Shear Buckling of Sandwich, Fiber Composite and Lattice Columns, Bearings, and Helical Springs: Paradox Resolved

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As shown three decades ago, in situations where the initial stresses before buckling are not negligible compared to the elastic moduli, the geometrical dependence of the tangential moduli on the initial stresses must be taken into account in stability analysis, and the stability or bifurcation criteria have different forms for tangential moduli associated with different choices of the finite strain measure. So it has appeared paradoxical that, for sandwich columns, different but equally plausible assumptions yield different formulas, Engesser's and Haringx' formulas, even though the axial stress in the skins is negligible compared to the axial elastic modulus of the skins and the axial stress in the core is negligible compared to the shear modulus of the core. This apparent paradox is explained by variational energy analysis. It is shown that the shear stiffness of a sandwich column, provided by the core, generally depends on the axial force carried by the skins if that force is not negligible compared to the shear stiffness of the column (if the column is short). The Engesser-type, Haringx-type, and other possible formulas associated with different finite strain measures are all, in principle, equivalent, although a different shear stiffness of the core, depending linearly on the applied axial load, must be used for each. The Haringx-type formula, however, is most convenient because it represents the only case in which the shear modulus of the core can be considered to be independent of the axial force in the skins and to be equal to the shear modulus measured in simple shear tests (e.g., torsional test). Extensions of the analysis further show that Haringx's formula is preferable for a highly orthotropic composite because a constant shear modulus of the soft matrix can be used for calculating the shear stiffness of the column, and further confirm that Haringx's buckling formula with a constant shear stiffness is appropriate for helical springs and built-up columns (laced or battened). [DOI: 10.1115/1.1509486]

1 Introduction

During the 1960s, there used to be lively polemics among the proponents of different three-dimensional stability formulations associated variationally with different finite strain measures (see, e.g., the preface of Biot's book [1]), different objective stress rates, and different incremental differential equations of equilibrium (proposed by Hadamard, Biot, Trefftz, Truesdell, Pearson, Hill, Biezeno, Hencky, Neuber, Jaumann, Southwell, Cotter, Rivlin, Engesser, Haringx, etc.—see [2] (p. 732 and Chap. 11) and [3]). These polemics were settled in 1971 by the demonstration, [4], that all these formulations are equivalent because the tangential elastic moduli of the material cannot be taken the same but must rather have different values in each formulation. It was also concluded that these differences matter only if the initial stresses at the critical state of buckling are not negligible compared to the elastic moduli ([2], Sec. 11.4), [4].

For most buckling problems, the differences between various stability criteria are insignificant because the initial stresses are negligible compared to the tangential moduli. One exception is the buckling of rubber and other elastomers. Others are the buckling of composites with a highly orthotropic fiber reinforcement and a very soft matrix, or built-up columns (battened or lattice columns), large regular frameworks treated in a smeared manner as a continuum, and elastomeric bearings used for bridges and for seismic isolation of buildings.

For sandwich plates, which are very sensitive to buckling, [5–11], the initial axial stress in the skins of a sandwich column is negligible compared to the elastic modulus of the skins, and the initial axial stress in the foam core is zero. Consequently, it may at first seem that the shear stiffness of the core should not depend on the axial force in the skins, which would imply that there should be no differences among the critical load formulas associated with different finite strain measures.

So, it came as a surprise that the Engesser-type, [12–14], buckling formula for sandwich columns, which is associated with the Doyle-Ericksen finite strain tensor of order $m=2$, [2], gave, for short sandwich columns, much smaller critical loads than the Haringx-type, [15,16], formula, which is associated with the Doyle-Ericksen tensor of order $m=-2$. The discrepancy was vehemently debated at several recent symposia on composites (especially at those sponsored by ONR at the ASME Congresses in Orlando (2000) and in New York (2001)). Using constant and the same shear stiffness values for both formulas, Kardomateas [17–22] and Simitses and Shen [23] showed that the Haringx-type buckling formula gave results closer to the experiments on sandwich columns and to three-dimensional finite element simulations. But, in view of the smallness of stresses in both the core and the skins, the reason for the difference has been seen as a paradox. To explain it, is the purpose of this paper. The explanation will also clarify the shear stiffness to be used for buckling of highly orthotropic composites and explain why Haringx's formula is the correct buckling formula for helical springs.

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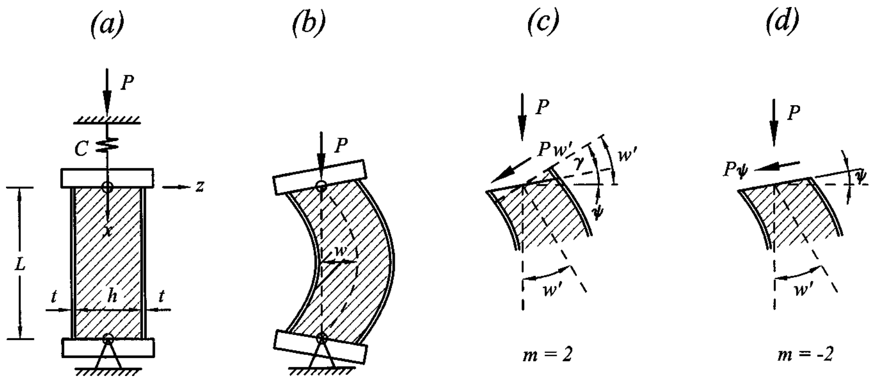

Fig. 1 Sandwich column in (*a*) initial state and (*b*) deflected state; (*c,d*) cross-section rotation, shear angle and shear force due to axial load

In applications to built-up (battened or laced (latticed)) columns and to fiber composites, the difference between the Engesser and Haringx shear buckling formulas has been discussed for about 60 years; e.g., [2,4,24–34]. Ziegler [27], for example, tended to favor Engesser's formula and Reissner, [28,29], Haringx's formula. However, no consensus on the theory has yet emerged, [25,26], although the experiments on helical springs, [16], elastomeric bearings, [35], and latticed columns, [32], clearly favor Haringx's formula.

2 Tangential Moduli Associated With Different Finite Strain Measures

To discuss buckling with shear, we need to recall the dependence of the tangential stiffness tensor of a material on the choice of the finite strain measure (Bažant [4]). A broad class of equally admissible finite strain measures which comprises practically all of those ever used is represented by the Doyle-Ericksen tensors $\boldsymbol{\epsilon}=(\mathbf{U}^m-\mathbf{I})/m$ (also called Hill's family of strains, see, e.g., [2], Section 9.1)); m can be any real number, $\mathbf{I}$=unit tensor, and $\mathbf{U}$=right-stretch tensor. The second-order approximation of these tensors is, in component form,

$$\epsilon_{ij}^{(m)}=e_{ij}+\frac{1}{2}u_{k,i}u_{k,j}-\alpha e_{ki}e_{kj}, \quad e_{ki}=\frac{1}{2}(u_{k,i}+u_{i,k}),$$
$$\alpha=1-\frac{1}{2}m. \tag{1}$$

Here e_{ij} is the small (linearized) strain tensor; the subscripts refer to Cartesian coordinates x_i, $i=1, 2, 3$) and repetition of tensorial subscripts implies summation. In all the formulations up to now, $-2\leq m\leq 2$.

