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Nonlocal Continuum Damage, Localization Instability and Convergence

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A recent nonlocal damage formulation, in which the spatially averaged quantity was the energy dissipated due to strain-softening, is extended to a more general form in which the strain remains local while any variable that controls strain-softening is nonlocal. In contrast to the original imbricate nonlocal model for strain-softening, the stresses which figure in the constitutive relation satisfy the differential equations of equilibrium and boundary conditions of the usual classical form, and no zero-energy spurious modes of instability are encountered. However, the field operator for the present formulation is in general nonsymmetric, although not for the elastic part of response. It is shown that the energy dissipation and damage cannot localize into regions of vanishing volume. The static strain-localization instability, whose solution is reduced to an integral equation, is found to be controlled by the characteristic length of the material introduced in the averaging rule. The calculated static stability limits are close to those obtained in the previous nonlocal studies, as well as to those obtained by the crack band model in which the continuum is treated as local but the minimum size of the strain-softening region (localization region) is prescribed as a localization limiter. Furthermore, the rate of convergence of static finite-element solutions with nonlocal damage is studied and is found to be of a power type, almost quadratric. A smooth weighting function in the averaging operator is found to lead to a much better convergence than unsmooth functions.

Introduction

Prediction of damage and failure of brittle heterogeneous materials such as concrete or rock requires a mathematically correct and physically realistic description of the strain-softening behavior (Bažant, 1986; Mazars and Pijaudier-Cabot 1986). Although it has been argued that strain-softening does not exist on the continuum level (Read and Hegemier, 1984), the macroscopic result of distributed microcracking or void growth is a behavior whose continuum description must incorporate strain-softening. Numerous attempts to describe this type of behavior by local inelastic continuum theories such as plasticity or continuum damage mechanics have been unsatisfactory because the phenomenon of strain localization caused by strain-softening cannot be captured objectively (Bažant, 1986). The principal fault of the local continuum models is that the energy dissipated at failure is incorrectly predicted to be zero, and the finite-element solutions converge to this incorrect, physically meaningless, solution as the mesh is refined.

To remedy the situation, one must introduce some form of a localization limiter (Bažant and Belytschko, 1987). Its simplest form is obtained by imposing a lower bound on the finite-element size, as is done in the crack band model. As a better localization limiter which permits arbitrary mesh refinement, one may adopt the nonlocal concept. Introduced into continuum mechanics long ago by Kröner (1967); Eringen (1972); Krumhansl (1968) and others, this concept was recently successfully applied to strain-softening (Bažant, et al., 1984). However, the formulation, in which all the state variables were nonlocal, turned out to be quite complicated. It required additional boundary and interface conditions, led to a nonstandard form of the differential equations of equilibrium, and the finite-element implementation required imbrication of the elements.

In the preceding study (Pijaudier-Cabot and Bažant, 1986) to be further expanded here, a new idea which turns out to bring considerable simplification was introduced. It was shown that it suffices to consider as nonlocal only the strain-softening damage, while the elastic behavior (including unloading and reloading) should be treated as local. This formulation was shown to require no element imbrication and no overlay with local continuum, which had to be previously used by Bažant, et al., (1984) in order to suppress certain periodic

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zero-energy modes in the imbricate, fully nonlocal solutions. The purpose of the present study is to show that, more generally, the key attribute of the nonlocal formulation for strain-softening is that the strain as a kinematic variable should be defined as local, and to study various aspects yet unexplored, including: (1) Static strain-localization instability; (2) Symmetry of the field operator; (3) Influence of various types of spatial averaging; (4) Convergence at mesh refinement.

Before embarking on our analysis, it should be mentioned that other forms of localization limiters are possible and deserve to be studied. The oldest idea, proposed already by L'Hermite (1952) in a study of concrete cracking due to shrinkage, is to introduce a dependence of the strength or yield limit on the strain gradient. In a general form, which may involve introduction of the strain gradient into the yield function, this idea has recently been developed by Floegl and Mang (1981), Schreyer and Chen (1984) and Mang and Eberhardsteiner (1986). Introduction of higher-order gradients into the differential equations of equilibrium or into the definition of strength was studied by Aifantis (1984) and also by Bažant (1984) and Bažant and Belytschko (1987). As another approach, Sandler (1984) as well as Needleman (1987) showed that the introduction of viscosity, whether real or artificial, may act in certain problems as a localization limiter, although this can be true only for a limited time period of response.

