JOURNAL OF
THE ENGINEERING
MECHANICS DIVISION

Total Strain Theory and Path-Dependence of Concrete

By Zdeněk P. Bažant,[1] F. ASCE and Tatsuya Tsubaki[2]

Introduction

Triaxial constitutive relations continue to pose the greatest challenge of continuum mechanics, and this is especially true of concrete. While for uniaxial behavior all theories are equivalent, diverse theories are possible for the triaxial behavior. They are basically of two types:

1. The incremental theories, which include the incrementally linear ones, such as classical incremental plasticity (1,2,14,15,20,47), the direct hypoelastic approach (6), and the recent plastic-fracturing theory (11), as well as the incrementally nonlinear ones, which are represented by the endochronic theory (7,9,10).
2. The total strain theory [known in metal plasticity as the deformation theory (18,23)], in which one postulates the existence of an algebraic relationship between the total strains and stresses rather than their increments.

It is the latter type which will be the objective of this study.

The fact that the total stress-strain relations are defined by algebraic rather than differential or integral equations is one obvious and generally accepted advantage. This makes the formulation conceptually much simpler, and more direct, and more easily understandable. For this reason, this approach has been popular and various useful models have been developed, first with elastic volume changes (43,44,46,48) and later with inelastic volume changes (13,28,29,31,36,37). We will see, however, that the aforementioned advantage does not imply greater simplicity of structural analysis if the nonlinear algebraic relationship between strains and stresses cannot be put into an explicit form. The explicitness must indeed be abandoned if we want to model, as we do, a broad range of behavior of concrete, especially the strain-softening and the inelastic volume change. To be able to use the nonexplicit formulation in structural analysis, we must first differentiate it, getting an incremental (incrementally linear) form which we then treat in the same manner as an incremental plastic constitutive relation. On the other hand, if we would stick to explicitness, it seems we could never describe the behavior of concrete quite realistically.

The total stress-strain formulation has other, not so widely appreciated, advantages. Namely, for a load increment "to the side," i.e., a load increment that follows a proportional loading path in the stress space and is normal to this path, the total stress-strain formulation gives an inelastic response with an incremental modulus equal to the secant modulus, whereas the incrementally linear theories such as the incremental plasticity theory give, for a load increment to the side, a perfectly elastic response, i.e., a response with an incremental modulus equal to the initial tangent modulus at zero load. This is a questionable feature of the incremental plasticity; for concrete, this response is always inelastic, i.e., softer than elastic, and this seems to be true of many inelastic materials (5,8,27,41). The recent incremental theories such as the plastic-fracturing theory and the endochronic theory do give an inelastic response for the first loading to the side. However, they are conceptually not as simple as the total stress-strain relation. Moreover, there are good theoretical reasons to believe that the incremental modulus indicated for loading to the side by the total stress-strain theory is the correct one (12,16,22,26). This is important for predictions of failure due to material instability, which often involve a loading path to the side.

By contrast, the total stress-strain relation cannot be correct for generally nonproportional loading paths other than the first load increment to the side (21). This is because this formulation predicts a path-independent response, while in reality the response is known to be strongly path-dependent. We will therefore also attempt to generalize the total stress-strain relation so as to make it path-dependent.

Compared to the existing total stress-strain relations (13,28,29,31,36,37), we will aim to cover a much broader range of the behavior of concrete, especially the stress peak and the subsequent strain-softening, inelastic dilatancy, inelastic compaction, and pressure sensitivity. However, only monotonic loading will be considered.

