Nonlocal Damage Theory Based on Micromechanics of Crack Interactions

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Abstract: A nonlocal continuum model for strain-softening damage is derived by micromechanics analysis of a macroscopically nonhomogeneous (nonuniform) system of interacting and growing microcracks, using Kachanov's simplified version of the superposition method. The homogenization is obtained by seeking a continuum field equation whose possible discrete approximation coincides with the matrix equation governing a system of interacting microcracks. The result is a Fredholm integral equation for the unknown nonlocal inelastic stress increments, which involves two spatial integrals. One integral, which ensues from the fact that crack interactions are governed by the average stress over the crack length rather than the crack center stress, represents short-range averaging of inelastic macro-stresses. The kernel of the second integral is the long-range crack influence function which is a second-rank tensor and varies with directional angle (i.e., is anisotropic), exhibiting sectors of shielding and amplification. For long distances r, the weight function decays as r^{-2} in two dimensions and as r^{-3} in three dimensions. Application of the Gauss-Seidel iteration method, which can conveniently be combined with iterations in each loading step of a nonlinear finite element code, simplifies the handling of the nonlocality by allowing the nonlocal inelastic stress increments to be calculated from the local ones explicitly. This involves evaluation of an integral involving the crack influence function, for which closed-form expressions are derived. Because the constitutive law is strictly local, no difficulties arise with the unloading criterion or continuity condition of plasticity.

INTRODUCTION

The nonlocal continuum—a concept introduced in elasticity by Eringen (1965, 1966), Kröner (1967), and others [see Bažant (1986)]—is a continuum in which the stress at a given point depends not only on the strain at that point but on the deformation of a certain neighborhood. As is now generally accepted, finite element analysis of distributed strain-softening damage, including its final localization into sharp fracture, requires the use of some type of nonlocal continuum (Bažant 1984; Bažant et al. 1984; Bažant 1986). An effective type is the nonlocal damage concept, in which the local damage or fracturing strain figuring in the incremental stress-strain relation is replaced by its spatial average (Pijaudier-Cabot and Bažant 1987; Bažant and Pijaudier-Cabot 1988; Bažant and Lin 1988a, 1988b; Bažant and Ožbolt 1990, 1992a, 1992b).

The argument for the nonlocal damage concept has been mainly computational—the need to limit localization of strain-softening damage to zones of nonzero volume. The physical explanation, on the other hand, has been mainly phenomenologic and empirical. Intuitively, it has been expected that the main source of nonlocality must be the interactions among adjacent microcracks. Certain micromechanics arguments based on a system of microcracks have been shown to lead to the nonlocal damage concept (Bažant 1987, 1991). However, interpretation of these arguments for the purpose of finite element analysis has not been clear. The interactions among the microcracks with simultaneous crack growth during the loading steps have not been taken into account, and the form of the spatial integral characterizing the nonlocal continuum has not been physically justified. The crack interactions have recently been analyzed by Pijaudier-Cabot and Bažant (1991), and Bažant and Tabbara (1992). However, the problem of determining the form of the spatial integral for nonlocal continuum has not been addressed in that context. It will be in this paper [the contents of which have been summarized at a recent conference (Bažant 1992a,b)].

A special case of nonlocal continuum models for strain softening, which will not be studied here, are the gradient models, which can be obtained from a Taylor series expansion of the nonlocal spatial integral (Bažant 1984). Much attention has recently been devoted to gradient-dependent plasticity of micropolar (or Cosserat) type (de Borst 1990, 1991; de Borst and Sluys 1991; Mühlhaus and Aifantis 1991; Vardoulakis 1989; Sluys 1992; Dietsche and William 1992). These models, however, have so far been justified mainly by the need to regularize the boundary value problem, while a physical justification from micromechanics is still lacking. Some microstructural physical arguments for micropolarity have been offered, but they have been vague and unconvincing.

Important contributions to micromechanics of cracking and damage have been made by Kachanov (1985, 1990), Chudnovsky et al. (1987), Ju and Lee (1991), Lee and Ju (1991), Ju (1990, 1991), Krajcinovic and Fonseka (1981), Benveniste et al. (1989) and others; see also the review in Bažant (1986). Most studies have so far been limited to the special problem of determining the effective elastic moduli of randomly microcracked solids that are on the macroscale in a statistically homogeneous state [for an excellent review, see Kachanov (1992)]. For this special problem, it has been possible to apply the homogenization methods for composites, such as Hill's self-consistent model, methods of periodic cells, methods of composite cylinders or composite spheres, variationally based bounds such as Hashin-Shtrickman bounds, statistical models for macrohomogeneous crack arrays, etc.

However, homogenization is not the principal, most difficult issue. Rather, it is the continuum smearing of damage that is spatially nonuniform (statistically nonhomogeneous). The smearing must preserve the essential interactions of cracks or other microdefects that govern localization of strain. This issue cannot be handled by homogenization methods because they apply only to macroscopically uniform fields. A different type of continuum model is required to handle localization. Such a model will be proposed in this paper.

Numerical studies with a finite element program are beyond the scope of this paper, but are already in progress (in collaboration with J. Ožbolt, using the microplane constitutive law). The results indicate that structural failures dominated by tension, shear or compression can all be modeled using the same nonlocal material characteristics, especially the same characteristic length. This has not been possible with the previous nonlocal models.

LOCAL AND NONLOCAL MACROSCOPIC STRESS-STRAIN RELATIONS

Finite element analysis of inelastic solids is generally carried out in small loading steps. For each of them the local constitutive law may be written in the incremental form

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Note. Discussion open until August 1, 1994. To extend the closing date one month, a written request must be filed with the ASCE Manager of Journals. The manuscript for this paper was submitted for review and possible publication on July 28, 1992. This paper is part of the *Journal of Engineering Mechanics*, Vol. 120, No. 3, March, 1994. ©ASCE, ISSN 0733-9399/94/0002-0593/$2.00 + $.25 per page. Paper No. 4481.

$$\Delta\boldsymbol{\sigma} = \mathbf{E} : (\Delta\boldsymbol{\varepsilon} - \Delta\boldsymbol{\varepsilon}'') = \mathbf{E} : \Delta\boldsymbol{\varepsilon} - \Delta\mathbf{S} \quad (1)$$

Here $\Delta\boldsymbol{\sigma}$, $\Delta\boldsymbol{\varepsilon}$ = increments of the stress and strain tensors, $\mathbf{E}$ = fourth-rank tensor of elastic moduli of uncracked material, $\Delta\boldsymbol{\varepsilon}''$ = inelastic strain increment tensor, and $\Delta\mathbf{S}$ = inelastic stress increment tensor. In a nonlocal continuum formulation, (1) is replaced by

$$\Delta\boldsymbol{\sigma} = \mathbf{E} : \Delta\boldsymbol{\varepsilon} - \Delta\bar{\mathbf{S}} \quad (2)$$

$\Delta\bar{\mathbf{S}}$ = the nonlocal inelastic stress increment tensor, which has been defined in recent works by the spatial averaging integral:

$$\Delta\bar{\mathbf{S}}(x) = \int_V \alpha(x, \xi)\Delta\mathbf{S}(\xi)\, dV(\xi) \quad (3)$$

V = volume of the body; $\mathbf{x}$, $\boldsymbol{\xi}$ = coordinate vectors; and $\alpha(\mathbf{x}, \boldsymbol{\xi})$ = given nonlocal weight function. When $\Delta\mathbf{S}(\mathbf{x})$ = a uniform field, $\Delta\bar{\mathbf{S}}(\mathbf{x}) = \Delta\mathbf{S}(\mathbf{x})$ must represent a possible solution. Hence the normalizing condition

$$\int_V \alpha(\mathbf{x}, \boldsymbol{\xi})\, dV(\boldsymbol{\xi}) = 1 \quad (4)$$

NONLOCALITY CAUSED BY INTERACTION OF GROWING MICROCRACKS

The main source of postpeak strain softening is the gradual spread of distributed microcracking. Accordingly, consider an increment of prescribed loads or boundary displacements for an elastic solid that contains, at the beginning of the load step, many microcracks numbered as $\mu = 1, \ldots, N$. On the macroscale, the microcracks are considered to be smeared, as required by a continuum model. Exploiting the principle of superposition, we may decompose the loading step into two substeps:

1. In the first substep, the cracks (already opened) are imagined temporarily "frozen" (or "filled with a glue"), that is, they can neither grow and open wider nor close and shorten. Also, no new cracks can nucleate. The stress increments, caused by strain increments $\Delta\varepsilon$ and transmitted across the temporarily frozen (or glued) cracks (I in Fig. 1), are then simply given by $\mathbf{E} : \Delta\boldsymbol{\varepsilon}$. This is represented by the line segment $\overline{13}$ (Fig. 2) having the slope of the initial elastic modulus E.
2. In the second substep, the prescribed boundary displacements and loads are held constant, the cracks are "unfrozen" (or "unglued"), and the stresses transmitted across the cracks are relaxed, which is equivalent to applying pressures (surface tractions) on the crack faces (II in Fig. 1). In response to this pressure, the cracks are now allowed to open wider and grow (remaining critical according to the crack propagation criterion), or to close and shorten. Also, new cracks are now allowed to nucleate.

