in a recent anniversary volume (Bažant 2001a,b) and also in the monograph by Bažant 2002b] attempts to determine these laws, while a separate study (Bažant and Guo 2002) deals with a similar behavior in floating sea ice plates. The present paper (Part I) develops the theory, and Part II (Bažant 2003) which follows discusses various practical implications.

Routine though the finite element analysis of inelastic beam structures has become, this numerical approach, when applied to structures with softening hinges, does not readily furnish a clear and simple picture of various influences on the size (and shape) effects, needed for design and structural optimization. It does not reveal simple asymptotic laws and may run into convergence difficulties.

As known from previous studies, the analytical solution of redundant (statically indeterminate) structures in which several inelastic hinges are softening (e.g., Maier 1986) is quite involved and irreducible to simple formulas. The basic idea of the present analysis is not to bother with exact analytical solutions for such situations and focus instead on the asymptotic cases of very large and very small structures, even though such asymptotic cases may be hypothetical situations far outside the practical size range.

The point is that the asymptotic cases happen to be simple to solve. In this paper [whose idea is briefly outlined in a recent anniversary volume (Bažant 2001a,b) and is discussed in Sec. 2.8 of Bažant 2002b], an approximate solution for the entire size range is obtained by fitting these asymptotic solutions with a smooth formula. Such an approach is known in general as asymptotic matching. It has been widely used in fluid mechanics (e.g., Sedov 1959; Bender and Orszag 1978; Barenblatt 1979; Hinch 1991) and has also been exploited to derive the size effect law for failures due to propagation of one dominant crack (Bažant 1984, 1997, 2002b; Bažant and Planas 1998).

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One well-known consequence of hinge softening in continuous beams and frames is a limitation of the capability of a statically indeterminate structure to redistribute its internal forces prior to reaching its maximum load. This limitation is reflected in the design code provisions for reinforced concrete. However, the size dependence of this limitation is ignored. For very small structures, no such limitation on plastic moment redistribution is in fact needed even if the material is brittle, while the severity of the limitation ought to increase as the structure gets larger. In other words, the larger the structures, the smaller is the degree of moment (or internal force) redistribution that can be permitted. The present solution of the nominal strength will provide answer to this problem.

Depending on the physical source, three types of postpeak softening in the moment-rotation diagram of an inelastic hinge may be distinguished:

1. One type is the softening caused by tensile fracture. This is typical of plain concrete beams or ice plates, and includes the vertical bending fracture of arch dams, as well as fracture of unreinforced retaining walls, foundations plinths, and tunnel linings. On the other hand, reinforced concrete beams that fail by yielding of the tensile steel exhibit no postpeak softening and thus no size effect.
2. A second, very important, type of postpeak softening in the moment-rotation diagram is caused by compression fracture (Bažant and Xiang 1997; Bažant and Planas 1998). This is typical of eccentrically loaded reinforced concrete columns failing on the compression side, as well as reinforced low-rise arches, shells, and frames with a large horizontal thrust. This second type of softening also occurs in prestressed concrete beams, and in reinforced concrete beams that are overreinforced. Although overreinforcement is prohibited by the design codes, it is often inevitably engendered by a retrofit with a laminate bonded to the tensile face. Although the softening in compression fracture is mitigated by confining reinforcement (stirrups), it is still significant for the normal amounts of such reinforcement (Bažant and Caner 2001, estimate for one type of concrete that the volume of a circular spiral that entirely prevents postpeak softening must be no less than about 16% of the volume of column).
3. A third type of post-peak softening in the moment-rotation diagram may occur in steel beams, being caused by a local nonlinear postbuckling response of compressed thin flanges (Maier and Zavelani 1970).

The concept of compression fracture, once regarded with suspicion, has by now become well established thanks to clarification of its various mechanisms, both microscopic (e.g., Biot 1965; Bažant 1967; Kendall 1978; Horii and Nemat-Nasser 1985, 1986, Ashby and Hallam 1986; Sammis and Ashby 1986; Shetty et al. 1986; Bažant et al. 1993) and macroscopic (Bažant and Chen 1997; Bažant and Xiang 1997; Bažant and Planas 1998; Brocca and Bažant 2001).

The study of softening in hinge regions of redundant beams has had a long and colorful history (e.g., Barnard 1965; Rosenblueth and Diaz de Cossio 1965; Barnard and Johnson 1966; Bažant 1976; Bažant and Cedolin 1991). Interest in the subject was stimulated by the so-called Wood–Roberts paradox (Wood and Roberts 1965; Wood 1968)—the fact that once the strain softening begins in one cross section of a fixed-end beam, the adjacent cross sections unload and thus cannot enter the strain-softening regime, which seems to indicate that a finite rotation in the hinge region could never develop (this phenomenon was, of course, a "paradox" only within the local theory and was removed by the nonlocal theory, after the spurious localization of strain softening became clarified; Bažant 1976).

The aforementioned early studies suggested that the solutions are meaningful only if the hinging region is assumed to have a certain fixed length (which was at first called the "contamination length" or "discontinuity length"). This is of course equivalent to abandoning the calculations of the hinge rotation from a softening stress–strain diagram, and determining this rotation directly, either by experiment, or theoretically—by fracture mechanics or nonlocal damage mechanics. That is the reason why the moment-rotation diagram rather than the stress-strain diagram must serve as the basis of our analysis.

The behavior of structures with softening hinges and the size effect were analyzed in Bažant et al. (1987a,b). However, the asymptotic behavior was not addressed and analytical formulas for the size effect were not obtained.

A related problem is the post-peak softening and snapback in the axial force-displacement diagram of inelastic columns. As shown by Horne and Merchant (1965); de Donato (1967); and Horne (1971), the geometrically nonlinear effects of buckling at large deflections can engender such axial softening even if the material is plastic-hardening (see also Bažant and Cedolin 1991, Sec. 8.6). In reinforced concrete columns, in which the material itself is softening, the axial softening of column gets intensified by buckling effects. Such a softening causes in redundant trusses and frames similar size effects as those which will be demonstrated here. This interesting subject, however, is beyond the scope of this paper.