It was shown in [4] and, didactically, in [2], Sec. 11.4, that the stability criteria expressed in terms of any of these strain measures are mutually equivalent if the tangential moduli associated with different m-values are related as follows:

$$C_{ijkl}^{(m)}=C_{ijkm}+\frac{1}{4}(2-m)(S_{ik}\delta_{jl}+S_{jk}\delta_{il}+S_{il}\delta_{jk}+S_{jl}\delta_{ik}) \tag{2}$$

([4] and [2] p. 727). Here C_{ijkl} are the tangential moduli associated with Green's Lagrangian strain ($m=2$), and S_{ij}=current stress (Cauchy stress). Obviously, the differences among the $C_{ijkl}^{(m)}$ values for different m are insignificant if the (suitable) norms

$$\|S_{ij}(\mathbf{x})\|\ll\|C_{ijkl}^{(m)}(\mathbf{x})\| \quad \text{(for every } \mathbf{x}\text{; } m \text{ bounded)} \tag{3}$$

where $\mathbf{x}$=coordinate vectors of points in the structure. This inequality is satisfied for sandwich columns, which is why the discrepancy between the two existing sandwich buckling formulas has seemed paradoxical.

3 Classical Paradox in Buckling of Columns Weak in Shear

Engesser in 1889 [12–14] and Haringx in 1942 [15] presented different formulas for the first critical load in buckling of columns exhibiting significant shear deformations (Fig. 1(*a,b*)). They read

$$P_{cr}=\frac{P_E}{1+(P_E/GA)} \quad \text{(Engesser)} \tag{4}$$

$$P_{cr}=\frac{GA}{2}\left(\sqrt{1+\frac{4P_E}{GA}}-1\right) \quad \text{(Haringx)} \tag{5}$$

(see also [24]); E,G=elastic Young's and shear moduli, $P_E=(\pi^2/l^2)EI$=Euler's critical load, l=effective buckling length, and EI,GA=bending stiffness and shear stiffness of the cross section (note that, in general, $A=\kappa A_0$ where A_0=actual cross-section area and κ=Timoshenko's shear correction factor, which is greater than but close to 1; for a sandwich $\kappa\approx 1$). Each of these two formulas can be regarded as a different and equally plausible generalization of the Timoshenko beam theory ([36]), which does not deal with finite strain effects and applies only to beams carrying negligible axial force.

The discrepancy between these two formulas used to be, until 1971, regarded as a paradox. Then it was shown, [2,4], that this classical paradox is caused by a dependence of the tangential shear modulus $C_{1212}=G$ on the axial stress $S_{11}=-P/A$, which inevitably is different for different choices of the finite strain measure, i.e., for different m. It turns out that Engesser's formula corresponds to Green's Lagrangian strain tensor ($m=2$), and Haringx's formula to the Lagrangian Almansi strain tensor ($m=-2$). Properly the shear moduli in (4) and (5) should be labeled as $G^{(2)}$ and $G^{(-2)}$, respectively, and (2) indicates that

$$G^{(2)}=G^{(-2)}+P/A. \tag{6}$$

Equation (2) further indicates a difference in the E-values, however, that difference can be neglected because the axial stress is always negligible compared to E. Replacing G in Engesser's formula with $G+P_{cr}/A$ and solving P_{cr} from the resulting equation, one obtains Haringx's formula ([2], p. 738), which makes the equivalence blatant.

Equation (6) shows that if the shear modulus is constant (independent of stress $-P/A$) for one formula, it cannot be considered constant for the other formula. For a homogeneous material for which these two buckling formulas give different results, the case of a constant shear modulus would be one chance among infinitely many possible stress-dependences of the shear modulus, and so a constant shear modulus is highly unlikely for either formula.

The difference in shear moduli in (6), of course, becomes significant only if the axial stress $S_{11}=-P/A$ is not negligible compared to G. Such a situation arises for built-up columns (consist-ing either of a regular pin-jointed lattice or a regular moment-resisting framework) approximated by a homogenizing continuum, or for highly orthotropic columns, e.g., columns made of a fiber composite with a very soft matrix. With a proper definition of the dependence of G on the axial stress, both formulas are equivalent. However, even though the equivalence of both formulas along with (6) was demonstrated three decades ago, a false perception of contradiction between these formulas has been widespread.

The relationship between the G values for two different m is linear in stress $S^0 = -P/A$. Thus, if the dependence of $G^{(m)}$ on S^0 is linear, there must exist one m value for which $G^{(m)}$ is constant. The formulation for this m is the the most convenient one for practical use. On the other hand, if the dependence of G on S^0 were nonlinear, there would exist no m for which G could be constant.

4 New Apparent Paradox for Sandwich Buckling

In this study, we focus attention on elastic sandwich columns, for which a similar but not identical paradox has arisen as a consequence of various recent studies, [19–23,32], and was debated at several recent conferences (especially the ASME congresses in Orlando in 2000, and in New York in 2001). Explanation of this new apparent paradox is the objective of our analysis.

Let L denote the length of the column, l its effective length, and P the axial force. The core has thickness h and shear modulus G. The skins have thickness t and are, in general, orthotropic, with axial elastic modulus E (Fig. 1(a)). Since Young's modulus of the core is negligible compared to E for the skins, the entire axial force and bending moment are carried by the skins. On the other hand, since we may generally assume that $t \ll h$, the entire shear force is carried by the core. Therefore, one may substitute $EI = Ebt(h+t)^2/2 + Ebt^3/6 \approx Ebth^2/2$ = bending stiffness of the sandwich ($t \ll h$), and $GA = Gbh$ = shear stiffness of the sandwich, b being the cross section width in the y-direction. With these substitutions, the Engesser and Haringx formulas become

$$P_{cr} = \frac{P_E}{1 + (P_E/Gbh)} \quad \text{(Engesser type)} \tag{7}$$

$$P_{cr} = \frac{Gbh}{2}\left[\sqrt{1 + \frac{4P_E}{Gbh}} - 1\right] \quad \text{(Haringx type)} \tag{8}$$

where P_E is the Euler load,

$$P_E = \frac{\pi^2}{l^2} EI \approx \frac{\pi^2}{l^2} \frac{Ebth^2}{2}. \tag{9}$$

Similar to (6), one may check that, by making the replacement

$$G_{\text{core}} \leftarrow G_{\text{core}} - \frac{2t}{h} \sigma_{\text{skins}} \tag{10}$$

where $\sigma_{\text{skins}} = -P_{cr}/2bt$ and $G = G_{\text{core}}$ = shear modulus of the core, the Engesser-type formula (7) gets transformed into the Haringx-type formula (8).

