Probabilistic Nonlocal Theory for Quasibrittle Fracture Initiation and Size Effect. I: Theory

By Zdeněk P. Bažant,[1] Fellow, ASCE, and Drahomír Novák[2]

Abstract: The nonlocal generalization of Weibull theory previously developed for structures that are either notched or fail only after the formation of a large crack is extended to predict the probability of failure of unnotched structures that reach the maximum load before a large crack forms, as is typical of the test of modulus of rupture (flexural strength). The probability of material failure at a material point is assumed to be a power function (characterized by the Weibull modulus and scaling parameter) of the average stress in the neighborhood of that point, the size of which is the material characteristic length. This indirectly imposes a spatial correlation. The model describes the deterministic size effect, which is caused by stress redistribution due to strain softening in the boundary layer of cracking with the associated energy release. As a basic check of soundness, it is proposed that for quasibrittle structures much larger than the fracture process zone or the characteristic length of material, the probabilistic model of failure must asymptotically reduce to Weibull theory with the weakest link model. The present theory satisfies this condition, but the classical stochastic finite-element models do not, which renders the use of these models for calculating loads of very small failure probabilities dubious. Numerical applications and comparisons to test results are left for Part II.

INTRODUCTION

The effect of randomness of material properties on the load capacity of brittle or quasibrittle structures and its statistical characteristics has been widely analyzed according to the stochastic finite-element models. Many studies have been devoted to these models, and significant achievements have been made (e.g., Schuëller 1997). This method is generally capable of predicting the loads of various specified failure probabilities.

Despite many years of investigations, however, it has apparently passed unnoticed that the stochastic finite-element models are fundamentally incapable of predicting the load with a given failure probability when the structure is very large. The reason is that when the ratio of the structure size D to the characteristic correlation length of the random field of local material strength tends to infinity (and when the geometry is positive, in the sense of fracture mechanics), the failure occurs right at the initiation of fracture from a microscopic flaw.

Consequently, Weibull theory based on the weakest link statistical model must be the limit case of the stochastic finite-element models. But this is not the case. In particular, in the special case of a very long cable made of a strain-softening random material, the stochastic finite-element method does not reduce to Weibull theory. This casts doubt on the basic soundness of the stochastic finite-element models in those problems that concern probabilistic prediction of load capacity of a brittle or quasibrittle structure.

Of particular importance to design is the prediction of a load with a very small failure probability, such as 10^{-7}. Only such predictions can be used as a direct check of design. When only the failure load with a large probability, such as 10^{-3}, is known, an empirical safety factor must still be used, and little is gained from stochastic structural analysis. In this regard it must be noted that the Weibull statistical model, based on the

[1] Walter P. Murphy Prof. of Civ. Engrg. and Mat. Sci., Northwestern Univ., Evanston, IL 60208.

[2] Assoc. Prof., Inst. of Struct. Mech., Facu. of Civ. Engrg., Tech. Univ. of Brno, 66237 Brno, Czech Republic; formerly, Visiting Fulbright Scholar, Northwestern Univ.

theory of extreme value distribution (weakest link model), is inevitable. Only this theory can predict the statistics of a load in a long cable giving an extremely low failure probability.

Probabilistic analysis using stochastic finite-element models focusing on the estimation of small probabilities is numerically feasible, but practically one faces difficulties in applying the method for complex models (e.g., to capture size effect phenomena). Crude Monte Carlo simulation, the subsequent curve fitting to get a theoretical model of the probability distribution function for structural response, and the estimation of failure probability from this theoretical model (such as Gaussian, log-normal, etc.) cannot be used for small probabilities, as the distribution support is only around the mean value of structural response. Advanced Monte Carlo simulation techniques (e.g., Schuëller et al. 1989) are then called for (they are discussed later in the paper).

The aim of this study is to present an alternative general method that provides the correct large size asymptotic behavior and thus offers a sound basis for statistical prediction of failure loads. This method combines the nonlocal concept of strain-softening damage with the Weibull weakest link model, in the general manner proposed by Bažant and Xi (1991) in their study of size effect in quasibrittle structures with large cracks or notches (see also Bažant and Planas 1998, chapter 11). The present paper will describe how to apply this concept in general situations, including unnotched structures that fail at fracture initiation from a smooth surface—a practically important case.

A typical example of such failures is the modulus of rupture test, on which an extensive experimental basis permitting statistical analysis has become available by now. Therefore, the implementation and numerical application will be focused on this test. The size effect on this test will receive the main attention since it is the principal consequence of both the strain-softening damage and the strength randomness. The general theory will be presented in Part I, and the application will follow in Part II (Bažant and Novák 2000).

NONLOCAL WEIBULL THEORY: STATISTICAL SIZE EFFECT

According to the weakest-link model (Tippett 1925; Fréchet 1927; Fischer and Tippett 1928) underlying the classical Weibull theory, the theoretical failure probability of a structure with a continuously variable uniaxial stress $\sigma(\mathbf{x})$ is

$$p_f = 1 - \exp\left\{ -\int_V \left\langle \frac{\sigma(\mathbf{x}) - \sigma_u}{\sigma_0} \right\rangle^m \frac{dV(\mathbf{x})}{V_r} \right\} \qquad (1)$$

(e.g., Bažant et al. 1991; Bažant and Planas 1998). Here V = volume of structure; σ_0, σ_u, and m are the parameters of Weibull probability distribution of the strength of the material (scale parameter, strength threshold, and shape parameter); V_r is the representative volume of the material, $\mathbf{x}$ is the coordinate vector of the material point; and $\langle\ \rangle$ denotes the positive part of the argument (used because only positive tensile stresses contribute to failure probability).

The integral in (1) diverges (for realistic m values) if the singular stress field of a sharp crack or notch is substituted. This means that the classical Weibull theory cannot be applied to failure (stability loss) that occurs only after large stable crack growth. To overcome this problem, a nonlocal continuum approach has been introduced (Bažant and Xi 1991; Bažant and Planas 1998). In this approach, the stress at a point depends not only on the strain at that point but also on the strain field within a certain neighborhood of that point. In the simplest version, it depends on the weighted spatial average of the strain in that neighborhood, which stands for the representative volume of the material. In the case of materials with strain-softening, the nonlocal concept is necessary to regularize the boundary value problem (prevent spurious localization with failure at zero energy dissipation and avoid spurious mesh sensitivity and unobjectivity). Within the framework of the Weibull integral, such averaging introduces, in a statistical sense, a spatial correlation (this is also demonstrated by the numerical simulations of Breysse and Fokwa 1992).

