Probabilistic Nonlocal Theory for Quasibrittle Fracture Initiation and Size Effect. II: Application

By Zdeněk P. Bažant,[1] Fellow, ASCE, and Drahomír Novák[2]

Abstract: The nonlocal probabilistic theory developed in Part I is applied in numerical studies of plain concrete beams and is compared to the existing test data on the modulus of rupture. For normal size test beams, the deterministic theory is found to dominate and give adequate predictions for the mean. But the present probabilistic theory can further provide the standard deviation and the entire probability distribution (calculated via Latin hypercube sampling). For very large beam sizes, the statistical size effect dominates and the mean prediction approaches asymptotically the classical Weibull size effect. This is contrary to structures failing only after the formation of a large crack, for which the classical Weibull size effect is asymptotically approached for very small structure sizes. Comparison to the existing test data on the modulus of rupture demonstrates good agreement with both the measured means and the scatter breadth.

INTRODUCTION

In the preceding Part I (Bažant and Novák 2000), a probabilistic nonlocal theory for quasibrittle structures exhibiting strain-softening damage due to cracking has been developed. In the present Part II, this theory will be applied in numerical studies of plain concrete beams, and the numerical results will be compared to the existing test data on the size effect on the modulus of rupture.

NUMERICAL STUDIES AND STATISTICAL ANALYSIS OF SIZE EFFECT

Input Data and Spatial Distributions

Three-point symmetric bending of a beam with a span-to-depth ratio $L/D = 4$ is considered first. The ratio of the modulus of rupture to the direct tensile strength, f_r/f'_t, is calculated for beam depths D spanning a very broad size range from $D = 0.01$ m (which is a hypothetical value, smaller than the maximum aggregate size) to $D = 10$ m. The following material properties are assumed: Modulus of elasticity $E = 27$ GPa, tangential softening modulus $E_t = 15$ GPa, tensile strength $f'_t = 2.8$ MPa, maximum aggregate size $d_a = 12.7$ mm (0.5 in.), characteristic length $l = 3d_a$, Weibull modulus $m = 24$, and Weibull scaling parameter $\sigma_0 = 0.9 f'_t$ (note that E_t is a function of the fracture energy of the material, G_f and l; Bažant and Planas 1998). The beam width is $b = 1$.

The selection of m is a crucial point for the statistical size effect. The higher the m-value, the milder this effect is. According to Zech and Wittmann (1977), $m = 12$. But that value was derived from the coefficient of variation of beam strength for one beam size and one shape, and was based on a rather limited set of test data. Comparisons of test data with the present numerical calculations showed that $m = 12$ gives an unrealistically strong size effect. A detailed study justification is given in a forthcoming paper (Bažant and Novák 1999).

Because of symmetry, only one-half of the beam needs to be analyzed. It is subdivided by a regular rectangular 50×50 mesh. To achieve the prescribed failure probability (target) $p_{f,t}$, the calculations were iterated, starting with the initial estimate of load P and observing the estimated lower and upper limit on load P. The criterion to terminate the iterations was $|p_{f,t} - p_f|/p_{f,t} \leq 0.02$. The number of iterations for reasonable heuristically selected initial starting loads was about 10. The starting load values P_1 and P_2 should be selected to satisfy two conditions: $p_{f,1}\ (P_1) \leq p_{f,t}$ and $p_{f,2}(P_2) \geq p_{f,t}$. Iterations (in which the load is varied according to the values of the probabilities) are then performed until the desired accuracy is achieved.

Fig. 1(a–c) shows 3D plots of the calculated distributions of stresses $\bar{\sigma}(x, y)$, $\sigma''(x, y)$ and $\varepsilon''(x, y)\ E$ over the beam of size $D = 0.1$ m for the three alternatives. Note that, despite the negativeness of strain in the compression part, the nonlocal inelastic stress can in the same place be positive because of averaging. Fig. 1(d–f) shows the corresponding plots of the integrand of formula (6) of Part I representing the density of contribution to failure probability from various points of the beam. Obviously by far the highest contribution comes from the small region near the tensile face and near the midspan. Note that, from the viewpoint of visualization, a sharp boundary of the contribution appears as a result of using a logarithmic scale for p_f.

Median of Modulus of Rupture and Its Size Effect

The most simple statistic to calculate is the median of the modulus of rupture. To this end, the failure probability is specified as $p_f = 0.5$, and the load P and corresponding f_r are obtained by iterative calculations. The values of the relative boundary layer thickness l_f/D at midspan for averaging alternative I are plotted as a function of the relative beam depth D/l in Fig. 2. For small sizes D, l_f reaches almost half of the beam depth. For sizes $D \geq 16l$ (or 0.6 m), approximately, l_f is negligible, which means that the size effect should become predominantly statistical because the peak load is reached when the cracking starts, and no significant stress redistribution occurs prior to the peak load. For smaller D/l, the size effect is predominantly deterministic, being caused mainly by stress redistribution that is a consequence of a large thickness of the boundary layer.

The dimensionless ratio of the median of modulus of rupture f_r to the standard tensile strength, f'_t is plotted in Figs. 3(a and b) for the three types of averaging as a function of the dimensionless relative size D/l. It can be seen that, for all the alternatives, the statistical size effect begins to dominate for relative sizes, approximately, $D/l \geq 16$ (or $D \geq 0.6$ m). For very large sizes, it asymptotically approaches the classical Weibull size effect represented in a doubly logarithmic plot by an asymptote of slope $-n_d/m = -2/24 = -1/12$ (n_d = number of

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[1]Walter P. Murphy Prof. of Civ. Engrg. and Mat. Sci., Northwestern Univ., Evanston, IL 60208.

[2]Assoc. Prof., Inst. of Struct. Mech., Facu. of Civ. Engrg., Tech. Univ. of Brno, 66237 Brno, Czech Republic; formerly, Visiting Fulbright Scholar, Northwestern Univ.

Note. Associate Editor: Gilles Pijaudier-Cabot. Discussion open until July 1, 2000. Separate discussions should be submitted for the individual papers in this symposium. To extend the closing date one month, a written request must be filed with the ASCE Manager of Journals. The manuscript for this paper was submitted for review and possible publication on August 20, 1999. This paper is part of the ***Journal of Engineering Mechanics***, Vol. 126, No. 2, February, 2000. ©ASCE, ISSN 0733-9399/00/0002-0175–0185/$8.00 + $.50 per page. Paper No. 21662.