Although the foregoing replacement works, it is, however, purely formalistic, with no physical basis. It is certainly paradoxical that the shear modulus in the *core* should depend on the axial stress in the *skins*. Therefore, the reason for the discrepancy between these two formulas cannot be caused by the differences in the shear modulus G of the core material, as given by (6). Besides, there is no reason for the G-moduli associated with different strain measures to differ because the axial stress in the core is negligible compared to the shear modulus of core.

We thus have a different kind of paradox, which we must explain. To this end, we must not limit consideration to the material level, as in (10). Rather, we must consider from the outset a sandwich column constrained by the hypothesis that (in slender enough columns) the cross sections of the core must remain plane.

5 Adaptation of Previous General Analysis to Sandwich Column

To clarify the differences between the Engesser-type and Haringx-type formulas, Bažant's [4] general analysis of a column with shear needs to be adapted to a sandwich column. In stability analysis, the incremental potential energy of the column must be expressed accurately up to the second order in displacement gradients. Since the critical axial stress S^0 is not small, finite strain expressions that are accurate up to the second order must be used in the incremental energy expression.

To check whether there is any difference between the P_{cr}-values for axial loads P applied under load control (e.g., gravity) and under displacement control (i.e., with a prescribed axial displacement u_0 at column top), it is convenient to consider a general loading in which P is applied through a spring of stiffness C, attached on top (Fig. 1(a)). We introduce Cartesian coordinates x_i ($i = 1,2,3$, $x_1 = x$, $x_2 = y$, $x_3 = z$), positioned so that x_1 = axial coordinate of column (Fig. 1(a)). The components of the incremental displacements from the stressed initial undeflected state are u_i; $u_3 = w(x)$ = small lateral deflection (displacement of the neutral axis in the direction of coordinate x_3), and $u_1 = u(x,y,z)$ = small incremental axial displacement; ψ a small rotation of the cross section, assumed to remain plane but generally not normal to the deflected beam axis (Fig. 1(c,d)). The shear angle $\gamma = \theta - \psi$ (Fig. 1(c,d)) where $\theta = w'$ = slope of the deflection curve (the primes denote derivatives with respect to x).

Obviously, the incremental axial strain in the neutral axis is distributed uniformly along the column; $e_{11}^0 = u_0/L$. The second-order incremental potential energy $\delta^2\mathcal{W}$ for small deflections $w(x)$ and small axial displacements $u(x)$ is

$$\delta^2\mathcal{W} = \int_0^L \int_A \left[S^0(y,z)(\epsilon_{11}^{(m)} - e_{11}) + \frac{1}{2} E^{(m)}(y,z) e_{11}^2 + \frac{1}{2} G^{(m)}(y,z)\gamma^2 \right] \mathrm{d}A \ \ \mathrm{d}x + \int_A \int_0^L \frac{1}{2} E^{(m)}(y,0)(u_0/L)^2 \mathrm{d}A \ \ \mathrm{d}x + \frac{1}{2} C u_0^2 \tag{11}$$

where the factor $(\epsilon_{11}^{(m)} - e_{11})$ is justified in [2] (Chap. 11); $y = x_2$ and $z = x_3$ = coordinates of the cross section whose area is A; $\mathrm{d}A = \mathrm{d}y\mathrm{d}z$; $S^0(y,z)$ = initial axial normal stress in the straight column before deflection; $E^{(m)}(y,z)$ and $G^{(m)}(y,z)$ are the tangential elastic moduli at point (y,z); $E^{(m)}(y,z)$ = axial elastic modulus (taken, in the case of a sandwich, to be nonzero only for the skins, which normally are orthotropic), $G^{(m)}(y,z)$ = shear modulus (taken into account, for a sandwich, only for the core); and $E^{(m)}(y,0)$ is the value at the neutral plane (midthickness of core). The superscript $^{(m)}$ indicates that, for different m, these values may in general be different, as implied by (2).

Since the skin thickness $t \ll h$, we consider the skins to possess only axial stiffness; the bending and shear stiffnesses of the skins are negligible, and we may also consider $\gamma = 0$ within the skins. Accordingly,

$$u = u_1 = -z\psi, \quad u_{1,3} = -\psi, \quad u' = u_{1,1} = e_{11} = -z\psi', \tag{12}$$

$$u_{1,3} = -\psi, \quad u_{3,1} = w' = \theta, \quad 2e_{13} = 2e_{31} = u_{1,3} + u_{3,1} = \gamma = w' - \psi \tag{13}$$

$$\epsilon_{11}^{(m)} - e_{11} = \frac{1}{2}(u_{1,1}^2 + u_{3,1}^2) - \alpha(e_{11}^2 + e_{31}^2) = \frac{1}{2}(z^2\psi'^2 + w'^2) - \alpha\left[z^2\psi'^2 + \frac{1}{4}(w' - \psi)^2\right] \tag{14}$$

(since we consider deflections from the initial state of loaded column before buckling, the expression $u_1 = -z\psi$ does not include

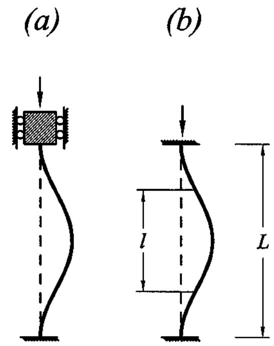

Fig. 2 Column loaded (*a*) under load control (e.g., by gravity) and (*b*) displacement control

the axial displacement at column axis, corresponding to displacement u_0 at the top). After these substitutions, (11) becomes

$$\delta^2\mathcal{W}=\frac{1}{2}\int_0^L\int_A\Big\{[(m-1)S^0(y,z)+E^{(m)}(y,z)]z^2\psi'^2 +S^0(y,z)w'^2 +\left[G^{(m)}(y,z)-\frac{1}{4}(2-m)S^0(y,z)\right](w'-\psi)^2\Big\}\mathrm{d}A\,\mathrm{d}x +\int_0^L\int_A\frac{1}{2}E^{(m)}(y,z)(u_0/L)^2\mathrm{d}A\,\mathrm{d}x+\frac{1}{2}Cu_0^2. \quad (15)$$

Note that none of the terms containing δu_0 contains also $w(x)$. Thus it is clear that the stiffness of the spring will have no effect on the critical load for lateral buckling. So we will from now on ignore the terms containing C, which is equivalent to considering $C\rightarrow\infty$, or to setting $u_0=\delta u_0=0$. As the displacement control (Fig. 2) of axial loading of the column is equivalent to $C\rightarrow\infty$ and the load control to $C=0$ (Fig. 1(*a*,*b*)), it follows that the critical loads of lateral buckling are the same for both types of load control (which is, in principle, well known).