Because very different strength thresholds (with very different m-values) can usually give equally good representations of test data, a zero threshold, $\sigma_u = 0$, is assumed as a result. Then, if the stresses are at the same time replaced by the nonlocal stresses, as proposed by Bažant and Xi (1991), the multidimensional generalization of (1) may be written as

$$p_f = 1 - \exp\left\{ -\int_V \sum_{i=1}^{n} \left\langle \frac{\bar{\sigma}_i(\mathbf{x})}{\sigma_0} \right\rangle^m \frac{dV(\mathbf{x})}{V_r} \right\} \qquad (2)$$

where n is the number of dimensions, σ_i are the principal stresses ($i = 1, \ldots n$), and an overbar denotes nonlocal averaging. The failure probability now depends not on the local stresses $\sigma_i(\mathbf{x})$ but on the nonlocal stresses $\bar{\sigma}_i(\mathbf{x})$, which are the results of some form of spatial averaging.

In the case of an unreinforced simply supported symmetric beam [Fig. 1(b)] with a symmetric uniaxial stress field, (2) becomes

$$p_f = 1 - \exp\left\{ -\frac{2}{V_r} \int_0^{L/2} \int_{-s}^{h/2} \left[\frac{\bar{\sigma}(x, y)}{\sigma_0} \right]^m dx\, dy \right\} \qquad (3)$$

Here the integration over y is limited to the domain of positive stresses, that is, from $-s$ to $h/2$, where s is the shift of the neutral axis caused by cracking at the tensile face [Figs. 1(b) and 1(e)].

The nonlocal averaging in (3) is 2D and thus independent of the width of the beam. This appears reasonable because the available test results do not reveal any systematic dependence on the width.

ALTERNATIVES OF NONLOCAL AVERAGING

Three types of nonlocal averaging will be explored:

Alternative I

Averaging the stresses directly would make no sense because, for a given stress value, the strain and damage level are
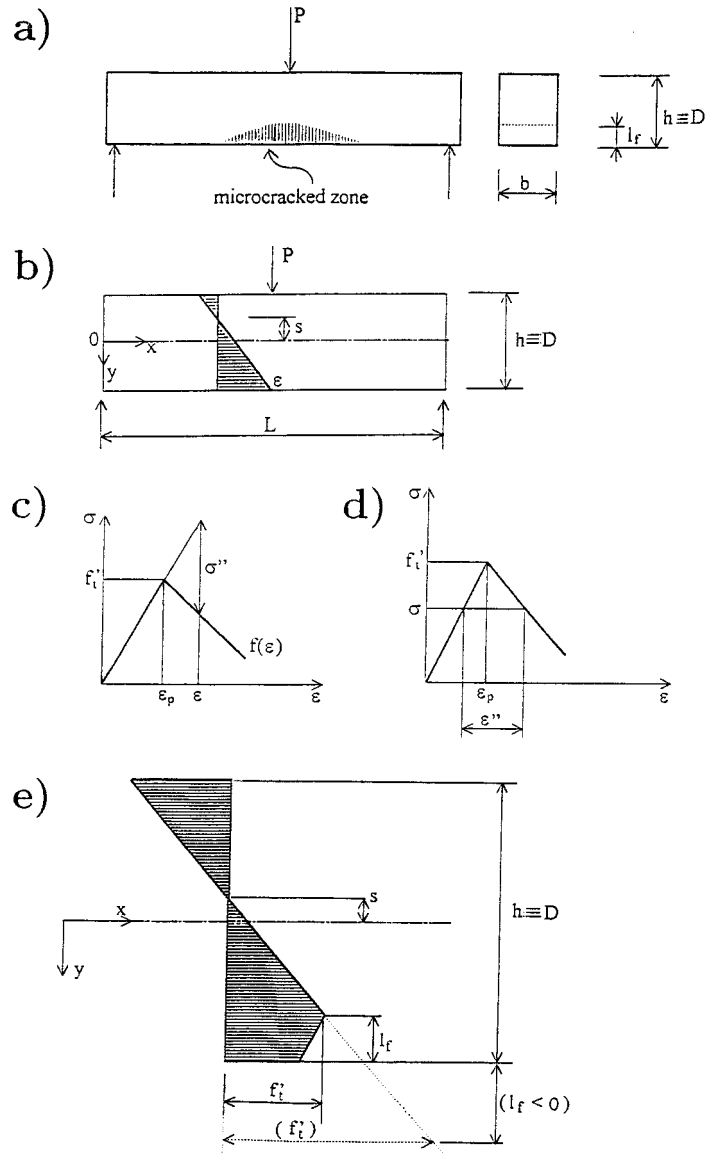
FIG. 1. (a) Microcracked Zone in Bažant and Li's (1995) Model; (b) Strain Distribution with Shifted Neutral Axis; (c) Stress-Strain Relation and Inelastic Stress; (d) Inelastic Strain; and (e) Stress Redistribution Due to Strain-Softening in Boundary Layer of Thickness l_f

nonunique (one prepeak and one postpeak strain correspond to the same given stress). One meaningful and simple formulation consists of a direct averaging of the local strains $\varepsilon(x', y')$, with weights given by a bell-shaped function $\alpha(x, y)$, as considered by Bažant and Xi (1991) and Bažant and Planas (1998). The nonlocal strains $\bar{\varepsilon}(x, y)$ are obtained according to the formula:

$$\bar{\varepsilon}(x, y) = \frac{1}{\bar{\alpha}(x, y)} = \int_0^L \int_{-h/2}^{h/2} \alpha(x' - x, y' - y)\varepsilon(x', y')\, dx'\, dy' \qquad (4)$$

where the weights are normalized by the function

$$\bar{\alpha}(x, y) = \int_0^L \int_{-h/2}^{h/2} \alpha(x' - x, y' - y)\, dx'\, dy' \qquad (5)$$

The nonlocal stresses to be used in calculating the generalized Weibull integral (3) are then simply evaluated from the nonlocal strains as the corresponding elastic stresses:

$$\bar{\sigma}(x, y) = E\bar{\varepsilon}(x, y) \qquad (6)$$

Averaging of the total strain was shown to require suppression of certain periodic zero-energy modes of instability (Bažant et al. 1984; Bažant and Cedolin 1991, chapter 13), which complicated the formulation. In the present model, though, such modes cannot arise. The reason is that we employ the nonlocal averaging of strain to calculate only the failure probability of a small material element rather than the stresses that are subject to the differential equation of equilibrium. In Weibull-type theory, we do not need to formulate a nonlocal constitutive equation to be used in solving the boundary value problem with strain-softening damage.