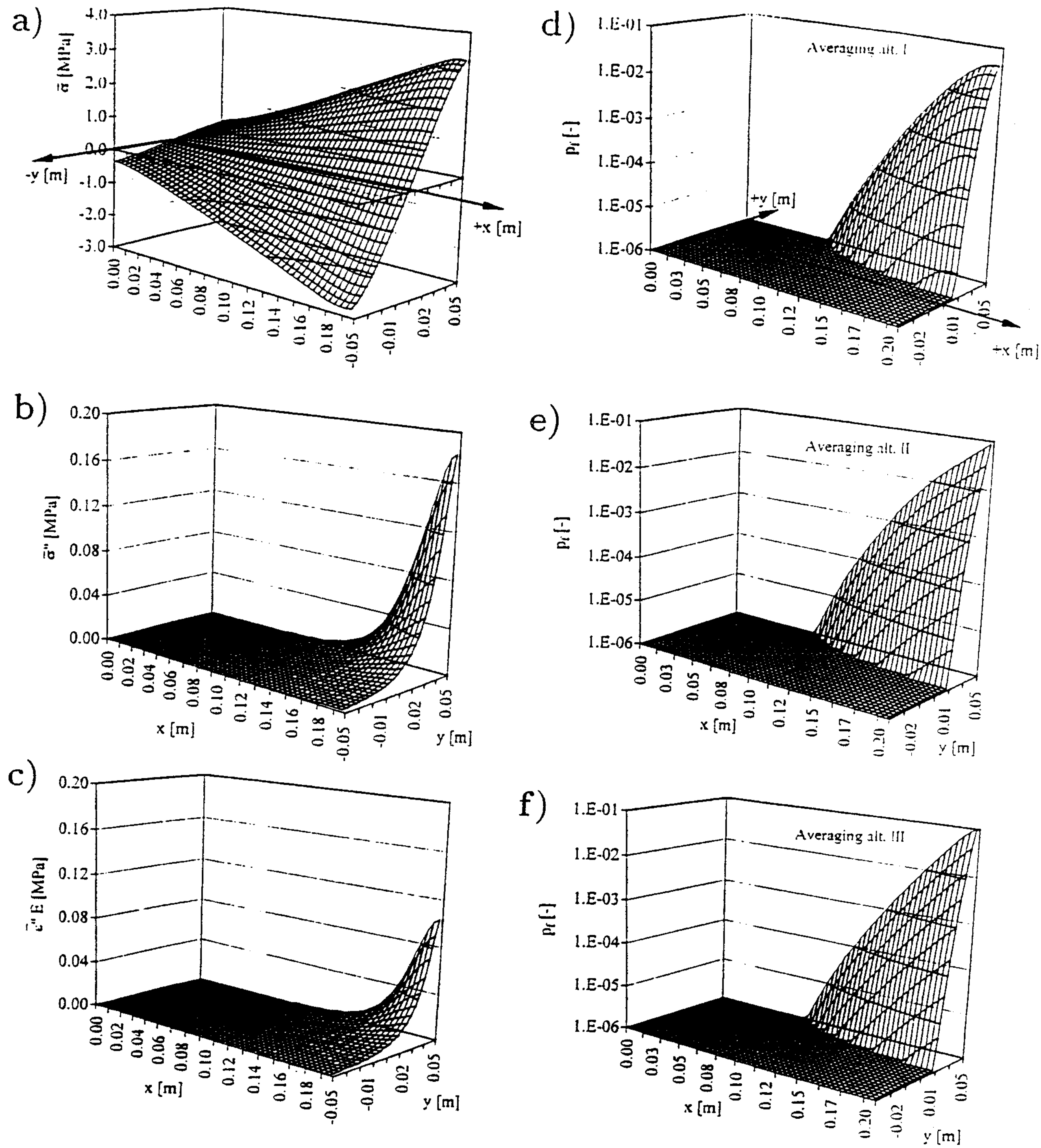

FIG. 1. (a–c) 3D Views of Distributions of Nonlocal Averaged Strain $\bar{\varepsilon}$ Times E, Inelastic Stress $\bar{\sigma}''$, and Inelastic Strain $\bar{\varepsilon}''$ Times E; (d–f) Density Distribution of Corresponding Contribution to Failure Probability, Given by Integrand of Formula (6) of Part I

dimensions). For large sizes, the ratio f_r/f'_t drops below 1, which is the asymptotic value for the deterministic size effect models of Bažant and Li (1995) and Planas et al. (1995). The range of sizes shown in the figures is deliberately far broader than the range of practical interest. In the range, approximately 0.1 m $\leq D \leq$ 0.6 m or 2.6 $\leq D/l \leq$ 16, the present statistical analysis gives about the same results as the deterministic formula, which means that the role of randomness of strength is negligible.

Among the three types of averaging, the closest to the deterministic prediction are alternatives II and III. These alternatives resulted in almost the same size effect curves, with negligible differences. The size effect curves obtained for the three averaging alternatives are plotted in Fig. 3(c). All the curves have been constructed from points calculated for the sizes D = 0.01, 0.05, 0.1, 0.2, . . . 0.9, 1, 5, and 10 m.

In one sense, the role of material strength randomness is opposite to that established for structures that are either notched or fail after large stable crack growth. In that case, not only the size effect for normal sizes but also the asymptotic size effect for large sizes is predominantly deterministic, while the statistical size effect dominates only for extremely small (hypothetical) sizes below the normal size range (Bažant and Xi 1991; Bažant and Planas 1998).

The physical reason for this difference is that, in the case of a preexisting crack or notch, there is virtually no chance for the final fracture to occur away from the notch tip or crack tip (whereas in the present case of a smooth surface with a

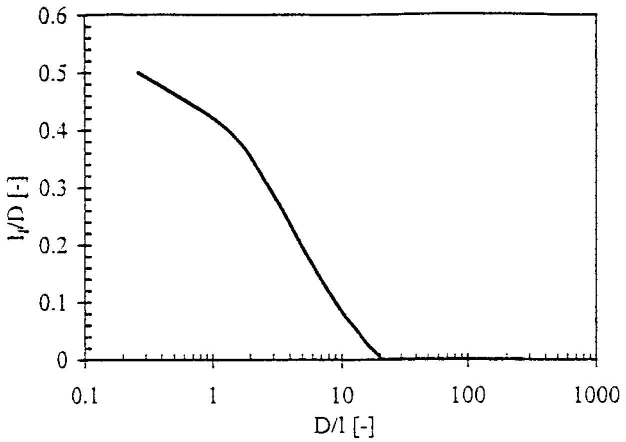

FIG. 2. Thickness l_f of Boundary Layer versus Beam Depth D

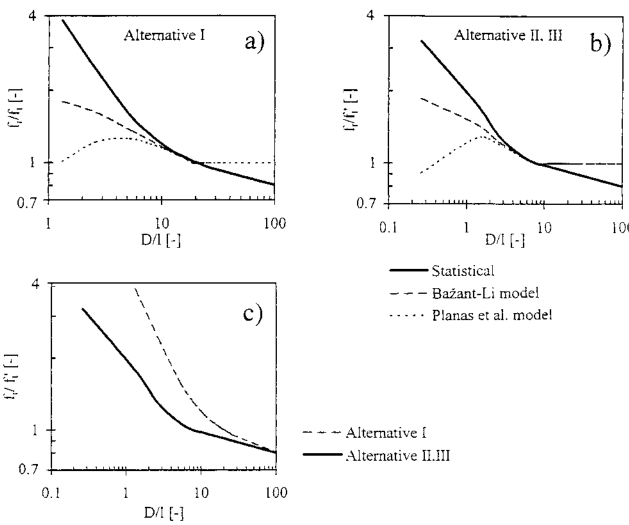

FIG. 3. Median of Modulus of Rupture versus Relative Size D/l. (a) Averaging Alternative I; (b) Averaging Alternatives II and III; (c) Comparison of Averaging Alternatives

boundary layer of cracking, the final fracture can initiate in a relatively large region). Attached to the tip is a fracture process zone whose size is approximately constant. Because of the stress concentration near the tip, the failure probability integral receives a significant contribution only from this fracture process zone, and since the size of this zone is independent of D, there can be no appreciable size effect due to material strength randomness, except when the structure is smaller than the fully developed process zone (Bažant and Xi 1991; Bažant and Planas 1998).

Probability Distribution Function

The Weibull-type integral makes it possible to estimate the failure probabilities corresponding to different load levels. Covering the full range of probabilities, one can estimate the probability distribution function for the modulus of rupture. Proper load levels are such that the entire range of the cumulative probability distribution function from 0 to 1 could be covered almost regularly. Thus, it is efficient to use the idea of the stratified sampling called Latin hypercube sampling (McKay et al. 1979), whose effective utilization was suggested by Novák et al. (1997). The range of the probability distribution function from 0 to 1 is divided into N equal nonoverlapping intervals of equal probability $1/N$ (N is thus the sample size). The centroids of these intervals are then used to get the values of the sample. The advantage of this strategy is obvious: The regularity of the probability intervals on the probability distribution function ensures good quality sampling, even for a very small sample size. This approach is known to lead to relatively good estimates of the statistical characteristics. It appears more efficient than selecting many load levels, even though the former approach, unlike the latter, necessitates iteration for every value of target failure probability. The failure probability values are chosen as

$$p_f(i) = \frac{2i - 1}{2N} \quad (i = 1, 2, \ldots N) \tag{1}$$

The sample size $N = 16$ has been chosen for calculations. In the case of Latin hypercube sampling, this represents a sufficient sample for obtaining good-quality estimates of basic statistical characteristics. Note that this sampling method is not used fully here: Only one random variable is characterized by samples of equal probability contents corresponding to 16 different probabilities, which are taken as the input into the nonlocal Weibull model.