If no cracks grew or closed (nor new cracks nucleated), the unfreezing (or unglueing) at prescribed increments of loads or boundary displacements that cause macrostrain increment $\Delta\boldsymbol{\varepsilon}$ would engender the stress drop $\overline{34}$ down to point 4 on the secant line $\overline{01}$ (Fig. 2). The change of state of the solid would then be calculated by applying the opposite of this stress drop onto the crack surfaces. However, when the cracks propagate (and new

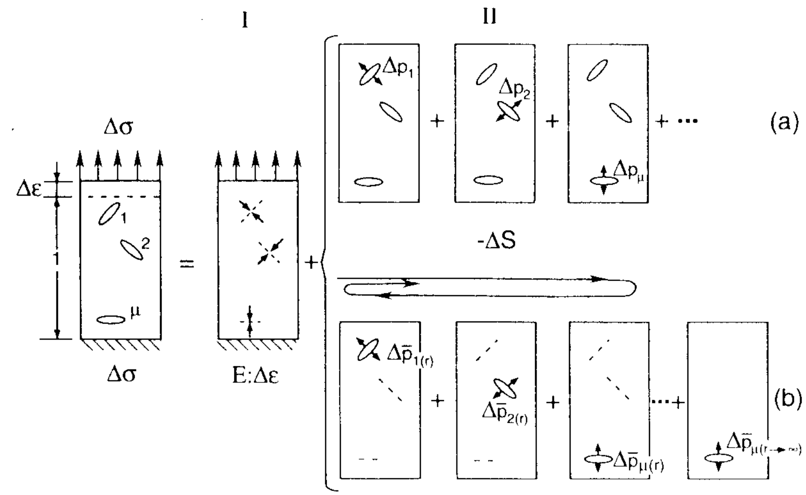

FIG. 1. Superposition Method for Solid with Many Cracks (*a* and *b* are Two Alternatives)

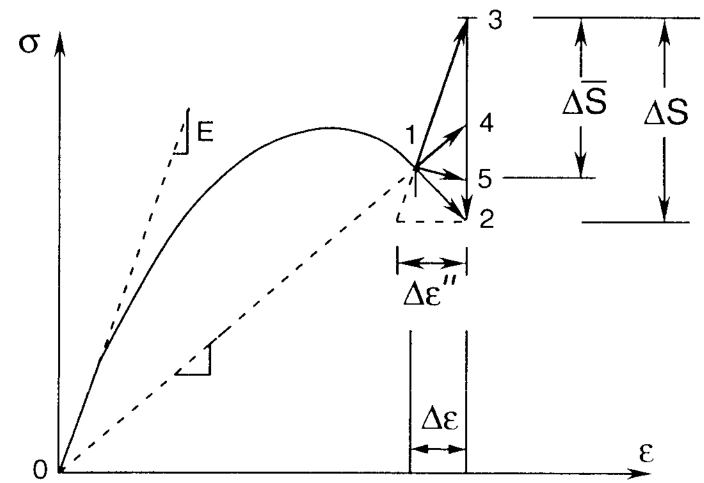

FIG. 2. Local and Nonlocal Inelastic Stress Increments during Loading Step

cracks nucleate), a larger stress drop defined by the local strain-softening constitutive law and represented by the segment $\Delta S = \overline{32}$ in Fig. 2 takes place. Thus, the normal surface tractions

$$\Delta p_\mu = \mathbf{n}_\mu \Delta\mathbf{S}_\mu \mathbf{n}_\mu \quad (5)$$

representing the normal component of tensor $\Delta\mathbf{S}_\mu$, must be considered in the second substep as loads Δp_μ that are applied onto the crack surfaces (Fig. 1), the unit normals of which are denoted as $\mathbf{n}_\mu$ (a product with no product sign denotes here a product of tensors contracted on one index; often it is written as the dot product, but here we omit the dot). Note also that for mode II or III cracks, a similar equation could in general be written for the tangential tractions on the crack faces.

Now we introduce two simplifying hypotheses:

1. Although the stress transmitted across each temporarily frozen crack varies along the crack, we consider only its average, i.e., Δp_μ is constant along each crack [Fig. 3(*a*)]. This approximation, which is crucial for our formulation, was introduced by Kachanov (1985, 1987). He discovered by numerical calculations that the error is negligible except for the rare case

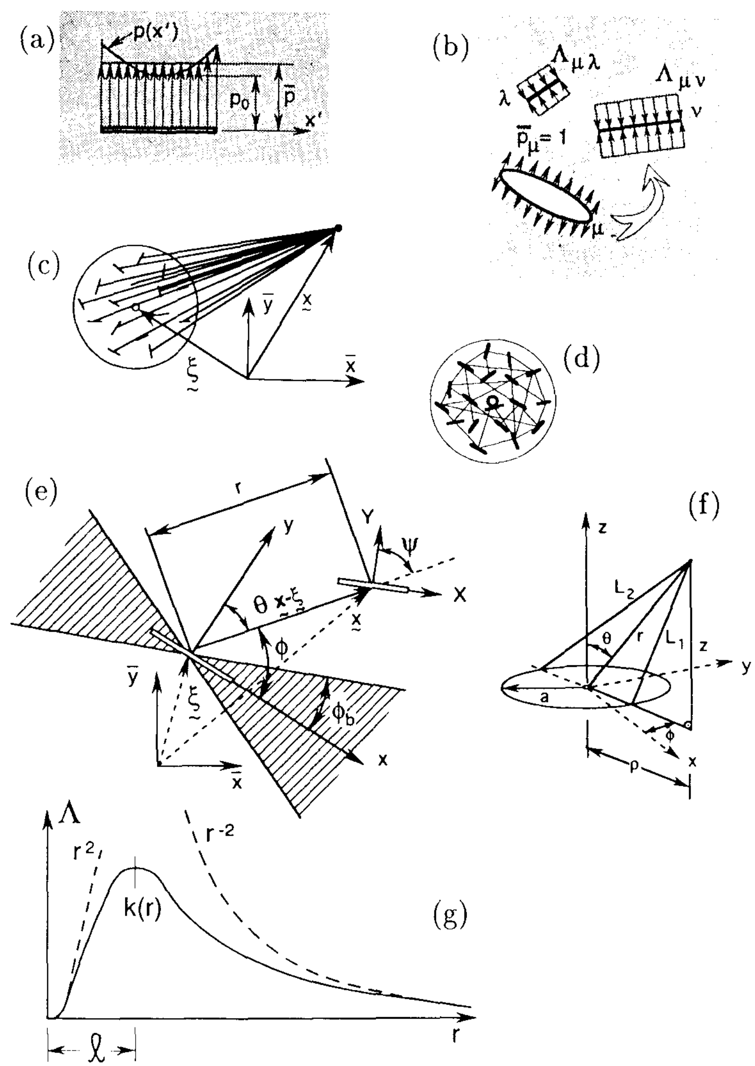

FIG. 3. Crack Interactions of Various Types, Their Radial and Angular Dependences, and Coordinates

when the distance between two crack tips is at least an order of magnitude less than their size.

2. We consider only mode I crack openings, i.e., neglect the shear modes (modes II and III). This is often justified, for instance in materials such as concrete, by a high surface roughness which prevents any significant relative slip of the microcrack faces (the mode II or III relative displacements that can occur on a macroscopic crack are mainly the result of mode I openings of microcracks that are inclined with respect to the macrocrack).

A simple-minded kind of superposition method would be to unfreeze all the cracks, load by pressure only one crack at a time, and then superpose all the cases (Fig. 1*a*). In this approach, the pressure on each crack, Δp_μ, would be known. But one would still have to solve a body with many cracks.

A better kind of superposition method is that adopted by Kachanov (1985, 1987), which was also used by Datsyshin and Savruk (1973), Gross (1982), Chudnovsky and Kachanov (1983), Chudnovsky et al. (1987), Chen (1984), and Horii and Nemat-Nasser (1985), and in a displacement version was introduced already by Collins (1963). In this kind of superposition, one needs to have the solution of the given body for the case of only *one* crack, with all the other cracks considered frozen (Fig. 1*a*). The cost to pay for this advantage is that the pressures to be applied at the cracks are unknown in advance and must be solved. By virtue of Kachanov's approximation, we apply this kind of superposition to the average crack pressures only. The opening and the stress intensity factor of crack μ are approximately characterized by the uniform (average) crack pressure $\Delta\bar{p}_\mu$ that acts on a single crack within the given solid that has elastic moduli E and contains no other crack. This pressure is solved from the superposition relation:

$$\Delta\bar{p}_\mu = \langle\Delta p_\mu\rangle + \sum_{\nu=1}^{N} \Lambda_{\mu\nu}\Delta\bar{p}_\nu \qquad \mu = 1, \ldots, N \quad \ldots\ldots\ldots (6)$$

$\langle \ldots \rangle$ is the averaging operator over the crack length; $\Lambda_{\mu\nu}$ = the crack influence coefficients representing the average pressure [Fig. 3(*a*)] at the frozen crack μ caused by a unit uniform pressure applied on unfrozen crack ν, with all other cracks being frozen [Fig. 3(*b*)]; and $\Lambda_{\mu\mu} = 0$ because the summation in (6) must skip $\nu = \mu$. The reason for the notation for $\Delta\bar{p}_\mu$ with an overbar instead of the operator $\langle \ldots \rangle$ is that the unknown crack pressure is uniform and thus its distribution over the crack area never needs to be calculated and no averaging operation actually needs to be carried out.

Note that the exact solution requires considering pressures $\Delta p_\mu(x')$ and $\Delta\bar{p}_\mu(x')$ that vary with coordinate x' along each crack. In numerical analysis, the crack must then be subdivided into many intervals. This could hardly be reflected on the macroscopic continuum level, but is doubtless unimportant at that level.

Substituting (5) into (6), we obtain

$$\Delta(\mathbf{n}_\mu\bar{\mathbf{S}}_\mu\mathbf{n}_\mu) = \langle\Delta(\mathbf{n}_\mu\mathbf{S}_\mu\mathbf{n}_\mu)\rangle + \sum_{\nu=1}^{N} \Lambda_{\mu\nu}\Delta(\mathbf{n}_\nu\bar{\mathbf{S}}_\nu\mathbf{n}_\nu) \quad \ldots\ldots\ldots (7)$$

The values of $\Delta\bar{\mathbf{S}}_\mu$ are graphically represented in Fig. 2 by the segment $\Delta\bar{S} = \overline{35}$. This segment can be smaller or larger than segment $\overline{32}$.