Alternative II

The aforementioned instability modes are avoided if the nonlocal constitutive model for damage subjects to nonlocal averaging not the total strain but only some measure of damage or inelastic strain (Pijaudier-Cabot and Bažant 1987; Bažant and Pijaudier-Cabot 1988; Bažant and Planas 1998). Thus, as the second alternative, we choose to subject to spatial averaging the inelastic stress $\sigma''(x', y')$, defined as the elastic stress corresponding to the given strain minus the actual stress for the same strain according to the given stress-strain diagram [Fig. 1(c)]. So we calculate first the nonlocal inelastic stress

$$\bar{\sigma}''(x, y) = \frac{1}{\bar{\alpha}(x, y)} \int_0^L \int_{-h/2}^{h/2} \alpha(x' - x, y' - y)\sigma''(x', y') \, dx' \, dy' \tag{7}$$

where

$$\sigma''(x', y') = E\varepsilon(x', y') - f[\varepsilon(x', y')] \tag{8}$$

The nonlocal stress at point (x, y) is then obtained as

$$\bar{\sigma}(x, y) = E\varepsilon(x, y) - \bar{\sigma}''(x, y) \tag{9}$$

Alternative III

Another type of averaging that avoids the zero-energy instability modes of a nonlocal constitutive model for damage is the spatial averaging of the inelastic strain, $\varepsilon''(x', y')$, representing the difference of the current strain and the elastic strain corresponding to the same stress [Fig. 1(d)]. So we choose the third alternative as

$$\bar{\varepsilon}''(x, y) = \frac{1}{\bar{\alpha}(x, y)} \int_0^L \int_{-h/2}^{h/2} \alpha(x' - x, y' - y)\varepsilon''[\sigma(x', y')] \, dx' \, dy' \tag{10}$$

and then calculate the nonlocal stress as

$$\bar{\sigma}(x, y) = E[\varepsilon(x, y) + \bar{\varepsilon}''(x, y)] \tag{11}$$

Alternative I is precisely the averaging used in Bažant and Xi's (1991) statistical size effect model for large cracks (also Bažant and Planas 1998). Because that model applies averaging to the strain field of a large sharp crack in an elastic material with no distributed damage, the other types of averaging cannot be applied to that model, unless its formulation were generalized.

The spatial averaging integral is approximated by a finite sum over all the points of the structure. For all three types of averaging considered here, the computer implementation of the integrations over x' and y' involves, for every mesh point, four nested loops over the mesh points at which the strains and stresses are evaluated. This considerably increases the amount of computations, but is not a problem if a powerful computer is used. The computer time can be reduced by using a weight function that vanishes at a finite distance.

Different choices of the weight function $\alpha(x, y)$ for spatial averaging are possible, but the results are not too sensitive to the choice made. For instance, one may simply choose a uniform weight function that is nonzero only over a certain representative volume such as a circle (or for convenience a square). However, according to the computational experience, a better convergence is achieved by using a smooth, bell-shaped weight function, which also appears more realistic (Bažant et al. 1984; Bažant and Lin 1988; Bažant and Ožbolt 1990). One suitable form of such a weight function, which is adopted here, is the modified bivariate normal (Gaussian) probability distribution function suggested by Bažant et al. (1984):

$$\alpha(x, y) = e^{-2\sqrt{x^2 + y^2}/l} \tag{12}$$

Here l is the characteristic length of the nonlocal continuum (or material length), which characterizes the size of the representative volume V_r. It represents the diameter of a cylinder of height 1 that has the same volume as the weight function. This condition requires that $V_r = \pi l/8$. For concrete, the characteristic length can be taken approximately as $l = 3d_a$, which is the band width used for the crack band model (d_a = maximum aggregate size).

For $l \to 0$, the material becomes local and (3) becomes the classical Weibull probability integral. So the classical Weibull theory is a special case of the present theory. For beams so large that the characteristic length is negligible compared to the depth of the cross section, the classical Weibull size effect on the modulus of rupture must be approached (as observed by Petersson 1981).

MODULUS OF RUPTURE TEST

The structure size effect on the modulus of rupture of plain concrete beams, as well as other quasibrittle materials such as rocks, composites, ceramics, or ice, has been known for a long time. Its classical explanation was statistical—the randomness of material strength (e.g., Bažant and Planas 1998), as suggested already by Mariotte (1686) and mathematically described in a final form by the theory of Weibull (1939).

As revealed, however, by the finite-element calculations of Hillerborg et al. (1976) and thoroughly demonstrated by Petersson (1981), the mean observed size effect can be described deterministically by the cohesive (or fictitious) crack model, as a consequence of stress redistribution in the cross section prior to maximum load. A simple analytical formula based on this redistribution was derived by Bažant and Li (1995) and was shown to match all the important test data reasonably well. The same formula was proposed earlier on an empirical basis by Rokugo et al. (1995) (Bažant and Li 1996). It was also shown that this formula can be derived from fracture mechanics if the nonnegligible size of the fracture process zone near the tensile face of the beam is taken into account [Bažant 1997(a)].

The size effect on the modulus of rupture, and generally the size effect for all failures occurring at crack initiation from a smooth surface, is of a different kind than the size effect on failures that occur after a long stable crack growth and are typical of reinforced concrete structures. Until the mid 1980s, that size effect was also generally believed to be statistical, but it is now widely accepted that its cause is primarily deterministic—the energy release due to crack growth (Bažant 1984).

The maximum load of plain concrete beams occurs before a continuous crack initiates. But it occurs only after a boundary layer of distributed cracking of a certain critical thickness develops at the tensile face of beam. For beams of different sizes made of the same concrete, the thickness of this layer appears to be about the same, dictated by the maximum aggregate size. Formation of this boundary layer, representing a fracture process zone, is the principal reason why the direct

tensile strength f_t' differs from the modulus of rupture, f_r, which is defined (for an unreinforced beam of a rectangular cross section) as

$$f_r = \frac{6M_u}{bD^2} \tag{13}$$

where M_u = ultimate bending moment, D = characteristic size of the structure, chosen to coincide with the beam depth (often denoted as h), and b = beam width.

The randomness of the heterogeneous microstructure of concrete and of its strength must nevertheless have at least some influence, as demonstrated by stochastic finite elements, including nonlocal finite-element simulations with random spatially correlated strength (e.g., Breysse 1990; Breysse and Fokwa 1992; Breysse et al. 1994; Breysse and Renaudin 1996). For quasibrittle structures failing after large stable crack growth, this question was studied by Bažant and Xi (1991) and Bažant and Planas (1998). They presented a generalization of Weibull-type theory in which the material failure probability depends not only on the local continuum stress but also on the average strain of a characteristic volume of the material. Their analysis, later slightly simplified and refined by Bažant and Planas (1998), led to a size effect formula that agrees well with test results and approaches the original size effect law as its deterministic limit.

The key point in Bažant and Xi's analysis (1991), which allows handling of the crack tip singularity, is the introduction of the nonlocal continuum concept for determining the failure probability of a material element. If the Weibull probability integral is applied to the redistributed stress field, the dominant contribution comes from the fracture process zone at the crack tip. The contribution from the rest of the structure is nearly vanishing, which is explained by the fact that the fracture cannot occur outside the process zone. Because the process zone size is nearly independent of structure size D, no statistical size effect can thus take place except when the structure is so small that its boundaries restrict full development of the process zone.

In the case of the modulus of rupture, the Weibull-type size effect can dominate only in unreinforced beams that are far deeper than the boundary layer of distributed cracking and thus fail right at crack initiation, as suggested by Petersson (1981). The beam depth, however, would have to exceed about a few meters, which is hardly a realistic test. Besides, good practice requires designing structures so as not to fail at crack initiation.