The probability distribution functions of the ratio of modulus of rupture to strength, calculated for averaging alternative III, are plotted in Fig. 4 for different sizes. As expected, the steepness increases with increasing size, which means that the scatter decreases with size. This agrees with the well-known fact that the statistical correlation of strength imposed by averaging has a major influence only for small sizes. The stronger the correlation on the input, the greater the statistical variability on the output, as known from reliability theory (because random deviations from the mean in a given set of variables have less of a chance to cancel each other if they are correlated). Such trends for the distribution functions were already in general sketched by Shinozuka (1972) (although no mutual dependence of the strengths of the reference volume elements was considered in his early work).

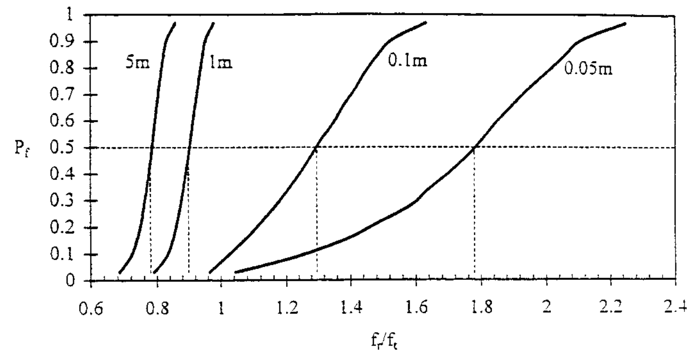

FIG. 4. Probability Distributions of Modulus of Rupture for Different Sizes of D

Mean Value, Variance, Coefficient of Skewness

The aforementioned Latin hypercube samples establishing the distribution functions for the modulus of rupture ratio have been statistically evaluated. The basic statistics of these samples are shown here for the sizes 0.05, 0.1, 1.0, and 5 m, corresponding to D/l = 1.3, 2.6, 26, and 130.

Fig. 5(a) shows the mean values for the modulus of rupture ratio as a function of the relative size D/l. The medians, obtained directly for $p_f = 0.5$, are also plotted. Their differences from the means are seen to be insignificant.

The statistical variability as characterized by the standard deviation is plotted in Fig. 5(b). A decreasing trend with the size can be observed that is the logical result of disappearing statistical correlation for large sizes, as already mentioned. In spite of the small sample size (N = 16), the coefficients of skewness have also been calculated [Fig. 5(c)].

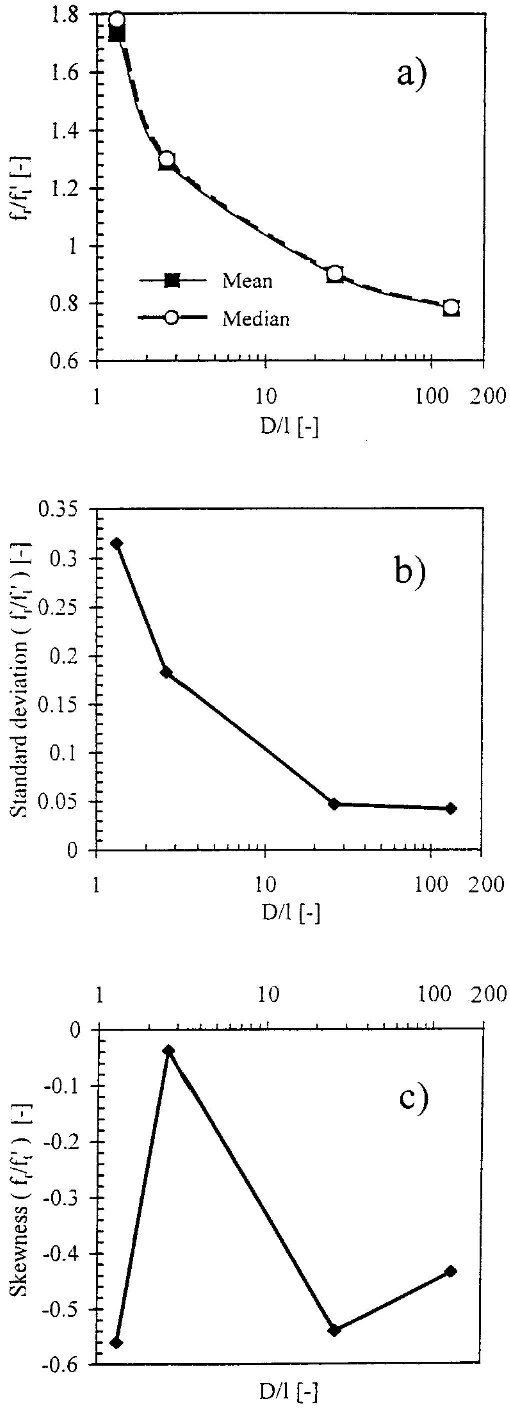

FIG. 5. (a) Mean and Median of Modulus of Rupture versus Relative Size *D/l*; (b) Standard Deviation of Modulus of Rupture versus Relative Size *D/l*; (c) Coefficients of Skewness of Modulus of Rupture versus Relative Size *D/l*

Statistical Size Effect for Different Span-to-Depth Ratios

Fig. 6(a) shows the median of the modulus of rupture as a function of the relative size D/l for various slenderness (span-to-depth ratios) L/D. As L/D increases, the modulus of rupture decreases. But the decrease is mild; compared to the size effect of the depth D, it is very small.

Statistical Size Effect for Different Types of Loading

In addition to the three-point bending, beams with four-point bending (with two loads at distance $L/3$ from the supports), as well as beams with a uniform distributed load p,

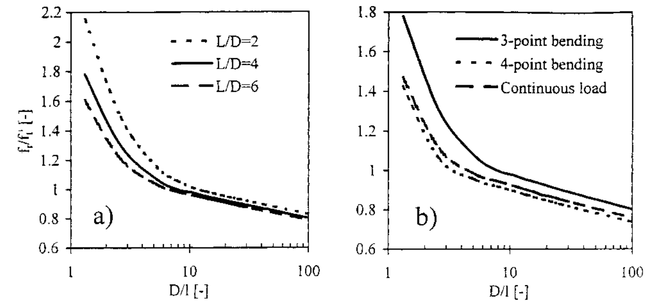

FIG. 6. (a) Median of Modulus of Rupture for Different Span-to-Depth Ratios *L/D*; (b) Median of Modulus of Rupture for Different Types of Loading

have been analyzed using the averaging alternative III. The results for the median modulus of rupture are shown in Fig. 6(b). The three-point bending yields the maximum ratios f_r/f'_t. This agrees with the observations in laboratory tests with three-point and four-point bending, for example, those of Wright (1952).

COMPARISON WITH EXISTING EXPERIMENTAL DATA

The present theory has further been compared with most important data sets found in the literature. They are described in Appendix I and listed in Tables 1 and 2. The averaging alternative III was again used; it agrees with test data better than alternative I. The corresponding values calculated by the present theory are also listed in the tables. The data points for the medians and for the 5th and 95th percentiles directly estimated from the various data sets reported in the literature are plotted in Figs. 7 and 8.

Fig. 7 shows asymptotic behavior beyond test data sizes (same vertical axis scale); in Fig. 8, the range of test data is zoomed and only ranges of reasonable sizes are plotted (different vertical axis scale). The calculated curves of the corresponding values seen in the figures reveal a satisfactory agreement, not only for the medians but also for the statistical measures of random scatter. It should be emphasized that the curves were obtained by numerical analysis simulating the available test data rather than by regression of experimental results.