Maximum Bending Moment According to Current Codes

The maximum bending moment M_0 that can be reached in a cross section of a beam is controlled by either tensile or compression failure. For a rectangular cross section failing by tension, the bending stress formula yields for the maximum bending moment the expression

$$M_0=\tfrac{1}{6}bD^2f_r \tag{1}$$

in which D, b = cross section depth and width and f_r = modulus of rupture (or flexural strength) of the material. For beams so deep that the size of the fracture process zone (FPZ) is negligible, the value of f_r is constant if strength randomness is neglected. For smaller beams, for which the FPZ size is not negligible, f_r depends on beam depth D.

For an overreinforced concrete beam, one may assume an equivalent rectangular stress block and a no-tension material, and use the American Concrete Institute code assumption that the compression strain magnitude at compression face at the moment of concrete crushing is 0.003. From the strain compatibility condition, a formula similar to Eq. (1) ensues

$$M_0=\tfrac{1}{6}bD^2\overline{f_r}, \quad \overline{f_r}=0.018E_s\rho(1-\tfrac{1}{2}\beta_1\gamma)(1-\gamma)/\gamma \tag{2}$$

where $\overline{f_r}$ = apparent modulus of rupture for compression failure; E_s = Young's modulus of tensile steel; ρ = reinforcement ratio; $\gamma = c/D$ = size-independent ratio in which c = depth to neutral axis; and β_1 = coefficient defining the depth $a=\beta_1 c$ of Whitney's equivalent rectangular stress block, as specified by the code. The value $\overline{f_r}$ needs to be corrected for the size effect, in view of the size effect on compression strength documented by many authors (first reported by Gonnermann 1925; see the works cited in Hillerborg 1989, 1990 and Bažant and Xiang 1997).

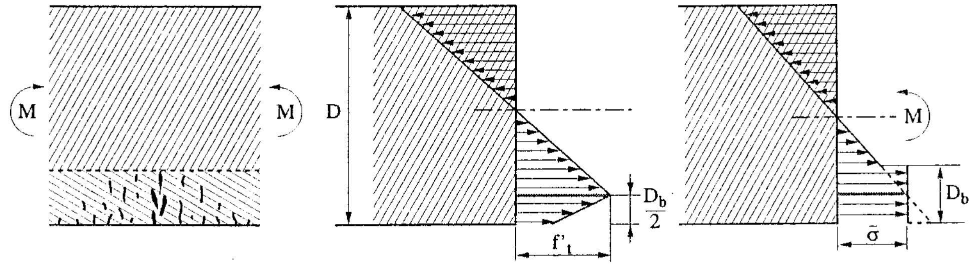

Fig. 1. Stress redistribution in boundary layer of distributed cracking and its approximations

Stress Redistribution Caused by Boundary Layer of Cracking

The size effect can be defined in terms of the maximum stress of the structure only if there are no singular points for which elastic analysis gives infinite stress. Therefore, the size effect is commonly, and more simply, defined in terms of the nominal strength of structure, defined for example as $\sigma_N = P/bD$, where D = chosen characteristic size (dimension) of the structure and b the characteristic width. According to elasticity, as well as the plastic limit analysis underlying the current design code philosophy, there is no size effect, i.e., σ_N is independent of structure size D when geometrically similar structures are compared (e.g., Bažant 2002b).

One, perhaps the easiest, way to derive the deterministic size effect formulas for the maximum bending moment M_0 is to take into account the stress redistribution caused by distributed cracking in a boundary layer of cracking at the tensile face (Fig. 1 left). Due to the heterogeneity of the material, the thickness D_b of this layer is finite. In view of the strain localization instability (Bažant and Cedolin 1991, Chap. 13) and in the spirit of nonlocal damage models, D_b may be assumed to be constant, i.e., independent of cross section depth D (this is verified by the fact that the predictions based on the hypothesis of constant D_b agree well with numerous test data; Bažant and Li 1995, 1996, Bažant and Planas 1998; Bažant and Novák 2000a,b,c). The strain softening due to distributed cracking causes approximately the bending stress redistribution with a shifted neutral axis shown in Fig. 1 (middle). For the sake of simplicity, this distribution may be replaced by the linear-rectangular stress distribution in Fig. 1 (right), in which the shift of the neutral axis is neglected and the maximum stress corresponds to the stress calculated from the elastic bending stress formula at a point of distance $D_b/2$ from the tensile face. Thus

$$\bar{\sigma} = \frac{M}{I}\left(\frac{D}{2} - \frac{D_b}{2}\right) \tag{3}$$

where D = beam depth; M = bending moment; and I = centroidal moment of inertia of cross section. Setting $\bar{\sigma}$ = constant = $f_{r\infty}$, and noting that the modulus of rupture f_r, chosen to represent the nominal strength ($\sigma_N = f_r$), is defined as the elastically calculated maximum stress in the beam, $f_r = \sigma_N = MD/2I$, we have

$$\sigma_N = f_{r\infty}\left(1 - \frac{D_b}{D}\right)^{-1} \quad (D \gg D_b) \tag{4}$$

where constant $f_{r\infty}$ has the meaning of the asymptotic nominal strength for infinite size, which is finite.

This size effect formula yields negative σ_N for $D \ll D_b$ and thus is meaningless for very small sizes D. In this regard it must be realized that the assumptions made are acceptable only for large enough D. It can be shown (Bažant and Li 1995, 1996) that the formula is correct only up to the first two terms of the asymptotic series expansion in terms of the powers of $1/D$ (Bažant and Li 1995, 1996). Therefore, any other formula that exhibits the same first two asymptotic terms is equally justified. This observation suggests the use of asymptotic matching, which is a technique to obtain an approximate solution for problems in which the extreme situations are much easier to solve, by "interpolating" between the opposite extremes, corresponding here to $D \to 0$ and $D \to \infty$ (Bender and Orszag 1978; Barenblatt 1979; Hinch 1991). In this spirit, we need to modify the formula such that the first two terms of the large-size asymptotic expansion would remain unaffected while a realistic small-size solution is matched.