Now we may integrate (15) over the cross section. Noting that the elastic modulus $E^{(m)}$ is negligibly small within the core, that the $G^{(m)}$ of the skins may be neglected because they carry a negligible portion of the shear force, and that the bending stiffness of the skins is negligible, we obtain in the integration process the following cross-section stiffness expressions and resultants:

$$\int_A E^{(m)}(y,z)z^2\mathrm{d}A=R^{(m)}=E^{(m)}\frac{1}{2}bth^2,$$

$$\int_A G^{(m)}(y,z)\mathrm{d}A=H^{(m)}=G^{(m)}bh \quad (16)$$

$$\int_A S^0(y,z)\mathrm{d}A=-P,\quad \int_A S^0(y,z)z^2\mathrm{d}A=-\frac{1}{2}Ph^2. \quad (17)$$

Here $E^{(m)}$ is the value for the skins and $G^{(m)}$ the value for the core; $R^{(m)}$ and $H^{(m)}$ are the bending and shear stiffnesses of the cross section. A simplification can be obtained by noting that

$$|(m-1)S^0(y,z)|\ll E^{(m)} \quad (18)$$

because the magnitude of the axial compressive stress in the skin is always negligible compared to the axial elastic modulus of the skin. Setting also $\alpha=1-m/2$, we thus obtain

$$\delta^2\mathcal{W}=\frac{1}{2}\int_0^L\Big\{R^{(m)}\psi'^2+\left[H^{(m)}+\frac{1}{4}(2-m)P\right](w'-\psi)^2 -Pw'^2\Big\}dx. \quad (19)$$

The necessary condition of stability loss and bifurcation is that the first variation of the second-order work $\delta^2\mathcal{W}$ during any kinematically admissible deflection variations $\delta w(x)$, $\delta u(x)$, δu_0 must vanish (Trefftz condition [2]). So,

$$\delta(\delta^2\mathcal{W})=\int_0^L\Big\{R^{(m)}\psi'\,\delta\psi'+\left[H^{(m)}+\frac{1}{4}(2-m)P\right](w'-\psi) \times(\delta w'-\delta\psi)-Pw'\,\delta w'\Big\}\mathrm{d}x=0. \quad (20)$$

Now we may integrate the first term by parts to eliminate the derivative of the variation $\delta\psi(x)$;

$$\delta(\delta^2\mathcal{W})=\int_0^L\Big\{\left[H^{(m)}+\frac{1}{4}(2-m)P\right](w'-\psi)-Pw'\Big\}\delta w'(x)\mathrm{d}x -\int_0^L\Big\{R^{(m)}\psi''+\left[H^{(m)}+\frac{1}{4}(2-m)P\right] \times(w'-\psi)\Big\}\delta\psi(x)\mathrm{d}x +[\ \ldots\ ]_0^L \quad (21)$$

where the boundary terms, $[\ \ldots\ ]_0^L$, need not be written out in detail for our purpose. The last variational equation must be satisfied for any kinematically admissible variations $\delta\psi(x)$ and $\delta w'(x)$. This condition requires that

$$Pw'-\left[H^{(m)}+\frac{1}{4}(2-m)P\right](w'-\psi)=0 \quad (22)$$

$$R^{(m)}\psi''+\left[H^{(m)}+\frac{1}{4}(2-m)P\right](w'-\psi)=0. \quad (23)$$

Consider now simple supports at ends (in which case $l=L$; Fig. 2). Upon adding the last two equations, we may integrate them and, in view of the boundary conditions of simple supports, we get:

$$R^{(m)}\psi'+Pw=0. \quad (24)$$

This equation together with (22) represents a system of two linear homogeneous first-order ordinary differential equations for $w(x)$ and $\psi(x)$. They can further be reduced to one second-order homogeneous equation by differentiating (22) and substituting ψ' expressed from (24). The result is

$$w''+k^2w=0 \quad (25)$$

where

$$k^2=\frac{P\left[H^{(m)}+\frac{1}{4}(2-m)P\right]}{R^{(m)}\left[H^{(m)}-\frac{1}{4}(2+m)P\right]}. \quad (26)$$

The solution of differential Eq. (25) satisfying the boundary conditions of a simply supported column (Fig. 1(*b*)) is $w(x)=a\sin kx$ where a is any real number. The boundary conditions require that $k^2=\pi^2/L^2$ for the first critical load. If we substitute this into (26), we acquire the following equation for the first critical load P at which buckling takes place:

$$\frac{1}{4}(2-m)P^2+\left[H^{(m)}+\frac{1}{4}(2+m)P_E^{(m)}\right]P-H^{(m)}P_E^{(m)}=0 \quad (27)$$

where P_E is the Euler load, i.e.,

$$P_E^{(m)}=\pi^2 R^{(m)}/L^2. \tag{28}$$

Equation (27) is a quadratic equation, which has for $m=2$ and $m=-2$ the following positive solutions analogous to Engesser's and Haringx's formulas:

$$\text{for } m=2: \quad P_{cr}=\frac{P_E^{(2)}}{1+(P_E^{(2)}/H^{(2)})} \quad \text{with } P_E^{(2)}=\frac{\pi^2}{l^2}R^{(2)}$$

$$\text{for } m=-2: \quad P_{cr}=\frac{H^{(-2)}}{2}\left[\sqrt{1+\frac{4P_E^{(-2)}}{H^{(-2)}}}-1\right] \tag{29}$$

$$\text{with } P_E^{(-2)}=\frac{\pi^2}{L^2}R^{(-2)}. \tag{30}$$

It has been shown in 1971, [4], (and with more detailed explanations in [2]) that the case $m=2$ is associated by work with Truesdell's objective stress rate, and the case $m=-2$ with Cotter and Rivlin's (convected) objective stress rate.

One could further obtain from (27) an infinite number of sandwich buckling formulas, each associated with any chosen value of m. Curiously, however, no investigators proposed critical load formulas associated with other m values, although many investigators (e.g., Biot [1], Biezeno, Hencky, Neuber, Jaumann, Southwell, Oldroyd, Truesdell, Cotter, and Rivlin—see [2], Chap. 11) introduced formulations for objective stress rates, three-dimensional stability criteria, surface buckling, internal buckling, and incremental differential equations of equilibrium associated with $m=1$, 0 and -1.