Randomness of the material must nevertheless be expected, in the case of the modulus of rupture, to play at least some role. To explore it is the objective of this two-part study. In the first part, a statistical theory that has the correct deterministic limit will be formulated, and the asymptotic behavior for very small and very large sizes will be determined. The role of Weibull-type statistical size effect in the modulus of rupture test will be analyzed using a simple nonlocal strain-softening model that is of the same type as introduced for failure after large crack growth in the Weibull theory generalization of Bažant and Xi (1991). The boundary layer of distributed cracking will play a role analogous to the crack-tip fracture process zone. However, by contrast to Bažant and Xi's (1991) statistical analysis of large continuous cracks, the reason for nonlocal strain averaging will not be the suppression of stress singularity but the necessity to impose spatial statistical correlation. The second part that follows will present numerical applications with comparisons to test results.

A SIMPLE DETERMINISTIC ENERGY-BASED SIZE EFFECT FORMULA

At maximum load, a boundary layer of distributed cracking having a nonnegligible thickness l_f [Figs. 1(a) and 1(e)] may be assumed to exist at the tensile face of the beam. With this assumption, and assuming further the beam theory (cross sections remaining plane) and a linear postpeak softening stress-strain diagram, Bažant and Li (1995) derived the approximate formula:

$$\frac{f_r}{f_t'} = 1 + 2\frac{l_f}{D} \tag{14}$$

where D = beam depth. The same formula was also derived by an energetic argument upon truncating an LEFM-based asymptotic series expansion in terms of l_f/D after the linear term (Bažant 1997a). Neither plane cross sections nor any postpeak stress-strain diagram has been assumed.

In spite of the fact that (14) contains no term nonlinear in l_f/D, it agrees well with the existing test results. Because the decline of the statistical density of microcracks with an increasing distance from the beam face is gradual rather than sudden, l_f should be interpreted as a certain effective thickness of the boundary layer of cracking, representing the fracture process zone. We will find it convenient to use l_f as a parameter of the statistical generalization.

The exact solution according to beam bending theory under the aforementioned simplifying assumptions of beam theory yields a slightly more involved formula (Planas et al. 1995):

$$\frac{f_r}{f_t'} = 1 + 2\frac{l_f}{D} - \frac{4k}{k + (D/l_f - 1)^2}; \quad k = \frac{E_t}{E} \tag{15}$$

E = Young's modulus of elasticity and E_t = softening modulus. This formula, however, does not yield appreciably better fits of test results, which is due to inevitable experimental scatter and even more to simplifications in the basic assumptions.

COMPUTATIONAL MODEL FOR MODULUS OF RUPTURE TEST

The constitutive model for failure of concrete must exhibit strain softening, that is, a postpeak decrease of stress at increasing strain. The simplest but adequate choice is a bilinear stress-strain diagram [Fig. 1(c)]. For the three-point beam bending test of the modulus of rupture, the statistical analysis proceeds as follows.

It might seem that the analysis of strain softening would call for using finite elements, as was done in studies by the cohesive crack model (Petersson 1981). In the present problem of beam bending, however, this is unnecessary because only the states before a crack forms are of interest. The softening zone, restrained by adjacent concrete in an elastic state, does not yet localize, remaining distributed over a long portion of the beam. Therefore, the classical hypothesis of cross sections remaining planar is a good approximation. It implies a linear distribution of strains within the cross section.

Because of strain softening (due to distributed cracking) near the tensile face, the neutral axis is shifted upward by distance s, which is an unknown parameter to be determined for each cross section where microcracking occurs [Fig. 1(b)]. The linear strain distribution within the cross section x may be written as

$$\varepsilon(x, y) = \kappa(x)[y + s(x)] \tag{16}$$

where $\kappa(x)$ = curvature of the beam, which is the second unknown. The unknown parameters s and κ can be solved from the equilibrium conditions for bending moment $M = Px/2$ (in the case of three-point bending) and axial force $N = 0$;

$$\int_{-h/2}^{h/2} yf\{\kappa(x)[y + s(x)]\} \, dy = \frac{Px}{2} \tag{17}$$

$$\int_{-h/2}^{h/2} f\{\kappa(x)[y + s(x)]\} \, dy = 0 \tag{18}$$

where the stress-strain relation is considered as $\varepsilon = f(\sigma)$. For the bilinear stress-strain diagram in Fig. 1(c), these equations become

$$\int_{-h/2}^{y_p} yE\kappa(x)[y + s(x)]\,dy$$

$$+ \int_{y_p}^{h/2} \{-E_t[\kappa(x)(y + s(x)) - \varepsilon_p] + f_t'\}y\,dy = \frac{Px}{2} \quad (19)$$

$$\int_{-h/2}^{y_p} E\kappa(x)[y + s(x)]\,dy$$

$$+ \int_{y_p}^{h/2} \{-E_t[\kappa(x)(y + s(x)) - \varepsilon_p] + f_t'\}\,dy = 0 \quad (20)$$

Evaluation of the integrals yields the following system of two nonlinear equations for s and κ:

$$E\kappa(y_p^3/3 + sy_p^2/2 + h^3/24 - sh^2/8) - E_t(\kappa h^3/24 + \kappa sh^2/8$$

$$- \varepsilon_p h^2/8 - \kappa y_p^3/3 - \kappa sy_p^2/2 + \varepsilon_p y_p^2/2) + f_t'(h^2/8 - y_p^2/2)$$

$$- Px/2 = 0 \quad (21)$$

$$E\kappa(y_p^2/2 + sy_p + h^2/8 - sh/2) - E_t(\kappa h^2/8 + \kappa sh/2$$

$$- \varepsilon_p h/2 - \kappa y_p^2/2 - \kappa sy_p + \varepsilon_p y_p) + f_t'(h/2 - y_p) = 0 \quad (22)$$

in which argument x is omitted, and $y_p = \varepsilon_p/\kappa(x) - s(x) = y$-coordinate of the boundary of the elastic zone [ε_p = strain at peak stress, Fig. 1(c)]. This system can be solved iteratively by the Newton-Raphson method or even more efficiently by the Levenberg-Marquardt nonlinear optimization algorithm. The solution for the shift of neutral axis s, however, does not always imply realistic values for the boundary layer thickness l_f. The value of l_f ensues from the condition that, for $y = h/2 - l_f$, $\varepsilon = \varepsilon_p$ and $\sigma = f_t'$ [Fig. 1(e)];

$$l_f(x) = \max\left(s(x) + \frac{h}{2} - \frac{f_t'}{E\kappa(x)}, 0\right) \quad (23)$$

The "max" condition (equivalent to taking the positive part $\langle\ \rangle$ of the first argument) must be introduced here to make the solution realistic. It prevents negative l_f values, which result from the solution of (21) and (22) when the entire cross section is elastic (because, in that case, the tensile strength is reached hypothetically outside the cross section). When $l_f = s = 0$, the curvature κ must be recalculated from the elastic bending theory.