The comparison of published experimental results with the nonlocal Weibull theory results is made for $p_f = 0.5$ (median) (the mean value is inconvenient, as it necessitates a tedious calculation of the points on the cumulative distribution function). The 5th and 95th percentiles of the measured values of the modulus of rupture were approximately obtained as the mean ± 1.64 standard deviation under the assumption of normal probability distribution (the means and standard deviations were based on reported data). In the case of nonlocal Weibull theory, the calculations were made for prescribed probabilities 0.05 and 0.95. Despite many uncertainties in both the experiments and the computations, the experimental and calculated statistics characterizing the limit scatter agree reasonably well.

Prediction of the classical, purely statistical size effect of the Weibull type is one benefit of the present theory. It is practically relevant only for the bending of very large unreinforced concrete structures such as arch dams, foundations, and earth-retaining structures (this was already suggested by Petersson, 1981, based on his analysis of size effect on the modulus of rupture with the cohesive crack model). By contrast, the prediction of the probability distribution that characterizes the random scatter of the modulus of rupture is relevant also to the typical sizes of unreinforced beams encountered in practice.

An important new source of statistical information are Koide et al.'s (1998) and Koide's (personal communication,

TABLE 1. Mean and 5th and 95th Percentiles of Modulus of Rupture for Various Test Data, Compared to Median and Percentiles Calculated by Present Theory for Various Beam Depths

Size D(mm)	TEST 5%	TEST Mean	TEST 95%	THEORY 5%	THEORY Median	THEORY 95%
Reagel & Willis (1931), 4-point bending						
10				11.00	14.00	
50				5.24	9.47	11.11
101.6	5.41	5.94	6.47	4.91	6.66	8.31
152.4	5.35	5.74	6.13	4.76	5.62	7.09
203.2	5.15	5.45	5.75	4.66	5.21	6.27
254	4.59	5.26	5.57	4.58	5.12	5.96
500				4.29	4.81	5.15
1000				4.05	4.53	4.84
Wright (1952), 3-point bending						
10				6.62	9.08	10.27
50				3.01	5.14	6.22
76.2	3.47	4.13	4.79	2.51	4.20	5.05
101.6	3.51	3.82	4.13	2.36	3.59	4.43
152.4	2.66	2.96	3.26	2.25	2.98	3.68
203.2	2.53	2.76	2.99	2.19	2.70	3.32
500				2.03	2.27	2.53
1000				1.91	2.14	2.28
Wright (1952) , 4-point bending						
10				5.20	7.11	7.79
50				2.48	4.07	4.80
76.2	2.88	3.21	3.54	2.19	3.29	4.09
101.6	2.77	2.94	3.11	2.14	2.90	3.54
152.4	2.30	2.60	2.90	2.07	2.43	2.96
203.2	2.19	2.31	2.43	2.01	2.25	2.67
500				1.86	2.09	2.24
1000				1.75	1.97	2.10
Nielsen (1954), 3-point bending						
10				3.55	5.31	6.09
50				2.83	4.25	5.23
100	3.29	3.57	3.85	2.73	3.66	4.59
150	2.90	3.16	3.42	2.68	3.37	4.16
200	3.01	3.30	3.59	2.65	3.21	3.94
500				2.55	2.90	3.38
1000				2.47	2.77	3.11
Lindner & Sprague (1956), 4-point bending						
10				8.52	10.26	11.50
50				4.26	7.57	9.07
152.4	3.53	4.48	5.43	3.64	4.54	5.66
228.6	3.27	4.07	4.87	3.51	3.94	4.72
304.8	3.05	3.93	4.81	3.42	3.84	4.32
457.2	3.40	3.79	4.12	3.31	3.70	3.96
1000				3.09	3.46	3.70
Walker & Bloem (1957), 4-point, da = 1 in.						
10				8.84	11.71	12.90
50				4.26	7.66	9.20
101.6	4.39	4.70	5.01	3.73	5.16	6.42
152.4	4.22	4.50	4.78	3.64	4.55	5.60
203.2	4.07	4.25	4.43	3.55	3.97	5.03
254	4.07	4.27	4.47	3.48	3.90	4.62
500				3.28	3.67	3.93
1000				3.09	3.46	3.70
Walker & Bloem (1957), 4-point, da = 2 in.						
10				7.00	11.22	12.35
50				4.85	7.90	9.70
101.6	4.06	4.68	5.30	3.27	5.48	6.87
152.4	4.06	4.34	4.62	3.19	4.82	6.10
203.2	3.71	4.15	4.59	3.11	4.00	5.31
254	3.43	3.74	4.05	3.05	3.61	4.77
500				2.88	3.22	3.61
1000				2.71	3.03	3.25
Sabnis & Mirza (1979), 4-point bending						
10		8.8		7.08	9.12	10.34
19.1		6.9		5.30	7.95	9.36
38.1		5.6		3.83	6.03	7.35
50				3.65	5.39	6.53
76.2		4.8		3.52	4.47	5.43
152.4		4.3		3.31	3.71	4.30
500				2.99	3.35	3.59
1000				2.81	3.15	3.37
Rokugo (1995), 4-point bending						
10				5.10	6.95	8.35
50		4.35		3.13	5.08	6.28
100		4.04		2.95	3.73	4.55
200		3.66		2.78	3.11	3.60
300		3.46		2.68	3.01	3.30
400		3.30		2.62	2.93	3.13
500				2.53	2.87	3.07
1000				2.42	2.70	2.90
Rocco (1997), 3-point bending						
10				8.40	10.33	12.65
17	6.34	7.04	7.74	6.54	8.69	11.01
37	6.27	6.52	6.77	4.92	6.31	8.11
75	5.16	5.60	6.03	4.48	4.97	6.35
150	4.78	5.12	5.47	4.22	4.34	5.34
300	4.60	4.67	4.75	3.97	4.03	4.80
500				3.80	3.87	4.56
1000				3.57	3.64	4.28

1999) three series of tests of 279 plain concrete beams in four-point bending, aimed at determining the influence of the beam length L on the flexural strength of beams of three depths D (Appendix I). Unfortunately, no tests of tensile strength and modulus of elasticity have been reported, and so their values are estimated as follows: $f_t' = 2.8$ MPa for series A and B, $f_t' = 2.2$ MPa for series C; modulus of elasticity $E = 35$ GPa with softening modulus $E_t = 10$ MPa for series A and B; and $E = 30$ GPa with $E_t = 8$ MPa for series C.

Koide's excellent data allow comparing the probability distribution function of maximum bending moment M_{max} corresponding to failure load over its full range (Koide, personal communication, 1999). The measured mean values and the calculated means obtained by nonlocal Weibull simulation are compared in Fig. 9. To calculate the probability density functions by the nonlocal approach, the efficient method of Latin hypercube sampling was again adopted. To cover the probability range (0, 1) efficiently, it was divided into 16 intervals of equal probability content, characterized by 16 prescribed probability values. A good agreement with Koide et al.'s (1998) data has been achieved. The calculations indicate a decrease of the flexural strength as the span increases. Information on the size effect, however, is missing since only one cross-section size was used by Koide et al. (1998).