6 Paradox Resolution: Shear Stiffness Definition for Stressed Sandwich

In analogy to (6) and in similarity to (10), one may expect the shear stiffnesses for the Engesser's and Haringx's formulas to be related as

$$H^{(2)}=H^{(-2)}+Ph/2t. \tag{31}$$

Indeed, when this relation is substituted into (29) and the resulting equation is solved for $P=P_{cr}$, formula (30) results. However, unlike homogeneous columns weak in shear, the foregoing transformation cannot be physically justified in the sense of (10), i.e., on the basis of the general transformation of tangential moduli in (2) nor its special case in (6). The reason is that the axial stress S^0 in the core in much less than $E^{(m)}$ in the core and negligible. From this viewpoint, the transformation appears illogical: Why should the shear modulus of the core be adjusted according to the axial stress in the skins?

This has become a new apparent paradox, which must be resolved. To this end, we need take a closer look at the definition of the shear stiffness H of a sandwich, which we do next.

Let us imagine a homogeneous pure shear deformation of an element Δx of the sandwich column;

$$u_1=u_{1,1}=u_{1,3}=e_{11}=0, \quad u_{3,1}=\gamma, \quad e_{13}=e_{31}=\gamma/2. \tag{32}$$

After substitution into (11), the second-order incremental potential energy of the element is obtained as

$$\delta^2\mathcal{W}=\Delta x\int_A\left[-\frac{P}{2bt}\left(\frac{1}{2}u_{k,1}u_{k,1}-\alpha e_{k1}e_{k1}\right)+\frac{1}{2}G_m\gamma^2\right]\mathrm{d}A \tag{33}$$

or

$$\delta^2\mathcal{W}=bh\Delta x\left(G^{(m)}-\frac{2+m}{4}\frac{P}{bh}\right)\frac{\gamma^2}{2}. \tag{34}$$

In particular, for $m=2$ (Engesser type) and $m=-2$ (Haringx type),

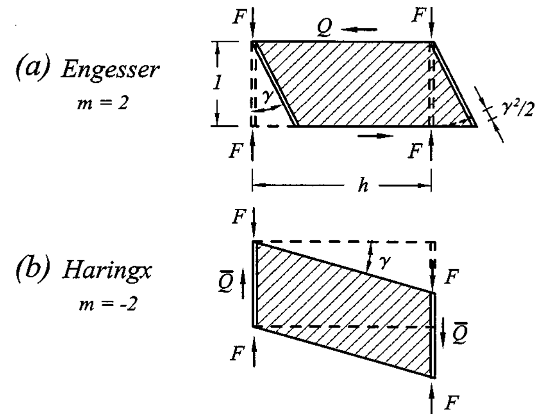

Fig. 3 Shear deformation of an element of sandwich column under initial axial forces $F=P/2$; (*a*) with second-order axial extension $\gamma^2/2$, and (*b*) at no axial extension

$$\delta^2\mathcal{W}=\begin{cases} bh\Delta x\ [G^{(2)}-(P/bh)]\gamma^2/2 & \text{(Engesser type } G) \\ bh\Delta x\ G^{(-2)}\gamma^2/2 & \text{(Haringx type } G) \end{cases}. \tag{35}$$

Since the foam core in an axially loaded sandwich column carries no appreciable axial stresses, it is convenient to use that definition of $G^{(m)}$ for which the shear stiffness of the core requires no correction for the effect of the axial force P carried by the skins. As we see, that is the latter, Haringx-type, expression (for $m=-2$). In that case, the shear modulus $G^{(-2)}$ is the same as that obtained in a pure shear test without normal stress, for example, in the torsion test of thin-wall tube made of the rigid foam.

The solutions for $m\neq-2$, including the Engesser-type formula, are of course equivalent. But if they are used the shear modulus of the core must be corrected for the effect of the axial forces $F=P/2$ carried by the skins. It would be wrong to use in them the G value measured in a pure shear test of the foam, in which no normal force acts on the shear plane.

Intuitive understanding can be gained from Fig. 3, which shows two kinds of shear deformation of an element (of height $\Delta x=1$) of a sandwich column. In the first kind (Fig. 3(*a*)), corresponding to the deformation described by (32), small shearing of the element is accompanied by a second-order small axial extension of the skins, equal to $1-\cos\gamma\approx\gamma^2/2$ (per unit height). If the initial forces F were negligible, this second-order small extension would make no difference but since they are not, one must take into account the work of the initial forces of F on this extension, which is $(2F\gamma^2/2)bh$ or $-bhS^0(-\gamma^2/2)$ (per unit height, $\Delta x=1$). This work must be added to the work of the shear stresses, $(G\gamma^2/2)bh$, in order to obtain the complete second-order work expression. In the second kind of shear deformation (Fig. 3(*b*)), the initial forces F do no work. So, the incremental second-order work expressions for these two kinds of shear deformation, respectively, are

$$\delta^2\mathcal{W}=\begin{cases} bh(G^{(2)}+S^0)\gamma^2/2 & \text{(case a)} \\ G^{(-2)}\gamma^2/2 & \text{(case b)}. \end{cases} \tag{36}$$

These two cases (Fig. 3) give the same incremental second-order work if $G^{(2)}=G^{(-2)}-S^0$ or $G^{(2)}=G^{(-2)}+2F/bh$. We see that these relations coincide with (6).

From the foregoing comparisons and the discussion of Fig. 3, it is now obvious that a constant shear modulus G, equal to the shear modulus measured in a shear test of the foam (e.g, a torsional test of a hollow tube), can be used only in the Haringx-type formula $(m=-2)$.

Recently Simitses and Shen [23], Kardomateas [19,20,22] and Kardomateas and Huang [21], studied the differences between the Engesser-type and Haringx-type formulas experimentally and by finite element analysis. They concluded that the Haringx-type formula gives better predictions. Since they tacitly adopted a constant value of incremental modulus G, this is indeed the conclusion that they should have obtained. The present theoretical analysis explains why.

When can the differences between the column solutions for different m, and particularly between the formulas of Engesser and Haringx, be ignored? They can if

$$P \ll H^{(m)} = G^{(m)} bh. \tag{37}$$

7 Ambiguity in Deriving Differential Equations of Equilibrium

The Engesser and Haringx formulas can also be derived from the differential equations of equilibrium. This is discussed for a homogeneous column weak in shear on p. 738 in [2], and we will now indicate it for a sandwich. Fig. 1(c,d) shows two kinds of cross sections of a sandwich column in a deflected position: (a) the cross section that is normal to the deflected column axis, on which the shear force due to axial load is

$$Q = Pw' \tag{38}$$

and (b) the cross section that was normal to the column axis in the initial undeflected state, on which the shear force due to axial load is

$$\bar{Q} = P\psi. \tag{39}$$

From equilibrium, for a simply supported (hinged) column, the bending moment is $M = -Pw$ in both cases. The force-deformation relations are $M = Ebth^2\psi'/2$ and Q or $\bar{Q} = Gbh\gamma = Gbh(w' - \psi)$ in case a or b, respectively. Eliminating M, γ, ψ and Q or $\bar{Q}$ from the foregoing relations, we get a differential equation of the form $w'' + k^2 w = 0$, same as (25), where

$$k^2 = \frac{GbhP}{E(bth^2/2)(Gbh - P)} \tag{40}$$

or

$$k^2 = \frac{P^2 + GbhP}{E(bth^2/2)Gbh}, \tag{41}$$

respectively. Setting again $k = \pi/l$ and solving for P, we find the former equation to lead to Engesser's formula (4) and the latter to Haringx's formula (5).

