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RANDOM GROWTH OF CRACK WITH *R*-CURVE: MARKOV PROCESS MODEL

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Abstract—A probabilistic model for the randomness of the progressive crack growth in a quasi-brittle material such as concrete is presented. The model consists of a Markov chain adapted to *R*-curve behavior. It yields the crack propagation probability in any loading step as well as the probability of failure at any stage of the fracture process. The *R*-curve is obtained from the given test data on the effect of structure size on the maximum loads. The standard deviation of the peak load is the minimum statistical information required. According to the available test results, this standard deviation is approximately a linear function of the crack propagation distance. The parameter estimation method is formulated and some applications are illustrated. © 1997 Elsevier Science Ltd

1. INTRODUCTION

WHILE THE fracture properties of all materials are random at least to some extent, the randomness is quite pronounced for highly heterogeneous materials such as concrete or other quasi-brittle materials, including various composites, rocks and ceramics. The previous investigations of fracture in concrete structures concentrated mainly on the deterministic and statistical behaviors at the maximum load[1–4]. A realistic theory is needed to predict the probabilistic nature of the steps in the crack growth process before the maximum load. This requires following the incremental jumps of the fracture process in a probabilistic manner.

A theoretical model for progressive crack growth is also an important topic in the reliability evaluation of existing concrete structures, in which two kinds of problems need to be solved. (1) If a given structure with a crack of a certain length has survived under the current loading level, then what is the failure probability for a given load increment? This problem is frequently encountered when the load in a building is to be changed or a traffic load heavier than the daily load must travel through a bridge. (2) If the structure has survived the given load increment, what is the probability that an existing crack in the structure grows by a certain length under the load increment? To answer these questions, a stochastic process model for crack growth is required.

There are many probabilistic models for random crack growth, such as the Markov chain models of Bogdanoff and Kozin[5] or Yuasa *et al.*[6], as well as other models[7]. These models, however, were mainly concerned with fatigue crack propagation in metals. But fatigue is a phenomenon of secondary interest for concrete structures, while crack growth under monotonic loading is of primary interest, and in any cases needs to be analyzed first. The simplest and most effective deterministic model for the monotonic crack growth in concrete is the *R*-curve (resistance curve) model. The model used in the present study takes into account the nonlinearity of monotonic fracture growth caused by the existence of a fracture process zone of a non-negligible size.

For monotonic loading, the weakest-link model has already been combined with the *R*-curve behavior, see Refs[8–10]. But these models took into account only the randomness of materials properties in the direction of specimen thickness. Another interesting probabilistic model has been developed by Chudnovsky and Kunin [11] and Mull and Chudnovsky [12], who assumed the surface energy of the material to be a random field. In this model, the crack propagates only from the weakest point in front of the crack tip. Experimental evaluation of the par-

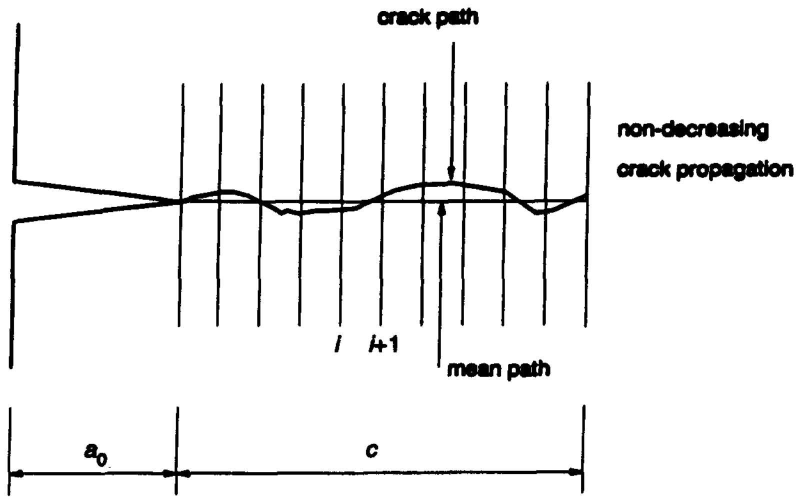

Fig. 1. Random crack growth in concrete.

ameters characterizing the random field of surface energy is required; however, this turns out to be a difficult problem for certain kinds of materials, such as cement-based materials.

The purpose of this paper is to develop a probabilistic model for quasi-brittle material such as concrete that can evaluate the probabilities of the crack growth under a given increment of loading as well as the probabilities of failure for a given increment of loading. The method will consist of a Markov chain model combined with the R-curve concept. The parameter estimation method for the model parameters will be formulated, and some applications will be presented.

2. MARKOV CHAIN MODEL FOR RANDOM CRACK GROWTH

The application of the Markovian hypothesis to damage evolution problems for fatigue cracks has been justified by many other authors [5–7]. The Markovian approach is based on the similarity between the process of progressive crack growth at monotonic loading and the process of random walk. Consider a crack of initial length a_0 and a crack extension of length c, as shown in Fig. 1. Only one of the possible crack paths is drawn in the figure, and the straight line indicates the mean crack path. As an example, taking the mean path as one realization of many possible crack paths, suppose the crack tip is at position i for the given applied load X. Then, for an infinitesimal load increment ΔX, the crack tip may at random either stay at point i or jump forward to point $i + 1$. The same happens for the next loading increment ΔX. The random nature of the crack propagation is due to the randomness of material resistance to crack propagation, which is caused primarily by the heterogeneity of the material. If the fracture resistance of the material were of deterministic nature, then for any ΔX there would be a corresponding increment of crack extension, Δc.

For a material with probabilistic properties, the crack propagation is a nondecreasing random process, because the crack tip can only stay fixed or move forward, but cannot move back. So, the nature of this process is the same as that of a one-way random-walk process. This process is illustrated in Fig. 2, where the state point can only stay still or move ahead at each stage. As one can see, such a process can be described by the Markov chain model.

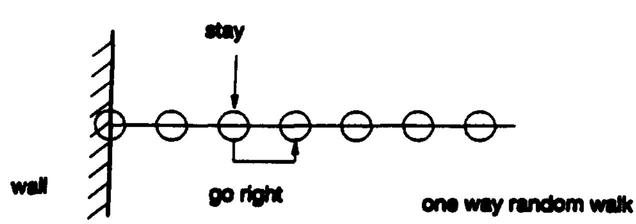

Fig. 2. One-way random walk model.

The Markov chain model is a stochastic model commonly used to analyze many kinds of accumulative damage processes[5]. In this paper, the damage of concrete is considered to be the crack. A typical set of sample curves of progressive crack growth is shown later in Fig. 5, where a, called the damage state, is an observable measure of the crack, which can be represented by the crack length or the crack mouth opening displacement (CMOD); a is a discrete variable ($a = 1,2,...,B$); B denotes the failure state, and X denotes the load (or a load parameter proportional to the load, such as the nominal stress).