Noting the binomial series expansion, one may check that by making the replacement

$$(1 - D_b/D)^{-1} \approx (1 + rD_b/D)^{1/r} \tag{5}$$

where r = any positive constant, the first two terms of the asymptotic expansion remain unaffected while at the same time σ_N is positive, finite, and monotonically decreasing through the entire range of D. This leads to the size effect expression:

$$\sigma_N = f_r = f_{r\infty} q(D), \quad q(D) = \left(1 + \frac{rD_b}{D}\right)^{1/r} \tag{6}$$

where r = empirical positive constant and $q(D)$ = positive dimensionless decreasing function of size D having a finite limit for $D \to \infty$.

A more realistic starting hypothesis is to consider the bilinear stress distribution with a shifted neutral axis shown in Fig. 1 (middle). However, we need the resulting formula to be accurate only up to the second term in the asymptotic power series expansion of σ_N in terms of $1/D$. It so happens that, up to the first two asymptotic terms, the result is the same as in Eqs. (4) and (6) (Bažant and Li 1995).

Alternatively, formula (6) also ensues from the asymptotic expansion of the energy release function of linear elastic fracture mechanics, upon assuming that an equivalent linear elastic crack at the tensile face has, at maximum M, a fixed depth D_b (see Bažant and Li 1996; more generally Bažant 1998; and with stochastic extension Bažant and Novák 2000a,b,c). The fracture mechanics approach also provides the geometry dependence of D_b and $f_{r\infty}$.

A further generalization is provided by the formula

$$\sigma_N = f_{r\infty} q(D), \quad q(D) = \left(1 + \frac{rD_b}{D + rsD_b}\right)^{1/r} \tag{7}$$

where s = nonnegative constant. This formula is, for large sizes, asymptotically equivalent to the original formula (4), up to the second term of expansion in $1/D$. One can verify it by the following approximations based on the binomial power series expansion, which are accurate up to the second term of the expansion in ξ (defined as $\xi = D_b/D$);

$$\frac{f_{r\infty}}{\sigma_N} = \left(\frac{1+rs\xi}{1+r(s+1)\xi}\right)^{1/r} \approx \frac{1+s\xi}{1+(s+1)\xi}$$

$$\approx (1+s\xi)[1-(s+1)\xi]$$

$$\approx 1 - \frac{D_b}{D} \tag{8}$$

To estimate s, note that, asymptotically for a vanishing D, the softening cohesive crack is equivalent to a perfectly plastic cohesive interface (i.e., a very thin plastic layer, Bažant 2002b) between two elastic solids. Thus, theoretically for $D \to 0$, the modulus of rupture f_r should agree with the prediction of plastic analysis in which the stress distribution is uniform, i.e., rectangular, throughout the whole cross section and, to balance the bending moment M, there is a concentrated compressive force at the face of beam (since no limit on the compressive strength across the crack is implied by the cohesive crack model). For a rectangular cross section, it can be deduced from this assumption that $M = f_{r\infty} bD^2/2$ while $M = f_r bD^2/6$, according to the definition of the modulus of rupture f_r. Equating these two expressions, one gets for the limit case of a plastic crack in a rectangular cross section the ratio

$$\lim_{D \to 0} f_r/f_{r\infty} = 3 \tag{9}$$

This limit can be used to calibrate the value of s in Eq. (7).

Extrapolation of the cohesive crack model to zero size is of course fictitious. Its purpose is to enable asymptotic matching.

According to computational experience with the crack band model as well as the studies of the existing numerous test data on the modulus of rupture (or flexural strength) of concrete (e.g., Bažant and Planas 1998; Bažant and Li 1995), the thickness D_b of the boundary layer of cracking appears to be about $2d_a$, d_a being the maximum aggregate size. We may assume that the plastic behavior is closely approached for beam depth $D \approx 2d_a$, which is just about the shallowest beam or plate that can be cast from concrete. Therefore $D_b/D \approx 2$. Knowing that $f_r/f_{r\infty} = 3$, and assuming that $r = 1$, we may solve s from Eq. (7). It so happens that the solution is $s = 0$, which may be used as a reasonable estimate.

This suggests that the simpler formula (6) ought to be adequate for bridging the brittle small-size asymptotics and the plastic small-size asymptotics. Nevertheless, formula (7) offers greater freedom, which should allow closer adjustment to test data of a very broad size range once they become available.

Correction Due to Weibull Probability Distribution of Random Strength

In the case that the maximum bending moment is governed by tensile strength, its randomness can amplify the size effect on the mean nominal strength. Generalizing Weibull theory to a nonlocal form, Bažant and Novák (2000a,b,c) studied this problem numerically and then derived by asymptotic analysis a simple broad-range size effect formula for the modulus of rupture. Based on the same arguments, it transpires that the probabilistic generalization of Eq. (7) giving the size effect on the mean nominal strength ought to have the form (Bažant 2001a,b):

$$q(D) = \left[\left(\frac{D_b}{D+rsD_b}\right)^{rn/m} + \frac{rD_b}{D+rsD_b}\right]^{1/r} \tag{10}$$