Nonlocal Generalization of Continuum Damage Mechanics

The principal idea for the treatment of softening is that only those variables which cause softening may be considered as nonlocal while the model must reduce to a local one for the special case of elastic response, which also includes unloading and reloading. This condition can be satisfied by a nonlocal formulation in which the strain, when used as a kinematic variable, is local. To implement this condition it is convenient, albeit not necessary, to use continuum damage mechanics because in this theory the strain-softening is characterized by a distinct single variable ω, called damage. For the nonlocal generalization, we adopt the simple, scalar damage formulation, although the same concept could be implemented in a similar manner in the anisotropic damage models, derived, e.g., by Mazars and Pijaudier-Cabot (1986) and Ladevèze (1983), which are more realistic. Various other constitutive theories can also be generalized to such a nonlocal form, particularly those which use fracturing strain or a degrading yield limit. (Bažant, et al. 1987). Unimportant from the viewpoint of the type of localization studied here, plastic strains will be omitted from the formulation; their inclusion, however, would require no conceptual changes.

As usual in continuum damage mechanics, we may introduce the relation between the strains ϵ_{ij} and the stresses σ_{ij} in the form (Lemaitre and Chaboche, 1985)

$$\sigma_{ij} = (1-\Omega)C_{ijkm}\epsilon_{km} \tag{1}$$

in which C_{ijkm} are the elastic constants of the material and Ω is the damage. We assume Ω to be nonlocal, defined by spatial averaging as follows

$$\Omega(\mathbf{x}) = \bar{\omega}(\mathbf{x}) = \frac{1}{V_r(\mathbf{x})}\int_V \alpha(\mathbf{s}-\mathbf{x})\omega(\mathbf{s})dV(\mathbf{s}) \tag{2}$$

in which

$$V_r(\mathbf{x}) = \int_V \alpha(\mathbf{s}-\mathbf{x})dV(\mathbf{s}). \tag{3}$$

Superimposed bar denotes the spatial averaging operator, $\mathbf{x}$ and $\mathbf{s}$ are the coordinate vectors, V = volume of the body, and α = given weighting function. Initially $\Omega=\omega=0$, and always $0 \le \Omega \le 1$, $0 \le \omega \le 1$.

As the simplest form of the weighting function, one may consider $\alpha=1$ within a certain representative volume V_r centered at points $\mathbf{x}$, and $\alpha=0$ outside of it. As we will see, however, a uniform weighting function over a finite domain does not yield the best convergence. A much better convergence on mesh refinement is obtained when function $\alpha(\mathbf{x}-\mathbf{s})$ decays smoothly with the distance from point $\mathbf{x}$. A suitable form is the Gaussian (normal) distribution function, previously used for nonlocal elasticity by Eringen (1972);

$$\alpha(\mathbf{x}) = \exp[-(k|\mathbf{x}|/l)^2] \tag{4}$$

in which we have, for one, two and three dimensions

$$\begin{aligned} &\text{1D:}\quad |\mathbf{x}| = x && k=\pi^{1/2}\\ &\text{2D:}\quad |\mathbf{x}| = (x^2+y^2)^{1/2} && k=2\\ &\text{3D:}\quad |\mathbf{x}| = (x^2+y^2+z^2)^{1/2} && k=(6\sqrt{\pi})^{1/3}. \end{aligned} \tag{5}$$

x, y, z are the Cartesian coordinates.

The expressions in equation (5) have been determined from the condition that the integral of $\alpha(\mathbf{x})$ for an infinite body be equal for one dimension to the length l (line segment), for two dimensions to the area of a circle of diameter l, and for three dimensions to the volume of a sphere of diameter l.