Now we adopt a third simplifying hypothesis: In each loading step, the influence of the microcracks at point ξ of the macrocontinuum upon the microcracks at point $\mathbf{x}$ of the macrocontinuum is determined only by the dominant microcrack orientation. This orientation is normal to the unit vector $\mathbf{n}_\mu$ of the maximum principal inelastic macrostress $\Delta\bar{S}^{(1)}$ at the location of the center of microcrack μ. We use the definition:

$$\Delta\bar{S}_\mu^{(1)} = \Delta(\mathbf{n}_\mu\bar{\mathbf{S}}_\mu\mathbf{n}_\mu) = \langle\mathbf{n}_\mu\bar{\mathbf{S}}_\mu\mathbf{n}_\mu\rangle_{new} - \langle\mathbf{n}_\mu\bar{\mathbf{S}}_\mu\mathbf{n}_\mu\rangle_{old} \quad (8)$$

The subscripts "new" and "old" denote the values at the beginning and end of the loading step, respectively. According to this hypothesis, the dominant crack orientation generally rotates from one loading step to the next. Eq. (7) may now be written as:

$$\Delta\bar{S}_\mu^{(1)} - \sum_{\nu=1}^{N} \Lambda_{\mu\nu}\Delta\bar{S}_\nu^{(1)} = \langle\Delta S_\mu^{(1)}\rangle \quad (9)$$

Alternatively, one might assume $\mathbf{n}_\mu$ to approximately coincide with the direction of the maximum principal strain. Such an approximation is simpler to use in finite element programs. It might be realistic enough, especially when the elastic strains are relatively small.

When the principal directions of the inelastic stress tensor $\mathbf{S}$ do not rotate, the increment operators Δ can of course be moved inside each product in (7), i.e., $\Delta(\mathbf{n}_\mu\bar{\mathbf{S}}_\mu\mathbf{n}_\mu) = \mathbf{n}_\mu\Delta\bar{\mathbf{S}}_\mu\mathbf{n}_\mu$, and so forth. One might wonder whether this should not be done even when these directions rotate (i.e., when $\mathbf{n}_\mu$ varies), which would correspond to crack orientations being fixed when the cracks begin to form. But according to the experience with the so-called rotating crack model, empirically verified for concrete, it is more realistic to assume that the orientation of the dominant cracks rotates with the principal direction of $\mathbf{S}$.

It might seem we should have taken in the foregoing equations only the positive part of $\Delta\mathbf{S}_\mu$. But this is not necessary since the unloading criterion prevents $\Delta\mathbf{S}_\mu$ from being negative.

FIELD EQUATION FOR NONLOCAL CONTINUUM

Now comes the most difficult step. We need to determine the nonlocal field equation for the macroscopic continuum, which represents the continuum counterpart of (9). The homogenization theories as known are inapplicable, because they apply only to macroscopically uniform fields while the nonuniformity of the macroscopic field is the most important aspect for handling localization problems. The following simple concept is proposed:

The continuum field equation we seek is an equation whose discrete approximation can be written in the form of the matrix crack interaction relation (9).

This concept leads us to propose the following field equation for the continuum approximation of microcrack interactions:

$$\Delta\bar{S}^{(1)}(\mathbf{x}) = \int_V \Lambda(\mathbf{x}, \xi)\Delta\bar{S}^{(1)}(\xi)\, dV(\xi) = \langle\Delta S^{(1)}(\mathbf{x})\rangle \quad (10)$$

because an approximation of the integral by a sum over the continuum variable values at the crack centers yields (9). Here we denoted $\Lambda(\mathbf{x}_\mu, \xi_\nu) = \mathscr{E}(\Lambda_{\mu\nu})/V_c$ = crack influence function, V_c = a constant that may be interpreted roughly as the volume per crack, and $\mathscr{E}$ = statistical averaging operator which yields the average (moving average) over a certain ap-

propriate neighborhood of point $\mathbf{x}$ or ξ. Such statistical averaging is implied in the macrocontinuum smoothing and is inevitable because in a random crack array the characteristics of the individual cracks must be expected to exhibit enormous random scatter.

It must be admitted that the sum in (9) is an unorthodox approximation of the integral from (10) because the values of the continuum variable are not sampled at certain predetermined points such as the chosen mesh nodes but are distributed at random, that is, the microcrack centers. Another point to note is that (9) is only one of various possible discrete approximations of (10). Since this approximation is not unique, at uniqueness of (10) as a continuum approximation is not proven. Therefore, acceptability of (10) will also depend on computational experience (which has so far been favorable).

When (10) is approximated by finite elements, it is again converted to a matrix form similar to (9). However, the sum then runs over the integration points of the finite elements. This means the crack pressures (or openings) that are translated into the inelastic stress increments are only sampled at these integration points, in the sense of their density, instead of being represented individually as in (9). Obviously, such a sampling can preserve only the long-range interactions of the cracks and the averaging. The individual short-range crack interactions will be lost, but they are so random and vast in number that aspiring to represent them in any detail would be futile anyway.

For macroscopic continuum smearing, the averaging operator $\langle. . .\rangle$ over the crack length now needs reinterpretation. Because of the randomness of the microcrack distribution, the macrocontinuum variable at point $\mathbf{x}$ should represent the spatial average of the effects of all the possible microcrack realizations within a neighborhood of point $\mathbf{x}$ whose size is roughly equal to the spacing l of the dominant microcracks (which is in concrete approximately determined by the spacing of the largest aggregates); hence

$$\langle\Delta S^{(1)}(\mathbf{x})\rangle = \int_V \Delta S^{(1)}(\xi)\alpha(\mathbf{x}, \xi)\, dV(\xi) \quad (11)$$

The weight function $\alpha(\mathbf{x}, \xi)$ is analogous to that in (3). It should vanish everywhere outside a domain of a diameter roughly equal to l. For computational reasons, it seems preferable that α have a smooth bell shape. Because of randomness of the microcrack distribution, function $\alpha(\mathbf{x}, \xi)$ may be considered as rotationally symmetric (i.e., same in all directions, or isotropic). Strictly speaking, the macroscopic averaging domain could be a line segment in the direction of the dominant microcrack [that is, normal to $\Delta S^{(1)}(\mathbf{x})$], or an elongated (roughly elliptical) domain. However, using a line segment seems insufficient for preventing damage localization into a line in the case of a homogeneous uniaxial tension field, and it would also be at variance with the energy release argument for nonlocality of damage presented in Bažant (1987, 1991).

Eq. (10) represents a Fredholm integral equation (i.e., an integral equation of the second kind with a square-integrable kernel) for the unknown $\Delta\bar{S}^{(1)}(\mathbf{x})$, which corresponds in Fig. 2 to the segment $\overline{35}$. The inelastic strain increment tensors $\Delta S^{(1)}(\mathbf{x})$ on the right-hand side, which correspond in Fig. 2 to the segment $\overline{32}$, are calculated from the strain increments using the given local constitutive law (for example the microplane model, continuum damage theory, plastic-fracturing theory, or plasticity with yield limit degradation).

SOME ALTERNATIVE FORMS AND PROPERTIES OF CONTINUUM MODEL

The solution of (10) can be written as:

$$\Delta\bar{S}^{(1)}(\mathbf{x}) = \langle\Delta S^{(1)}(\mathbf{x})\rangle - \int_V K(\mathbf{x}, \boldsymbol{\xi})\langle\Delta S^{(1)}(\boldsymbol{\xi})\rangle \, dV(\boldsymbol{\xi}) \quad \text{(12)}$$

in which function $K(\mathbf{x}, \boldsymbol{\xi})$ is the resolvent of the kernel $\Lambda(\mathbf{x}, \boldsymbol{\xi})$. (This resolvent could be calculated numerically in advance of the nonlocal finite element analysis, but it would not allow a simple physical interpretation and a closed-form expression.) With the notation

$$\Psi_{\mu\nu} = \delta_{\mu\nu} - \Lambda_{\mu\nu} \quad \text{(13)}$$

where $\delta_{\mu\nu}$ = Kronecker delta, (9) can be transformed to

$$\sum_\nu \Psi_{\mu\nu}\Delta\bar{S}^{(1)}_\nu = \langle\Delta S^{(1)}_\mu\rangle \quad \text{(14)}$$

The macrocontinuum counterpart of this discrete matrix relation is

$$\int_V \Psi(\mathbf{x}, \boldsymbol{\xi})\Delta\bar{S}^{(1)}(\boldsymbol{\xi}) \, dV(\boldsymbol{\xi}) = \langle\Delta S^{(1)}(\mathbf{x})\rangle \quad \text{(15a)}$$

$$= \int_V \Delta S^{(1)}(\boldsymbol{\xi})\alpha(\mathbf{x}, \boldsymbol{\xi}) \, dV(\boldsymbol{\xi}) \quad \text{(15b)}$$

which represents an integral equation of the first kind for the unknown function $\Delta\bar{S}^{(1)}(\boldsymbol{\xi})$. Obviously,

$$\Psi(\mathbf{x}, \boldsymbol{\xi}) = \delta(\mathbf{x} - \boldsymbol{\xi}) - \Lambda(\mathbf{x}, \boldsymbol{\xi}) \quad \text{(16)}$$

where $\delta(\mathbf{x} - \boldsymbol{\xi})$ = Dirac delta function in two or three dimensions; indeed, substitution of this expression into (15) yields (10).