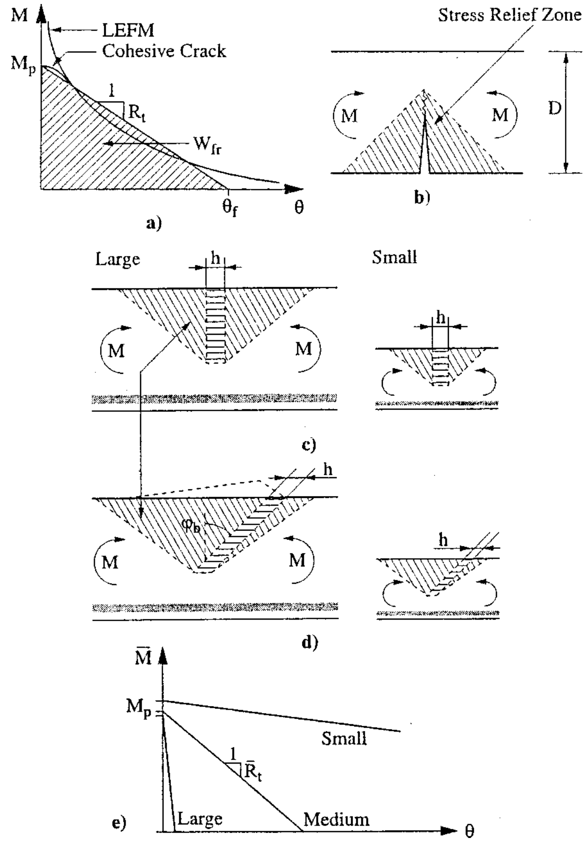

Fig. 2. Softening inelastic hinge: (a) its idealized moment rotation diagram; (b) hinge in brittle beam or plate (e.g., ice plate) failing by tensile crack; (c) hinge in reinforced concrete beams failing by compression fracture band; (e) size effect on moment-rotation diagram; and (f) unloading of partially softened hinge

where n = number of spatial dimensions ($n = 1$, 2, or 3), and m = material constant = Weibull modulus. This formula has been derived so as to satisfy three required asymptotic properties:

1. For $D \to \infty$, it must asymptotically approach the Weibull statistical size effect on the mean (or median) nominal strength, $\sigma_N \propto D^{-n/m}$ (see, e.g., Bažant and Planas 1998);
2. for $m \to \infty$, it must reduce to the deterministic formula (7); and
3. for $D \to 0$, it must have the same limit as the deterministic formula (7).

Extensive study of the data sets on the modulus of rupture (or flexural strength) available in the literature showed that the deterministic formulas (6) and (7) are adequate for normal-size concrete beams, with cross sections up to 1 or 2 m in depth. In other words, the Weibull statistical contribution to the size effect on modulus of rupture was found significant only for deeper beams. This means that the statistical generalization Eq. (10) is unnecessary for most practical applications except concrete structures as thick as arch dams, and foundation plinths or retaining walls.

Softening in Inelastic Hinges

Postpeak softening of inelastic hinges in brittle beams or plates [Fig. 2(a)] can be engendered by either tensile or compression fracture. While the effective length of the yielding zone of plastic (non-softening) hinges is proportional to the characteristic size of beam, which may be taken as the beam depth D, the effective length of softening fracturing hinges [Figs. 2(b and c)] is not proportional to D but is close to being approximately constant, fixed by the characteristic length of the material. The cause is the localization instability of strain softening (Bažant and Cedolin 1991, Chapt. 13), and one consequence is the size effect.

Hinge softening due to a crack propagating from the tensile face [Fig. 2(b)] is exhibited by the bending of plain (unreinforced) concrete beams. The examples, already mentioned, are a foundation plinth, a retaining wall, an arch dam behaving as a horizontal arch, or a tunnel lining behaving as a ring. The energy dissipated by the crack per unit area of the cross section is the fracture energy of the material, G_f.

Hinge softening in reinforced concrete beams is caused by compression fracture (Bažant and Xiang 1997) and occurs if the bending moment is accompanied by a sufficient axial compression force, as is the case for prestressed concrete beams, for columns with a large enough axial force, and for frames or arches with a large enough horizontal thrust. Beams without axial force must be designed, according to the code, as underreinforced, to ensure failure by yielding of tensile steel rather than compression crushing, and in that case the moment-rotation diagram of a hinge has a long plateau instead of softening. However, when such beams are retrofitted with a fiber laminate bonded to the tensile face, the failure mechanism can involve compression crushing [Figs. 2(c and d)] and hinge softening may then again take place.

The compression fracture in a large enough beam consists of an inclined band of axial splitting microcracks, called the crushing band. Such fracture is often regarded as a shear failure in compression, even though shear slip becomes possible only after the stress in the band gets reduced much below the peak; e.g., Bažant and Xiang 1997). The plane of the band is typically inclined, by some angle ϕ_b, with respect to the orthogonal cross section and can intersect the cross section in a line that is vertical [see the side view in Fig. 2(c) where the inclination is not seen; it would be visible by looking from top down] or horizontal [Fig. 2(d)]. The energy, G_b, dissipated by formation of the band, per unit area of the cross section (unit area of the projection of the compression fracture onto the critical cross section) plays the role of the fracture energy of the band, considered to be a material constant (Bažant and Xiang 1997, Bažant 2002b).

The assumption of constant G_b is indicated theoretically by studies of localization of softening damage (Bažant and Cedolin 1991, Chap. 13). However, the existing experimental evidence is not unambiguous. Mattock (1964) and Corley (1966) tested beams with inelastic hinges failing by compression crushing but did not report any size effect on the apparent uniaxial stress–strain relation. Yet Hillerborg (1989, 1990), reanalyzing Corley's data, concluded that they did not disprove the size effect and were consistent with assuming the hinging region to have the length $k_f D$ and the postpeak compressive uniaxial stress-strain diagram in this region to depend on beam depth D as $\epsilon = \sigma/E + k_f f(\sigma)/D$, where E is the Young's modulus of concrete, k_f is a dimensionless empirical constant, and $\delta = f(\sigma)$ represents the softening stress–displacement relation of a cohesive crack, as known from tests of notched fracture specimens. Recently, the question was revisited by Ala et al. (1997) who did not detect in the statistical scatter of test results any size effect; however, the range of his tests was rather limited and the largest beams may have been too small for the development of softening in the compression layer of beam, restrained against damage localization by the underlying elastic layer. By contrast, Kim et al. (1999, 2000) found a clear size effect in their tests of geometrically similar eccentrically compressed prismatic specimens of various sizes, failing by compression crushing. This is in agreement with the size effect on postpeak softening observed by van Mier (1986) in his tests of centrically compressed prismatic plain concrete specimens of various sizes (and later shown by Bažant (1989) to be in agreement with the crack band model or the series coupling model for strain localization; see also Bažant and Cedolin 1991, Sec. 13.2).