General Total Stress-Strain Relation for Isotropic Material

We assume the stress-strain relation to consist of an algebraic tensorial equation involving stress tensor σ_{ij} and strain tensor ϵ_{ij}, i.e., $f_{ij}(\sigma_{km}, \epsilon_{km}) = 0$. Restricting attention to monotonic loading, we may assume, in accordance with experience, that a single tensor σ_{ij} corresponds to a given ϵ_{ij}, i.e.:

$$\sigma_{ij} = F_{ij}(\epsilon_{km}) \quad (1)$$

in which F_{ij} is a continuous and smooth tensorial function. Note that it is not possible to assume that $\epsilon_{ij} = F_{ij}(\sigma_{km})$ because two values of ϵ_{ij} exist for a given σ_{ij} if the stress-diagram exhibits a peak and a decline of stress at increasing strain (strain-softening). Also note that the diagrams of σ_{11} versus

Note.—Discussion open until May 1, 1981. To extend the closing date one month, a written request must be filed with the Manager of Technical and Professional Publications, ASCE. This paper is part of the copyrighted Journal of the Engineering Mechanics Division, Proceedings of the American Society of Civil Engineers, Vol. 106, No. EM6, December, 1980. Manuscript was submitted for review for possible publication on September 25, 1979.

[1] Prof. of Civ. Engrg., Northwestern Univ., Evanston, Ill.
[2] Grad. Research Asst., Northwestern Univ., Evanston, Ill.

ϵ_{kk}, in uniaxial and triaxial tests, which exhibit two σ_{11} values for a given ϵ_{kk}, do not violate Eq. 1 because the values of individual strain components, ϵ_{ij}, are different for these two σ_{11} values.

To satisfy the invariance requirements for isotropic materials, Eq. 1. must split into one tensorial relation (with scalar coefficients) between deviatoric tensors, and one scalar relation involving traces of tensors. Following Prager (39) and Rivlin (40), we may invoke the Cayley-Hamilton theorem of matrix theory, which states that the cubic-order and higher-order powers of a (3 × 3) matrix (or tensor) can be expressed as linear combinations of the first and second powers of the matrix (tensor). Therefore, the stress-strain relation may involve at most quadratic tensor terms in ϵ_{ij}. So, the most general possible algebraic stress-strain relation for monotonic loading can be written in the form:

$$s_{ij} = 2Ge_{ij} + Mf_{ij}; \quad \sigma = 3K\epsilon \quad \text{(2)}$$

in which $$f_{ij} = e_{ik}e_{kj} - \frac{1}{3}e_{km}e_{km}\delta_{ij} = \epsilon_{ik}\epsilon_{kj} - \frac{1}{3}\epsilon_{km}\epsilon_{km}\delta_{ij} \quad \text{(3)}$$

Here, G, K, M are functions of the stress and strain invariants, G and K representing the secant shear and bulk moduli; $s_{ij} = \sigma_{ij} - \delta_{ij}\sigma$; $e_{ij} = \epsilon_{ij} - \delta_{ij}\epsilon$ = deviators of stress and strain; δ_{ij} = Kronecker's delta; $\sigma = \sigma_{kk}/3$; $\epsilon = \epsilon_{kk}/3$ = volumetric (mean) stress and strain; subscripts of tensors refer to cartesian coordinates $x_i (i = 1, 2, 3)$ and repetition of subscripts implies the summation rule; and f_{ij} = deviator of the quadratic tensor $\epsilon_{ik}\epsilon_{kj}$. In principal strains $(\epsilon_1, \epsilon_2, \epsilon_3)$, we have $f_{11} = (2\epsilon_1^2 - \epsilon_2^2 - \epsilon_3^2)/3$ with f_{22}, f_{33} ensuing by cyclic permutation of indices.

Because $e_{12} = e_{23} = e_{31} = 0$ implies that $f_{12} = f_{23} = f_{31} = 0$ and Eq. 2 then indicates that $s_{12} = s_{23} = s_{31} = 0$, we see that the principal directions of σ_{ij} and ϵ_{ij} coincide. This is a rather significant limitation of the isotropic total stress-strain relations in general. It is well known, e.g., that if the principal stress axes rotate the stress and strains tensors for an inelastic initially isotropic material are generally not coaxial, and incremental theories such as plasticity (e.g., von Mises plasticity or plastic-fracturing theory) exhibit this effect. But without path-dependence this effect cannot be modeled.