The result of the foregoing calculation is the 2D local strain field $\varepsilon(x, y)$, to which the spatial averaging according to one of the aforementioned alternatives needs now to be applied to obtain the nonlocal stress field $\bar\sigma(x, y)$. In the statistical sense, this averaging is equivalent to imposing spatial statistical correlation, characterized by the characteristic length l, which is a material property of concrete.

The calculated nonlocal stress field $\bar\sigma(x, y)$ is then substituted into the generalized Weibull-type integral (3). For prescribed load P, the integral can be evaluated numerically, furnishing the failure probability p_f. For prescribed p_f, the load P needs to be solved iteratively.

The foregoing procedure makes it possible to study the full probability range for the modulus of rupture (ratio f_r/f_t') at different beam sizes. It can provide estimates of the mean values, variances, and coefficients of skewness, and it also makes it possible to test the probability distribution function of the theoretical model. Once a data sample of the modulus of rupture is obtained, a theoretical model of the probability distribution function can be assigned to the sample. The most suit-

able theoretical model can be selected using standard statistical tests, such as the Kolmogorov-Smirnov test or the chi-square test.

The presence of a characteristic length in the problem is essential, not only from the viewpoint of statistical correlation. It is now well established that the very concept of strain softening per se is meaningless without reference to a certain non-zero characteristic length. In the deterministic model of Bažant and Li (1995), the role of the characteristic length is played by the boundary layer thickness l_f, which is in that model an assumed material parameter.

The field of nonlocal stresses needed for evaluating the generalized nonlocal Weibull probability integral (2) or (3) requires solving the deterministic boundary value problem for nonlocal strain-softening material behavior. This usually necessitates finite-element analysis. But in the case of the bending beam test of the modulus of rupture, the solution may be carried out quite simply according to the beam theory.

Compared to the stochastic finite-element approaches (see, for example, the review by Breysse et al. 1994) or the "numerical concrete" model (Roelfstra et al. 1985), an important feature that brings about great simplification is that the nonlocal structural analysis with strain softening, or the structural analysis with a cohesive crack, can be carried out deterministically, that is, independently of the probability analysis. However, iterations of the deterministic solution are required if the failure probability of the structure is specified. But the number of these iterations is very small compared to the classical approaches of reliability engineering (where calculations of failure probability usually require thousands of repetitive deterministic solutions using advanced Monte Carlo-type simulation techniques).

ILLUSTRATION OF SPATIAL DISTRIBUTION OF CONTRIBUTIONS TO FAILURE PROBABILITY

To clarify the basic concept, it is helpful to present at this point Fig. 2, which shows the succession of breaks of material points according to the spatial distribution of the contributions to failure probability entering the integral (3). This figure has been calculated using the material parameters of "average concrete," as will be described in Part II of this paper, where the spatial distribution of the contributions to failure probability also will be plotted.

Both the three-point bending and the four-point bending cases are studied in Fig. 2. For the input, consisting of the number of failed points (indicated at the top-left corner of each quarter of the beam), pure Monte Carlo simulation is performed according to the distribution of probabilities. Naturally, the first failed points appear near the midspan, in the case of three-point bending, or near the bottom face within the maximum moment region of the beam, in the case of four-point bending. As the number of failed points increases, the development of the shape of the fracture process zone, visualized by different levels of probabilities, can be observed. In the case of four-point bending, a large bending span (60% of the total beam span) was selected. It should be kept in mind that the figure does not portray the sequence of failures associated with the formation of a real crack. Rather, it shows merely the distribution of the contributions to failure probability intended to provide better insight into the nonlocal Weibull theory.

COMPARISON WITH STOCHASTIC FINITE-ELEMENT MODELS

Applications of the theory of random fields to the finite-element method have led during the last 15 years to the development of the stochastic finite-element method (SFEM). Several effective alternatives have been proposed by Schuëller

Three-point bending   P ↓   P ↓   Four-point bending

Failure probability   • $p_f > 10^{-2}$ ⊕ $10^{-4} < p_f < 10^{-2}$ • $10^{-6} < p_f < 10^{-4}$ ○ $p_f < 10^{-6}$
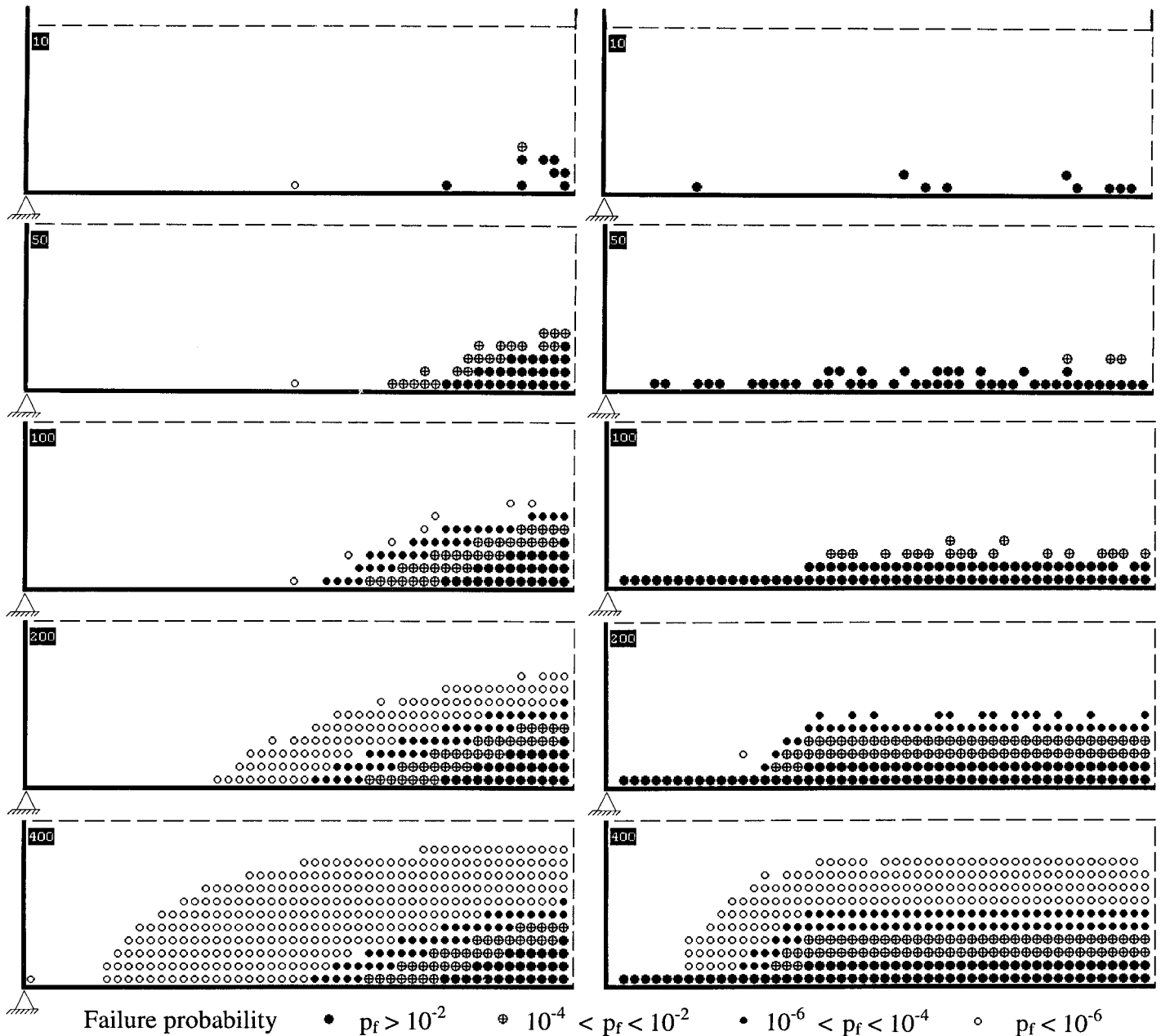
FIG. 2. Monte Carlo Simulation of Failed Points According to Spatial Distribution of Contributions to Failure Probability