So it appears reasonable to assume that the postpeak compression crushing does localize into a crushing band of an approximately constant width h in the axial direction (which is consistent with the aforementioned Hillerborg formula). Crude though this assumption may be, the opposite assumption of a softening damage in compression remaining distributed, with no localization at all, would definitely be wrong. Only for such an opposite assumption, the size effect would be altogether absent. Such an assumption would imply the area under the stress-strain curve, including its softening tail which represents the energy per unit volume, W_f, to be constant (which is known to be impossible for reasons of localization instability, Bažant and Cedolin 1991, Chap. 13), while the case of localization implies G_b, the energy per unit area in the cross section plane, to be constant.

Size Effect on Postpeak Softening of Inelastic Hinges

According to linear elastic fracture mechanics (LEFM), the moment-rotation diagram for an inelastic hinge that is softening as a result of tensile fracture propagation is a curve descending from infinity, as sketched in Fig. 2(a). If the nonlinear cohesive nature of a tensile crack, or the finite size of the FPZ of a tensile crack or a compression crushing band, is taken into account, the diagram starts its descent from a finite value, M_0, representing the maximum (or nominal) bending moment whose calculation was already discussed.

Strictly speaking, a size effect may also exist in the initiation of tensile or compression fracture, i.e., the flexural strength (modulus of ruptures) may be size dependent. But this kind of size effect would be of a different type; it would be weaker than that shown in Fig. 2(a), and would disappear for larger sizes. Here it will be neglected.

For the sake of simplicity, we will idealize the moment-rotation diagram as linear [triangular, Fig. 2(a)], i.e.,

$$M = R_t(\theta_f - \theta) = M_0 - R_t\theta \tag{11}$$

where $R_t = M_0/\theta_f$ is minus the tangent stiffness of the hinge (representing the slope of the $M-\theta$ diagram) and θ_f = hinge rotation at complete break [Fig. 2(a)]. The energy W_f dissipated by a total break of the cross section is given by the area under this diagram [Fig. 2(a)]

$$W_f = \tfrac{1}{2}M_0\theta_f = M_0^2/2R_t \tag{12}$$

Attention will now be restricted to beams of rectangular cross sections and to plates (although generalization to arbitrary cross sections would not be difficult). The energy dissipated over the whole cross section upon reaching a complete break must be equal to

$$W_f = G_f bD \tag{13}$$

where D = depth and b = width of a rectangular cross section (in the case of a hinge in a plate, we consider a unit width $b=1$); G_f may now be interpreted not only as the fracture energy of concrete, for the case of tensile failure, but also as the energy of the crushing band, for the case of compression fracture. To guarantee energy conservation, we set both expressions for W_f equal and thus we get

$$\theta_f=\frac{2bG_fD}{M_0}=\frac{12G_f}{f_{r\infty}}\frac{q(D)}{D} \tag{14}$$

where $f_{r\infty}$ = large-size asymptotic flexural strength.

To bring to light the scaling, it is helpful to introduce the dimensionless bending moment and the dimensionless tangential softening stiffness of the softening inelastic hinge

$$\bar{M}=\frac{M}{EbD^2},\quad \bar{R}_t=\frac{R_t}{EbD^2}=\frac{M_0}{EbD^2\theta_f} \tag{15}$$

where E = Young's modulus. For nonsoftening elasto–plastic materials, the diagram of $\bar{M}$ versus θ is independent of beam depth D, and so any change in this diagram as a function of structure size reveals a size effect. For our softening material and for a rectangular cross section, we find by substituting Eqs. (1) and (14) into Eq. (15) that the dimensionless stiffness is given by $\bar{R}_t=[bD^2f_r/6]/[EbD^2(12\,G_f/f_rD)]=f_r^2D/72EG_f$ or

$$\bar{R}_t=D/72l_f,\quad l_f=EG_f/f_r^2 \tag{16}$$

Here l_f represents an Irwin-type characteristic length of the material (e.g., Bažant and Planas 1998).

As we see from Eqs. (16) and (14), the dimensionless softening stiffness of the hinge increases in proportion to size D, and the rotation at full break decreases at inverse proportion to D [Fig. 2(e)]. As the structure size approaches infinity, $\bar{R}_t$ becomes infinite (a vertical drop). As the structure size tends to zero, $\bar{R}_t$ becomes zero (a horizontal line) and so the softening hinge becomes equivalent to a plastic hinge (in plasticity, there is no size effect because $\bar{R}_t=R_t=0$ for any size, and $\theta_f\rightarrow\infty$). The dimensionless moment-rotation diagram is given by

$$\bar{M}=\bar{M}_0(D)-\theta D/72l_f$$
$$\bar{M}_0(D)=f_{r\infty}q(D)/6E \tag{17}$$

The effect of size D on this diagram is shown in Fig. 2(e). In plasticity, this diagram is the same for all D.

Note: Instead of $\bar{M}$ and $\bar{R}_t$ we could work with the nominal bending moment $M_N=M/D^2$ and the nominal tangent stiffness $R_{tN}=R_t/bD^2$. This would be more in line with the entrenched usage of nominal strength σ_N instead of the dimensionless strength $\bar{\sigma}=\sigma_N/E$. But there is no precedent for using M_N and R_{tN}, and for the analysis of size effect it makes no difference.

Size Effects on Load Peaks and Troughs for Very Large Structures

As already emphasized, the difficulty of the direct analytical solution for the usual sizes for which more than one hinge soften simultaneously will be circumvented by taking again the asymptotic matching approach, which uses "interpolation" between the easily soluble asymptotic behaviors for very large and very small structures. So we will now focus on the asymptotic cases, even though they may usually lie far outside the size range of structures that are, or could be, built.