The data points in Fig. 10 show the empirical cumulative probability density functions for three different spans obtained from all the series of test beams of Koide et al. (1998) and Koide (personal communication, 1999), and the solid lines show the corresponding results of the present probabilistic nonlocal theory. As can be seen, the calculated probability density functions exhibit trends similar to those of data points. These functions were obtained by optimizing only the values of tensile strength f_t' and Weibull scale parameter σ_0, the latter being restricted to the range $0.9 f_t' \leq \sigma_0 \leq 1.15 f_t'$. The Weibull modulus was not optimized but kept as $m = 24$, which is a value suitable for all concretes on the average (as shown in

TABLE 2. Koide's (1999) Test Data on Maximum Loads in kN (Reprinted with Permission)

Test No	Series A Bending span 5cm	7cm	9cm	Series B Bending span 5cm	10cm	20cm	Series C Bending span 20cm	40cm	60cm
1	2.340	1.773	1.833	5.559	6.585	5.565	7.251	6.742	6.677
2	2.127	2.150	1.643	5.169	5.125	5.310	7.546	7.132	6.807
3	2.000	1.971	1.765	6.205	5.814	5.919	7.713	7.031	6.424
4	2.164	2.200	1.895	5.596	6.335	5.718	6.609	6.591	7.090
5	1.936	2.172	1.666	5.986	6.025	5.119	7.567	6.716	5.861
6	1.631	2.086	1.987	5.809	5.498	5.411	6.937	6.338	6.786
7	1.985	1.732	2.041	6.191	5.785	5.393	8.140	6.559	6.603
8	2.323	1.634	2.076	5.910	7.093	5.153	7.530	7.091	6.673
9	2.008	1.935	2.110	6.118	6.103	5.879	7.346	7.662	6.436
10	2.090	2.004	2.009	6.147	5.698	5.306	6.638	6.069	4.958
11	1.840	1.895	1.880	5.919	6.724	5.399	7.721	4.378	5.072
12	2.050	1.788	1.751	6.687	5.966	5.407	7.494	6.384	5.957
13	1.998	1.876	2.233	7.015	6.708	4.813	7.703	6.659	6.210
14	2.070	1.894	1.722	6.482	7.066	5.708	7.656	5.184	4.896
15	2.475	1.791	2.046	6.218	6.831	5.717	7.418	5.928	4.948
16	2.135	2.130	2.125	5.949	6.017	6.271	7.813	5.374	5.112
17	2.234	1.855	1.998	7.126	6.103	6.523	6.569	5.082	6.000
18	2.380	2.064	2.046	6.173	6.305	6.102	7.433	6.223	5.745
19	2.447	1.943	1.623	6.390	7.099	6.080	7.569	5.680	4.449
20		2.154	1.833	6.710	5.801	5.943	7.269	5.867	5.927
21		2.160	.815	6.927	5.480	6.651	5.942	6.447	5.609
22		1.868	1.733	6.869	6.255	6.099	6.686	6.679	5.297
23		2.210	1.936	7.311	6.203	5.735	5.294	6.777	5.897
24		1.999	1.749	6.172	5.823		6.020	6.220	6.498
25			1.991	6.544	5.839		6.406	6.191	6.412
26			2.060	5.826	6.279		7.137	6.662	5.694
27			1.983	6.786	6.427		6.731	6.402	5.809
28			1.570	7.436	6.219		6.099	6.525	6.253
29			1.889	6.141	6.136		5.940	5.452	5.634
30			1.898	5.344			6.241	6.280	7.442
31			1.874	6.435			6.496	6.100	6.358
32				6.495			6.475	6.535	6.665
33							6.646	6.837	6.998
34							6.425	7.732	6.798
35							6.930	6.147	6.504
36							7.122	7.849	
37							6.571	6.668	
38							7.536	6.668	
39							7.873	6.506	
40							6.689	7.313	
41							6.873		
42							7.071		
43							6.962		
44							7.734		
45							7.458		
46							7.567		

Bažant and Novák 1999). The elastic modulus E and the softening modulus E_t were not optimized either. The statistics for the data and the theory are summarized in Table 3.

The values of Weibull modulus m and scale parameter σ_0 are found to have a paramount influence on the statistical scatter, particularly on the shape of the probability density function. The measured probability density functions could be matched better, and in fact very closely, if the m and σ_0 values were also optimized for Koide et al.'s (1998) data set. But the intent was to show how close the match is when m and σ_0 values suitable for all concretes on the average are used (m = 24).

Modulus of rupture tests were performed by the Portland Cement Association (PCA) (1966) in order to study the effect of slenderness L/D and the differences between three types of beam bending: three-point, four-point, and cantilever. Even though no detailed information on the concrete properties, beam geometry, supports, load application, number of tests, and statistics was reported, the information on the differences in f_r between these three types of loading is unique and rather interesting, and so it has been decided to check whether it can be explained by the present theory.

The results of the fitting of the PCA (1966) data are shown in Fig. 11, where the solid lines represent the medians and the dashed lines the 5th and 95th percentiles. As seen, the difference between the three- and four-point loading is well captured by the present statistical theory (while the deterministic theory, $m \to \infty$, predicts no difference).

Comparing the numerical simulation of three-point and cantilever loading, in the former case the nonlocal averaging volume protrudes into the other symmetric half of beam, but in the latter case it does not protrude into the support. This leads to only a minute difference in the response curves, as seen in Fig. 11. The difference in the reported test data is much larger, in fact too large for being matched by the present theory. However, this difference is well within the calculated scatter band bounded by the 5th and 95th percentiles, and so it may well be assigned to the scatter of test data, especially since the number of tests was probably very small. On the other hand, this difference might result from inadequate supports causing different stress concentrations or from various other systematic differences.

SUMMARY AND CONCLUSIONS FROM PARTS I AND II

1. In the nonlocal generalization of Weibull theory for quasibrittle materials such as concrete, previously proposed by Bažant and Xi (1991) for notched specimens or structures with a large crack at the moment of failure, the failure probability of a small material element is a function of nonlocal (spatially averaged) continuum variables rather than the local stress. This generalization can also be applied to unnotched specimens or structures failing at the initiation of macroscopic fracture, and in particular to the test of modulus of rupture (flexural strength).
2. The nonlocality is needed not only to prevent spurious localization of cracking but also to introduce spatial correlation of random material strength, governed by a certain finite characteristic length of the material.
3. As in the previous model for structures containing notches or large cracks at the moment of failure, the size effect on the mean or median of modulus of rupture is, for normal size beams, essentially deterministic. However, the size range in which the statistical size effect dominates is different from that in the previous model—it is the asymptotic range of very large sizes, which are often beyond the range of practical interest, while in the previous model it is the range of very small sizes that happens to lie below the range of practical interest.
4. Compared to the existing stochastic finite-element approaches, great simplification is achieved by the fact that the nonlocal structural analysis with strain softening can be deterministic because the probability analysis is separated from the stress analysis, in a manner similar to the classical Weibull theory. Yet, just as in these existing approaches, the present approach is general and can be applied not only to quasibrittle failures occurring at crack initiation (as in the modulus of rupture test), but also to quasibrittle failures occurring after long stable crack growth (typical for reinforced concrete structures or dams). Thus, the statistical theory of both becomes unified.
5. Nonlocality is required to avoid the spurious localization and mesh sensitivity inferred from strain softening, and to introduce spatial correlation. Three simple alternatives for the nonlocality are studied. The failure prob-