The basic simple relations for the Markov chain model will now be introduced for the reader's convenience (details can be found in Refs [5, 13]. The well-known basic evolution equation for the Markov chain model is

$$\underset{\sim}{p}_x = \underset{\sim}{p}_0 \underset{\sim}{P}^x \tag{1}$$

where x is an integer, denoting the value of X (which is thus allowed to take only discrete values), and $\underset{\sim}{p}_0$ is the vector of initial state probability

$$\underset{\sim}{p}_0 = \{\pi_1, \pi_2, \ldots, \pi_B, 0\} \text{ with } \sum \pi_j = 1 \tag{2}$$

in which π_j is the probability that damage state j is initially occupied. Equation (1) means that the probability of crack advance depends only on the current state, i.e. is independent of the preceding states (the history). In the present study, we always assume that initially $\pi_1 = 1$, with all other $\pi_j = 0$, which means the crack or damage always starts from state 1. This assumption is close to reality when a problem such as concrete structures with existing major cracks is under consideration. $\underset{\sim}{p}_0$ is the vector of the damage state probability

$$\underset{\sim}{p}_x = \{p_x(1), p_x(2), \ldots, p_x(B)\} \tag{3}$$

in which $p_x(j)$ is the probability that damage state j is occupied at loading level x; $\underset{\sim}{P}$ is the probability transition matrix

$$\underset{\sim}{P} = \begin{bmatrix} p_{1,1} & p_{1,2} & p_{1,3} & \cdots & p_{1,B} \\ 0 & p_{2,2} & p_{2,3} & \cdots & p_{2,B} \\ 0 & 0 & p_{3,3} & \cdots & p_{3,B} \\ 0 & 0 & 0 & p_{B-1,B-1} & p_{B-1,B} \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \tag{4}$$

with

$$\sum_{j=1}^{n} p_{i,j} = 1, \quad i = 1, 2, \ldots, B-1 \tag{5}$$

where $p_{i,i}$ is the probability of remaining in state i during one loading step, $p_{i,j}$ is the probability that, in one loading step, the damage goes from state i to state j. This is the so-called multi-jump model since the crack tip can jump $j - i$ elements in one loading step. From eq. (4) one can see that there are $[(B^2 + B)/2] - (B - 1) - 1$ independent parameters needed to establish a transition matrix for a multi-jump model. These, however, are too many for experimental calibration.

A commonly used simplification is to limit the multi-jump model to a unit-jump model. In that case, eq. (4) becomes

$$\underset{\sim}{P} = \begin{bmatrix} p_{1,1} & p_{1,2} & 0 & \cdots & 0 \\ 0 & p_{2,2} & p_{2,3} & \cdots & 0 \\ 0 & 0 & p_{3,3} & \cdots & 0 \\ 0 & 0 & 0 & p_{B-1,B-1} & p_{B-1,B} \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}. \tag{6}$$

Then we have only $(B - 1) + 1 - 1$, i.e. $B - 1$, parameters to be evaluated in eq. (6). For convenience, we rewrite eq. (6) in the form

$$\underset{\sim}{P} = \begin{bmatrix} p_1 & q_1 & 0 & \dots & 0 \\ 0 & p_2 & q_2 & \dots & 0 \\ 0 & 0 & p_3 & \dots & 0 \\ 0 & 0 & 0 & p_{B-1} & q_{B-1} \\ 0 & 0 & \dots & 0 & 1 \end{bmatrix} \tag{7}$$

where p_i is the probability of remaining in state i during one loading step, and q_i is the probability that in one loading step the damage moves from state i to state $i + 1$.

A simpler model can be obtained if the process is assumed to be state independent. Then eq. (7) becomes

$$\underset{\sim}{P} = \begin{bmatrix} p & q & 0 & \dots & 0 \\ 0 & p & q & \dots & 0 \\ 0 & 0 & p & \dots & 0 \\ 0 & 0 & 0 & p & q \\ 0 & 0 & \dots & 0 & 1 \end{bmatrix} \tag{8}$$

where $p + q = 1$. As one can see, only two independent parameters, namely p and B, need to be evaluated.

Another concept that needs to be introduced is the failure rate function or hazard function, which is useful for reliability analysis. The definition of the hazard function is

$$h_x(x) = 1 - \frac{\bar{F}_x(x)}{\bar{F}_x(x-1)} \tag{9}$$

in which function $\bar{F}_x(x)$ is defined as

$$\bar{F}_x(x) = 1 - F_x(x) \tag{10}$$

$$F_x(x) = P\{X<x\} = p_x(B) \tag{11}$$

where $p_x(B)$ is the probability that the failure state B occurs at the loading level x; $F_x(x)$ and $\bar{F}_x(x)$ are the failure probability and the survival probability at loading level x, respectively. The physical meaning of the hazard function is that $h_x(x)\Delta x$ is the probability that the specimen fails in the loading increment $[x, x + \Delta x]$ given that it has survived through the loading $(0, x]$. As is clear from eqs (1)–(11), the transition matrix $\underset{\sim}{P}$ is the fundamental characteristic of the Markov chain model; it characterizes all of the statistical structure during the loading history.

Let the random variable $X_{1,B}$ denote the load at failure that is reached by starting in damage state 1 from the initial value $X = 0$. Then the mean and the first few central moments of $X_{1,B}$ are found to be[5]

$$E\{X_{1,B}\} = \sum_{j=1}^{B-1}(1 + r_j) \tag{12}$$

$$\text{Var}\,\{X_{1,B}\} = \sum_{j=1}^{B-1} r_j(1 + r_j) \tag{13}$$

$$\mu_3\{X_{1,B}\} = \sum_{j=1}^{B-1} r_j(1 + r_j)(1 + 2r_j) \tag{14}$$

$$\mu_4\{X_{1,B}\} = \sum_{j=1}^{B-1} r_j(1 + r_j)(1 + 2r_j)(1 + 3r_j) + \sum_{j=1}^{B-1} r_j^2(1 + r_j) + 3[\text{Var}\,\{X_{1,B}\}]^2 \tag{15}$$

where

$$r_j = \frac{p_j}{q_j}, \quad p_j = \frac{r_j}{1+r_j}, \quad q_j = \frac{1}{1+r_j}. \tag{16}$$

3. CRACK PROPAGATION IN CONCRETE

For concrete specimens, unfortunately, a precise measurement of the distance of crack propagation (i.e. the crack tip location) appears to be impossible because of tortuosity of the crack path, crack bridging and insufficient accuracy of the techniques for measuring the crack tip location. It has been attempted to overcome this problem by doing the calculations for an effective crack length defined in terms of linear elastic fracture mechanics (LEFM), that is, on the basis of the measured compliance. But the problem persists since it is unclear which one among several possible definitions of the effective crack length should be used (e.g. equal load-point compliance or equal crack mouth opening compliance). In fact, there are only very limited test results available for the load-crack extension relationship, while a large amount of test data is available for applied loads or failure loads in terms of CMOD or load-point displacement. So, instead of dwelling on how we should define the crack tip location and how precisely we should measure the crack extensions, we will pay more attention to the statistical scatter of the failure load.