The function in equation (4) decays so rapidly that for points $\mathbf{s}$ whose distance from point $\mathbf{x}$ exceeds $2l$ one may set $\alpha=0$. Calculation of $V_r(\mathbf{x})$ according to equation (3) is needed for the treatment of boundaries. Function α in general extends beyond the boundaries of the body. The domain beyond the boundary is simply deleted from integration, but the weighting function is scaled so that the integral of all effective weights $\alpha'(\mathbf{x},\mathbf{s}) = \alpha(\mathbf{s}-\mathbf{x})/V_r(\mathbf{x})$ over the body should be exactly 1 for any $\mathbf{x}$. In numerical programming, the integrals in equations (2) and (3) are approximated as finite sums. The values of $\alpha'(\mathbf{x},\mathbf{s})$ are generated for all combinations of all integration points of all elements in advance of the finite-element analysis. The programming of the averaging integral, which was demonstrated by Bažant, et al. (1987) in a study of cave-in of a tunnel due to compressive strain-softening for a nonlocal finite-element system with up to 3248 unknowns, is easier when the integral in equation (2) extends over the entire body. When a finite averaging domain is used, it is important to closely match by finite elements the precise boundaries of this domain, but this is difficult to implement (Saouridis, et al., 1987).

Length l, called the characteristic length, represents a material property and is of the same order of magnitude as the maximum size of material inhomogeneities. Length l has been determined experimentally (Bažant and Pijaudier-Cabot, 1987*a*) by comparing the responses of specimens in which the damage (e.g., microcracking) remains distributed with the responses of fracture specimens, in which damage localizes. For concrete, such experiments indicated that l is roughly equal to 2.7-times the maximum aggregate size.

Length l can be determined also by micromechanics analysis. In a parallel study (Bažant, 1987), a local homogeneous elastic continuum, containing an array of growing circular cracks with periodic spacing l and quasi-periodic crack sizes, was analyzed. It was shown that by applying the usual homogenization conditions, one obtains a nonlocal continuum with local strain and a nonlocal cracking strain, for which the weighting function is uniform and the characteristics length is equal to crack spacing. For the case when the crack spacing is not constant but randomly distributed, this indicates that one should superpose at each point a set of uniform weighting functions with various characteristics lengths l. This, however, appears to be approximately equivalent to using a nonuniform weighting function.

For a uniform weighting function, volume V_r has a similar meaning as the representative volume in the statistical theory of heterogeneous materials. However, its size l, is smaller than that of the representative volume and is too small to obtain statistical averages of the microstructure. The representative volume in the statistical theory must be several times larger.

The specific free energy $\rho\psi$ per unit volume (with ρ being the material mass density) and the energy dissipation rate ϕ may be expressed as

$$\rho\psi = \tfrac{1}{2}(1-\Omega)C_{ijkm}\epsilon_{ij}\epsilon_{km} \tag{6}$$

$$\phi = -\partial(\rho\psi/\partial t) = -\dot{\Omega}\partial(\rho\psi)/\partial\Omega = \dot{\Omega}Y$$

in which Y, called the damage energy release rate, is

$$Y = -\partial(\rho\psi)/\partial\Omega = \tfrac{1}{2}C_{ijkm}\epsilon_{ij}\epsilon_{km}. \tag{7}$$

The damage evolution is characterized in general by an equation of the type $\dot{\omega} = f(\epsilon_{ij},\omega)$. An integrable special form of the damage evolution law has been used in all computations;

$$\omega = g(Y) = 1 - [1 + b(Y - Y_1)]^{-n} \tag{8}$$

in which b, n = positive material constants, $n > 2$, and Y_1 = local damage threshold.

Damage is assumed to grow only for loading. For unloading or reloading, $\dot{\omega} = 0$, which means the response is elastic. The loading criterion and the nonlocal damage Ω are defined as follows

$$\left.\begin{array}{l}\text{If } F(\bar{\omega}) = 0 \text{ and } \dot{F}(\bar{\omega}) = 0, \text{ then } \dot{\Omega} = \dot{\bar{\omega}} \\ \text{If } F(\bar{\omega}) < 0, \text{ or if } F(\bar{\omega}) = 0 \text{ and } \dot{F}(\bar{\omega}) < 0, \text{ then } \dot{\Omega} = 0.\end{array}\right\} \tag{9}$$

Function $F(\bar{\omega})$ represents the loading function and is defined as $F(\bar{\omega}) = \bar{\omega} - \kappa(\bar{\omega})$, where $\kappa(\bar{\omega})$ is a softening parameter which is set to be equal to the maximum value of $\bar{\omega}$ achieved up to the present. The initial value of $\kappa(\bar{\omega})$ is zero. The damage expression in equation (8) was found to approximate closely the behavior of concrete, provided that different local damage thresholds Y_1 are introduced for tension and compression (Mazars and Pijaudier-Cabot, 1986).