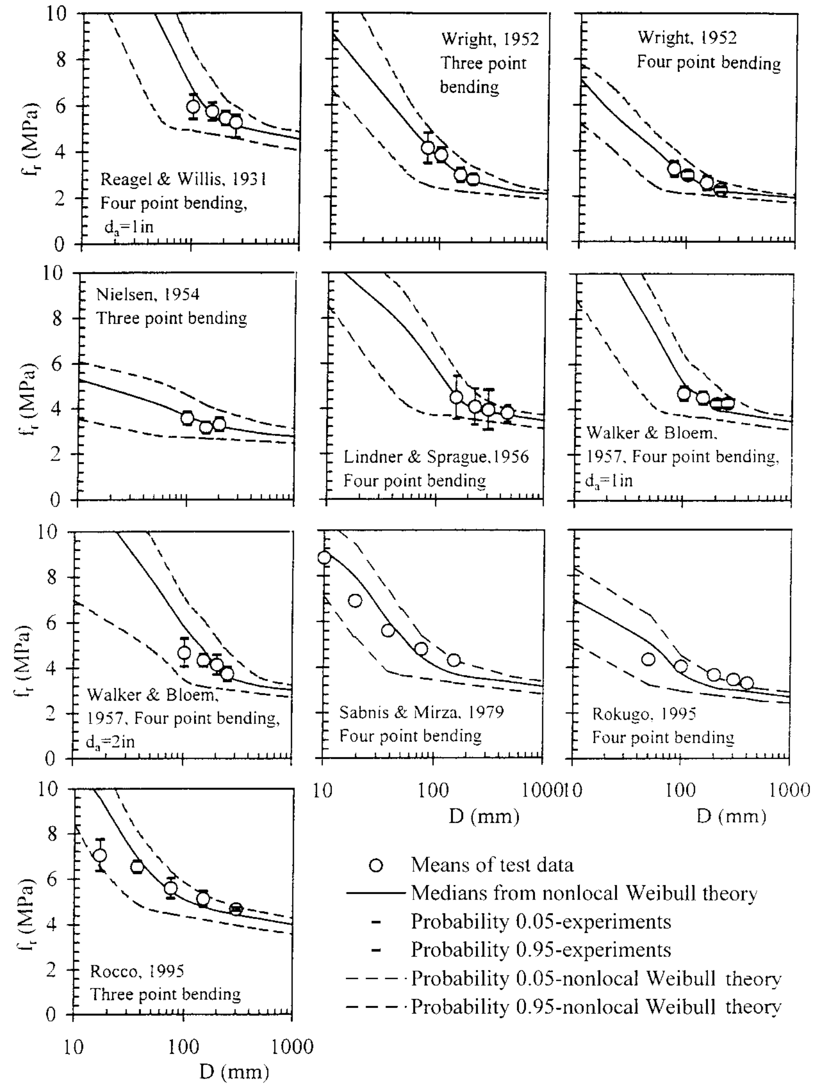

FIG. 7. Data Points for Means, and for 5th and 95th Percentiles, from Various Test Data in the Literature, Compared to Corresponding Curves Calculated for Very Broad Size Range

ability of a small material element is considered to depend, in the same manner as in Weibull theory, on (alternative I) the strain averaged over a certain neighborhood whose size is determined by the characteristic length of the material; (alternative II) the averaged inelastic stress; or (alternative III) the averaged inelastic strain. All three give similar results, but the third yields the mildest size effect and seems closest to the test data. For alternatives II and III, the size effect is quite similar, but slightly stronger for alternative I.

6. The redistribution of stresses due to strain softening in the boundary layer of cracking may be approximately taken into account in the manner of the deterministic size effect model of Bažant and Li (1995), based on the hypothesis of plane cross sections.
7. The present model agrees well with the test data sets found in the literature.
8. The main benefit of the present theory is the possibility of predicting, for various structure sizes (and shapes), the full probability distribution of structural strength, and in particular the modulus of rupture. Examples of calculating the 5th and 95th percentile probabilities of structural failure are given.
9. The calculated size dependence confirms that standard deviation characterizing the scatter of the modulus of rupture decreases with increasing beam size.
10. As a fundamental check of soundness, the classical Weibull theory with weakest link model (extreme value distribution) should be recovered as the asymptotic limit when the size of a quasibrittle or strain-softening structure tends to infinity. Satisfying this requirement has been the main objective in developing the present theory. The stochastic finite-element method, however, does not satisfy this basic requirement, which casts doubts on its applicability, especially when one needs the load of a very small failure probability, requiring the use of extreme value distribution.

APPENDIX I. INFORMATION ON TEST DATA USED

Reagel and Willis (1931). Span length l = 0.4572, 0.6096, 0.7620, 0.9144 m (18, 24, 30, 36 in.). The span/depth ratio was not constant, but varied from 4.5 to 3.6. Width $b = D$ (square cross sections); depth D = 10.16, 15.24, 20.32, 25.4 cm (4, 6, 8, 10 in.). The direct tensile strength f_t' = 5.3 MPa, modulus of elasticity E = 50.00 GPa, and softening modulus E_t = 35.00 GPa had to be intuitively estimated here. Maximum

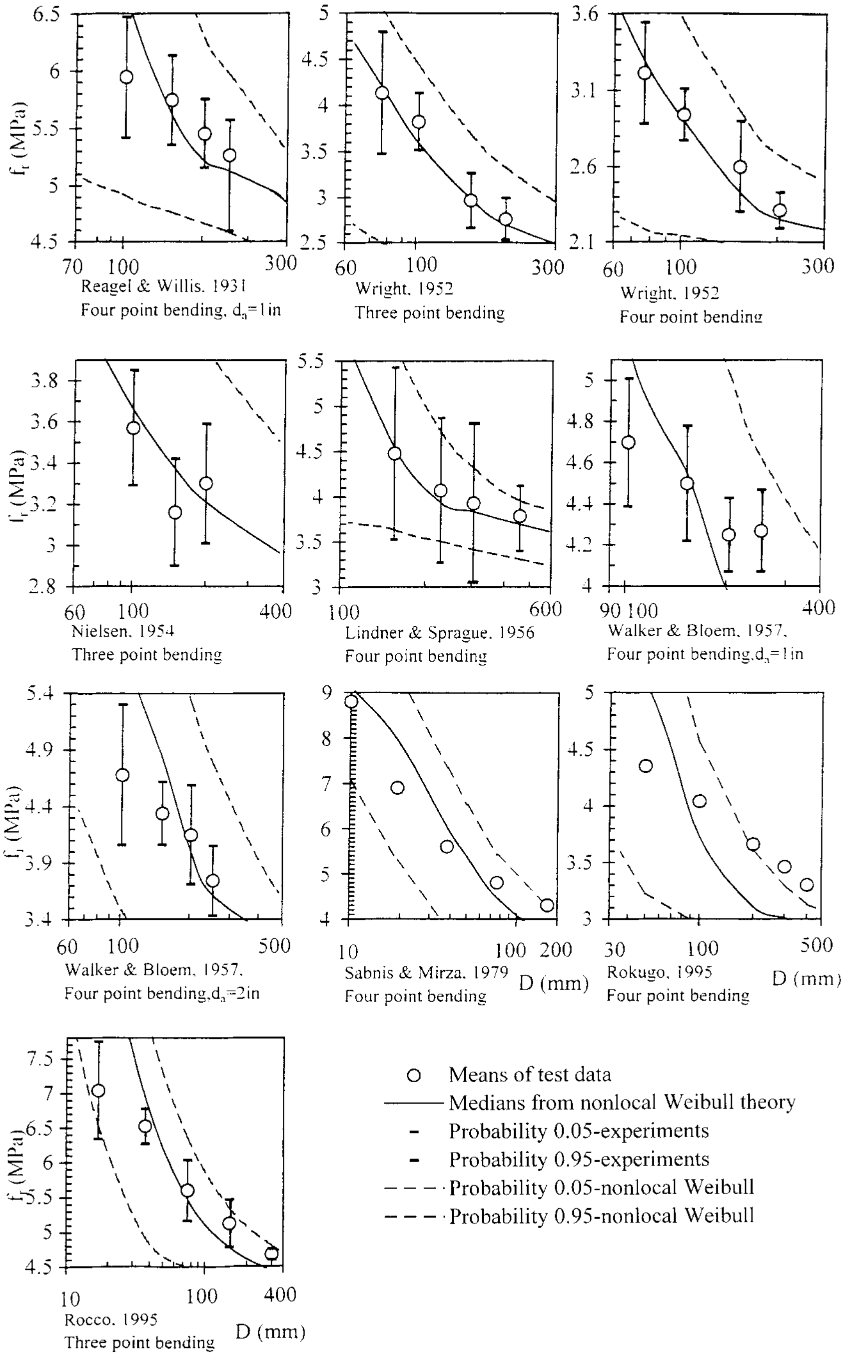

FIG. 8. Data Points for Means and for 5th and 95th Percentiles, from Various Test Data in Literature, Compared to Corresponding Curves Calculated for Range of All Test Data Only (Same as Fig. 7, but Zoomed)

aggregate size d_a = 25.4 mm (1 in.); number of test specimens for each depth = 64.

Wright (1952). Span length l = 0.2286, 0.3048, 0.4572, 0.6096 m (9, 12, 18, 24 in.). Only the results for a constant span-depth ratio l/D = 3 are considered here; width $b = D$; square cross sections; depth D = 7.62, 10.16, 15.24, 20.32 cm (3, 4, 6, 8 in.). The direct tensile strength f_t' = 2.3 MPa, modulus of elasticity E = 40.00 GPa, and softening modulus E_t = 30.00 GPa all had to be intuitively estimated for the present analysis. Maximum aggregate size d_a = 19.05 mm (3/4 in.), river gravel; number of test specimens for each depth = 5 or 6.