First consider the simplest case, eq. (8). One of the advantages of using a state-independent model is that we do not need to know the crack length at the maximum load. From eqs (8), (12) and (13), one can find the mean of the maximum load, $\bar{X}_{max}$, and its standard deviation, s_{max}, which suffice to determine the two parameters r ($r = p/q$) and B (failure state) of the model because

$$\bar{X}_{max} = (B-1)(1+r) \tag{17}$$

$$s^2_{max} = (B-1)r(1+r). \tag{18}$$

Therefore

$$r = \frac{s^2_{max}}{\bar{X}_{max}} \tag{19}$$

$$B = \frac{\bar{X}^2_{max}}{\bar{X}_{max} + s^2_{max}} + 1. \tag{20}$$

But this two-parameter model does not define the evolution of the cumulative damage process, because r and B characterize just the mean and the standard deviation of the load at the failure state but tell us nothing about the states between the initial state and the failure state. Figure 3(a) illustrates that different mean curves can reach the same final state. Two sets of sample curves in Fig. 3(b) and Fig. 3(c) may have the same r and B but represent completely different damage evolution (a_{max} is the crack length corresponding to the maximum load). Therefore, in the two-parameter model there are no parameters that can take into account the shape of the damage evolution curves.

To determine damage evolution before the peak load, that is to single out the unique mean curve representing a specific damage evolution, one must use a state-dependent process, i.e. Equation (7), in which parameters r_j ($j = 1,\ldots,B-1$) and B determine the shape of the mean curves. Figure 4 shows the two curves of load vs damage level determined by the two mean sample curves, for which either $r_1 < r_2 < \ldots r_{B-1}$ or $r_1 > r_2 > \ldots r_{B-1}$. Within the two curves in Fig. 4, there are $(B-1)!$ sample curves characterized by $r_1, r_2, \ldots, r_{B-1}$ in all possible different orders (combinations). Determination of the parameters r_j ($j = 1,\ldots,B-1$) and B requires knowledge of the mean curve and the standard deviations during the entire damage evolution process.

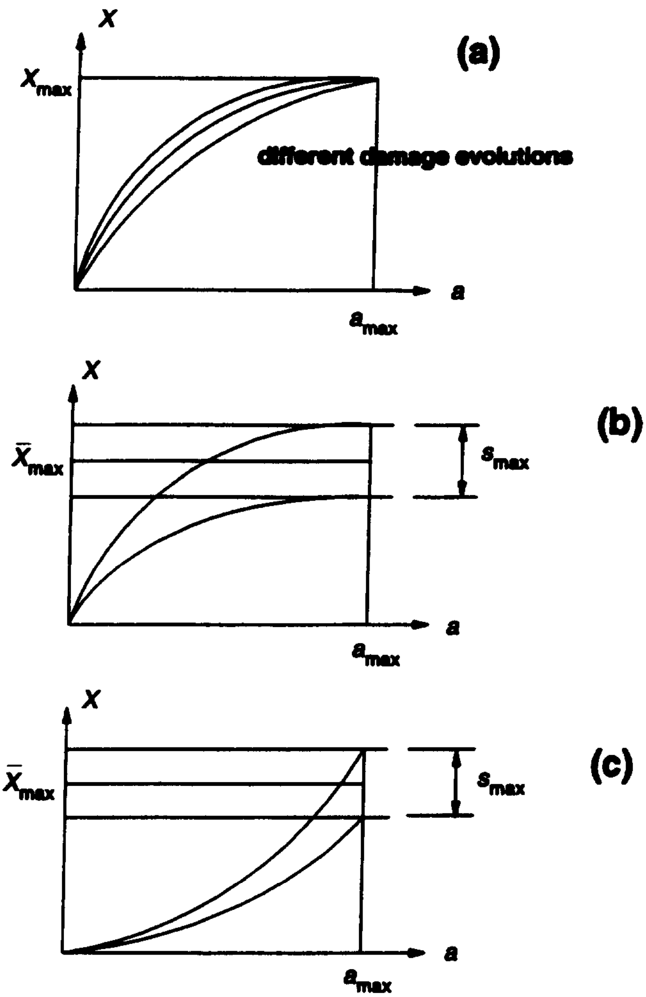

Fig. 3. Damage evolution.

In the case of real engineering problems, the maximum load and its deviation are the data most likely to be available, especially the failure loads measured on small specimens. So, a deterministic equation must be employed as the mean curve. One must realize that the deterministic equations obtained from fracture mechanics handbooks cannot be easily used for this purpose. The equation must be calibrated from the maximum load test results and averaged over all the specimens of different sizes. For quasi-brittle materials for which the dependence of the nominal strength on the specimen size is transitional between the strength theory and linear elastic fracture mechanics, the size effect law proposed by Bazǎnt [14] provides a suitable deterministic basis for the aforementioned purpose.

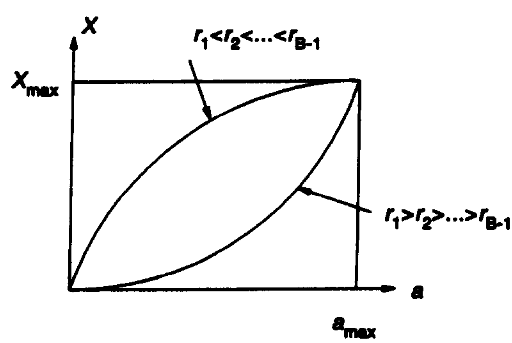

Fig. 4. Two bounds for damage evolution.

4. DETERMINISTIC R-CURVE

The deterministic equation for the nominal stress may generally be written in the form

$$\bar{X} = \frac{\sqrt{R(a-a_0)E_c}}{\sqrt{\pi a}F(a/d)} \tag{21}$$

where $\bar{X}$ represents the mean nominal stress (which is proportional to the applied load), E_c is the initial elastic modulus, $R(a-a_0)$ is the R-curve which represents the energy required for crack growth as a function of the crack extension c ($c = a - a_0$), $F(a/d)$ is a geometry-dependent function available for many different geometries of specimens from fracture mechanics handbooks (e.g. Ref. [15]), a is the current crack length, and a_0 is the initial crack length, that is, the length of the traction-free crack (or notch). The R-curve can be obtained using the following size effect law (proposed by Bazănt[14]):