(1997); Ghanem and Spanos (1991); Hisada and Nakagiri (1985); Liu et al. (1986); and others (for a literature review, see Brenner 1991). The objective of these models is to determine the statistics of the response, such as the deflection, stress, failure load, and, consequently, the probability of failure. Similar approaches, such as the stochastic lattice models, can physically simulate material heterogenity and deal with the spatial variability of material strength (Breysse 1990; Breysse and Fokwa 1992), including stochastic lattice models [Jirásek and Bažant 1995 (a,b)]. The method generally requires repetitive solutions of the finite-element system equations using Monte Carlo simulation. The failure probability within the framework of SFEM can be calculated in one of the following ways:

1. The FORM (first-order reliability method) or SORM (second-order reliability method) approximation technique (e.g., Madsen et al. 1986), normally used within the framework of the SFEM. This technique requires applying a suitable optimization method to find the so-called design point. For highly nonlinear problems, however, the accuracy of the calculation of failure probability is not good (e.g., Schuëller et al. 1989).

2. Direct Monte Carlo simulation. Even if advanced simulation techniques, such as importance sampling and adaptive sampling (e.g., Schuëller et al. 1989), are applied, a large number of simulations are still necessary for accurately estimating a low failure probability, such as 10^{-7}, which makes applications of complex computationally intensive finite-element models practically impossible.

3. The response surface techniques (e.g., Bucher and Bourgund 1987), with the consequent application of advanced simulation methods. This approach is feasible for large

systems because it is based merely on a polynomial approximation of the failure surface. It requires only a limited number of finite-element calculations (from tens to hundreds). Once the polynomial approximation is obtained, then any standard technique for failure probability calculation can be applied, and thousands of simulations can be performed easily.

The term ''stochastic finite elements'' is used in the sense of both random variables and random fields. The latter naturally require representing the random field by a large set of random variables, and the size of the problem increases even if one uses some special technique for reducing the number of random variables used as a discrete approximation of the field.

Asymptotic Case of Very Large Quasibrittle Structure

The problem of failure of a structure with distributed damage is one of many physical phenomena whose mathematical modeling is very difficult for the practical range but easy for the asymptotic cases. An approximate model is usually good if it is made to match the asymptotic cases. This approach, in which the model for the intermediate range is anchored at the infinities so as to provide ''interpolation'' between the opposite infinities, is known as asymptotic matching. It has been used with great success in fluid mechanics, and its emulation has been proven useful for developing approximate scaling laws for the nominal strength of quasibrittle structures.

When the ratio of the structure size D to the effective size c_f of the fracture process zone (or damage zone) tends to infinity, the failure behavior approaches that of linear elastic fracture mechanics (LEFM), and when this ratio (D/c_f) tends to zero, the behavior approaches that of plasticity. These asymptotic behaviors should be satisfied not only by deterministic failure models but also by probabilistic ones. This asymptotic condition has not previously been considered in connection with the probabilistic models of failure. It is proposed here to adopt it as a basic *check of the soundness* of such models, including the stochastic finite-element models.

The characteristic length ℓ of the autocorrelation function of the random field of local material strength, which is a basic characteristic of stochastic finite-element models, is proportional to c_f, which in turn is proportional to the nonlocal characteristic length ℓ in the present nonlocal Weibull theory (as well as to the boundary layer thickness l_f).

When D/l tends to infinity or zero, so do D/c_f and D/ℓ (or D/l_f). For $D/c_f \to 0$, the damage cannot localize, and so the small size asymptotic behavior ought to approach plasticity. Both the present theory and SFEM satisfy this condition.

When $D/c_f \to \infty$, the front of the zone of distributed cracking or strain softening is unstable and must therefore localize into a single point (in structures of positive geometry). This implies that such a structure must fail right at the initiation of cracking or strain softening at one point of the structure (this is intuitively most obvious when the structure is a long tensioned bar). Such points of initial cracking or strain softening play a role similar to that of the material flaws (or points of a low local strength) in the Weibull theory.

Consequently, for $D/\ell \to 0$, a sound stochastic finite-element model must reduce to the classical Weibull theory, based on the weakest-link statistical model (or extreme value statistics). In particular, the size effect of classical Weibull theory ought to be the limiting case of a sound stochastic finite-element model.

The nonlocal Weibull theory proposed here obviously satisfies this condition. However, the SFEM, as presently known, does not. In particular, it cannot model the brittle failures of very long bars under uniaxial tension, for which Weibull the-

ory is inevitable. Recognition of this fundamental drawback stimulated the development of the present theory.

Other Aspects of Comparison

Aside from the asymptotic case of a very large structure, the comparison of both approaches may be summarized as follows:

1. One advantage of the SFEM is that any number of variables or random fields can be used to simulate the uncertainties of material, environmental, and geometric parameters. In the present nonlocal Weibull approach, the reliability problem is reduced to one dominant random variable (strength). Randomness of elastic moduli and other parameters of a nonlinear strain softening constitutive law can nevertheless be taken into account by standard Monte Carlo simulation.

2. The nonlocal Weibull theory is conceptually transparent and simple. In calculating the failure load probability, the stochastic finite-element models are considerably more complicated than the present nonlocal Weibull approach and, despite their many achievements, cannot yet handle really complex structures because of the tremendous amount of computational effort required.

3. Applying commercial finite-element codes to the stochastic finite-element models would be difficult since a special interface to the reliability software would have to be developed. The nonlocal Weibull theory, by contrast, can use the commercial codes without any significant modification.