Note that Eq. 1 brings about a great simplification compared to the relation $f_{ij}(\sigma_{km}, \epsilon_{km}) = 0$. For that relation, the Cayley-Hamilton theorem permits also the quadratic tensor $\sigma_{ik}\sigma_{kj}$ as well as mixed terms such as $\sigma_{ik}\epsilon_{kj}$, $\sigma_{ik}\epsilon_{km}\epsilon_{mj}$, $\sigma_{ik}\epsilon_{km}\sigma_{mp}\epsilon_{pi}$, etc. All these terms may be omitted on the basis of Eq. 1.

A quadratic deviatoric tensor analogous to f_{ij}, namely $t_{ij} = \sigma_{ik}\sigma_{kj} - \sigma_{km}\sigma_{km}\delta_{ij}/3$ was used by Prager (39) in a total strain theory for small strains of isotropic metals, which do not exhibit strain-softening. He showed that a relation analogous to Eq. 2, but with t_{ij} instead of f_{ij} and with purely elastic volume change, was the most general possible. Our simplification of the stress-strain relation on the basis of isotropy is similar to that first made by Rivlin (40) and others (17).

Due to the fact that K, G, and M are general functions of stress and strain invariants, terms such as $\epsilon_{kk}e_{ij}$, $\sigma_{kk}e_{ij}$, $e_{km}e_{km}e_{ij}$, etc. are implied by Eq. 2.

For the sake of simplicity, and because of the lack of requisite test data, we will assume in the following that $M = 0$. Eq. 2 with $M = 0$, in which both K and G are functions of either stress invariants or strain invariants, was apparently first proposed by Newmark (35).

Squaring Eq. 2 (for $M = 0$) we have $s_{ij}s_{ij} - 2Ge_{ij}e_{ij}$ or $\tau = 2G\gamma$ in which $\tau = (s_{ij}s_{ij}/2)^{1/2} = \sqrt{J_2}$ = stress intensity, and $\gamma = (e_{ij}e_{ij}/2)^{1/2} = \sqrt{J_2^\epsilon}$ = strain intensity (where J_2 and J_2^ϵ are the second invariants of the deviators of σ_{ij} and ϵ_{ij}). Since G and K depend on stress and strain invariants we may set $(2G)^{-1} = (2G_0)^{-1} + \gamma''/\tau$ in which G_0 = initial shear modulus (a constant); and γ'' = function of stress and strain invariants. Eq. 2 (for $M = 0$) and the secant relation $\gamma = \tau/2G$ which follows from Eq. 2 may then be written as

$$e_{ij} = \frac{s_{ij}}{2G_0} + e''_{ij}; \quad e''_{ij} = \frac{s_{ij}}{\tau}\gamma'' \quad \text{(4)}$$

and $$\gamma = \frac{\tau}{2G_0} + \gamma'' \quad \text{(5)}$$

in which e''_{ij} represents the deviator of inelastic strains ϵ''_{ij}. By squaring the expression for e''_{ij} we find that $\gamma'' = (e''_{ij}e''_{ij}/2)^{1/2}$. Introducing the initial bulk modulus K_0, we may also rewrite the relation $\epsilon = \sigma/3K$ (Eq. 2) in the form

$$\epsilon = \frac{\sigma}{3K_0} + \epsilon'' \quad \text{(6)}$$

in which ϵ'' = volumetric inelastic strain.

Constitutive Relation Representing Proportional Test Data

Based on the main trends observed in experiments as well as intuitive understanding of inelastic microstructural phenomena, the inelastic strains may be expected to be essentially functions of the form

$$\epsilon'' = \lambda + \lambda'; \quad \lambda = f_1(\gamma, \sigma, \tau); \quad \lambda' = f_2(\sigma); \quad \gamma'' = f_0(\gamma, \sigma, \tau) \quad \text{(7)}$$

whose purpose and form we will now analyze. The deviatoric inelastic strain is produced chiefly by microcracking which corresponds to distortion. The main influencing factor in γ'' must be γ rather than τ because during strain-softening, γ'' increases while τ decreases. So, function f_0 must increase with γ, and it must also increase with τ, although this is a secondary factor. The effect of hydrostatic pressure $p = -\sigma$ is to reduce inelastic deviatoric strains, and so function f_0 must decrease with p.