Consider a statically indeterminate (redundant) beam structure which requires N inelastic hinges to form in order to collapse (e.g., four hinges in Fig. 3). Let the inelastic hinges be numbered

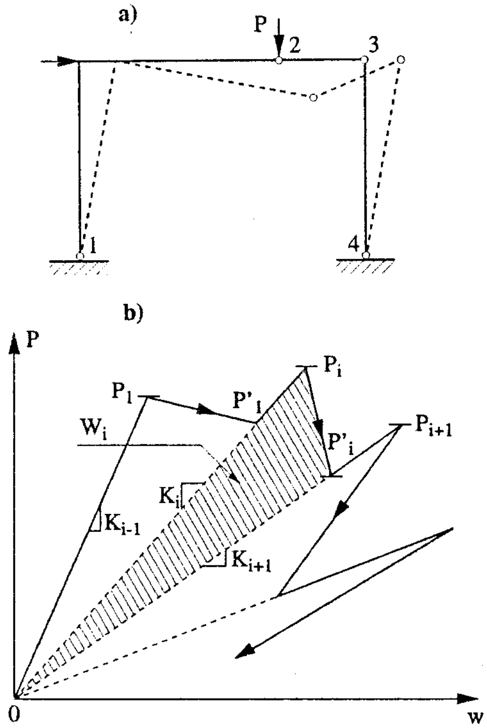

Fig. 3. (a) Frame failing by softening inelastic hinges and (b) load–deflection diagram when one and only one hinge is softening at time

as $j=1,2,\ldots,N$ in the sequence in which they form as the load-point displacement w is increased. Denote as K_i the stiffness associated with the applied load P if hinges $j=1,2,\ldots,i-1$ have completely softened (i.e., $M=0$ and $\theta\geqslant\theta_f$ in these hinges) and hinges $j=i,\ i+1,\ldots,N$ have not yet started to form (i.e., $M\leqslant M_0$ and $\theta=0$ in these hinges). Obviously, $K_1>K_2>K_3>\cdots>K_N>0$.

In the theoretical asymptotic case of a zero size ($D\rightarrow 0$), the slope of the diagram of dimensionless moment $\bar{M}$ versus rotation θ tends to horizontal. Consequently, the plastic limit analysis, in which the moment-rotation diagram has a long horizontal plateau, is applicable. In Fig. 2(a), the limiting zero-size response consists of line segments parallel to the rays of slopes K_i ($i=1,2,\ldots,N$) emanating from the origin (note the double dashes marking the pairs of parallel lines). Such a response is easy to calculate by well-known methods (e.g., Jirásek and Bažant 2002).

Let us now consider the simple (hypothetical) case in which all the hinges $j=1,2,\ldots,i-1$ have softened to a zero moment and hinge i has not yet started to form. In that case, the load-point deflection w is decided solely by K_i and the start of softening in the next hinge i is decided by a critical stress σ_i representing the stress at the tensile face in the case of tensile fracture or at the compression face in the case of compression fracture; therefore

$$P=K_iw,\quad \sigma_i=S_iw \tag{18}$$

where S_i = constants. Restricting again attention, for the sake of simplicity, to rectangular cross sections, one may introduce dimensionless structure stiffness $\bar{K}_i$ and dimensionless critical stress $\bar{S}_i$ in hinge i, such that

$$K_i=\bar{K}_iEb,\quad S_i=\bar{S}_iE/D \tag{19}$$

where D = characteristic size of the structure, considered as two-dimensional, and b = characteristic thickness in the third dimension.

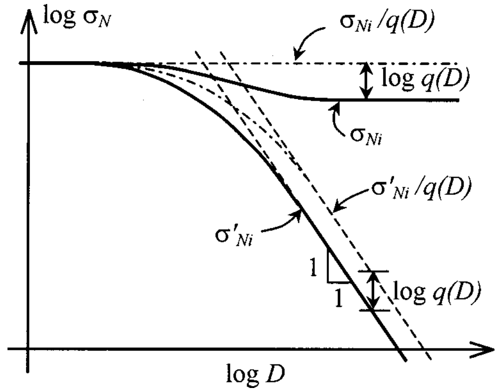

Fig. 4. Size effects on nominal strengths at peak and at trough, matched to small-size plastic limit (Weibull statistical size effect for very large sizes is ignored)

In the assumed case, for which the size may be larger than the size of any real structure, the hinges form and fully soften *one by one*, i.e., no two hinges are softening at the same time. Then the load–deflection diagram must look as shown in Fig. 3(b), where the slope of each ray emanating from the origin is K_i and the maximum load on each ray, marked as P_i ($i=1,2,\ldots,$), corresponds to the start of softening of the next hinge. The trough P_i' on each ray is the load at which the softening of each hinge gets completed (i.e., $M=0$ and $\theta=\theta_f$). Since the moment-rotation diagram of the hinges was assumed to be linear, the load–deflection diagram from P_i to P_i' must be a straight line.

The load peaks P_i are determined from the condition $\sigma_i=f_r$ $=$modulus of rupture (flexural strength). Noting that σ_i $=PS_i/K_i$, we find

$$P_i=f_rK_i/S_i \tag{20}$$

The corresponding nominal stresses are

$$\sigma_{Ni}=P_i/bD=f_r\overline{K}_i/\overline{S}_i=f_{r\infty}(\overline{K}_i/\overline{S}_i)q(D) \tag{21}$$