The formulation of the loading function automatically satisfies the dissipation inequality. The density of the energy dissipation rate due to damage is $\phi = \dot{\Omega}Y$, and since $\dot{\Omega} \geq 0$, we have $\phi \geq 0$.

In view of the use of a loading function, it might be more appealing to introduce some potential function, similar to plasticity. However, a formulation with a potential function has not yet been developed in the literature on continuum damage mechanics of concrete or geomaterials.

The fact that equation (1) uses nonlocal rather than local damage, and that the unloading condition is stated in terms of the nonlocal damage, represents all that is different from the classical local damage theory.

The original formulation of the nonlocal damage theory in the previous study by Pijaudier-Cabot and Bažant (1986) used a somewhat different definition of nonlocal damage. The averaged quantity was the damage energy release rate Y rather than the local damage, and so the nonlocal damage was determined as $\Omega = g(\bar{Y})$. This original formulation can be regarded as a simplified approximation of the present formulation. The numerical results for these two different definitions of nonlocal damage appear to be quite close.

The strains as well as ω have at least a C_0-continuity (i.e., they could consist of Dirac delta functions). According to equations (9) and (2) and the fact that $\phi = Y\dot{\Omega}$, the dissipation rate density is given by a spatial averaging integral over ω. Consequently, ϕ must have at least a C_1-continuity. This simple argument proves that the energy dissipation cannot localize into a zone of zero volume, and numerical solutions confirm it.

In general, one could show that by introducing any variable which causes the dissipation rate to be nonlocal (e.g., nonlocal plastic-strain path in plasticity with softening yield limit Bažant, et al., 1987), the energy dissipation is prevented to localize into a vanishing volume.

Differential Equation of Equilibrium and Boundary Conditions

In the original formulation of the nonlocal continuum for strain-softening in which nonlocal strains $\bar{\epsilon}_{ij}$ are used (Bažant, et al., 1984), the virtual work expression for a body of volume V_b and surface S_b must involve $\delta\bar{\epsilon}_{ij}$ rather than $\delta\epsilon_{ij}$

$$\delta W = \int_{V_b} (\sigma_{ij}\delta\bar{\epsilon}_{ij} - f_i\delta u_i)dV - \int_{S_b} p_i\delta u_i dS \tag{10}$$

in which u_i = displacement components in Cartesian coordinates x_i $(i = 1,2,3)$, and f_i, p_i = distributed volume and surface forces. The fact that $\delta\bar{\epsilon}_{ij}$ involves the spatial averaging operator defined by equation (2) complicates the derivation of the differential equation of equilibrium and boundary conditions and yields a nonstandard form of these equations (Bažant, 1984).

For the present nonlocal theory, ϵ_{ij} is local and so $\delta\bar{\epsilon}_{ij}$ in equation (10) must be replaced by $\delta\epsilon_{ij}$. But then the variational derivation of the differential equation of equilibrium and the boundary conditions is the same as usual (Bažant and Pijaudier-Cabot, 1987). This shows that, as long as the strains are defined as local, the differential equations of equilibrium, as well as the boundary conditions or the interface conditions, will have the standard form. This further means that the finite-element discretization can be of the same type as for the usual, local continuum. Therefore we may conclude that the key simplifying feature of a continuum model for damage is the use of a nonlocal continuum in which the strains are local. In other words, the continuum should not be fully, but only partially nonlocal. It is also clear that if the strain is local then the elastic behavior, including the behavior at unloading and reloading, will be local.