We see that Engesser's formula ($m=2$) is obtained when the shear deformation γ is assumed to be caused by the shear force acting on the cross section that is normal to the deflected axis of column, and Haringx's formula ($m=-2$) when caused by the shear force acting on the rotated cross section that was normal to the beam axis in the initial state.

The foregoing equilibrium derivation, however, does not show that the values of shear stiffness in both formulas must be different. Especially, it does not show that the shear stiffness in the direction of the rotated cross section can be kept constant, while the shear stiffness in the direction of normal to the deflected axis must be considered to depend on the axial force. This has been a perennial source of confusion. To dispel it, the work of the shear forces must be considered. So, an energy approach is appropriate.

8 Implications for Highly Orthotropic Composites

Orthotropic composite plates or columns, reinforced by fibers in one or two directions, can have a shear modulus G much smaller then the axial elastic modulus E (typically 25 times smaller). The shear modulus of the composite can be determined in two ways: (1) by calculation from the measured shear and axial moduli G_p, E_p of the polymeric matrix and the elastic modulus E_f of the fibers, or (2) by direct testing of the composite in tension and torsion.

In the former way, moduli G_p, E_p, and E_f may be taken as constant because the stresses in the matrix and fibers are too small compared to the respective moduli. Since in Fig. 3(a,b) we can imagine the skins acting like two adjacent fibers in an element of a composite column, and the core like the matrix between these fibers, the situation is similar to that analyzed for the sandwich. It immediately follows that Haringx's formula (5), with a constant value of G_p, is the appropriate one. Engesser's formula (4) could nevertheless be used to get identical results if G_p of the matrix were transformed as a linear function of the initial stress S_f in the fibers, similar to formula (36a) or (10).

In the latter way, the incremental $G^{(m)}$ value of the composite depends in general on the initial axial force in the tube being tested, if any is applied. However, the special case of Haringx's formula ($m=-2$) employs a G modulus that corresponds, as already shown, to the shearing in torsion at constant length of tube (Fig. 3(a)), in which the axial stress S_f in the fibers does no work. So, in that special case, the incremental $G^{(m)}(m=-2)$ should be independent of S_f.

Therefore, the G value in Haringx's formula (5) for an orthotropic composite can be taken as constant. On the other hand, the G value in Engesser's formula must be considered to depend on P_{cr} (linearly, in the manner of (31)). This makes P_{cr} an unknown, and so the formula becomes an equation (a quadratic one) for P_{cr}. The solution of course leads to the Haringx-type formula.

9 Remarks on Helical Springs, Built-Up Columns, and Bridge Bearings

As another obvious ramification, the present analysis explains why Haringx's formula, [15], is the correct one for helical springs, which were the objective of Haringx's original study. It suffices to note that, in the case of springs, the rotated, initially normal, cross section lies symmetrically within a single pitch of the spiral, halving the separation between the pitches at the point diametrically opposite to the point of intersection of the cross section with the deformed spiral (see "Haringx" in Fig. 4). The stiffness for this cross section can be calculated easily. By contrast, a cross section normal to the deflected axis of the helix does not exhibit this kind of symmetry and may even cut through more than one pitch of the helix (see "Engesser" in Fig. 4). The shear stiffness for such a cross section must depend on the axial force in the spring. Its calculation would be messy and unsuitable for practice.

In bridge bearings that consist of a stack of horizontal steel plates separated by bonded elastomeric layers (Fig. 4 right), the shear force that determines the shear deformation of each elastomeric layer is parallel to the steel plate, and thus to the cross section that was horizontal before deflection, and not to the cross section that is normal to the current deflected axis of the bearing. This again implies that a constant shear modulus can be used only with Haringx's formula (provided, of course, that the layers of elastomer behave elastically).

The built-up columns are normally approximated in a smeared manner as continuous columns (Fig. 5). They can consist of (i) a single-bay regular rectangular frame (flanges joined by battens), which resists the shear force predominantly through the bending of the flanges and the batten in each repetitive cell of the column, or (ii) a lattice, which is idealized as pin-jointed and resists the shear force predominantly by axial forces in the members of each lattice cell.

For both cases, the equivalent shear stiffness H of the continuum approximation of a built-up column may be calculated from the shear deformation of one repetitive cell of the column ([2], p. 739, Fig. 11.6b,c). A constant shear stiffness can be used if the shear deformation of the cell is calculated at constant length of the vertical flange, as shown in Fig. 5 (bottom). In that case, Haringx's formula is appropriate.

On the other hand, if the shear stiffness of the cell is calculated from the shear deformation in Fig. 5, the work that forces $F=$

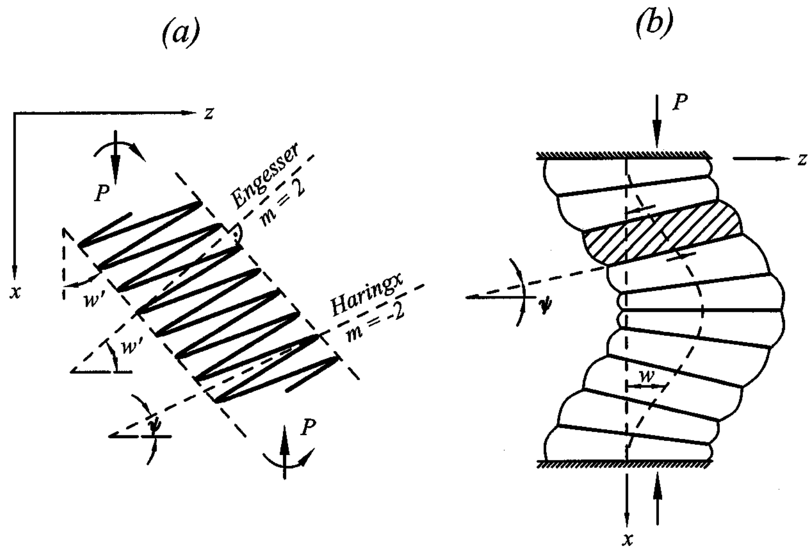

Fig. 4 (*a*) Lateral view of a helical spring and cross sections on which the shear force is defined in Haringx and Engesser theories; (*b*) shear buckling of an elastomeric bridge bearing

$-P/2$ in the flanges do on the second-order axial extension $\gamma^2/2$ must be taken into account. This leads to Engesser's formula. But the shear stiffness in that formula must of course be considered to depend on P.