Nielsen (1954). Span length l = 1.0 m (kept constant); width b = 0.15 m (kept constant); beam depths D = 10, 15, 20 cm. The direct tensile strength f_t' = 2.8 MPa was estimated from the compressive strength at 27 days; f_c' = 47.3 MPa by the CEB-FIP formula $f_t' = 0.22\ (f_c')^{2/3}$. Modulus of elasticity E = 40.6 GPa, measured at 27 days [the mean value from all results obtained as (40.8 + 40.9 + 40.1)/3 GPa]. The softening modulus E_t = 27.0 GPa was estimated assuming E/E_t = 1.5. Maximum aggregate size d_a = 26 mm was estimated from the published grading curve of the aggregate. The number of test specimens for each depth = 4. Curing conditions: 21 days in water and 7 days in air. The mean values were obtained from only four measured values for each depth. It must be stressed that geometrical similarity was not maintained in these tests as the beam length was kept constant (l = 1 for all the small and large sizes). The calculations were made for span-to-depth ratios D/l, varying from 1 to 100. This is doubtless the reason that the statistical size effect for large sizes is less pronounced.

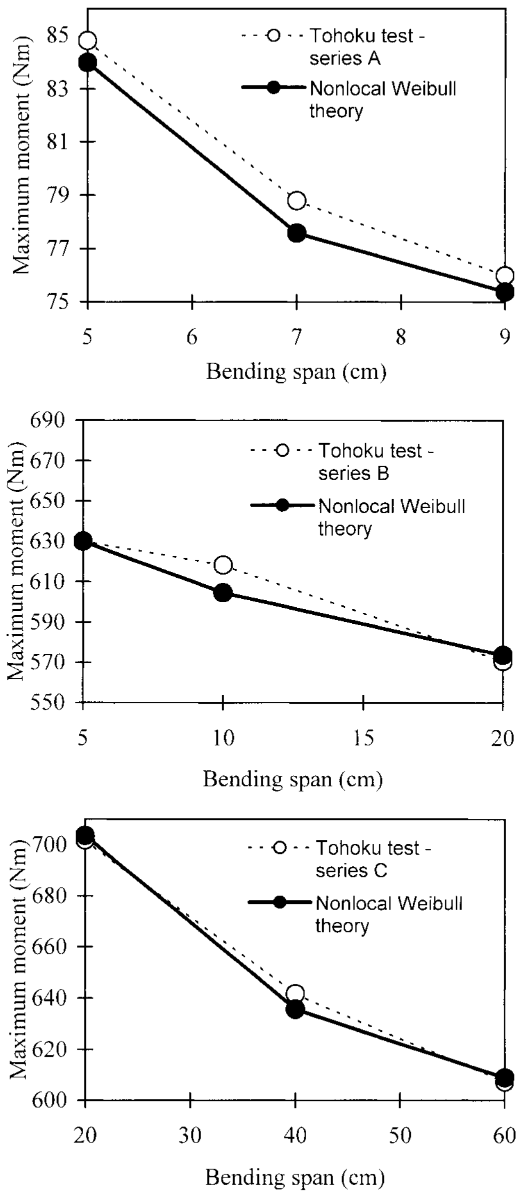

FIG. 9. Comparison of Means of Koide's (Personal Communication, 1999) Three Series of Four-Point Bending Tests and Means of Nonlocal Weibull Calculation, for Different Bending Spans

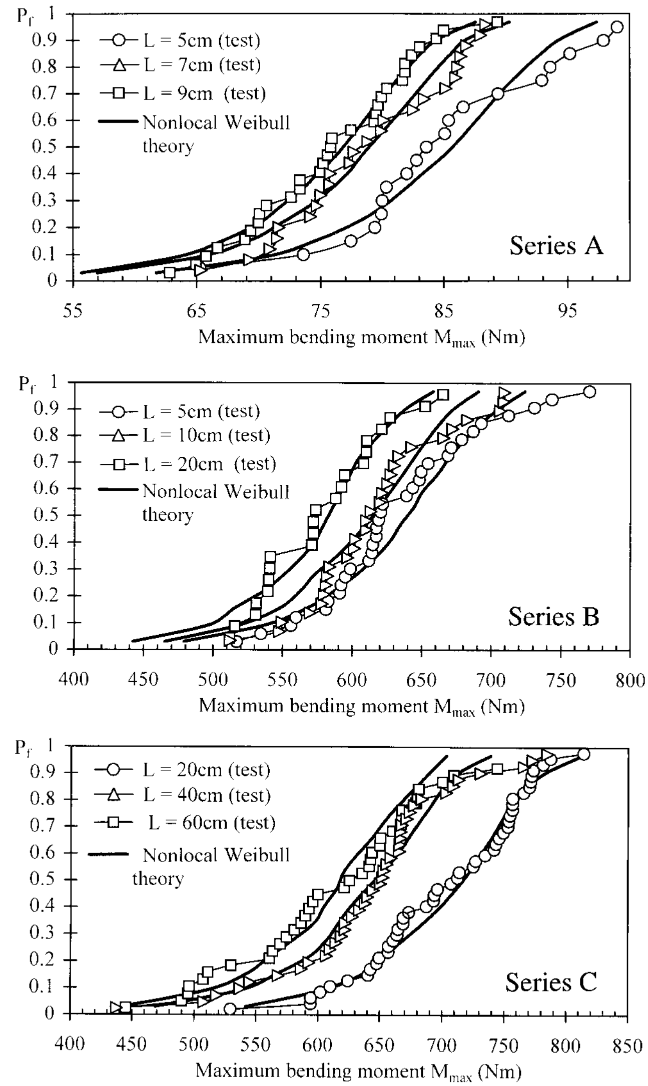

FIG. 10. Comparison of Probability Distribution Functions of Maximum Bending Moment M_{max} from Koide's (Personal Communication, 1999) Three Series of Four-Point Bending Tests and from Probabilistic Nonlocal Theory

Lindner and Sprague (1956). Span length l = 0.4572, 0.6858, 0.9144, 1.3716 m; span/depth ratio = 3; width $b = D$ (square cross sections); beam depth D = 15.24, 22.86, 30.48, 45.72 cm. The direct tensile strength f_t' = 4.0 MPa, modulus of elasticity E = 21.00 GPa, softening modulus E_t = 15.00, and maximum aggregate size d_a = 25.4 mm (1 in.) had to be estimated. Number of test specimens for each depth = 8 to 24.

Walker and Bloem (1957). Span length 1 = 0.381, 0.4572, 0.6096, 0.762 m (15, 18, 24, 30 in.). The span/depth ratio was l/D = 3, except for the smallest size. The beam width varied as b = 3, 6, 8, 10 in. The direct tensile strength f_t' = 4.0 MPa (but 3.4 MPa for d_a = 2 in.), modulus of elasticity E = 40.00 GPa, and softening modulus E_t = 30.00 GPa all had to be intuitively estimated for the present analysis. Maximum aggregate size d_a = 25.4, 50.8 mm (1, 2 in.); number of test specimens for each depth = 10 for every aggregate size and every depth.

Sabnis and Mirza (1979). Span length l = 0.0381, 0.0764, 0.1524, 0.3048, 0.6096 m; span/depth ratio = 4; width b = (2/3) D; beam depth D = 0.953, 1.91, 3.81, 7.62, 15.24 cm. The direct tensile strength f_t' = 3.8 MPa, modulus of elasticity E = 30.00 GPa, softening modulus E_t = 20.00 GPa, maximum aggregate size d_a = 12.7 mm (1/2 in.) all had to be estimated here. The number of test specimens for each depth was taken as 1 because no statistics were reported.