As may have been expected, peaks σ_{Ni} exhibit only the size effect of fracture initiation (Fig. 4). This size effect is the same as that on the modulus of rupture, and is neglected here (and in Fig. 4). The overall maximum strength is

$$\sigma_N=\max_i \sigma_{Ni} \tag{22}$$

As discussed later, not only the peaks but also the troughs are important for design. To determine the troughs P_i', consider the area of the shaded triangle in Fig. 3(b) between the rays of slopes K_i and K_{i+1}; the area is

$$W_i=\frac{P_iP_i'}{2}\left(\frac{1}{K_{i+1}}-\frac{1}{K_i}\right) \tag{23}$$

W_i represents the work dissipated when hinge i softens from M_0 to 0. Since, in the simple case considered now, no other hinge is assumed to be softening simultaneously with hinge i, energy conservation requires W_i to be equal to the work dissipated by hinge i;

$$W_i=G_fb_iD_i=G_f\beta_i\delta_iD^2$$

with

$$\beta_i=b_i/D, \quad \delta_i=D_i/D \tag{24}$$

where b_i and D_i=width and depth of the cross section at hinge i; and β_i and δ_i=parameters (β_i would be constant for structures geometrically similar in three dimensions). Setting this equal to Eq. (23), we can solve for P_i' and obtain:

$$\frac{\sigma_{Ni}'}{\sigma_{Ni}}=\frac{P_i'}{P_i}=\frac{2EG_f}{\sigma_{Ni}^2}\left(\frac{1}{\overline{K}_{i+1}}-\frac{1}{\overline{K}_i}\right)^{-1}\frac{1}{D}=\frac{\text{const.}}{D} \quad (\text{for large } D) \tag{25}$$

Factor $1/D$ represents the large-size asymptotic size effect on the troughs (Fig. 4). It is a very strong size effect, much stronger than the LEFM size effect of large cracks, which is of the type $1/\sqrt{D}$. For small sizes, this size effect is amplified by the size dependence of f_r in Eqs. (6) or (7).

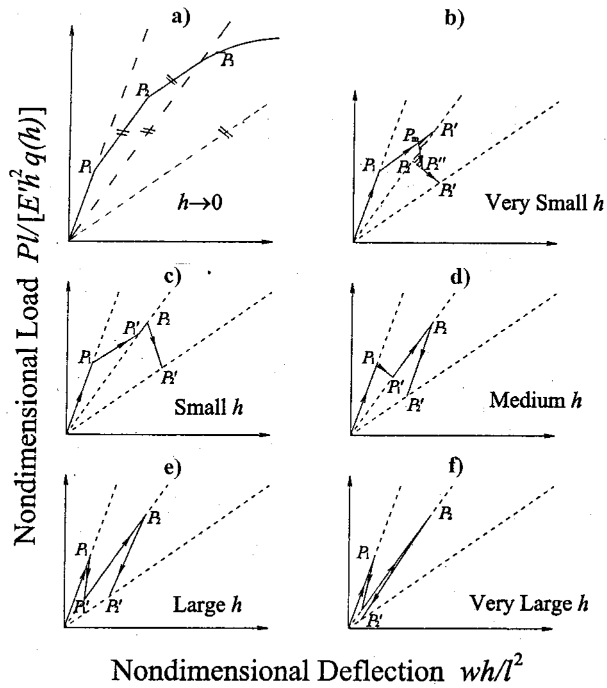

Fig. 5. Evolution of load–displacement diagrams (in terms of dimensionless nominal strength σ_N/E and relative deflection w/D) with increasing size D of structure failing by softening hinges (in such coordinates, secant stiffness slopes, as well as peak points, do not change with size). Parallel dashes in (a) mark parallel lines (h $\sim D$ is cross section depth)

Fig. 5 shows a sequence of typical load–deflection diagrams for geometrically similar structures of various sizes D. These diagrams are plotted in dimensionless coordinates w/D and σ_N/E. To show clearly the size effect brought about by R_t, the size effect on the peaks, stemming from the size dependence of bending stress $\sigma=f_r$ at crack initiation, is omitted from these plots [i.e., $q(D)\equiv 1$]. The dimensionless plots have two advantages: (1) the slopes corresponding to K_i (and also the points corresponding to the peaks P_i) are independent of the size, and (2) the diagrams for structures of different sizes become identical in the limiting case of an elasto–plastic material (for which the size effect is known to be absent). Thus, any change in these diagrams due to a change in the structure size signifies a size effect.

The scaling of the troughs P_i', as described by Eq. (25), indicates that when the structure size increases, the response consists of narrower and narrower spikes (Fig. 5). The descending part of each spike in Fig. 5(e) is unstable for any type of load control (control of P). After the descent switches to a line of positive slope, the structure develops a snapback instability, i.e., the response becomes unstable for both the load control and the displacement control (e.g., Bažant and Cedolin 1991). We could have intuitively expected such behavior on the basis of the fact that the diagram of $\overline{M}$ versus θ approaches a vertical drop as $D \to \infty$.

When the structure size D is reduced, the line from P_i to P_i' eventually changes its slope from negative to positive, i.e., the load increases during hinge softening, and P_i' ceases to be a trough and P_i a peak. For a certain size reduction, P_i' eventually becomes coincident with the next peak P_{i+1}. At that moment the simple limiting case of hinges softening one by one, which has been considered so far, ceases to apply.

For still smaller structures sizes D, there exists, during the loading process, a period in which two hinges, say i and $i+1$, function in the softening regime simultaneously. For reasons of bifurcation and stability of response path (Bažant and Cedolin 1991), one hinge in that case softens further while the other must unload [Fig. 1(f)]. Once the softening of one hinge has become complete (i.e., the hinge moment reduced to zero), the other hinge may reload and enter a period of further softening. Assuming the unloading or reloading diagram to also be linear, the load-deflection diagram during such simultaneous rotation of two hinges is linear, too [lines 67 and 89 in Fig. 5(b)], but different for each different combination of softening hinges and of their loading and unloading regimes. The calculation, unfortunately, is more messy and not amenable to simple formulas. For even smaller structure sizes, more and more hinges, or all the hinges—some loading, some unloading—are found in the softening regime at the same time.