One-Dimensional Strain-Localization Instability

Consider for the sake of simplicity the one-dimensional problem of a bar loaded through two springs of spring constant C (Fig. 1). This problem was used by Bažant (1976) to demonstrate the localization instability due to strain-softening and was recently solved exactly with a fully nonlocal theory (Bažant and Zubelewicz, 1986). The length coordinate is x, the

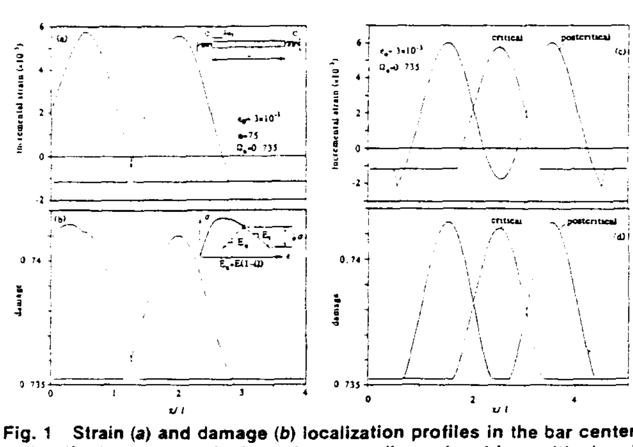

Fig. 1 Strain (*a*) and damage (*b*) localization profiles in the bar center and at the boundary, calculated for a one-dimensional bar; critical and postcritical strain and damage localization profiles (*c*, *d*)

bar length is L, and the bar ends are $x=0$ and $x=L$. The bar is initially in a state of uniform strain ϵ_0 and stress σ_0, with uniform damage $\Omega_0=\omega_0$. The initial state is in the strain-softening range and satisfies the relation $\sigma_0=(1-\Omega_0)E\epsilon_0$. If the constitutive relation for damage is given in the form of equation (8), we have $\omega_0 = 1-[1+b(\frac{1}{2}E\epsilon_0^2-Y_1)]^{-n}$.

We now consider a small deviation from the initial state caused by incremental variation of the load at the bar ends. Let $\delta\epsilon(x)=\eta(x)$ = incremental strain and $\delta\sigma=\tau$ = incremental stress. To maintain equilibrium, it is necessary that τ = constant along the bar. The compatibility condition for fixed supports requires that

$$\int_0^L \eta(x)dx+\frac{2\tau}{C}=0. \tag{11}$$

Taking the one-dimensional form of the constitutive law in equation (1) with the averaging according to equation (2) and the loading criterion according to equation (9), we find

For $\int_0^L \alpha(s-x)\eta(s)ds>0$: $\quad \sigma_0+\tau$

$$=\left[1-\frac{1}{l'(x)}\int_0^L \alpha(s-x)\omega(s)ds\right]E(\epsilon_0+\eta(x)); \tag{12}$$

Otherwise: $\tau=(1-\omega_0)E\eta(x)$

in which $l'(x)=\int_0^L\alpha(s-x)dx$ = effective averaging length for point x. For infinitely small increments $\eta(x)$, the equations can be simplified by incremental linearization. To this end, we may introduce the linear approximation $\omega(x)=\omega_0+\omega_\epsilon\eta(x)$, with $\omega_\epsilon=\partial\omega/\partial\epsilon$ for $\omega=\omega_0$. Substituting into equation (12), neglecting the quadratric term $\eta(x)\eta(s)$, and subtracting the equation $\sigma_0=E(1-\Omega_0)\epsilon_0$, we can reduce equation (12) to the form

$$\tau=(1-\omega_0)E\eta(x)-\frac{k}{l'(x)}<\int_0^L \alpha(s-x)\eta(s)ds> \tag{13}$$

with $k=E\epsilon_0\omega_\epsilon$. The symbol $< >$, which introduces the loading criterion, is defined as $<x>=x$ if $x>0$ and otherwise $<x>=0$. For the special damage constitutive law in equation (8), we may evaluate $\omega_\epsilon=E\epsilon_0\omega_Y$, with $\omega_Y=bn[\frac{1}{2}E\epsilon_0^2-Y_1]^{n-1}[1-b(Y-Y_1)]^{-2n}$.