In the inelastic volumetric behavior, two distinct phenomena may be distinguished: The dilatancy, λ, which is prominent when the ratio of shear strain magnitude to hydrostatic pressure is high as, e.g., in the post-peak uniaxial response, and the compaction, λ', which is produced by hydrostatic pressure as long as its ratio to the shear strain magnitude is sufficiently large and as long as the hydrostatic pressure is not too large so as to lead to closing of pores. The dilatancy originates from microcracking caused by shear strain, and the compaction originates chiefly from collapse of the pores due to hydrostatic pressure.

Function f_1 for dilatancy must vanish when $\gamma = 0$ and must grow with γ until a certain limiting possible dilatancy value is approached. Since the microcracks get closed under hydrostatic pressure, function f_1 must decrease with p. The rise of function f_1 for dilatancy must not begin until the peak point of the response diagram is approached, as indicated by tests. In pure hydrostatic loading (20), compaction does not begin until the pressure reaches about 0.4 f'_c (where f'_c = standard uniaxial strength), and it ends when p exceeds about 5 f'_c for which most large pores have already been closed. At still higher p, the volumetric response stiffens again (see Fig. 1).

Guided by considerations of the preceding type, expressions for ϵ'' and γ'' were selected and they were then successively refined as needed for fitting

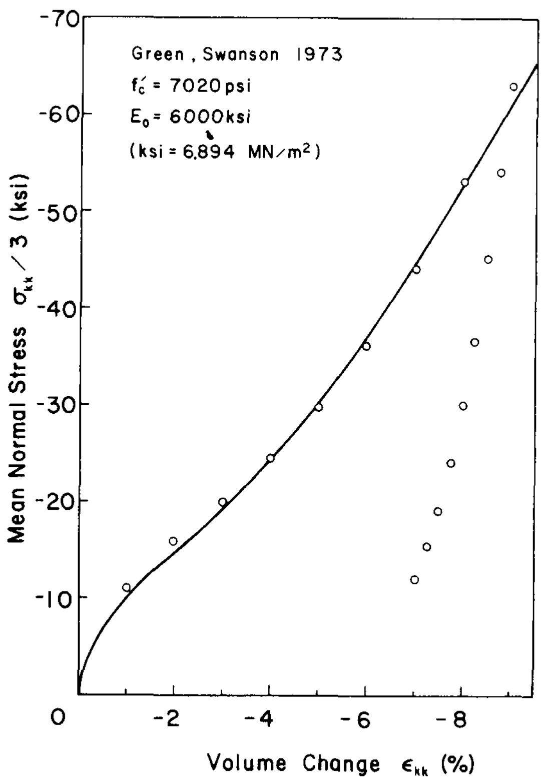

FIG. 1.—Fit of Hydrostatic Loading

test data. The following expressions which give good fits of available test data have been identified:

$$\gamma'' = \gamma_p \frac{q^{n+1} + r q^2}{q^n + rq + n - 1}; \quad \gamma_p = \gamma_0 [(1 + a_1 F_1^a) + a_2 F_3^{1/4}];$$

$$r = a_6 \left(1 + a_3 F_1^b + \frac{a_4 F_3^c}{1 + a_5 F_4}\right); \quad q = \frac{\gamma}{\gamma_p} \quad \ldots\ldots\ldots (8)$$

$$\epsilon'' = \lambda + \lambda'; \quad \lambda = \frac{b_1}{F_2} \frac{\gamma_p q^{m+1}}{q^m + F_5};$$

$$\lambda' = -[1 + b_4 F_2 (1 - b_5 F_2)](b_2 P + b_3) \frac{P^k}{P^k + b_6}; \quad P = \frac{p}{p_0} \quad \ldots\ldots\ldots (9)$$