Defining the inverse square matrix:

$$[B_{\mu\nu}] = [\Psi_{\mu\nu}]^{-1} \quad \text{(17)}$$

we may write the solution of the equation system (14) as

$$\Delta\bar{S}^{(1)}_\mu = \sum_\nu B_{\mu\nu}\langle\Delta S^{(1)}_\nu\rangle = \sum_\lambda C_{\mu\lambda}\Delta S^{(1)}_\lambda \quad \text{(18a)}$$

$$C_{\mu\lambda} = \sum_\nu B_{\mu\nu}\alpha_{\nu\lambda} \quad \text{(18b)}$$

with $\alpha_{\nu\lambda} = \alpha(\mathbf{x}_\nu, \boldsymbol{\xi}_\lambda)$. The macrocontinuum counterpart of the last equation is

$$\Delta\bar{S}^{(1)}(\mathbf{x}) = \int_V B(\mathbf{x}, \boldsymbol{\xi})\langle\Delta S^{(1)}(\boldsymbol{\xi})\rangle \, dV(\boldsymbol{\xi}) = \int_V C(\mathbf{x}, \boldsymbol{\xi})\Delta S^{(1)}(\boldsymbol{\xi}) \, dV(\boldsymbol{\xi}) \quad \text{(19)}$$

where $B(\mathbf{x}_\mu, \boldsymbol{\xi}_\nu) = \mathscr{E}(B_{\mu\nu})/V_c$ and $C(\mathbf{x}, \boldsymbol{\xi}) = \int_V B(\mathbf{x}, \boldsymbol{\xi})\alpha(\boldsymbol{\xi}, \mathbf{x}) \, dV(\boldsymbol{\xi})$. The kernel $B(\mathbf{x}, \boldsymbol{\xi})$ represents the resolvent of the kernel $\Psi(\mathbf{x}, \boldsymbol{\xi})$ of (15). Furthermore

$$B(\mathbf{x}, \boldsymbol{\xi}) = \delta(\mathbf{x} - \boldsymbol{\xi}) - K(\mathbf{x}, \boldsymbol{\xi}) \quad \text{(20)}$$

because substitution of this equation into (19) furnishes (12).

With (19), we have reduced the nonlocal formulation to the same form as (3) for the previous nonlocal damage formulation (Pijaudier-Cabot and Bažant 1987; Bažant and Pijaudier-Cabot 1989; Bažant and Ožbolt 1990, 1991, 1992). However, the presence of the Dirac delta function in the last equation makes (19) inconvenient for computations. Aside from that, it seems inconvenient to calculate in finite element codes function $B(\mathbf{x}, \boldsymbol{\xi})$. Another difference is that the weight function (i.e., the kernel) is anisotropic (and, in the present simplification, associated solely with the principal inelastic stresses).

Note also that if we would set $\Lambda(\mathbf{x}, \boldsymbol{\xi}) = 0$, the present formulation would become identical to the aforementioned previous nonlocal damage model. But this would not be realistic. The interactions characterized by $\Lambda(\mathbf{x}, \boldsymbol{\xi})$ appear to be essential.

Because the nonlocal integral in (22) is additive to the local stress ΔS, the present nonlocal model can be imagined as an overlay of two solids that are forced to have equal displacements at all points: (1) The given solid with all the damage due to cracks, but local behavior (no crack interactions); and (2) an overlaid solid that describes crack interactions only. The nonlocal stress $\Delta\bar{\mathbf{S}}$ represents the sum of the stresses from both solids. It is the stress that is to be used in formulating the differential equilibrium equations for the solid.

For the sake of simplicity, we have so far assumed that the influence of point $\boldsymbol{\xi}$ on point $\mathbf{x}$ depends only on the orientation of the maximum principal inelastic stress at $\boldsymbol{\xi}$. Since at $\boldsymbol{\xi}$ there might be cracks normal to all the three principal stresses (denoted now by superscripts i = 1, 2, 3 in parentheses), it might be more realistic to consider that each of them separately influences point $\mathbf{x}$. In that case, (9) and (10) can be generalized as follows:

$$\Delta\bar{S}^{(i)}_\mu - \sum_{\nu=1}^{N}\sum_{j=1}^{3} \Lambda^{(ij)}_{\mu\nu}\Delta\bar{S}^{(j)}_\nu = \langle\Delta S^{(i)}_\mu\rangle \quad \text{(21)}$$

$$\Delta\bar{S}^{(i)}(\mathbf{x}) - \int_V \sum_{j=1}^{3} \Lambda^{(ij)}(\mathbf{x}, \boldsymbol{\xi})\Delta\bar{S}^{(j)}(\boldsymbol{\xi}) \, dV(\boldsymbol{\xi}) = \langle\Delta S^{(i)}(\mathbf{x})\rangle \qquad (j = 1, 2, 3) \quad \text{(22)}$$

Similar generalizations can be made in the subsequent equations, too. Note that when the body is infinite, all the summations or integrations in this paper are assumed to follow a special path labeled by $\odot$, which will be defined in the next section.

The heterogeneity of the material, such as the aggregate in concrete, is not specifically taken into account in our equations. Although the heterogeneity obviously must influence the nonlocal properties [e.g., Pijaudier-Cabot and Bažant (1991)], this influence is probably secondary to that of microcracking. The reason is that the prepeak (hardening) inelastic behavior, in which microcracking is much less pronounced than after the peak while the heterogeneity is the same, can be adequately described by a local continuum. The main effect of heterogeneity (such as the aggregates in concrete, or grains in ceramics) is indirect; it determines the spacing, orientations, and configurations of the microcracks.

ADMISSIBILITY OF UNIFORM INELASTIC STRESS FIELDS

In the previous nonlocal formulations, the requirement that a field of uniform inelastic stress and damage must represent at least one possible solution led to the normalizing condition [(4)]. Similarly, we must now require that the homogeneous stress field $\Delta\bar{S}^{(1)} = \langle\Delta S^{(1)}\rangle$ satisfy (9) and (10) identically. This yields the conditions that the integral of $\Lambda(\mathbf{x}, \boldsymbol{\xi})$ or the sum of $\Lambda_{\mu\nu}$ over an infinite body vanish. However, the asymptotic behavior of $\Lambda(\mathbf{x}, \boldsymbol{\xi})$ for $r \to \infty$ which will be discussed later causes this integral or sum to be divergent. Therefore, the conditions must be imposed in a special form—the integral in polar coordinates is required to vanish only for a special path, labeled by $\odot$, in which the angular integration is completed before the limit $r \to \infty$ is calculated, that is

$$\int_V^{\odot} \Lambda(\mathbf{x}, \boldsymbol{\xi})\, dV(\boldsymbol{\xi}) = \lim_{R\to\infty} \int_0^R \left[\int_0^{2\pi} \Lambda(\mathbf{x}, \boldsymbol{\xi}) r\, d\phi\right] dr = 0 \quad \text{(for 2D)} \quad . (23a)$$

$$\int_V^{\odot} \Lambda(\mathbf{x}, \boldsymbol{\xi})\, dV(\boldsymbol{\xi}) = \lim_{R\to\infty} \int_0^R \left[\int_0^{2\pi}\int_0^{\pi} \Lambda(\mathbf{x}, \boldsymbol{\xi}) r^2 \sin\theta\, d\theta\, d\phi\right] dr = 0$$

(for 3D) . . . . (23b)

r and ϕ = polar coordinates, r, θ, and ϕ are spherical coordinates. Furthermore, labeling again by $\odot$ a similar summation path (or sequence) over all the cracks ν in an infinite body, the following discrete condition needs to be also imposed:

$$\sum_{\nu}^{\odot} \Lambda_{\mu\nu} = 0 \quad . . . . (24)$$

This condition applies only to an array of infinitely many microcracks that are, on the macroscale, perfectly random and distributed statistically uniformly over an infinite body (or are periodic). By the same reasoning, we must also have

$$\int_V^{\odot} K(\mathbf{x}, \boldsymbol{\xi})\, dV(\boldsymbol{\xi}) = 0 \quad . . . . (25)$$

$$\int_V^{\odot} \Psi(\mathbf{x}, \boldsymbol{\xi})\, dV(\boldsymbol{\xi}) = \int_V^{\odot} B(\mathbf{x}, \boldsymbol{\xi})\, dV(\boldsymbol{\xi}) = \int_V^{\odot} C(\mathbf{x}, \boldsymbol{\xi})\, dV(\boldsymbol{\xi}) = 1 \quad . . . . (26)$$

and in the discrete form

$$\sum_{\nu}^{\odot} \Psi_{\mu\nu} = \sum_{\nu}^{\odot} \alpha_{\mu\nu} = \sum_{\nu}^{\odot} B_{\mu\nu} = \sum_{\nu}^{\odot} C_{\mu\nu} = 1 \quad . . . . (27)$$

For integration paths in which the radial integration up to $r \to \infty$ is carried out before the angular integration, the foregoing integrals and sums are divergent.

GAUSS-SEIDEL ITERATION APPLIED TO NONLOCAL AVERAGING

For the purpose of finite element analysis, we will now assume that subscripts μ and ν label the numerical integration points of finite elements, rather than the individual microcracks. This means that the microcracks are represented by their mean statistical characteristics sampled only at the numerical integration points.

In finite element programs, nonlinearity is typically handled by iterations of the loading steps. Let us, therefore, examine the iterative solution of (9) or (14), which represents a system of N linear algebraic equations for N unknowns $\Delta\bar{S}_\nu^{(1)}$ if $\Delta S_\mu^{(1)}$ are given. The matrix of $\Psi_{\mu\nu}$ is in general nonsymmetric (because the influence of a large crack on a small crack is not the same as the influence of a small crack on a large crack). This nonsymmetry seems disturbing until one realizes that this is so only because of our choice of variables $\Delta\bar{S}_\nu^{(1)}$ and $\langle\Delta S_\mu^{(1)}\rangle$, which do not represent thermodynamically conjugate pairs of generalized forces and generalized displacements. If $\langle\Delta S_\mu^{(1)}\rangle$ were expressed in terms of the average crack openings $\bar{w}_\nu$, then the equation system resulting from (9) or (14) would have a matrix which would have to be symmetric (because of Betti's theorem), and also positive definite (if the body is stable). These are the attributes mathematically required for convergence of the iterative solution by Gauss-Seidel method [e.g., Rektorys (1969); Collatz (1960); Korn and Korn (1968); Varga (1962); Fox (1965); Strang (1980)]. Aside from that, convergence of the iterative solution of (9) or (14) must also be expected on physical grounds (because it is mechanically equivalent to the relaxation method, which always converges for stable elastic systems).