Equation (13) has the form of a linear integral equation of the second kind. However, the problem is not simply that of solving an integral equation, because equation (13) is an integral equation only for those x for which loading takes place. Outside the loading region, equation (12) reduces to the usual linear elastic differential equation for displacements and the behavior is then local, elastic, and unaffected by the strain-localization in another segment of the bar. If we find one solution such that there exist elastic segments of finite length at both ends of the bar, then we can obtain another solution simply by shifting the softening segment along with the softening solution profile as a rigid body. This shift is arbitrary provided the entire original length of the softening region, along with a small neighborhood of points located just outside the softening zone, remains within the bar. Due to this fact, we can calculate the length of the softening region and the solution profile through it by analyzing any bar of a shorter length, provided the bar length exceeds the length of the softening region. The actual boundary conditions and the compatibility condition may be disregarded in such analysis and satisfied afterward. Obviously, one can have infinitely many solutions. Arbitrary shifts of the softening region, however, do not affect the overall response of the bar. The length of the softening region and the solution profile through it is nevertheless unique. The localization profiles terminating at the boundary points are different from the interior one and require a special analysis. For real materials, the actual location of the strain-softening segment that does not reach the boundary is decided by inevitable random fluctuations of material strength along the bar.

Now consider the alternative formulation previously proposed by Pijaudier-Cabot and Bažant (1986), in which the averaged variable is Y rather than ω. For small strain increments, equation (2) can be approximated as $\Omega(x)=\omega_0+\Omega_Y[Y(x)-Y_0]$, with $\Omega_Y=[\partial\Omega/\partial Y]_{\epsilon 0}=\omega_Y$ in which Y_0, Ω_0 and ϵ_0 are the values of Y, Ω and ϵ in the initial state. Linearization for small increments is then accomplished by writing

$$Y(x)=\frac{1}{l'(x)}\int_0^L \alpha(s-x)\,\frac{E}{2}\,(\epsilon_0^2+2\eta(s))\,\epsilon_0 ds. \tag{14}$$

Substituting this into the basic relation $\tau=(1-\Omega)E\eta(x)$, one obtains for $\eta(x)$ an equation identical to equation (13), in which $k=2Y_1\Omega_Y$. An equation of the same form would be obtained for various other possible types of averaging. For problems with nonuniform initial strain, however, different types of averaging do not yield the same field equations.

Equation (13) can be easily solved numerically. If we subdivide the bar length into N equal elements of length $\Delta x=L/N$, use for integration the trapezoidal rule, express τ from equation (11) and substitute it into equation (13), we may approximate the resulting equation as follows

$$\sum_{j=1}^N K_{ij}\eta_j=0 \quad \text{with}$$

$$K_{ij}=(1-\omega_0)E\delta_{ij}-\frac{k\Delta x}{l_i'}\,I_i\alpha(x_j-x_i)+\frac{C}{2}\,. \tag{15}$$

Subscripts i, j refer to centroids of elements number i or j, and $I_i=1$ if $\int_0^L\alpha(x_i-s)\eta(s)ds \simeq \Sigma_j\alpha(x_j-x_i)\eta_j\Delta x \geq 0$; otherwise $I_i=0$. Equation (15) represents a system of homogeneous linear algebraic equations for η_j. The critical state occurs at such ϵ_0 for which $\det(K_{ij})=0$. The loading segment, characterized by $I_i=1$, is of course unknown in advance, and so it must be determined iteratively. To search for the critical state with the smallest ϵ_0, the values of ϵ_0 were incremented in small loading steps and, for each ϵ_0, the following algorithm was used:

1 In the first iteration, we assume that only one element i undergoes loading (the central element). In the subsequent iterations, we increase the number of elements that undergo loading by one (either on the left or on the right), unless the number of loading elements already equals N, in which case we start a new loading step with a larger ϵ_0.

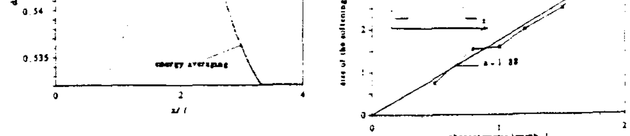

Fig. 2(a, b) Comparison of localization profiles of strain and damage for the damage and energy release rate averaging; (c, d) influence of the characteristic length l on the damage localization profiles, and the size of the softening zone