Haringx's formula with a constant effective shear stiffness is obviously also appropriate for the overall shear buckling of large regular multi-bay multi-story frames as used in tall buildings (Section 2.9 in [2]).

10 Summary and Conclusions

1. In the case where the initial stresses before buckling are not negligible in comparison to the elastic moduli, the dependence of the tangential moduli on the initial stresses must be taken into account in stability analysis, and the stability or bifurcation criteria have different forms for tangential moduli associated with different choices of the finite strain measures, [4]. It has been regarded as paradoxical that sandwich columns apparently defy this condition—equilibrium analyses based on different but equally plausible assumptions yield different formulas (Engesser's and Haringx' formulas) even though the axial stress in the skins is negligible compared to the axial and shear moduli of the skins and the axial stress in the core is negligible compared to the axial and shear moduli of the core. Here it is shown by variational energy analysis that the aforementioned condition for the stress dependence of the tangential moduli needed for stability analysis is only a sufficient condition but not a necessary one. Another condition applies to sandwich structures—if the normal stress in a stiff component of the cross section, the skins, is not negligible compared to the shear

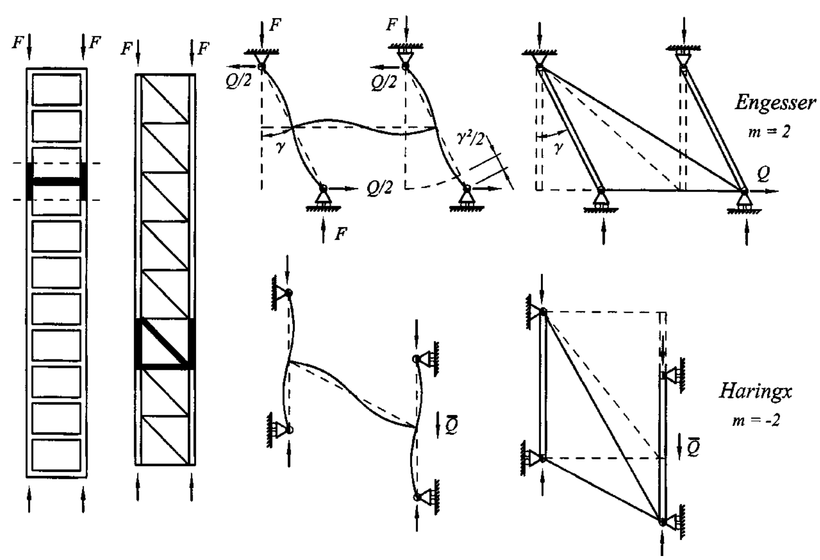

Fig. 5 Left: Column with battens and pin-jointed lattice column. Middle: Shearing of a cell of batten column. Right: Shearing of a cell of lattice column. Top: Shearing with second-order axial extension. Bottom: Shearing with no axial extension.

modulus in an adjacent soft component, the foam core, then this stress influences the shear stiffness of the sandwich cross section.

2. The shear stiffness of the core is in general a linear function of the axial forces carried by the skins, and this function is different for stability theories associated with different finite strain measures. The corresponding definitions of the shear force caused by the applied axial load are also different.
3. The Engesser-type buckling formula and the Haringx-type buckling formula are both, in principle, correct and mutually equivalent. But a different shear stiffness of the core of the sandwich column, in general depending linearly on the applied axial load, must be used in each.
4. The Haringx-type formula represents a special case in which the shear modulus of the foam core can be taken as independent of the axial force in the skins and equal to the shear modulus measured in a simple shear test (e.g., the torsional test of a thin-wall foam tube). For the Engesser-type formula, the shear stiffness of the core must be considered to depend on the unknown axial force. Therefore, the Haringx-type formula is more convenient for practice.
5. The foregoing conclusion is in agreement with the recent findings of Simitses, Kardomateas and co-workers who used a constant shear modulus for the core of sandwich columns in both the Engesser-type and Haringx-type buckling formulas and found that the latter gave results closer to experiments as well as three-dimensional finite element calculations.
6. An extension of the analysis further shows that Haringx's formula is preferable for highly orthotropic composites. If, and only if, that formula is used, a constant shear modulus of the soft matrix can be used for calculating the shear stiffness of column. For Engesser's formula, the shear modulus of the matrix must be considered to depend on the axial stress in the fibers.
7. As further ramifications, the effective shear stiffness of helical springs, elastomeric bridge bearings, built-up (battened or laced) columns and multi-bay multi-story frames can be considered to be constant only if Haringx's formula is used.
8. The difference between the Engesser-type and Haringx-type formulas for a sandwich (or orthotropic fiber composite) can be ignored only when the axial force carried by the skins (or the fibers) is much less then the shear stiffness of the core (or the matrix).
9. For orthotropic materials whose tangential shear modulus is a nonlinear function of the normal stress, both the Engesser-type and Haringx-type formulas necessitate the use of a variable shear stiffness, and in that case none of them is more convenient than the other.
10. For general structures, condition (3) is only a necessary one for making the differences among various choices of the finite strain measure $\epsilon^{(m)}$ irrelevant. A condition that is sufficient is

$$\max_{(\mathbf{x})}\|S_{ij}(\mathbf{x})\|\ll\min_{(\mathbf{x})}\|C^{(m)}_{ijkl}(\mathbf{x})\| \quad (m \text{ bounded}). \tag{42}$$

This condition appears to be both sufficient and necessary when the maximum and minimum are taken within any single cross section of a slender beam, but not necessary for a general body with the maximum and mimimum taken over the whole body. The reason that the inequality $\|S_{ij}(\mathbf{x})\|\ll\|C^{(m)}_{ijkm}(\mathbf{x})\|$ for a material point is insufficient is the interaction within the cross section, reflected in the hypothesis of cross sections of slender beams remaining plane.