$$F_1 = F_2^{-1/4} - 1; \quad F_2 = \frac{\tau^2}{3\sigma^2}; \quad F_3 = \frac{\sqrt{3}\,|I_3|}{9\tau^3}; \quad F_4 = \frac{I_3}{27\sigma^3};$$

$$F_5 = \frac{b_7}{1 + b_8 (F_1^{-1} - 1)^d} \quad \ldots\ldots\ldots (10)$$

in which $p = -\sigma = -\sigma_{kk}/3$; I_3 = Det $(\sigma_{ij}) = \sigma_1 \sigma_2 \sigma_3$; $a_1, \ldots, a_6, b_1, \ldots, b_8, a, \ldots, d, k, m, n$; and γ_0, and p_0 = material parameters given in the sequel. We now explain the purpose of the basic features of these expressions.

Variable, γ, which characterizes the magnitude of shear strain is monotonically increasing if the loading is monotonic. The expressions for γ'' and λ are chosen so that the effect of γ diminishes as p increases. However, γ'' is not taken to be inversely proportional to p since this would make γ'' decrease as the peak stress in uniaxial or standard triaxial loading is approached, which would disagree with experimental observations. Variable λ' which represents inelastic compaction due to the hydrostatic component of the stresses, is nonzero for pure hydrostatic loading, while dilatancy, λ, due to shear in this case vanishes. It is for this reason that λ' depends chiefly on p. It also depends, although mildly, on τ and I_3 which is described by nondimensional functions F_2, F_3, and F_4; τ is used in F_2 in order to express the weakening when τ is large compared to p. The third stress invariant, I_3, has a similar effect as p^3 but differs by the fact that it vanishes for uniaxial and biaxial tests. This is useful to adjust their relation to triaxial tests, which is why I_3 is used in F_3. Variable γ_p represents the value of γ at the peak stress. It is proportional to $1/F_2$ and F_3 so that γ_p would increase as the hydrostatic pressure increases. Variables γ_p, r, and n control the position of the peak, the magnitude of the peak stress, and the slope of the descending portion of the curve.

Since there exists no complete data set for one concrete, it has been impossible to determine the material parameters from the data for one concrete. So, it is necessary to introduce some parameters which characterize the type of concrete used in various tests. For simplicity, solely the standard uniaxial compressive strength, f'_c, has been chosen for this purpose. Other possible parameters, such as the mix ratio, elastic constants, and type of the aggregate, etc., are not taken into account because of limited amount of available data. Only a few material parameters are considered as functions of f'_c, and the following values of the parameters have been determined for the best fit of the experimental data for standard weight and normal strength concretes:

$$\gamma_0 = 1.715 \times 10^{-3}; \quad p_0 = 107.6 \left(\frac{f'_c}{\text{psi}}\right)^{0.5278}; \quad a_1 = 14.15; \quad a_2 = 10;$$

$$a_3 = 20.71; \quad a_4 = 8.272; \quad a_5 = 56.62; \quad a_6 = 4\sqrt{3}\,\frac{\gamma_0 G_0}{f'_c} - 3;$$

$$b_1 = 0.5; \quad b_2 = 5.819\left(\frac{f'_c}{\text{psi}}\right)^{-0.8409}; \quad b_3 = 0.005886; \quad b_4 = 6; \quad b_5 = 0.95;$$

$$b_6 = 1.841; \quad b_7 = 18.01; \quad b_8 = 0.004811; \quad a = 3.022; \quad b = 2.773,$$

$$c = 0.15; \quad d = 8.873; \quad k = 3; \quad m = 2; \quad n = 3 \quad \ldots \ldots \ldots \ldots (11)$$