Imagine, for example, that at point 1 in Fig. 5(b), hinge i starts softening while all the previously formed hinges have softened fully (i.e., to a zero moment). At point 2, hinge $i+1$ starts softening as well. For reasons of stability of response path (Bažant and Cedolin 1991, Secs. 10.2 and 13.4), hinge 2, only partially softened, must start unloading [Fig. 2(f)]. At point 3, hinge $i+1$ has softened fully and begins rotating as a real hinge ($M=0$), while hinge i starts reloading. At point 4, hinge i has reentered softening while hinge i has increased its rotation beyond θ_f [Fig. 2(f)] at zero moment. At point 5, hinge i has softened fully as well while hinge $i+2$ has not started softening yet. In that case, area 0123450 must be equal to $2G_f/Eh$, which is the same as the combined area of triangles 0180 and 0950 which correspond to the hypothetical case that hinges i and $i+1$ were forced to soften one after the other, regardless of their M_p value.

Calculation of the maximum moment in the foregoing case is not difficult conceptually but rather involved. It gets even more involved if hinge $i+2$ starts softening before hinge $i+1$ has softened fully, of if three hinges are found in the softening–unloading regime simultaneously. Obviously, simple formulas for the size effect cannot be obtained for such situations.

Therefore, the idea of the presently proposed approximation is to avoid the analysis such complicated situations—i.e., to skip the case of small structures in Fig. 2(b), solve only the cases in Figs. 2(a) and Figs. 2(c–f), and use instead the asymptotic matching, i.e., an approximate ''interpolation'' between the simple asymptotic cases developed later.

Brittle materials such as concrete (or ice) are not the only ones that lead to softening inelastic hinges. As shown by Maier and Zavelani (1970), the nonlinear geometric effect of local buckling of thin flanges in the plastic hinge region of large steel beams can cause sharp softening in the moment-curvature diagram. The foregoing analysis applies to that case. Obviously, the local buckling of flanges is much more dangerous in large steel structures than in small ones. In this connection, it deserves mention that the analysis of instabilities in frames due to softening hinges was pioneered by Maier (1971, 1986) and Maier et al. (1973). Maier (1986) particularly demonstrated how the softening segment of the load–deflection diagram of a structure of one size becomes steeper (and transits to a snapback) if a steeper descent of the moment-rotation diagram is assumed. Although he did not discuss the size effect, in the light of formula (16) this is an analogous phenomenon.

Asymptotically Matched Approximate Size Effect Formula for Troughs in Entire Size Range

We face here a typical problem which is hard and messy for the intermediate size range, in which many hinges soften simultaneously, but becomes easy for the asymptotic cases of very large and very small structures. In that case, a good approximate solution for the entire size range can simply be obtained by finding a smooth formula that has the required asymptotic properties. The philosophy of such ''interpolation'' between the opposite infinities in the $\log D$ scale, better regarded as ''asymptotic matching,'' is co-opted from fluid mechanics (Sedov 1959; Barenblatt 1979). It has been successfully employed for other size effect laws (Bažant and Planas 1998; Bažant and Chen 1997).

To match the large-size asymptotic size effect in Eq. (25) to the horizontal small-size asymptote, one may introduce the following simple approximate formulas for the nominal stress at the trough for all structures sizes:

$$\sigma_{Ni}' \approx \frac{\sigma_{0i}'}{1+D/D_{0i}'} \quad \text{or} \quad \frac{\sigma_{0i}'}{(1+D^s/D_{0i}'^{s})^{1/s}} \tag{26}$$

where σ_{0i}', D_{0i}', and $s=$positive constants.

If the structure size is large enough for the hinges to soften one by one, the hinge number i corresponding to the overall maximum load does not change. For smaller sizes, for which more than one hinge soften simultaneously, that hinge number may change with D and the overall maximum load cannot be correlated to any particular hinge [Figs. 5(a and b)].

It should now be noted that the size effects laws for the peaks as well as the troughs are very different from the size effect law proposed by Bažant (1984) and its various refinements (Bažant and Chen 1997; Bažant and Planas 1998). The cause of this difference is twofold: (1) A single crack, or a single softening hinge, is insufficient for failure of a statically indeterminate beam or frame; and (2) when the size is changed, the crack length in the softening hinge at maximum load does not remain geometrically similar.

The present assumptions and the ensuing size effect are appropriate for slender redundant beam structures requiring more than one inelastic hinge to fail. On the other hand, the assumptions of a single dominant fracture and of its approximate geometric similarity are appropriate for statically determinate structure and for many two- and three-dimensional problems, as shown by many types of experiments and numerical simulations (Bažant and Planas 1998).

The axial expansion caused by rotation of an inelastic softening hinge in a beam under compression is neglected in this paper. If the expansion is resisted by the structure, it may significantly influence the size effect on the nominal strength of redundant beams and frames (Bažant and Xiang 1997; Brocca and Bažant 2000). The expansion produces a dome effect in vertical penetration through a floating sea ice plate (Bažant 2002a).

One related type of size effect was recently brought to light by Jirásek (1997). He studied the effect of the number of columns in a horizontally loaded one-story multibay frame having fixed cross-sectional dimensions, a fixed height H, and a fixed bay length. For perfectly plastic hinges ($\theta_f \rightarrow \infty$), the limit load is proportional to the number of columns. Therefore, the actual collapse load divided by the plastic limit load can be considered as the dimensionless nominal collapse load. For perfectly brittle hinges ($\theta_f = 0$), the nominal collapse load is 1 for the one-bay (portal) frame with two columns; it drops down to 0.8 for the frame with three columns and then gradually increases with increasing number of columns. For inelastic softening hinges with $0 < \theta_f < \infty$, as shown by Jirásek, the dependence of the nominal collapse load on the number of columns is between the plastic and brittle limits. Jirásek characterized this dependence by the dimensionless ductility parameter l_f/H where l_f is the characteristic length defined in Eq. (16). The brittle limit is approached for small values of the ductility parameter, and the plastic limit for large values (note that the opposite could be said about a 'brittleness parameter' defined as H/l_f).

Closing Comment

As shown by the preceding analysis, statically indeterminate beams and frames with softening hinges exhibit deterministic size effects of a new kind, different from structures failing by one dominant fracture. Part II (Bažant 2003) which follows will examine some implications.

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