Rokugo (1995). Span length l = 0.15, 0.30, 0.60, 0.90, 1.2 m; span/depth ratio = 3; width $b = D$ (square cross sections); beam depth D = 5, 10, 20, 30, 40 cm. The direct tensile strength f_t' = 3.2 MPa was inferred from the splitting tensile strength. Modulus of elasticity E = 27.50 GPa. The softening modulus E_t = 20.00 GPa had to be intuitively estimated here. The maximum aggregate size was d_a = 15 mm. The number of test specimens for each depth was 8, but no statistics were reported in the paper, only the mean values.

Rocco (1995; personal communication, 1997). Span length l = 0.068, 0.148, 0.300, 0.600, 1.2 m; span/depth ratio = 4; beam width b = 0.50 m; beam depth D = 1.7, 3.7, 7.5, 15.0, 30.0 cm. The direct tensile strength f_t' = 4.6 MPa was estimated from the reported splitting tensile strength = 3.66 MPa; the modulus of elasticity E = 29.10 GPa; maximum aggregate size d_a = 5 mm. The softening modulus E_t = 20.00 GPa had to be intuitively estimated. Number of test specimens for each depth = 3 to 4.

TABLE 3. Statistics of Koide's (Personal Communication, 1999) Test Data and Nonlocal Weibull Theory Calculations

Series (1)	Bending span (cm) (2)	Mean (Nm) (3)	Median (Nm) (4)	Standard deviation (Nm) (5)	Coefficient of variation (6)	Skewness (7)
(*a*) Test Data						
A	5	84.7	83.6	8.3	0.098	−0.18
	7	78.8	78.3	6.4	0.082	−0.17
	9	75.8	75.8	6.5	0.086	−0.14
B	5	632.3	619.8	57.9	0.092	0.38
	10	616.8	612.0	48.6	0.079	0.27
	20	577.5	572.7	42.4	0.073	0.30
C	20	701.8	709.7	62.4	0.089	−0.52
	40	641.7	651.6	69.6	0.108	−0.50
	60	607.1	621.0	70.6	0.116	−0.39
(*b*) Nonlocal Weibull Theory						
A	5	84.0	85.7	9.0	0.108	−0.89
	7	77.6	79.0	8.3	0.107	−0.90
	9	75.4	76.8	8.0	0.106	−0.87
B	5	630.0	639.6	62.4	0.099	−0.81
	10	604.6	614.8	58.2	0.096	−0.78
	20	573.7	582.8	54.0	0.094	−0.79
C	20	703.7	716.5	67.8	0.096	−0.74
	40	635.8	645.7	67.7	0.106	−0.85
	60	608.8	617.9	66.9	0.110	−0.90

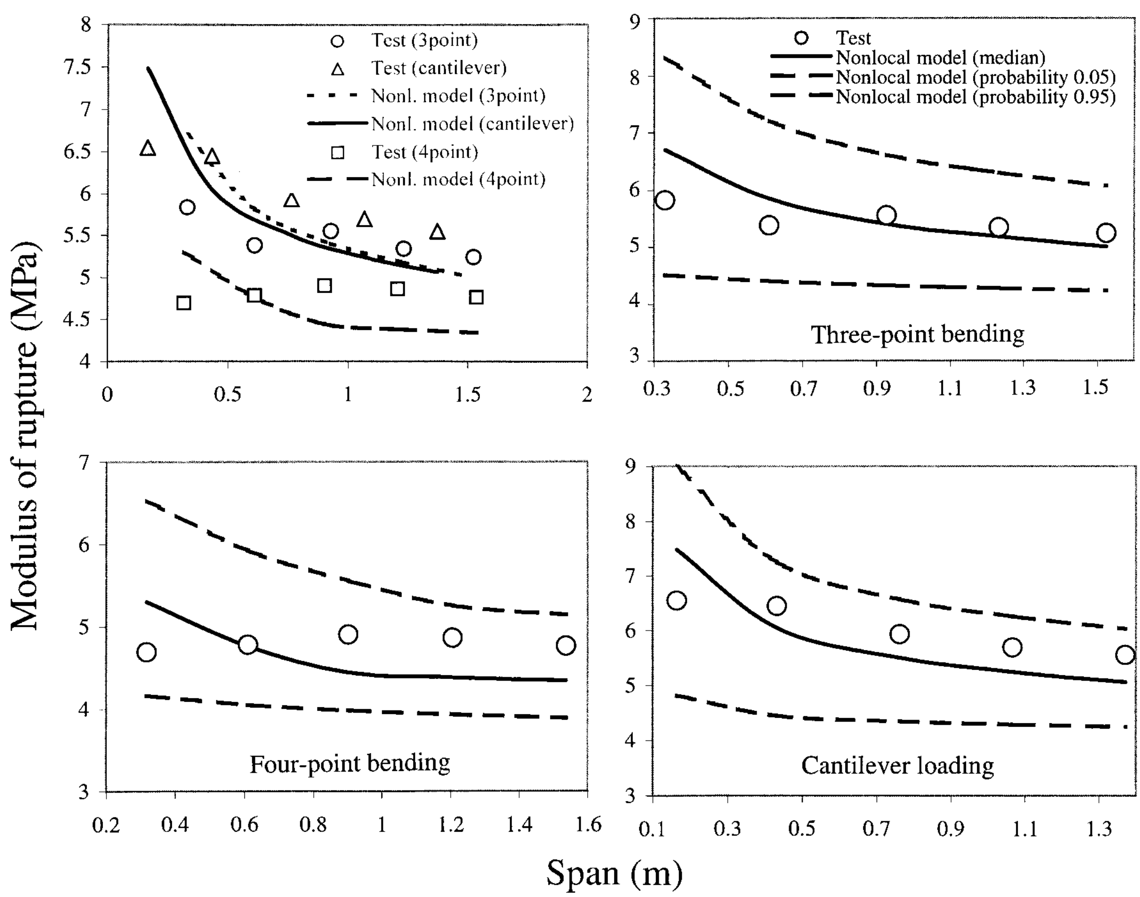

FIG. 11. PCA (1966) Bending Test Data Compared with Median (Solid Curves) and 5th and 95th Percentiles (Dashed Curves) of Modulus of Rupture for Different Types of Loading. Top Left: Median Only, Three Types of Loading, Rest: Medians and Percentiles for One Type of Loading Only

Koide et al. (1998, 1999). Three test series of 279 tests of beams in four-point bending (19 to 46 specimens for each span). Series A: cross section 4.5 × 4.5 cm and bending spans 5, 7, and 9 cm. Series B: cross section 8.5 × 8.5 cm and bending spans 5, 10, and 20 cm. Series C: cross section 10.0 × 10.0 cm and bending spans 20, 40, and 60 cm. Mean compressive strength of concrete: f'_c = 48.2 MPa in series A; 49.1 MPa in series B; and 30.0 MPa in series C. Maximum aggregate size d_a = 10 mm in series A and d_a = 20 mm in series B and C. All the specimens were cast from one and the same batch of concrete and were cured under identical environmental conditions. See Table 2 for the individual data.

Portland Cement Association (PCA, 1966). Bending tests for different spans. Cross section 15.24 cm × 15.24 cm (6 × 6 in.). Three types of loading were compared. Three-point bending, four-point bending and cantilever. No information on material properties, number of tests and statistics was given. The direct tensile strength f_t' = 4.5 MPa, modulus of elasticity E = 40.0 GPa, softening modulus E_t = 30.0 GPa, maximum aggregate size d_a = 25.4 mm, had all to be intuitively estimated.

DEDICATION

Dedicated to Prof. Alfredo H.-S. Ang on the occasion of a symposium in celebration of his career.

ACKNOWLEDGMENTS

The present research was funded under National Science Foundation (NSF) grant N00014-91-J-1109 to Northwestern University. The visiting scholar appointment of the second writer at Northwestern University was supported under NSF Grant INT-9531299 and a grant from Fulbright Foundation, and his research in Brno was funded under grant No. 103/97/K003 from the Grant Agency of Czech Republic and grant CEZ:J22/98:261100007.

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