In the rth iteration, the new, improved values of the unknowns, labeled by superscripts $[r + 1]$, are calculated from the previous values, labeled by subscript $[r]$, either according to the recursive relations:

$$\Delta\bar{p}_\mu^{[r+1]} = \langle\Delta p_\mu\rangle + \sum_{\nu=1}^{N} \Lambda_{\mu\nu}\Delta\bar{p}_\nu^{[r]} \quad . . . . (28)$$

$$\Delta\bar{S}_\mu^{(1)[r+1]} = \langle\Delta S_\mu^{(1)}\rangle + \sum_{\nu=1}^{N} \Lambda_{\mu\nu}\Delta\bar{S}_\nu^{(1)[r]} \qquad (\mu = 1, 2, \ldots, N) \quad . . . . . . (29)$$

or according to the recursive relations:

$$\Delta\bar{p}_\mu^{[r+1]} = \langle\Delta p_\mu\rangle + \sum_{\nu=1}^{\mu-1} \Lambda_{\mu\nu}\Delta\bar{p}_\nu^{[r+1]} + \sum_{\nu=\mu+1}^{N} \Lambda_{\mu\nu}\Delta\bar{p}_\nu^{[r]}$$

$(\mu = 1, 2, \ldots, N)$ . . . . (30)

$$\Delta\bar{S}_\mu^{(1)[r+1]} = \langle\Delta S_\mu^{(1)}\rangle + \sum_{\nu=1}^{\mu-1} \Lambda_{\mu\nu}\Delta\bar{S}_\nu^{(1)[r+1]} + \sum_{\nu=\mu+1}^{N} \Lambda_{\mu\nu}\Delta\bar{S}_\nu^{(1)[r]}$$

$(\mu = 1, 2, \ldots, N)$ . . . . (31)

Eq. (29), also known as the Gauss method or Jacobi method, is normally slightly less efficient than (31), in which the latest approximations are always used. The values of $\Delta S_\mu^{(1)}$ may be used as the initial values of $\Delta\bar{S}_\mu^{(1)[r]}$ in the first iteration.

It is possible to derive (30) more directly, rather than from (6). To this end, we note that the sequence of iterations is identical to a solution by the relaxation method in which one crack after another is relaxed (i.e., its pressure reduced to zero) while all the other cracks are frozen (which is a problem with one crack only), as illustrated in Fig. 1(b). Each relaxation produces pressure on the previously relaxed cracks. After relaxing, one by one, all the cracks, the cycle through all the cracks is repeated again and again. This kind of relaxations is known in mechanics to converge in general [this was numerically demonstrated for a system of cracks and inclusions by Pijaudier-Cabot and Bažant (1991)]. The solution to which the relaxation process converges is obviously that defined by (9). Note also that this relaxation argument in fact represents a simple way to prove the superposition equation [(6)].

For structural engineers, it is interesting to note the similarity with the Cross method (moment distribution method) for elastic frames. Relaxing the pressure at one crack while all the other microcracks are frozen (glued) is analogous to relaxing one joint in a frame while all the other joints are held fixed. Repeating this for each joint, and then repeating the cycles of such relaxations of all the joints, eventually converges to the exact solution of the frame.

The macrocontinuum counterpart of the Gauss-Seidel iterative method, which converges to the solution of the Fredholm integral equation [(10)], is analogous to (29) and is given by the following relation for successive approximations (iterations):

$$\Delta\bar{S}^{(1)[r+1]}(\mathbf{x}) = \langle\Delta S^{(1)}(\mathbf{x})\rangle + \int_V \Lambda(\mathbf{x}, \boldsymbol{\xi})\Delta\bar{S}^{(1)[r]}(\boldsymbol{\xi})\, dV(\boldsymbol{\xi}) \quad \text{.......... (32)}$$

The discrete approximation of the last relation is the equation that ought to be used in finite element programs with iterations in each step. We see that the form of averaging is different from that currently used, given by (3). There are now two additive spatial integrals, one for close-range averaging of the inelastic stresses from the local constitutive relation and one for long-range crack interactions based on the latest iterates of the inelastic stresses.

In programming, the old iterates need not be stored in the computer memory. So the subscripts $[r]$ and $[r + 1]$ may be dropped and (33) and (34) and may be replaced by the following assignment statements:

$$\Delta\bar{S}^{(1)}_{\mu} \leftarrow \langle\Delta S^{(1)}_{\mu}\rangle + \sum_{\nu=1}^{N} \Lambda_{\mu\nu}\Delta\bar{S}^{(1)}_{\nu} \qquad (\mu = 1, 2, \ldots N) \quad \text{.......... (33)}$$

$$\Delta\bar{S}^{(1)}(x) \leftarrow \langle\Delta S^{(1)}(\mathbf{x})\rangle + \int_V \Lambda(\mathbf{x}, \boldsymbol{\xi})\Delta\bar{S}^{(1)}(\boldsymbol{\xi})\, dV(\boldsymbol{\xi}) \quad \text{.......... (34)}$$

A strict implementation of Gauss-Seidel iterations suggests programming one iteration loop for (33) to be contained within another loop for the iterations of the loading step in which the displacement and strain increments in the structure are solved. However, one common iteration loop, which is computationally much more efficient, can serve both purposes. Then, of course, the iteration solution is not exactly the Gauss-Seidel method because the strains are also being updated during each iteration. There is already some computational experience showing that convergence can still be achieved.

The common iteration loop has the advantage that it permits the use of the explicit load-step algorithm for structural analysis. In a loading step of this algorithm, one evaluates in each iteration at each integration point the elastic stress increments $\mathbf{E}{:}\Delta\boldsymbol{\epsilon}$ and the local inelastic stress increments $\Delta\mathbf{S}$ from fixed strains $\Delta\boldsymbol{\epsilon}$; then one uses (33) to calculate from $\Delta\mathbf{S}$ the nonlocal inelastic stress increments $\Delta\bar{\mathbf{S}}$ for all the integration points, and solves new nodal displacements and strains by elastic structural analysis.

CRACK INFLUENCE FUNCTION

Cracks Far from Boundary in Two-Dimensional Body

By virtue of applying the Gauss-Seidel iterative method, coefficients $\Lambda_{\mu\nu}$ can be obtained from the stress field of only one pressurized crack in the given elastic solid. In practice, this solid is finite, and then $\Lambda_{\mu\nu}$ should in principle be calculated taking into account the geometry of the body. This means that for every different body shape and size and every different crack location, a new set of coefficients $\Lambda_{\mu\nu}$ would have to be calculated. This would be a preposterous task.

A simplification is suggested, however, by the decay of stresses with the distance from a pressurized crack. For practical purposes, the distance of most cracks from the boundary is such that the interference of the boundary with the stress field of the crack is negligible. So, except for points near the boundary, this field can be approximately calculated as if the crack were embedded in an infinite elastic solid.

For the purpose of macrocontinuum representation, some aspects of the stress field in an infinite body underlying the crack influence function Λ must be preserved while others must be simplified. Preserved must be the long-range asymptotic form of this field, because the long-range contributions to the integral that come from the neighborhood of a remote point $\boldsymbol{\xi}$ come to point $\mathbf{x}$ from nearly the same direction and nearly the same distance [Fig. 3(c)]. How to handle the close-range fields of the microcracks is a much more difficult question. Certain aspects must obviously be simplified. First, it is impossible to represent on the macroscale the microcracks as finite in size, having (in two dimensions) two distinct crack tips, and second, the singularities of the stress fields near the crack tips must be smeared at the macrolevel as a nonsingular, bounded field. The first condition is met by taking the long-range asymptotic field of a crack in infinite elastic solid. This field is easy to derive, as follows.

Consider now a crack in an infinite solid, subjected to uniform pressure σ [Fig. 3(b) and (e)]. According to Westergaard's solution [e.g., Brock (1987) and Hellan (1984)]

$$\sigma_{xx} = \mathrm{Re}\, Z - y\, \mathrm{Im}\, Z' - \sigma \quad \text{.......... (35a)}$$

$$\sigma_{xx} = \mathrm{Re}\, Z + y\, \mathrm{Im}\, Z' - \sigma \quad \text{.......... (35b)}$$

$$\tau_{xy} = -y\, \mathrm{Re}\, Z' \quad \text{.......... (35c)}$$

in which σ_{xx} and σ_{yy} = the normal stresses, τ_{xy} = the shear stress, and

$$Z = \sigma z(z^2 - a^2)^{-1/2} \qquad (z = re^{i\phi}) \quad \text{.......... (36)}$$

Here $2a$ = crack length, $i^2 = -1$, $Z' = dZ/dz$, and r and ϕ = polar coordinates with origin at the crack center and angle ϕ measured from the crack direction x. For $r >> a$ we have the approximation:

$$Z = \sigma\left(1 - \frac{a^2}{r^2 e^{2i\phi}}\right)^{1/2} = \sigma\left(1 + \frac{a^2}{2r^2}e^{-2i\phi} + \cdots\right) = \sigma\left(1 + \frac{a^2}{2z^2} + \cdots\right) \quad \text{.......... (37)}$$

From this, we calculate

$$\mathrm{Re}\, Z = \sigma\left(1 + \frac{a^2}{2r^2}\cos 2\phi + \cdots\right) \quad \text{.......... (38a)}$$

$$Z' = \sigma(-a^2 z^{-3} + \cdots) \quad \text{.......... (38b)}$$

$$y\, \mathrm{Im}\, Z' = \sigma a^2 r \sin\phi\, \mathrm{Im}(-r^{-3}e^{-3i\phi}) = -\sigma a^2 r^{-2}\sin\phi(-\sin 3\phi) \quad \text{.. (38c)}$$

Substituting this into (35) and using the formulas for products of trigonometric functions, we get the following simple result for the long-range ($r >> a$) asymptotic field:

$$\sigma_{xx} = \sigma k(r) \frac{\cos 4\phi}{2} \qquad (39a)$$

$$\sigma_{yy} = \sigma k(r) \left(\cos 2\phi - \frac{\cos 4\phi}{2} \right) \qquad (39b)$$

$$\tau_{xy} = \sigma k(r) \frac{\sin 4\phi - \sin 2\phi}{2} \qquad (39c)$$

where $k(r) = a^2/r^2$. Subscripts x and y refer to Cartesian coordinates with origin at point ξ coinciding with the crack center and axis y normal to the crack; σ_{xx} and σ_{yy} = the normal stresses, τ_{xy} = the shear stress; and ϕ = polar coordinates with origin at the crack center, with the polar angle ϕ measured from axis x. The principal stresses $\sigma^{(1)}$ and $\sigma^{(2)}$ and the first principal stress direction $\phi^{(1)}$ are given by:

$$\sigma^{(1)} = \sigma k(r) \left(\frac{\cos 2\phi}{2} + \sin \phi \right) \qquad (40a)$$

$$\sigma^{(2)} = \sigma k(r) \left(\frac{\cos 2\phi}{2} - \sin \phi \right) \qquad (40b)$$

$$\tan 2\phi^{(1)} = -\cot 3\phi \qquad (40c)$$

The foregoing expressions describe the long-range form of function $\Lambda(\mathbf{x}, \xi)$. It does not matter that they have a r^{-2} singularity at the crack center, because they are invalid for not too large r. Note that the average of each expression over the circle r = constant is zero, which is in fact a necessary property.