$$\sigma_N = \frac{Af_u}{\sqrt{1+\dfrac{d}{d_0}}} \tag{22}$$

where f_u is the tensile strength, d is the characteristic dimension, and A and d_0 are two constants which can be identified by linear regression of the test results. $\sigma_N = c_n P_u/(bd)$ is the nominal strength of the specimen; P_u is the maximum load; b, d are the width and height of the beam; and c_n is a convenience factor which can make σ_N match some formula for maximum stress; in a beam, for example, the maximum elastic bending stress for a simply supported beam is matched by choosing $c_n = 1.5L/d$, where L is the span of the beam and L/d is constant for geometrically similar specimens. The size effect law is useful for materials for which linear elastic fracture mechanics are not valid [14]. The main consideration leading to eq. (22) is that the crack tip is surrounded by a large fracture process zone, which is typical for all quasi-brittle materials, such as concrete, rock, ice and toughening ceramics. Once the parameters Af_u and d_0 in eq. (22) are known, the R-curve can be obtained[1] as follows:

$$R(a-a_0) = G_f \frac{g'(\gamma)c}{g'(\alpha_0)c_f} \tag{23}$$

$$\frac{c}{c_f} = \frac{g'(\alpha_0)}{g(\alpha_0)}\left[\frac{g(\gamma)}{g'(\gamma)} - \gamma + \alpha_0\right] \tag{24}$$

$$G_f = \frac{(Af_u)^2}{c_n^2 E} d_0 g(\alpha_0) \tag{25}$$

$$c_f = \frac{d_0 g(\alpha_0)}{g'(\alpha_0)} \tag{26}$$

in which $\alpha_0 = a_0/d$, $g(\alpha_0)$ is the nondimensionalized energy release rate obtained from handbooks, γ is a working variable, G_f is the fracture energy equal to the critical energy release rate for an infinitely large specimen, and c_f is the effective fracture process zone length for an infinite size of specimen. By choosing a series of γ values, the corresponding c values are obtained from eq. (24) and then, substituting each c into eq. (23), the corresponding R-curve values are calculated. Equation (21) will then represent the mean curve of the nominal stress as a function of the crack length.

5. MEASUREMENTS OF STANDARD DEVIATION OF NOMINAL STRESS BEFORE FAILURE LOAD

Consider now the standard deviation of load or nominal stress from the mean curve (that is the variance at each damage state) before the maximum load. The literature search has shown that there has not been much research on the statistical analysis of crack propagation for concrete. In order to determine the standard deviation, three-point bend tests of many concrete beams have been conducted in the laboratory. A detailed description of the tests can be found elsewhere[16, 17], only the relevant information will be presented in the present paper.

The geometry of the beams is shown later in Fig. 8. Beams of four different sizes were tested with beam depths of 5.08, 7.62, 10.16, and 15.24 cm (2, 3, 4, and 6 in). The thickness of all the concrete beams was 6.35 cm (2.5 in). This ratio was chosen mainly due to the restriction of the support span of the loading system. The ratio of the notch length to the beam depth was 0.25 for all specimens. The ratio of the total length of beam to the span was 1.2. In this way, all the beams are geometrically similar.

The aggregates used in the concrete consisted of gravel and river sand. The maximum aggregate size was 1.9 cm (3/4 in). The volume ratio of coarse to fine aggregate was 2, which was kept the same for all beams. In this manner, the effect of aggregate size on the fracture properties was eliminated. The water–cement ratio was 0.5 for all the beams. All specimens were tested after 14 days of curing in a fog room.

The three-point bend beam tests were performed with an Instron close-loop control testing system with crack mouth opening displacement (CMOD) control. The loading rate was controlled such that the beams reached their maximum loads in about 10 min. To examine the effect of aggregate contents, four different volume fractions were used: 0.45, 0.55, 0.65, and 0.75. Four sizes of beams were tested for each volume fraction of aggregate, and three specimens for each size, and thus 48 beams in total were tested. Since the aggregate volume fractions $V_a = 55\%$ and 65% are commonly used in the construction industry, the results of those beams with $V_a = 55\%$ and 65% are used in the present study. The typical load-CMOD curves with $V_a = 55\%$ are shown in Fig. 5.

To normalize the curves in Fig. 5, the load is divided by the average load and the CMOD is divided by the CMOD at the maximum load. Also, only the curves before the peak loads are shown since the present study concerns only the deviation in the ascending part of the curves.

From the normalized CMOD-load curves, the variance at different loading levels is computed as shown in Fig. 6. The basic trend observed from Fig. 6 is that, with increasing CMOD, the variance increases with certain fluctuation. At the initial state at which CMOD = 0, the variance can be considered to be zero, the small deviation at the initial state as shown in Fig. 6 is due to the pre-load before switching from the load control to the CMOD control in the loading process, which can also be seen in Fig. 5.

On the other hand, the R-curve approach described in the last section requires the crack extension c or the crack length a. This means that the variance at state j should be expressed as a function of the crack length a. However, as stated previously, the technique for testing frac-

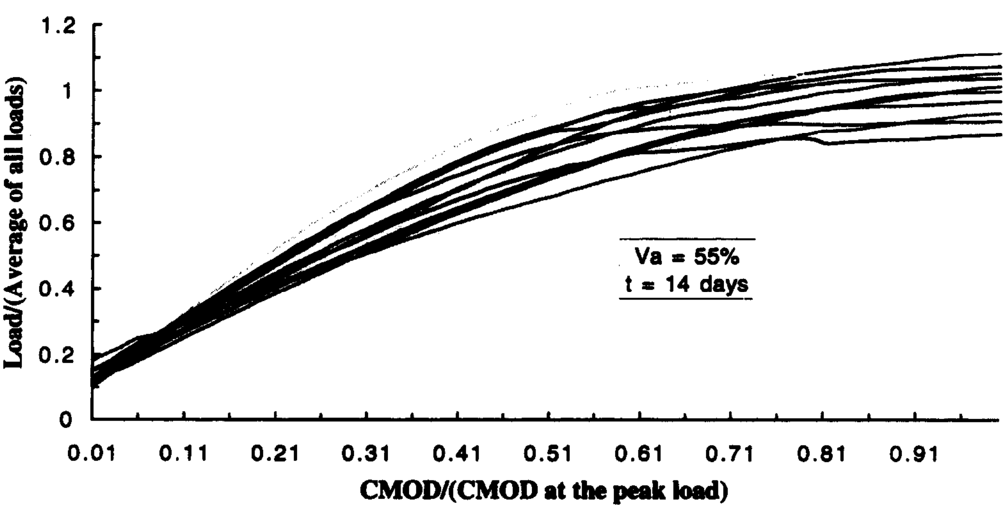

Fig. 5. Sample curves of CMOD – various loads.