4. Essential random field characteristics required as the input to the stochastic finite-element models, particularly the correlation length, type of correlation function, or spectral density of the field, are very difficult to determine in a rational manner and have generally been estimated heuristically, based on intuitive judgement. In the nonlocal Weibull theory, on the other hand, the parameter of spatial correlation is the characteristic length, which is the same as in the deterministic nonlocal damage theory and has an intimate relationship to the heterogenity of the material (it may be taken as several times the maximum aggregate size in concrete and may also be related to the fracture energy and strength of the material and calibrated by size effect tests).

5. Whereas the calculation of loads with a failure probability such as 5% doubtless gives reliable results, the calculation of loads with an extremely small failure probability, such as 10^{-7}, of which the main interest is for design, is probably unreliable. It thus still seems necessary to employ in design some safety factor based on experience and intuition. In principle, the prediction of failure should in some way be based on the extreme value statistics (Tippett 1925; Fisher and Tippett 1928) and the Weibull distribution, which can capture the fact that the failure is triggered by the weakest material elements in the critical region of the structure. The SFEM misses these fundamental features; it is based on full-range distributions such as Gaussian, which have physically meaningless far-off tails and are inherently inappropriate for treating the statistics of extremely unlikely breaks in the material responsible for extremely rare failures.

6. Not only the asymptotic limit of very large structures, but generally the incorporation of fracture mechanics into the stochastic finite-element models poses difficulties, and attempts to do so have been rare.

CLOSING COMMENTS

Note that the direct tensile strength f_t', as well as D, E, E_t, L, and d_a, is a deterministic parameter in the present model. It would of course be possible to generalize these parameters as random, select their random samples, run for each sample the present model in the sense of classical Monte Carlo simulation, and obtain the statistics of the responses. However, conceptually this seems inappropriate because the parameters of a statistical model are by definition deterministic, and the aforementioned parameters indeed play here precisely such a role.

The situation is analogous to the Weibull scale parameter σ_0, which is treated physically as deterministic in all Weibull theories. It could be argued that σ_0 is also random. This kind of argument could continue since it could be argued that a parameter of the distribution of σ_0, which must have a dimension of stress, should be physically random too. The parameters of the distribution of that parameter should again be random, and so forth, ad infinitum. Such induction reduces this kind of arguments to nonsense.

In this regard, note further that the present model has been verified and calibrated by the available test data considering σ_0 and other parameters to be deterministic. Obviously the model should be used in the same way as it has been verified and calibrated, which will be done in Part II.

The general approach has by now been presented; it remains to apply it in numerical calculations and compare it to the result of experiments. This task will be tackled in Part II, in which all the conclusions are formulated.

APPENDIX I.   GENERALIZED DETERMINISTIC ENERGY-BASED FORMULA

The derivations of formula (14) in both Bažant and Li (1995) and Bažant (1997a) are accurate only up to the first two terms of the asymptotic power series expansion of f_r in terms of $1/D$. As it turns out, however, the following more general formula has the same first two terms of the asymptotic expansion and is therefore equally justified:

$$\frac{f_r}{f_t'} = \left(1 + \frac{rD_b}{D} \right)^{1/r} \quad (r > 0) \tag{24}$$

Exponent r is related to the third term of the expansion.

It is interesting that a special case of formula (24) for $r = 2$ (as pointed out in Bažant 1998) is the empirical formula

$$f_r = \sqrt{A_1 + \frac{A_2}{D}} \tag{25}$$

where A_1, A_2 = constants. This special case, which was derived from fracture mechanics in Bažant (1998), is identical to the formula proposed on the basis of strictly geometric arguments by Carpinteri et al. (1994, 1995), which was named the "multifractal scaling law" (this term is, however, questionable because, if an energy-based mechanical analysis is carried out, a different law is found to ensue from the fractal hypothesis: Bažant 1997b, 1997c).

Exponent r could in theory be determined from the third derivative of the energy release function of LEFM, but in practice such determination of r is oversensitive to changes of geometry. Using the Levenberg-Marquardt algorithm to simultaneously fit the set of all the presently used data indicated $r = 1.47$ as the optimum value. But the coefficient of variation of the prediction errors was found to be only a little sensitive to the changes in r and was not much larger for $r = 1$. For this reason, (14) is used here instead of (24). Eq. (25) is studied in detail in a separate paper (Bažant and Novák 1999), along with its transition to Weibull theory at large sizes.

DEDICATION

This paper is dedicated to Prof. Alfredo H.-S. Ang on the occasion of a symposium in celebration of his career.

APPENDIX II.   REFERENCES

Bažant, Z. P. (1984). "Size effect in blunt fracture: Concrete, rock, metal." *J. Engrg. Mech.*, ASCE, 110, 518–535.

Bažant, Z. P. (1997a). "Scaling of quasibrittle fracture: Asymptotic analysis." *Int. J. Fracture*, 83(1), 19–40.

Bažant, Z. P. (1997b). "Scaling of quasibrittle fracture: Hypotheses of invasive and lacunar fractality, their critique and Weibull connection." *Int. J. Fracture*, 83(1), 41–65.

Bažant, Z. P. (1997c). "Statistical and fractal aspects of size effect in quasibrittle structures: Conspectus of recent results." *Proc., Int. Conf. on Struct. Safety and Reliability*, Balkema, Rotterdam, The Netherlands, 1255–1262.

Bažant, Z. P. (1998). "Size effect in tensile and compression fracture of concrete structures: Computational modeling and design." *Proc., 3rd Int. Conf., Fracture Mech. of Concrete Struct.*, H. Mihashi and K. Rokugo, eds., Aedificatio Publishers, Freiburg, Germany, 1905–1922.

Bažant, Z. P., Belytschko, T. B., and Chang, T.-P. (1984). "Continuum model for strain softening." *J. Engrg. Mech.*, ASCE, 110, 1666–1692.

Bažant, Z. P., and Cedolin, L. (1991). *Stability of structures: Elastic, inelastic, fracture and damage theories*. Oxford University Press, New York.

Bažant, Z. P., and Li, Z. (1995). "Modulus of rupture: Size effect due to fracture initiation in boundary layer." *J. Struct. Engrg.*, ASCE, 121(4), 739–746.

Bažant, Z. P., and Li, Z. (1996). "Discussion of 'Fracture mechanics approach to evaluation of flexural strength of concrete," by K. Rokugo, Y. Uchida, H. Katoh, and W. Koyanagi." *ACI Mat. J.*, 93(5), 507–510.

Bažant, Z. P., and Lin, F.-B. (1988). "Non-local yield limit degradation." *J. Numer. Methods in Engrg.*, 26, 1805–1823.

Bažant, Z. P., and Novák, D. (1999). "Energetic-statistical size effect in quasibrittle failure at crack initiation." *Rep.*, Dept. of Civ. Engrg., Northwestern University (*ACI Mat. J.*, in press).