Function $\Lambda(\mathbf{x}, \xi)$ can also be easily determined for small r. As intuitively suggested by Fig. 3(d), the short-range interactions go in all directions and should cancel each other. That this is indeed so is confirmed by Kachanov's (1992) numerical studies of interactions of randomly generated crack systems that are uniform over a large body. He found that for such systems the classical assumption of noninteracting cracks is very good, which means that all the interactions mutually cancel. It follows that for $r \rightarrow \infty$, the function $\Lambda(\mathbf{x}, \xi)$ should approach the asymptote $\Lambda = 0$.

For intermediate r, calculation of $\Lambda(\mathbf{x}, \xi)$ would need to take into account statistical interactions, which seems very difficult. Therefore, we propose to use a smooth empirical function that approaches for $r \rightarrow \infty$ and for $r \rightarrow 0$ the two asymptotic curves we established, as shown in Fig. 3(g). We also know the function must be bounded. The simplest expression to have these properties, which replaces a^2/r^2 in the foregoing expressions, is

$$k(r) = \left(\frac{\kappa l r}{r^2 + l^2} \right)^2 \qquad (41)$$

Here l = an empirical constant that represents the distance to the peak in Fig. 3(g). It may be identified with what has been called the characteristic length of the nonlocal continuum. Probably its value reflects the dominant

spacing of the microcracks, which in turn is determined by size and spacing of the dominant inhomogeneities such as aggregates in concrete, or grain or crystal size in ice, ceramics, and rocks. It may perhaps be taken equal to the larger of the crack size and the maximum inhomogeneity (aggregate) size. κ = an empirical constant such that κl roughly represents the average or effective crack size a for the macrocontinuum (in theory, it seems this value should be increased during the loading process since the cracks grow).

In the formalism we introduced previously, $\Lambda(\mathbf{x}, \xi)$ = a scalar. All the information on the relative crack orientations is embedded in the values of this function. The principal stress direction at point ξ, which can be regarded as the dominant crack direction at that location [Fig. 3(e)], is all the directional information needed to calculate the stress components at point $\mathbf{x}$; see (40), in which $r = \|\mathbf{x} - \xi\|$ = distance between points $\mathbf{x}$ and ξ. The value of $\Lambda(\mathbf{x}, \xi)$, needed for (32) or (9), may be determined as the projection of the stress tensor produced at point $\mathbf{x}$ onto the principal inelastic stress direction at that point. According to Mohr circle: $2\Lambda(\mathbf{x}, \xi) = (\sigma_{xx} + \sigma_{yy}) + (\sigma_{xx} - \sigma_{yy})\cos 2(\psi - \theta) - 2\tau_{xy} \sin 2(\psi - \theta)$ in which θ, ψ = angles of the principal stress directions at points ξ, $\mathbf{x}$, respectively, with the line connecting these two points (i.e., with the vector $\mathbf{x} - \xi$). Substituting here for σ_{xx}, etc., the expressions from (39), one obtains a trigonometric expression that [as J. Planas (personal communication, 1992) pointed out], can be brought by trigonometric transformations to the form:

$$\Lambda(\mathbf{x}, \xi) = -\frac{k(r)}{2l^2} [\cos 2\theta + \cos 2\psi + \cos 2(\theta + \psi)] \qquad (42)$$

where $\theta = 90° - \phi$. Note that the function $\Lambda(\mathbf{x}, \xi)$ is symmetric. This is of course a necessary consequence of the fact that the body is elastic.

Two properties contrasting with the previous nonlocal formulations should be noted: (1) The crack influence function is not axisymmetric (isotropic) but depends on the polar angle (i.e., is anisotropic); and (2) it exhibits a shielding sector and an amplification sector. We may define the amplification sector as the sector in which σ_{yy} (the same stress component as that applied at the crack faces) is positive, and the shielding sector as the sector in which σ_{yy} is negative. The amplification sector $\sigma_{yy} \geq 0$ is, according to (39), given by $\phi \leq \phi_b$, where

$$\phi_b = 55.740° \qquad (43)$$

The sector in which the volumetric stress $\sigma_{xx} + \sigma_{yy}$ (first stress invariant) is positive is $\phi < 45°$. The sector in which $\sigma_{xx} \geq 0$ is $\phi \leq 22.5°$ and $\phi \geq 67.5°$. The sector in which $2\tau_{max} = \sigma_{xx} - \sigma_{yy} \geq 0$ is $\phi \leq 45°$. The maximum principal stress $\sigma^{(1)}$ is positive for all angles ϕ, and the minimum principal stress $\sigma^{(2)}$ is positive for $\phi < 21.471°$.

The consequence of the anisotropic nature of the crack influence function is that interactions between adjacent cracks depend on the direction of damage propagation with respect to the orientation of the maximum principal inelastic stress. In a cracking band that is macroscopically of mode I [Fig. 4(a)], propagating in the dominant direction of the microcracks, the microcracks assist each other in growing because they lie in each other's amplification sectors. In a cracking band that is macroscopically of mode II [Fig. 4(b)], the microcracks are mutually in the transition between their amplification and shielding sectors, and thus interact little. Under compression, a band of axial splitting cracks may propagate sideways [Fig. 4(c)], and in that case the microcracks inhibit each other's growth because they

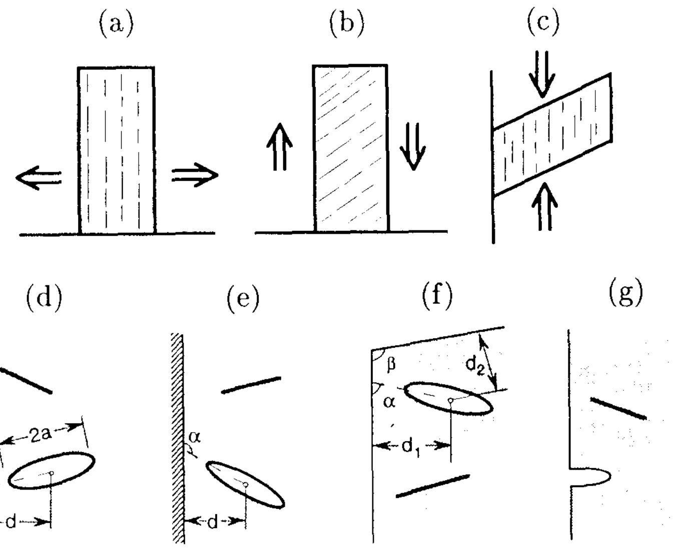

FIG. 4. Cracks near Boundary and Crack Bands

lie in each other's shielding sectors. Different interactions of this kind probably explain why good fitting of test data with the previous nonlocal microplane model required using a different material characteristic length for different type of problems (e.g., mode I fracture specimens versus diagonal shear failure of reinforced beam).

Cracks Far from Boundary in Three-Dimensional Body

The case of three dimensions (3D) is not difficult when the cracks are penny-shaped and the boundary is remote. The stresses around such cracks have been expressed as integrals of Bessel functions (Sneddon and Lowengrub 1969; Kassir and Sih 1975), which are cumbersome for calculations. Recently, however, Fabrikant (1990) ingeniously derived the following closed-form expressions:

$$\sigma_{xx} = \frac{\sigma_1 + \mathrm{Re}\,\sigma_2}{2} \quad \text{(44a)}$$

$$\sigma_{yy} = \frac{\sigma_1 - \mathrm{Re}\,\sigma_2}{2} \quad \text{(44b)}$$

$$\tau_{xy} = \frac{\mathrm{Im}\,\sigma_z}{2} \quad \text{(44c)}$$

$$\tau_{xz} = \mathrm{Re}\,\tau_z \quad \text{(44d)}$$

$$\tau_{yz} = \mathrm{Im}\,\tau_z \quad \text{(44e)}$$

in which

$$\sigma_z = \frac{2\sigma}{\pi}(B - D) \quad \text{(45a)}$$

$$\sigma_1 = \frac{2\sigma}{\pi}[(1 + 2\nu)B + D] \quad \text{(45b)}$$

$$\sigma_2 = e^{2i\phi}\frac{2\sigma}{\pi}\frac{al_1^2 l_3}{l_2^2 l_4^2}\left(1 - 2\nu + \frac{z^2[a^2(6l_2^2 - 2l_1^2 + \rho^2) - 5l_2^4]}{l_4^4 l_3^2}\right) \quad \text{(45c)}$$

$$\tau_z = -e^{i\phi}\frac{2\sigma}{\pi}\frac{zl_1[a^2(4l_2^2 - 5\rho^2) + l_1^4]l_3}{l_2 l_4^6} \quad \text{(45d)}$$

$$B = \frac{al_3}{l_4^2} - \arcsin\frac{a}{l_2} \quad \text{(45e)}$$

$$D = \frac{az^2[l_1^4 + a^2(2a^2 + 2z^2 - 3\rho^2)]}{l_4^6 l_3} \quad \text{(45f)}$$

$$l_1 = \frac{L_2 - L_1}{2} \quad \text{(45g)}$$

$$l_2 = \frac{L_2 + L_1}{2} \quad \text{(45h)}$$

$$l_3 = \sqrt{l_2^2 - a^2} \quad \text{(45i)}$$

$$l_4 = \sqrt{l_2^2 - l_1^2} \quad \text{(45j)}$$

$$L_1 = \sqrt{(a - \rho)^2 + z^2} \quad \text{(45k)}$$

$$L_2 = \sqrt{(a + \rho)^2 + z^2} \quad \text{(45l)}$$

in which a = crack radius [Fig. 3(f)]; r, θ, and ϕ = the spherical coordinates attached to Cartesian coordinates x, y, and z at point ξ, with angle θ measured from axis z, which is normal to the crack at point ξ; r = distance between points $\mathbf{x}$ and ξ; and ρ, ϕ, z = the cylindrical coordinates with origin at the crack center and ρ, ϕ as polar coordinates in the crack plane, angle ϕ being measured from axis x.