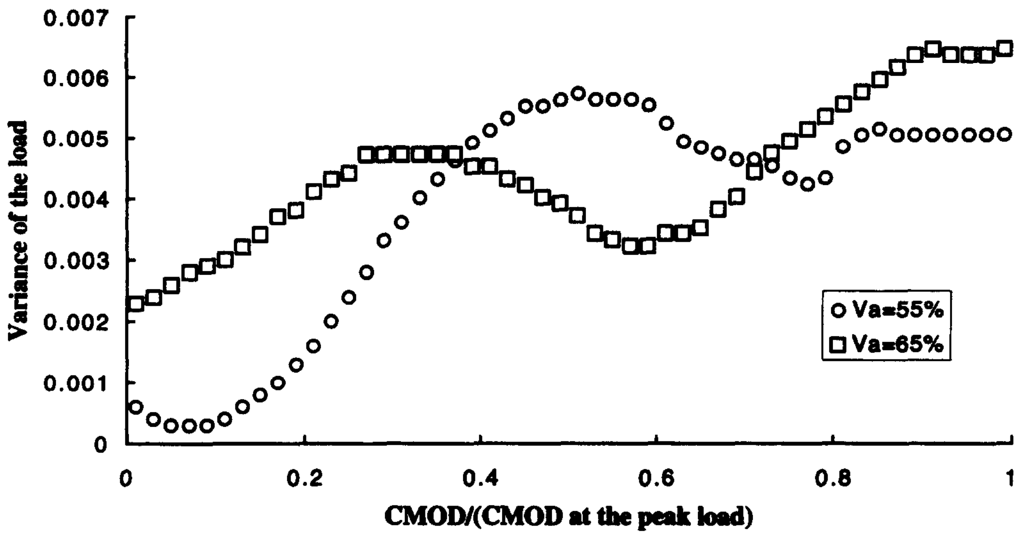

Fig. 6. Scattering of the load vs CMOD/(CMOD at peak load).

ture under crack length control is not available. Based on the statistical analysis of the measured CMOD-load curves and as a first approximation, we assume that the deviation of the load in terms of the crack extension would be similar to the CMOD-load curves. Then, an additional assumption may be introduced to set up the model, that is, the standard deviation of the load prior to the maximum load may be assumed to be a linear function of the crack extension (or crack length) $c = a - a_0$, i.e.

$$s_j^2 = \frac{(a_j - a_0)}{(a_{\max} - a_0)} s_{\max}^2 \tag{27}$$

where $a_{\max}$ can be obtained from eq. (21); $s_{\max}^2$ represents the variance of the maximum load, which is size dependent but may perhaps be considered for larger specimens to be approximately size independent. From eq. (27) the standard deviation of the entire process of crack propagation may be predicted solely from the standard deviation of the maximum loads of specimens, $s_{\max}$. This is very advantageous for practical applications because $s_{\max}$ is often the only information available.

6. MARKOV CHAIN MODEL COMBINED WITH R-CURVE

Based on eqs (21) and (27), we can derive the expression for the parameters in eq. (7) and damage state B_j. First, we divide the damage states from 1 to $B - 1$ into m groups as follows: $1,\ldots,B_1{-}1$; $B_1,\ldots,B_2{-}1$; and $B_{m-1},\ldots,B-1$. Then we assume r_1 for $1,\ldots,B_1{-}1$; r_2 for $B_1,\ldots,B_2{-}1$; and r_m for $B_{m-1},\ldots,B-1$. For $m = 1$, eqs (12) and (13) become

$$\bar{X}_1 = (B_1 - 1)(1 + r_1) \tag{28}$$

$$s_1^2 = (B_1 - 1)(1 + r_1)r_1 \tag{29}$$

where $\bar{X}_1$ is a given value and a_1, corresponding to $\bar{X}_1$, is given by eq. (21). After evaluating s_1^2 from eq. (27), B_1 and r_1 can be solved from eqs (28) and (29). Then, for $m = 2$

$$\bar{X}_2 = (B_1 - 1)(1 + r_1) + (B_2 - B_1)(1 + r_2) \tag{30}$$

$$s_2^2 = (B_1 - 1)(1 + r_1)r_1 + (B_2 - B_1)(1 + r_2)r_2 \tag{31}$$

B_2 and r_2 can now be solved. Continuing in this manner, we find the formula for any state j:

$$B_j = \frac{(\bar{X}_j - \bar{X}_{j-1})^2}{(\bar{X}_j - \bar{X}_{j-1}) + (s_j^2 - s_{j-1}^2)} + B_{j-1} \tag{32}$$

$$r_j = \frac{\bar{X}_j - \bar{X}_{j-1}}{B_j - B_{j-1}} - 1. \tag{33}$$

For $j = 1$ we have

$$B_1 = \frac{\bar{X}_1}{s_1^2 + \bar{X}_1} + 1 \tag{34}$$

$$r_1 = \frac{\bar{X}_1}{B_1 - 1} - 1. \tag{35}$$

After all the parameters, r_j and B_j, have been determined, the B_j need to be taken as integers and then eqs (1) and (7) can be used to calculate the damage state probability. Up to this point, we are able to answer the two questions posed in the Introduction of this paper. The first question is that, if a given structure with a crack of certain length survived under the current loading level, what is the failure probability for a given load increment? Now, the probability of reaching damage state j at stress level x is

$$F_x(x, j) = p_x(j). \tag{36}$$

When $j = B$, eq. (36) becomes eq. (11), which is the failure probability at load level x. The second question is that, if the structure survives the given load increment, what is the probability that an existing crack in the structure grows by a certain length under the load increment. Now, the hazard function is

$$h_x(x, j) = 1 - \frac{1 - p_x(j)}{1 - p_{x-1}(j)}. \tag{37}$$

This equation gives the probability that damage would develop from state j to state $j + 1$ if stress X increased from x to $x + 1$. When $j = B$, eq. (37) becomes eq. (9).

7. NUMERICAL EXAMPLE

Consider now, as an example, a notched three-point bend beam specimen of high strength concrete. The details of the tests can be found in ref. [2]. Based on the measured maximum loads of specimens of different sizes and eqs (21)–(26), the R-curve can be obtained and is shown in Fig. 7, and then the relation of the effective crack extension to the nominal stress can also be obtained and is shown in Fig. 8. From Fig. 8, the maximum nominal stress $X_{max} = 66$ psi, and the crack extension at the maximum load $c_{max} = 0.274$ in. The coefficient of variation of the maximum loads obtained from the test was about 15%. Therefore, the variance of the maximum loads, $s_{max}^2 = (0.15 \times 66)^2 = 98.01$. This value is used in eq. (27) for calculating the variances during the loading process.