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References

[1] Biot, M. A., 1965, *Mechanics of Incremental Deformations*. John Wiley and Sons, New York.
[2] Bažant, Z. P., and Cedolin, L., 1991, *Stability of Structures: Elastic, Inelastic, Fracture and Damage Theories*, Oxford University Press, New York (and 2nd updated Ed., Dover, New York, 2002).
[3] Bažant, Z. P., 1968, ''Conditions of Deformation Instability of a Continuum and Their Application to Thick Slabs and a Half Space'' (in Czech, with English summary), Stavebnícky Časopis (SAV, Bratislava), **16**, pp. 48–64.
[4] Bažant, Z. P., 1971, ''A Correlation Study of Incremental Deformations and Stability of Continuous Bodies,'' ASME J. Appl. Mech., **38**, pp. 919–928.
[5] Goodier, J. N., and Hsu, C. S., 1954, ''Nonsinusoidal Buckling Modes of Sandwich Plates,'' J. Aeronaut. Sci., **21**, pp. 525–532.
[6] Plantema, F. J., 1966, *Sandwich Construction: The Bending and Buckling of Sandwich Beams, Plates and Shells*, John Wiley and Sons, New York.
[7] Allen, H. G., 1969, *Analysis and Design of Sandwich Panels*, Pergamon Press, Oxford, UK.
[8] Kovařík, V., and Šlapák, P., 1973, *Stability and Vibrations of Sandwich Plates*, Academia, Prague (in Czech).
[9] Michiharu, O., 1976, ''Antisymmteric and Symmetric Buckling of Sandwich Columns Under Compressive Loads,'' Trans., Jap. Soc. of Aeronaut. Space Sci., **19**, pp. 163–178.
[10] Chong, K. P., Wang, K. A., and Griffith, G. R., 1979, ''Analysis of Continuous Sandwich Panels in Building Systems,'' *Building and Environment* 44.
[11] Frostig, Y., and Baruch, M., 1993, ''Buckling of Simply Supported Sandwich Beams With Transversely Flexible Core—A High Order Theory,'' J. Eng. Mech. **119**, pp. 955–972.
[12] Engesser, F., 1889, ''Die Knickfestigkeit gerader Stäbe,'' Zentralblatt des Bauverwaltung., **11**, p. 483–486.
[13] Engesser, F., 1889, ''Die Knickfestigkeit gerader Stäbe,'' Z. Architekten und Ing. Verein zu Hannover, **35**, p. 455.
[14] Engesser, F., 1891, ''Die Knickfestigkeit gerader Stäbe,'' Zentralblatt der Bauverwaltung, **11**, pp. 483–486.
[15] Haringx, J. A., 1942, ''On the Buckling and Lateral Rigidity of Helical Springs,'' *Proc., Konink. Ned. Akad. Wetenschap.*, **45**, p. 533.
[16] Haringx, J. A., 1948–1949, *Phillips Research Reports*, Vols. 3–4, Phillips Research Laboratories, Eindhoven.
[17] Kardomateas, G. A., 1995, ''Three Dimensional Elasticity Solution for the Buckling of Transversely Isotropic Rods: The Euler Load Revisited,'' ASME J. Appl. Mech., **62**, pp. 346–355.
[18] Kardomateas, G. A., and Dancila, D. S., 1997, ''Buckling of Moderately Thick Orthotropic Columns: Comparison of an Elasticity Solution With the Euler and Engesser/Haringx/Timoshenko Formulas,'' Int. J. Solids Struct., **34**(3), pp. 341–357.
[19] Kardomateas, G. A., 2001, ''Elasticity Solutions for Sandwich Orthotropic Cylindrical Shell Under External Pressure, Internal Pressure and Axial Force,'' AIAA J., **39**(4), pp. 713–719.
[20] Kardomateas, G. A., 2001, ''Three-Dimensional Elasticity Solutions for the Buckling of Sandwich Columns,'' *ASME Intern. Mechanical Engrg. Congress*, pp. 1–6.
[21] Kardomateas, G. A., and Huang, H., 2002, ''Buckling and Initial Postbuckling Behavior of Sandwich Beams Including Transverse Shear,'' AIAA J., in press.
[22] Kardomateas, G. A., Simitses, G. J., Shen, L., and Li, R., 2002, ''Buckling of Sandwich Wide Columns,'' Int. J. Non-Linear Mech., (special issue on ''Nonlinear Stability of Structures''), in press.
[23] Simitses, G. J., and Shen, L., 2000, ''Static and Dynamic Buckling of Sandwich Columns,'' *Mechanics of Sandwich Structures*, Y. D. S. Rajapakse et al., eds., ASME, New York, AD-Vol. 62/AMD-Vol. 245, pp. 41–50.
[24] Timoshenko, S. P., and Gere, J. M., 1961, *Theory of Elastic Stability*, McGraw-Hill, New York, pp. 135, 142.
[25] Bažant, Z. P., 1992, discussion of ''Stability of Built-up Columns,'' by A. Gjelsvik, J. Eng. Mech., **118**(6), pp. 1279–1281.
[26] Bažant, Z. P., 1993, discussion of ''Use of Engineering Strain and Trefftz Theory in Buckling of Columns,'' by C. M. Wang and W. A. M. Alwis ASME J. Appl. Mech., **119**(12), pp. 2536–2537.
[27] Ziegler, F., 1982, ''Arguments for and Against Engesser's Buckling Formulas,'' Ingenieur-Archiv, **52**, pp. 105–113.
[28] Reissner, E., 1972, ''On One-Dimensional Finite-Strain Beam Theory: The Plane Problem,'' J. of Applied Mathematics and Physics, **23**, pp. 795–804.
[29] Reissner, E., 1982, ''Some Remarks on the Problem of Column Buckling,'' Ingenieur-Archiv, **52**, pp. 115–119.
[30] Simo, J. C., and Kelly, J. M., 1984, ''The Analysis of Multilayer Elastomeric Bearings,'' ASME J. Appl. Mech., **51**, pp. 256–262.

[31] Simo, J. C., Hjelmstad, K. D., and Taylor, R. L., 1984, ''Numerical Formulation of Elasto-Viscoplastic Response of Beams Accounting for the Effect of Shear,'' Comput. Methods Appl. Mech. Eng., **42**, pp. 301–330.
[32] Gjelsvik, A., 1991, ''Stability of Built-Up Columns,'' J. Eng. Mech., **117**(6), pp. 1331–1345.
[33] Wang, C. M., and Alwis, W. A. M., 1992, ''Use of Engineering Strain and Trefftz Theory in Buckling of Columns,'' J. Eng. Mech., **118**(10), pp. 2135–2140.
[34] Attard, M. A., 2002, draft of a manuscript on ''Finite Strain Beam Theory,'' University of New South Wales, Australia (private communication to Bažant, 2002).
[35] Buckle, I., Nagarajaiah, S., and Ferell, K., 2002, ''Stability of Elastomeric Isolation Bearings: Experimental Study,'' J. Eng. Mech., **128**(1), pp. 3–11.
[36] Timoshenko, S. P., 1921, ''On the Correction for Shear in the Differential Equation of Transverse Vibrations of Prismatic Bars,'' Philos. Mag., **21**, p. 747.