Since the long-range asymptotic form of the foregoing stress field has not been given, we need to derive it. For this purpose, one needs to note that, for large r, $L_1 \approx r - a\sin\theta$, $L_2 \approx r + a\sin\theta$ [see the meaning of L_1 and L_2 in Fig. 3(f)], $l_1 \approx a\sin\theta$, $l_2 \approx r$ and, for $r \gg a$, $\arcsin(a/l_2) \approx [1 + (a^2/6l_2^2)]a/l_2$, $\sqrt{l_2^2 - a^2} \approx r[1 - (a^2/2r^2)]$. The result is the following long-range asymptotic field:

$$\sigma_{\rho\rho} = \sigma k(r)\left[(1 + 2\nu)\left(\sin^2\theta - \frac{2}{3}\right) + (1 - 2\nu - 5\cos^2\theta)\sin^2\theta\right] \quad \text{(46a)}$$

$$\sigma_{\phi\phi} = \sigma k(r)\left[(1 + 2\nu)\left(\sin^2\theta - \frac{2}{3}\right) - (1 - 2\nu - 5\cos^2\theta)\sin^2\theta\right] \quad \text{(46b)}$$

$$\sigma_{zz} = \sigma k(r)\left(\sin^2\theta - \frac{2}{3}\right) \quad \text{(46c)}$$

$$\sigma_{\rho z} = -\sigma k(r) \sin 2\theta (4 - 5 \sin^2 \theta) \quad \text{(46d)}$$

$$\sigma_{\rho\phi} = \sigma_{\phi z} = 0 \quad \text{(46e)}$$

in which, for three dimensions, $k(r) = a^3/(\pi r^3)$. For the same reasons as those that led to (41), this expression may be replaced by

$$k(r) = \frac{1}{\pi} \left(\frac{\kappa l r}{r^2 + l^2} \right)^3 \quad \text{(47)}$$

The crack influence function based on (46) satisfies again the condition that its spatial average over every surface r = constant be zero.

It is important to note that, asymptotically for large distances r, the crack influence function in three dimensions decays as r^{-3}, whereas in two dimensions it decays as r^{-2}. Again, in contrast to the previous formulations, the weight function (crack influence function) is not axisymmetric (isotropic) but depends on the polar or spherical angles (i.e., is anisotropic).

Note again that one can distinguish a shielding sector and an amplification sector. According to the change of sign of σ_{zz} in (46), the boundary of these sectors is given by the angle

$$\theta_b = \arcsin \sqrt{\frac{2}{3}} = 54.736° \quad \text{(48)}$$

or $90° - \theta_b = 35.264°$. Thus, the amplification sector $\theta \geq \theta_b$ is significantly narrower in three than in two dimensions.

In the case of a field translationally symmetric in z, one might wonder whether integration over z might yield the two-dimensional crack influence function. However, this cannot occur because the two-dimensional crack influence function corresponds in three dimensions to a field of strip cracks aligned in the z direction, which cannot yield the same properties as the penny-shaped cracks.

Cracks Near Boundary

When the boundary is near, the crack influence function should be obtained by solving the stress field of a pressurized crack at a certain distance d from the boundary [Fig. 4(d)]. Obviously, the function will depend on d as a parameter, i.e., $\Lambda(\mathbf{x}, \boldsymbol{\xi}, d)$. Functions Λ will be different for a free boundary, fixed boundary [Fig. 4(e)], sliding boundary, and elastically supported boundary or interface with another solid. When the crack is near a boundary corner [Fig. 4(f)], Λ represents the solution of the stress field of a pressurized crack in the wedge, and will depend on the distances from both boundary planes of the wedge. These solutions will be much more complicated than for a crack in infinite body, and simplifications will be needed. On the other hand, because of the statistical nature of the crack system, exact solutions of these problems are not needed. Only their essential features are.

A crude but simple approach to the boundary effect is to consider the same weight function as for an infinite solid, protruding outside the body. In the previous nonlocal formulations, based on the idea of spatial averaging, the same weight function as for the infinite solid has been used in the spatial integral and the weight function has simply been scaled up renormalized, so that the integral of the weight function over the reduced domain would remain 1. In the present formulation, such scaling would have to be applied

to all the weight functions whose integral should be 1, i.e., α, ψ, B, and C. For those weight functions whose integral should vanish, a different scaling would be needed to take the proximity of the boundary into account; for example, the values at the boundary should be scaled up so that the spatial integral would always vanish, as indicated in (23). As a reasonable simplification, this might perhaps be done by replacing the $\Lambda_{\mu\nu}$ values for the integration points ξ_ν of the boundary finite elements by $k_b \Lambda_{\mu\nu}$ where the multiplicative factor k_b is determined from the condition that $\sum_{\nu=1}^{N} \Lambda_{\mu\nu} = 0$ (with the summation carried over all the points in the given finite body)

$$k_b = -\frac{\sum_{\text{interior } \nu} \Lambda_{\mu\nu}}{\sum_{\text{boundary } \nu} \Lambda_{\mu\nu}} \quad \text{(49)}$$

LONG-RANGE DECAY AND INTEGRABILITY

Consider now an infinite two-dimensional elastic solid in which the stress, strain, and cracking are macroscopically uniform. All the microcracks are of the same size a, and the area per crack is s^2. The stress σ applied on each microcrack is the same. From (39) we calculate the contribution to the nonlocal integral from domain V_1 outside of a circle of radius R_1 that is sufficiently large for permitting the approximation $k(r) \approx a^2/r^2$

$$\int_{V_1}^{\odot} (\sigma_{xx} + \sigma_{yy})\, \mathrm{d}V = \lim_{R \to \infty} \int_{r=R_1}^{R} \int_{\phi=0}^{2\pi} \frac{\sigma a^2 \cos 2\phi}{2r^2} \frac{r\, \mathrm{d}\phi\, \mathrm{d}r}{s^2}$$

$$= \frac{\sigma a^2}{2s^2} \int_{r=R_1}^{\infty} \int_{\phi=0}^{2\pi} \frac{\cos 2\phi}{r}\, \mathrm{d}\phi\, \mathrm{d}r \quad \text{(50)}$$

Now an important observation, to which we already alluded: The last expression is an improper integral which is divergent (because it is divergent when the integrand is replaced by its absolute value [see, e.g., Rektorys (1969)]). This also means that the value of the integral depends on the integration path. For some path the integral may be convergent, and that path, shown in (50), has been labeled by $\odot$. So we must conclude that a homogeneous $\Delta \mathbf{S}$ field, that is, a field of uniform length increment of all the cracks in an infinite body that is initially in a statistically uniform state, is impossible.

But this is not all that surprising. As is known from analysis of bifurcation and stable equilibrium path, strain-softening damage (which is due to microcrack growth) must localize [e.g., Bažant and Cedolin (1991)]. So in practice the domain of the integrals such as the last one must not be infinite in two directions. It can only be finite or infinite in one direction only, as is the case for a localization band. The basic reason for this situation is that the asymptotic decay r^{-2}, which we have obtained, is relatively weak—much weaker than the exponential decay assumed in previous works (for an exponential decay, the integration domain could be infinite in all directions without causing this kind of problem).

A similar analysis of uniform damage can be carried out for an infinite three-dimensional solid, and the conclusion is that the integration domain, that is, the zone of growing microcracks, can only be finite or infinite in two directions only (a localization layer), but not in three.

A similar divergence of the integral over infinite space has been known to occur in other problems of physics, for example, in calculation of the stresses from periodically distributed inclusions, or the light received from an infinite number of statistically uniformly distributed stars. For a perspicacious mathematical study of this type of problem see Furuhashi, Kinoshita and Mura (1981).

GENERAL FORMULATION: TENSORIAL CRACK INFLUENCE FUNCTION

In (10), the principal stress orientations at points $\mathbf{x}$ and $\boldsymbol{\xi}$ are reflected in the values of the scalar function $\Lambda(\mathbf{x}, \boldsymbol{\xi})$. For the purpose of general analysis, however, it seems more convenient to use a tensorial crack influence function, referred to common structural Cartesian coordinates $X = X_1$, $Y = X_2$, $Z = X_3$, and transform all the inelastic stress tensor components to X, Y, Z. The local Cartesian coordinates $x = x_1$, $y = x_2$, $z = x_3$ at point $\boldsymbol{\xi}$ are chosen so that axis y coincide with the direction of the maximum principal value of the inelastic stress tensor $\bar{\mathbf{S}}(\boldsymbol{\xi})$, and axes x and z coincide with the other two principal directions (Fig. 4). Eqs. (33) and (34) may be rewritten in common structural coordinates as follows:

$$\Delta\bar{S}_{\mu IJ} \leftarrow \langle \Delta S_{\mu IJ} \rangle + \sum_{\nu=1}^{N} \sum_{i=1}^{3} R^{(i)}_{\nu IJkl} \Lambda^{(i)}_{\mu\nu kl} \Delta\bar{S}^{(i)}_{\nu}$$