The next step is to calculate damage state B_j by using eqs (32)–(35). First, one can assume a value for the total number of damage states B (70 for example), and the increment of the loading step can then be obtained by dividing the maximum load by B. At each loading step, the corresponding crack length (or crack extension) can be obtained by eq. (21) with the obtained R-curve, and the result is shown in Fig. 8. Secondly, using the obtained crack length at each loading step, a_j, and eq. (27), the variance of the load at each loading step, s_j^2, can be calculated. Then the parameters B_j and r_j can be determined from eqs (32) and (33) in which $\bar{X}_j$ is the load at loading step j. Since B_j must be integers, sometimes one perhaps has problems such as $B_j = B_{j-1}$, which leads to an infinite value of r_j as one can see from eq. (33). Computational experiences show that this problem may be avoided by reducing the assumed value of B and repeating the above described calculation until each B_j has different values. In this example, the proper value of B is 32. This means that when the load increases from 0 to 66 psi, c is changing

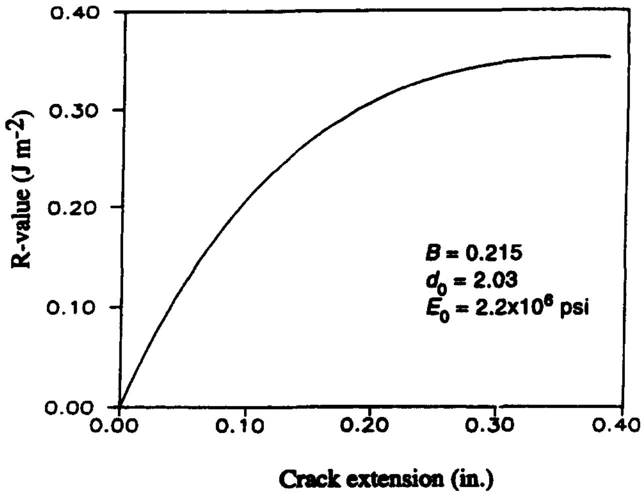

Fig. 7. R-curve.

from 0 to 0.274 in, and B_j is changing from 1 to 32. The relationship of B_j and the crack extension c (real damage state) obtained in this way is shown in Fig. 10.

The final step is to apply eq. (1) to calculate the probabilities for every loading step and every crack length. The result can be presented as a three-dimensional graph with the crack

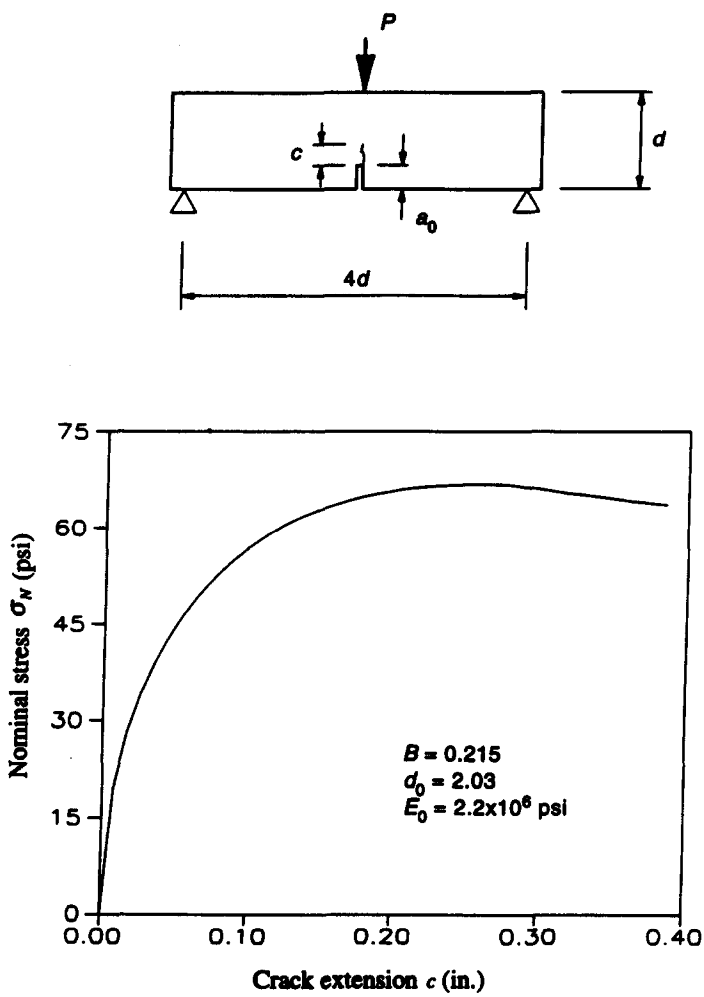

Fig. 8. Three-point bend fracture test.

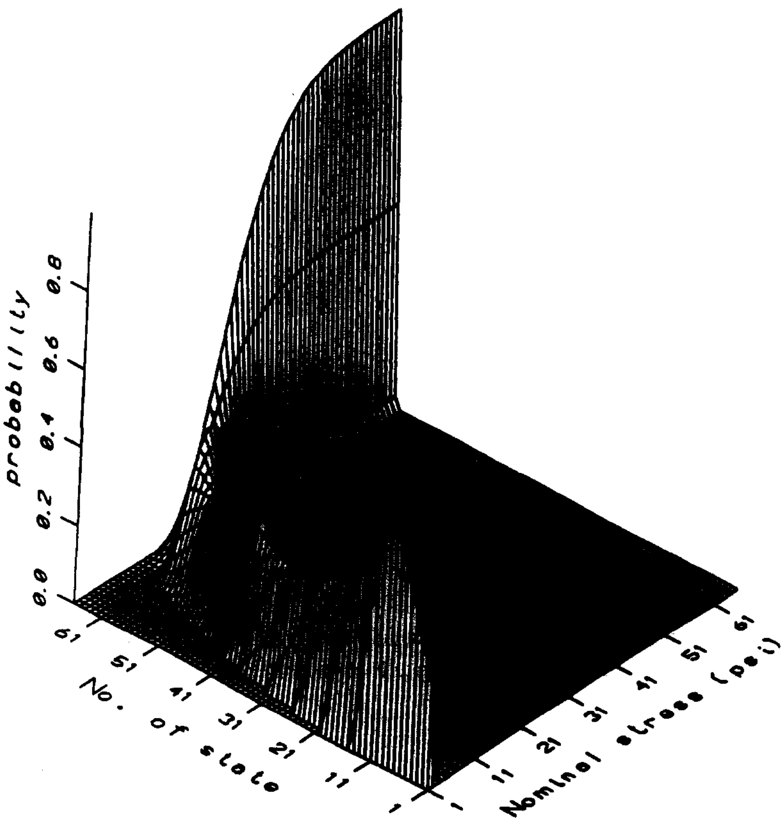

Fig. 9. Relation of the number of states in the crack extension process to the length of the crack extension.

extension c and the nominal stress X in a two-dimensional square mesh and the probability as the third dimension. The graph is shown in Fig. 10 in which the crack extension c is replaced by a sequence of integers called No. of state (No. of state = 1,2,...,66). The real crack length c can be obtained by c = (No. of state)c_{max}/X_{max}. For example, when No. of state = 66, $c = c_{max}$ (since X_{max} is 66 psi). The introduction of No. of state is just for the purpose of a better three-