Bažant, Z. P., and Novák, D. (2000). "Probabilistic nonlocal theory for quasibrittle fracture initiation and size effect. II: Application." *J. Engrg. Mech.*, ASCE, 126(2), 175–185.

Bažant, Z. P., and Ožbolt, J. (1990). "Nonlocal microplane model for fracture, damage, and size effect in structures." *J. Engrg. Mech.*, ASCE, 116(11), 2485–2505.

Bažant, Z. P., and Pijaudier-Cabot, G. (1988). "Nonlocal continuum damage, localization instability and convergence." *J. Appl. Mech.—T. ASME*, 55, 287–293.

Bažant, Z. P., and Planas, J. (1998). *Fracture and size effect in concrete and other quasibrittle materials*. CRC Press, Boca Raton, Fla.

Bažant, Z. P., and Xi, Y. (1991). "Statistical size effect in quasi-brittle structures. II: Nonlocal theory." *J. Engrg. Mech.*, ASCE, 117(11), 2623–2640.

Bažant, Z. P., Xi, Y., and Reid, S. G. (1991). "Statistical size effect in quasi-brittle structures. I: Is Weibull theory applicable?" *J. Engrg. Mech.*, ASCE, 117(11), 2609–2622.

Brenner, C. E. (1991). "Stochastic finite element methods (literature review)." *Working Rep. No. 35–91*, Inst. of Engrg. Mech., University of Innsbruck, Austria.

Breysse, D. (1990). "Probabilistic formulation of damage evolution law of cementitious composites." *J. Engrg. Mech.*, ASCE, 116(7), 1489–1510.

Breysse, D., and Fokwa, D. (1992). "Influence of disorder of the fracture process of mortar." *Proc., 1st Int. Conf., Fracture Mech. of Concrete Struct.*, Z. P. Bažant, ed., Elsevier, London, 536–541.

Breysse, D., Fokwa, D., and Drahy, F. (1994). "Spatial variability in concrete: Nature, structure, and consequences." *Appl. Mech. Rev.*, ASME, 47(1), 184–196.

Breysse, D., and Renaudin, P. (1996). "On the influence of local disorder on size effect." *Proc., IUTAM Symp., Size-Scale Effects in Failure Mechanisms of Mat. and Struct.*, A. Carpinteri, ed., E&FN Spon, London, 187–199.

Bucher, C. G., and Bourgund, U. (1987). "Efficient use of response surface method." *Working Rep. No. 9-87*, Inst. of Engrg. Mech., University of Innsbruck, Austria.

Carpinteri, A., Chiaia, B., and Ferro, G. (1994). "Multifractal scaling law for the nominal strength variation of concrete structures." *Size effect in concrete structures*, H. Mihashi, H. Okamura, and Z. P. Bažant, eds., E&FN Spon, London, 193–206.

Carpinteri, A., Chiaia, B., and Ferro, G. (1995). "Size effect on nominal tensile strength of concrete structures: Multifractality of material liga-

ments and dimensional transition from order to disorder.'' *Mat. Struct.*, Paris, 28, 311–317.

Fisher, R. A., and Tippett, L. H. C. (1928). ''Limiting forms of the frequency distribution of the largest and smallest member of a sample.'' *Proc., Cambridge Philosophical Society*, 24, 180–190.

Fréchet, M. (1927). ''Sur la loi de probabilité de l'écart maximum.'' *Ann. Soc. Polon. Math.* (Cracow) 6, 93.

Ghanem, R., and Spanos, P. D. (1991). *Stochastic finite elements: A spectral approach*. Springer, New York.

Hillerborg, A., Modéer, M., and Petersson, P.-E. (1976). ''Analysis of crack formation and crack growth in concrete by means of fracture mechanics and finite elements.'' *Cement and Concrete Res.*, 6(6), 773–782.

Hisada, T., and Nakagiri, S. (1985). ''The role of the stochastic finite element method in structural safety and reliability.'' *Proc., Int. Conf. on Struct. Safety and Reliability*, Vol. I, Elsevier, Amsterdam, 385–394.

Jirásek, M., and Bažant, Z. P. (1995a). ''Macroscopic fracture characteristics of random particle systems.'' *Int. J. Fracture*, 69(3), 201–228.

Jirásek, M., and Bažant, Z. P. (1995b). ''Particle model for quasibrittle fracture and application to sea ice.'' *J. Engrg. Mech.*, ASCE, 121(9), 1016–1025.

Liu, W. K., Belytschko, T., and Mani, A. (1986). ''Random field finite elements.'' *Int. J. Numer. Methods in Engrg.*, 23, 1831–1845.

Madsen, H. O., Krenk, S., and Lind, N. C. (1986). *Methods of structural safety*. Prentice-Hall, Englewood Cliffs, N.J.

Mariotte, E. (1686). *Traité du mouvement des eaux*, posthumously edited by M. de la Hire; English translation by J. T. Desvaguliers, London (1718), 249; also *Mariotte's collected works*, 2nd ed., The Hague (1740).

Petersson, P.-E. (1981). ''Crack growth and development of fracture zone in plain concrete and similar materials.'' *Rep. No. TVBM-1006*, Div. of Build. Mat., Lund Institute of Technology, Lund, Sweden.

Pijaudier-Cabot, G., and Bažant, Z. P. (1987). ''Nonlocal damage theory.'' *J. Engrg. Mech.*, ASCE, 113(10), 1512–1533.

Planas, J., Guinea, G. V., and Elices, M. (1995). ''Rupture modulus and fracture properties of concrete.'' *Fracture mechanics of concrete structures*, Vol. 1, F. H. Wittmann, ed., Aedificatio Publishers, Freiburg, Germany, 95–110.

Roelfstra, P. E., Sadouki, H., and Wittmann, F. H. (1985). ''Le béton numérique.'' *Mat. and Struct.*, 18, 327–335.

Rokugo, K., Uchida, Y., Katoh, H., and Koyanagi, W. (1995). ''Fracture mechanics approach to evaluation of flexural strength of concrete.'' *ACI Mat. J.*, 92(5), 561–566 (selected translation from Japan Concrete Institute).

Schuëller, G. I. (1997). ''Structural reliability—Recent advances.'' *Proc., Int. Conf. on Struct. Safety and Reliability*, Balkema, Rotterdam, The Netherlands, I, 3–35 (Freudenthal lecture).

Schuëller, G. I., Bucher, C. G., Bourgund, U., and Ouypornprasert, W. (1989). ''On efficient computational schemes to calculate structural failure probabilities.'' *Probabilistic Engrg. Mech.*, 4(1), 10–18.

Tippett, L. H. C. (1925). ''On the extreme individuals and the range of samples taken from a normal population.'' *Biometrika*, 17, 364–387.

Weibull, W. (1939). ''The phenomenon of rupture in solids.'' *Proc., Royal Swedish Inst. of Engrg. Res.* (Ingenioersvetenskaps Akad. Handl.) Stockholm, 153, 1–55.