$$(\mu = 1, 2, \ldots N;\ i = 1, 2, 3) \qquad (51)$$

$$\Delta\bar{S}_{IJ}(\mathbf{x}) \leftarrow \langle \Delta S_{IJ}(\mathbf{x}) \rangle + \int_V \sum_{i=1}^{3} R^{(i)}_{IJkl}(\boldsymbol{\xi}) \Lambda^{(i)}_{kl}(\mathbf{x}, \boldsymbol{\xi}) \Delta\bar{S}^{(i)}(\boldsymbol{\xi})\, \mathrm{d}V(\boldsymbol{\xi})$$

$$(i = 1, 2, 3) \qquad (52)$$

in which, similarly to (22), we included the influence of the cracks normal to all the principal stress direction at each point; $R^{(i)}_{IJkl}(\boldsymbol{\xi})$ or $R^{(i)}_{\nu IJkl} = c_{kI}c_{lJ}$ = fourth-rank coordinate rotation tensor (programmed as a square matrix when the stress tensors are programmed as column matrices) at point $\boldsymbol{\xi}$ or $\boldsymbol{\xi}_\mu$; c_{kI}, c_{lJ} = coefficients of rotation transformation of coordinate axes (direction cosines of new axes) from local coordinates x_i at point $\boldsymbol{\xi}$ (having in general a different orientation at each $\boldsymbol{\xi}$) to common structural coordinates $X_I[c_{kI} = \cos(x_k, X_I)$, $X_I = c_{kI}x_k$, $\sigma_{IJ} = c_{kI}c_{lJ}\sigma_{kl}]$; subscripts I, J, or k, l refer to Cartesian components in the common structural coordinates or in the local coordinates at $\boldsymbol{\xi}$; and $\Lambda^{(i)}_{\mu\nu kl}$ or $\Lambda^{(i)}_{kl}(\mathbf{x}, \boldsymbol{\xi})$ = components of a tensorial discrete or continuous nonlocal weight function (crack influence function, replacing the scalar function Λ), which are equal to r^{-2} times the Cartesian stress components σ_{kl} for $\sigma = 1$ as defined by (39) for two dimensions, or r^{-3} times such Cartesian components as defined by (46) for three dimensions (with $r = \|\mathbf{x} - \boldsymbol{\xi}\|$).

CONSTITUTIVE RELATION AND GRADIENT APPROXIMATION

As is clear from the foregoing exposition, the constitutive relation is defined only locally. It yields the inelastic stress increment $\langle \Delta S^{(1)}(\mathbf{x}) \rangle$, illustrated by segment 32 shown in Fig. 2. This contrasts with the previous nonlocal formulations, in which the nonlocal inelastic strain, stress, or damage was part of the constitutive relation. This caused conceptual difficulties as well as continuity problems with formulating the unloading criterion. Furthermore, in the case of nonlocal plasticity, this also caused difficulties with the consistency condition for the subsequent loading surfaces.

Here these difficulties do not arise, because the nonlocal spatial integral is separate from the constitutive relation. Thus the unloading criterion can, and must, be defined strictly locally. If plasticity is used to define the local stress-strain relation, the consistency condition of plasticity is also local.

Recently there has been much interest in limiting localization of cracking by means of the so-called gradient models. These models can be looked at as approximations of the nonlocal integral-type models, and can be obtained by expanding the nonlocal integral in Taylor series (Bažant 1984; Bažant and Cedolin 1991). Unlike the present model, there have been only scant and vague attempts of physical justifications for the gradient models, especially for aggregate-matrix composites such as concrete. It seems that the physical justification for the gradient models of such materials must come indirectly, through the integral-type model. However, if that is the case, the present conclusions signal a problem. If the spatial integral in (10) were expanded into Taylor series and truncated, the long-range decay of the type r^{-2} or r^{-3} could not be preserved. Yet it seems that this decay is for microcrack systems important. If so, then the gradient approximations are physically unjustified.

CONCLUSIONS

1. The inelastic stress increments correspond to the stresses that the load increment would produce on the cracks if they were temporarily "frozen" (or "glued"), i.e., prevented from opening and growing. The nonlocality arises from two sources: (1) Crack interactions, which means that application of the pressure on the crack surfaces that corresponds to the "unfreezing" (or "ungluing") of one crack produces stresses on all the other frozen cracks; and (2) averaging of the stresses due to unfreezing over the crack surface, which is needed because crack interactions depend primarily on the stress average over the crack surface (or the stress resultant) rather than the stress at the crack center. The crack interactions (source 1) can be solved by Kachanov's (1987) simplified version of the superposition method, in which only the average crack pressures are considered.

2. The resulting nonlocal continuum model involves two spatial integrals: One integral, which corresponds to source 1 and has been absent from previous nonlocal models, is long-range and has a weight function whose spatial integral is 0; it represents interactions with remote cracks and is based on the long-range asymptotic form of the stress field caused by pressurizing one crack while all the other cracks are frozen. Another integral, corresponding to source (2), is short-range, involves a weight function whose spatial integral is 1, and represents spatial averaging of the local inelastic stresses over a domain whose diameter is roughly equal to the spacing of major microcracks (which is roughly equal to the spacing of large aggregates in concrete).

3. As an approach to continuum smoothing when the macroscopic field is nonuniform, one may seek a continuum field equation whose possible discrete approximation coincides with the matrix equation governing a system of interacting microcracks.

4. The long-range asymptotic weight function of the nonlocal integral representing crack interactions (source 1) has a separated form that is calculated as the remote stress field of a crack in infinite body. It decays with distance r from the crack as r^{-2} in two dimensions and r^{-3} in three dimen-

sions. This long-range decay is much weaker than assumed in previous nonlocal models. In consequence, the long-range integral diverges when the damage growth in an infinite body is assumed to be uniform. This means that only the localized growth of damage zones can be modeled.

5. In contrast to the previous nonlocal formulations, the weight function (crack influence function) in the long-range integral is a tensor and is not axisymmetric (isotropic). Rather, it depends on the polar or spherical angle (i.e., is anisotropic) exhibiting sectors of shielding and amplification.

6. When an iterative solution of crack interactions according to the Gauss-Seidel iterative method is considered, the long-range nonlocal integral based on the crack influence function yields the nonlocal inelastic stress increments explicitly. This explicit form is suitable for iterative solutions of the loading steps in nonlinear finite element programs. The nonlocal inelastic stress increments represent a solution of a tensorial Fredholm integral equation in space, to which the iterations converge.

7. The constitutive law, in this new formulation, is strictly local. This is a major advantage, eliminating difficulties with formulating the unloading criterion and the continuity condition, experienced in the previous nonlocal models in which nonlocal inelastic stresses or strains have been part of the constitutive relation.

ACKNOWLEDGMENT

Financial support under AFOSR Grant 91-0140 to Northwestern University is gratefully acknowledged. Supplementary support for the analysis of cracking damage in concrete was received from the Center for Advanced Cement-Based Materials at Northwestern University. The general approach was conceived during a visiting appointment at Lehrstuhl für Mechanik (director Professor Horst Lippmann), Technische Universität München, supported under Humboldt Award of Senior U.S. Scientist. Stimulating discussions by graduate research assistant Milan Jirásek, as well as visiting scholars Joško Ožbolt (research engineer on leave from Stuttgart University) and Jaime Planas (professor of materials science on leave from Technical University, Madrid), are appreciated. Thanks for illuminating comments are due to Toshio Mura and John Dundurs, professors at Northwestern University.

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NONLOCAL DAMAGE THEORY BASED ON MICROMECHANICS OF CRACK INTERACTIONS[1]

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Errata and Addendum

Errata:

- On the line above (28), p. 604, replace the word 'subscript' by the word 'superscript'.
- On the second line above (33), p. 605, replace the word 'subscripts' by the word 'superscripts'.
- In equation (35b), p. 606, replace σ_{xx} by σ_{yy}.
- In the last sentence of the second paragraph after (40), replace $r \to \infty$ by $r \to 0$.

Addendum: A rigorous mathematical definition of the continuum crack influence function Λ has not been given in the paper. It can be given as follows. Function $\Lambda(0, \xi)$ represents the influence of a source crack at $x = 0$ on a target crack at ξ. At the macro-continuum point $x = 0$ there may or may not be a crack. To idealize the random two-dimensional arrangement of cracks, we may imagine that the center of the source crack influencing some target crack can occur randomly anywhere within the square $s \times s$ centered at point $x = 0$; s represents the typical spacing of the dominant cracks of length $2a$ near point $x = 0$ (in a material such as concrete, $s \approx$ average spacing of the largest aggregate pieces). The macroscopic crack influence function can describe the influence of the source crack only in the average sense. Therefore, $\Lambda(0, \xi)$ is defined as the mathematical expectation, $\mathcal{E}$, with regard to all the possible random realizations of the source crack center within the square $s \times s$; $\Lambda(0, \xi) = \langle \mathcal{E}[\sigma^{(1)}(\xi - x, \eta - y)] \rangle$ where the operator $\langle \quad \rangle$ represents averaging over length $2a$ of the target crack at ξ, and $(\xi - x, \eta - y) = r =$ vector from the center $x = (x, y)$ of a source crack to the center $\xi = (\xi, \eta)$ of the target crack. Choosing axis y to be normal to the source crack, we thus have the following statistical definition:

$$\Lambda(0, \xi) = \frac{1}{2a} \int_{-a}^{a} \frac{1}{s^2} \int_{-s/2}^{s/2} \int_{-s/2}^{s/2} w(x, y) \sigma^{(1)}(\xi - x, \eta - y)\, dx\, dy\, da' \tag{53}$$

where $\sigma^{(1)}$ is the principal stress at $\xi = (\xi, \eta)$ caused by a unit uniform pressure applied on the faces of a crack of length $2a$ centered at $x = (x, y)$, as given by equation (40); and $w(x, y)$ is a certain smooth bell-shaped weight function expressing the fact that the probability of a crack center occurring near the boundary of the square is less than near the center because the crack cannot occur in near contact with a crack centered in the adjacent square. However, (53) might be unnecessarily complicated. The simple approximation given in the paper on the basis of the asymptotic properties of this integral seems to be preferable for practical computations.

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[1] *J. of Engrg. Mech. ASCE*, 120 (3), March 1994, .

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