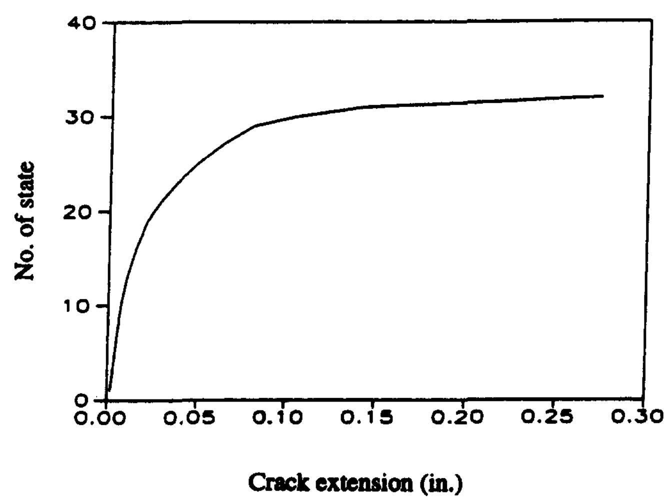

Fig. 10. Relation of the probability, nominal stress, and number of states in the crack extension process.

dimensional graphic presentation, because No. of state and the nominal stress X form a uniform square mesh.

One can see from Fig. 10 that when, for example, the nominal stress is 65 psi, which is near the maximum value of 66 psi, the probability of the occurrence of No. of state = 65 is very high, more than 90%. This is because No. of state = 65 represents the crack extension c = (No. of state)$c_{max}/X_{max} = 0.27$ in, which is almost the failure state. On the other hand, the probabilities of the occurrence of the lower damage states at the same nominal stress level are almost zero (as one can see from Fig. 10 for No. of state from 1 to 60, which corresponds to crack extension from 0.004 to 0.25 in). This must be true in reality, because at such a high loading level the probabilities of short crack extensions should be very small.

Figure 11 shows the cumulative density function for the maximum load, which is obtained from Fig. 10 by fixing No. of state as 66 (corresponding to $B_j = B$ and $c = c_{max}$). All the statistical information known prior to the present analysis has included only the statistics at the failure load, as shown in Fig. 11. By applying our analysis, we now know all the statistical information not only about the failure state but also about the process from the initial state up to the failure state, which is shown by Fig. 10. In the present example, the statistical information consists of the probability of the crack extension at each loading level (or at each nominal stress, which is proportional to the load), and of the probability of the crack length increment for a given increment of loading.

Another advantage of the present model is that sample curves of loading in terms of damage states can easily be simulated by computers. In this manner, the scatter band and the trend of damage evolution can be visualized. More importantly, one can see whether the developed model gives a realistic picture or not. The increment of the nominal stress, ΔX_j, between any two unit jumps of the damage state (as shown in Fig. 12(a)) is governed by a geometric distribution with parameter p_j. Then the sample curves can be generated as shown in Fig. 12(b). Figure 13 shows the relationship between the crack extension and the nominal stress. As one can see, the generated sample curves resemble the experimentally observed test curves quite well. This means that the present model can indeed realistically characterize the probabilistic structure for the entire loading history from the initial state up to the failure.

8. POSSIBLE GENERALIZATIONS

The foregoing analysis has dealt with the method to obtain the probabilistic structure prior to the maximum load by combining the Markov chain model with the maximum-load information alone, that is, with the *R*-curve determined solely from the maximum-load data. This approach is advantageous when the test data available for statistical analysis is limited. But

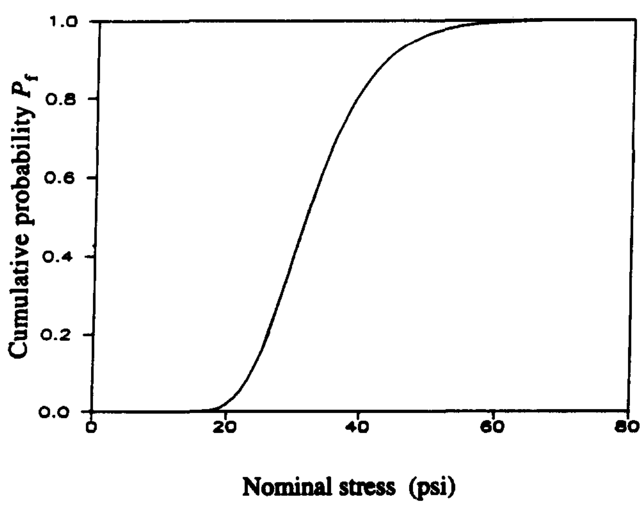

Fig. 11. Cumulative probability of failure load.

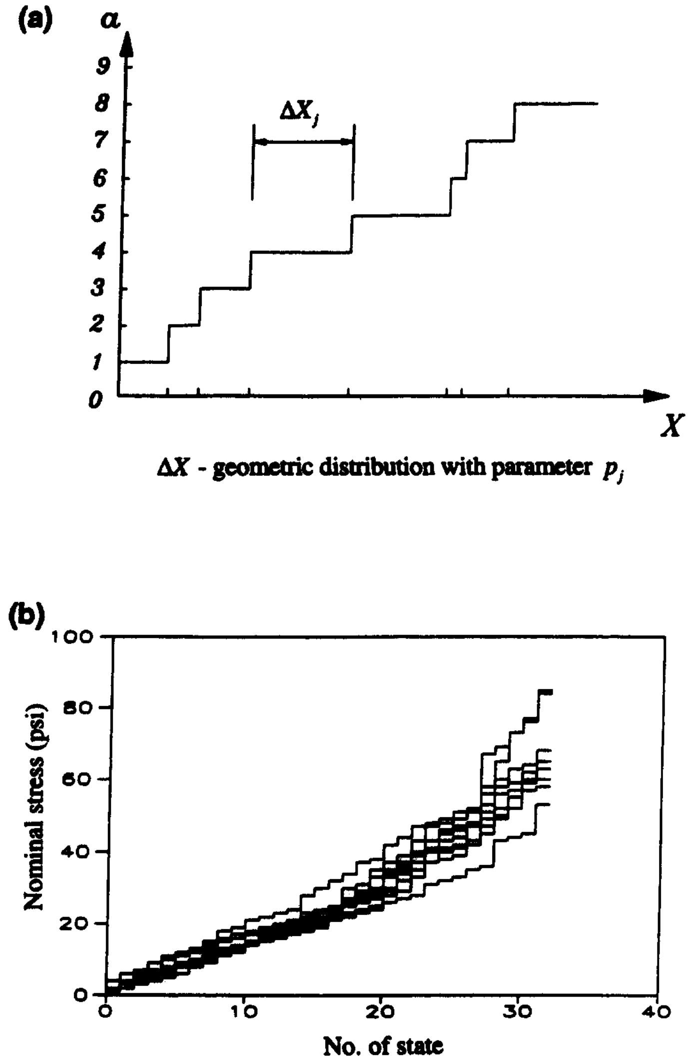

Fig. 12. Generated samples of nominal stress and number of states in the crack extension process.

when a large number of sample curves are available, it would be possible to approach the problem in the opposite way, that is to determine the R-curve behavior in the mean sense from the sample curves and the Markov chain model.

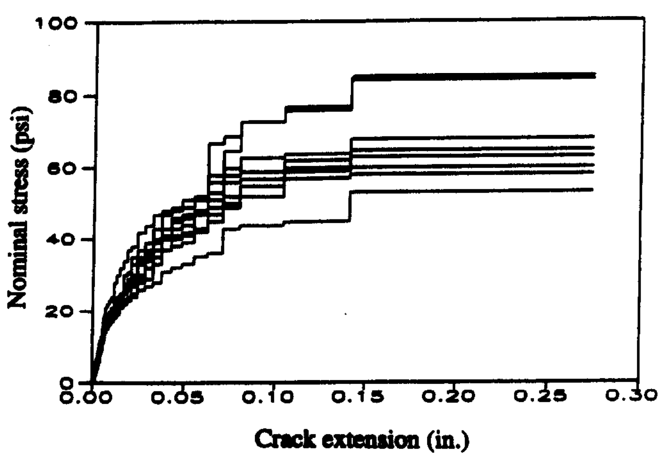

Fig. 13. Generated samples of nominal stress and crack extension.

There are other possible generalizations. The multi-jump model in eq. (4) could be used to simulate the activation energy and then investigate the statistical nature of the rate effect. Also, a two-way random walk model could be used to characterize material behaviors after the maximum load. But the solutions to all of these problems depend upon the development of the method for evaluating the model parameters from the available test data. These might be interesting topics for further research.

9. CONCLUSIONS

(1) The discrete Markov chain model can be used to evaluate the probabilistic structure of progressive cracking under monotonic loading in materials characterized by R-curve behavior. The crack propagation probability for an existing crack at any loading step before the maximum load and the probability of failure at any damage state can be calculated by the present method.

(2) The determination of model parameters in principle requires a large number of sample curves, which are difficult to obtain. Normally only the maximum load data are available and the number of maximum load samples alone does not suffice to obtain the statistical parameters required for the model. The basic idea to circumvent these limitations of the available data is to use, as a substitute, the mean curve obtained by fracture mechanics calculations.

(3) Three-point bend concrete beam tests are performed in order to determine the variation of the load along the loading path. The basic trend observed from the test results is that with increasing CMOD, the variance of the load increases with some fluctuations. At the initial state, at which CMOD = 0, the variance can be considered to be zero. Based on currently available test results, we introduce an assumption that the variance is a linear function of the CMOD.

(4) The R-curve obtained by size effect analysis of the measured maximum loads of geometrically similar specimens of different sizes is employed as the mean curve. In the framework of the R-curve model, the variance of the load needs to be considered as a function of the crack extension. Based on the statistical analysis of the CMOD-load curves and as a first approximation, we assume that the standard deviation of the load in terms of the crack extension evolves similarly to that of the CMOD-load curves. As a result, the standard deviation of the load prior to the maximum load may be assumed to be a linear function of the crack extension (or the crack length). Thus, the standard deviation of the entire process of crack propagation may be predicted solely from the standard deviation of the maximum loads of specimens, s_{max}. This is advantageous for practical applications because s_{max} is often the only information available.

(5) The probabilistic information on the damage evolution process predicted by the present model covers the entire monotonic loading history.

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REFERENCES

1. Bažánt, Z. P. and Kazemi, M. T., Size effect in fracture of ceramics and its use to determine fracture energy and effective process zone length. *Journal of American Ceramic Society*, 1990, **73**(7), 1841–1953.
2. Gettu, R., Bažánt, Z. P. and Karr, M. E., Fracture properties and brittleness of high-strength concrete. *ACI Material Journal*, 1990, **87**, 608–618.
3. Bažánt, Z. P. and Xi, Y., Statistical size effect in quasibrittle structures Part I. Is Weibull Theory applicable?. *Journal of Engineering Mechanics, ASCE*, 1990, **117**(11), 2609–2622.
4. Bažánt, Z. P. and Xi, Y., Statistical size effect in quasibrittle structures Part II. Nonlocal Theory. *Journal of Engineering Mechanics, ASCE*, 1990, **117**(11), 2623–2640.
5. Bogdanoff, J. L. and Kozin, F., *Probabilistic Models of Cumulative Damage*. John Wiley and Sons, New York, 1985.
6. Yuasa, M., Makagawa, T. and Tokuno, H., Analysis of fatigue crack propagation under random loading by Markov chain. *Proc. 5th Int. Conf. on Structural Safety and Reliability*, 1989, pp. 1531–1538.
7. Sobczyk, K. and Spencer Jr., B. F., *Random Fatigue: From Data to Theory*. Academic Press, New York, 1992.
8. Bruckner, A. and Munz, D., Scatter of fracture toughness in the brittle–ductile transition region of a ferritic steel. In *Advances in Probabilistic Fracture Mechanics, Pressure Vessels and Piping, 92*, ed. C. (Raj) Sundararajan, 1984, pp. 105–111.

9. Bruckner, A. and Munz, D., Prediction of failure probabilities for cleavage fracture from the scatter of crack geometry and of fracture toughness using the weakest link model. *Engineering Fracture Mechanics*, 1983, **18**(2), 359–375.
10. Sun, D., Dormagen, D. and Dahl, W., Statistical characterization of fracture in the brittle–ductile transition regime of structural steels. *Steel Research*, 1987, **58**(1), 40–45.
11. Chudnovsky, A. and Kunin, B., A probabilistic model of brittle crack formation. *Journal of Applied Physics*, 1987, **62**(10), 4124–4129.
12. Mull, M. A. and Chudnovsky, A., A probabilistic approach to the fracture toughness of composites. *Philosophical Magazine A*, 1987, **56**(3), 419–443.
13. Papoulis, A., *Probability, Random Variables, and Stochastic Processes.* McGraw-Hill Book Company, New York, 1984.
14. Bazănt, Z. P., Size effect in blunt fracture: concrete, rock, and metal. *Journal of Engineering Mechanics, ASCE*, 1984, **110**(4), 518–535.
15. Tada, H., *The Stress Analysis of Cracks Handbook.* Del Research Corp., St. Louis, MI, 1983.
16. Xi, Y., Amparano, F. E. and Li, Z., Effect of aggregate content on fracture behavior of concrete. *Proc. Materials Research Society Fall Meeting, Symposium Q*, 1995.
17. Xi, Y. and Amparano, F. E., Experimental study on the effect of aggregate content on fracture behaviors of concrete (in preparation).

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