in which psi = 6.895 KN/m^2. Furthermore, the initial Young's modulus is

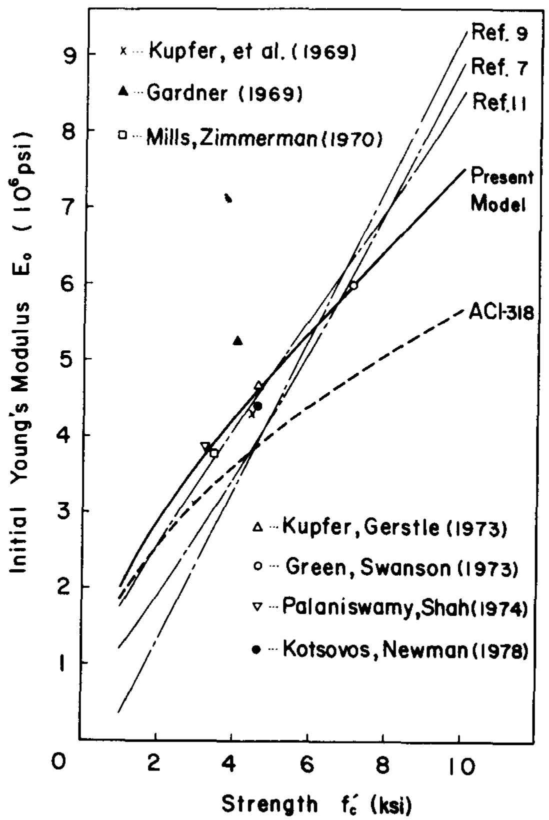

FIG. 2.—Dependence of Initial Young's Modulus E_0 on Strength f'_c

also taken as a function of f'_c, and the following expression has been determined:

$$E_0 = (1.09 \text{ psi} + 2.30 \times 10^{-5} f'_c)(57{,}000\sqrt{f'_c})(\text{psi})^{-1/2} \quad \ldots \ldots \ldots \ldots (12)$$

This expression represents a slight modification of the American Concrete Institute (ACI) formula (see Fig. 2). It must be kept in mind, however, that Eq. 12 is a crude approximation of limited applicability because it is known, e.g., that concretes of different mixes but equal strengths can have rather different values of E_0 (32). The fact that the elastic modulus is influenced by the strain rate is also neglected in the present model.

Fits of characteristic test data for proportional loading are shown by the solid lines in Figs. 1, 3–8. We see that a reasonable agreement has been achieved. The practical procedure to identify the functions which give the curves in Figs. 1–8 has been simplified by two advantages of the total stress-strain relations, which are as follows.

One advantage is that some response curves can be described by explicit algebraic functions. The total stress-strain model of the response curves under

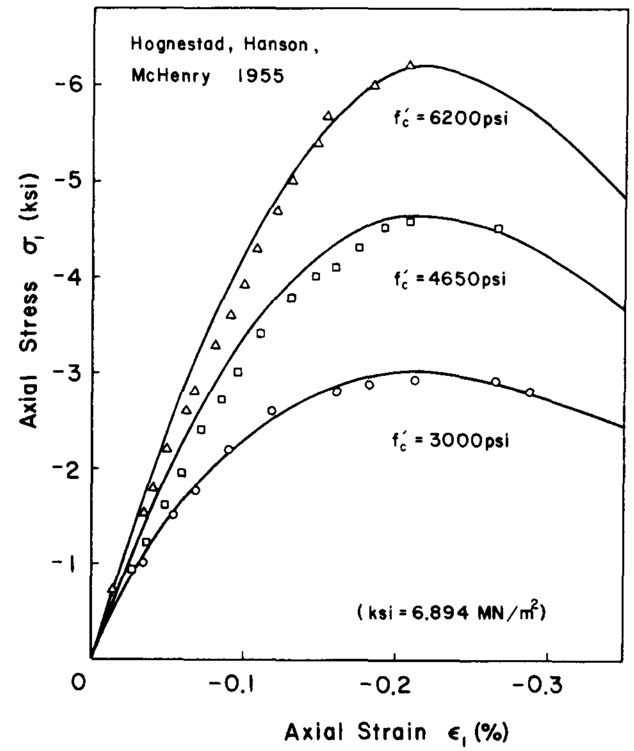

FIG. 3.—Fit of Uniaxial Test Data for Various Strength f'_c

pure hydrostatic pressure is given by a single algebraic expression, obtained by substituting $\tau = \gamma = 0$ in Eq. 9:

$$\epsilon = \frac{\sigma}{3K_0} - (b_2 P + b_3)\frac{P^k}{P^k + b_6}; \quad P = \frac{-\sigma}{p_0} \quad \ldots \ldots \ldots \ldots (13)$$

So this expression can be determined independently by a simple fitting of one hydrostatic response curve. The bracketed term involving functions F_2 in Eq. 9 is then determined as the adjustment that must be made to this curve (Eq. 13) to obtain ϵ for the states at $\tau \neq 0$. After the expressions for λ' are thus determined separately, the curve of λ versus γ may then be plotted. By fitting these curves, the expression for λ in Eq. 9 and function F_5 have been obtained. The function giving γ'' has been chosen similar to Saenz' expression (42) for the uniaxial stress-strain curve, based on the assumption that in a uniaxial test, the variables, τ and γ, increase roughly proportionally to σ_{11} and ϵ_{11}.

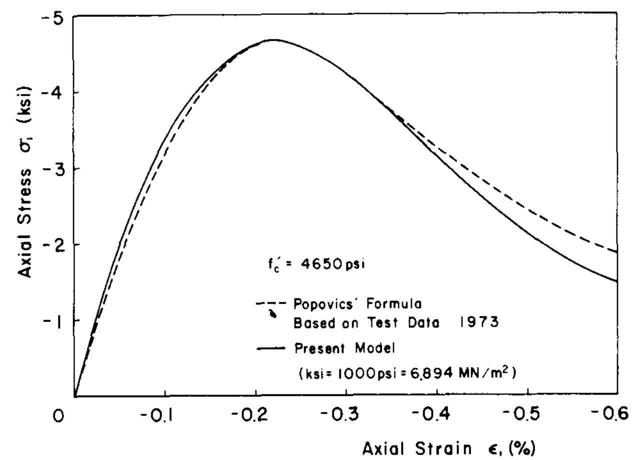

FIG. 4.—Fit of Empirical Formula by Popovics

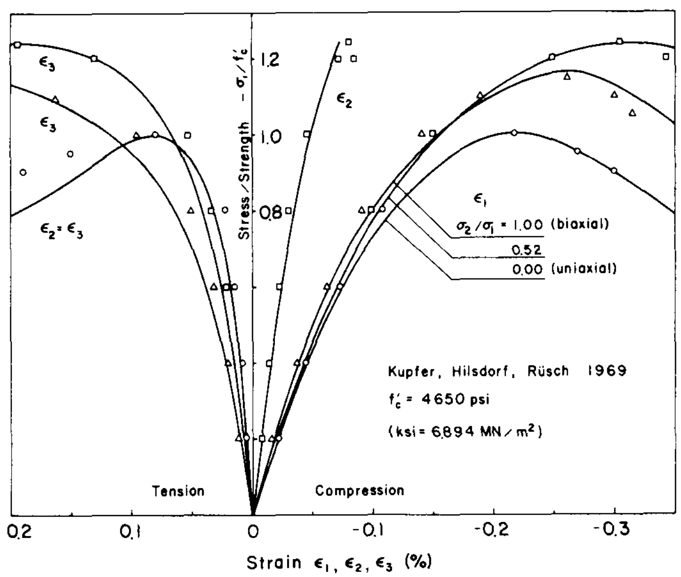

FIG. 5.—Fits of Uniaxial and Biaxial Test Data; Axial and Lateral Strains

The second advantage of the present theory, compared to incremental theories such as the plastic-fracturing theory or the endochronic theory is that it is possible to obtain algebraic expressions for the failure envelopes, such as those

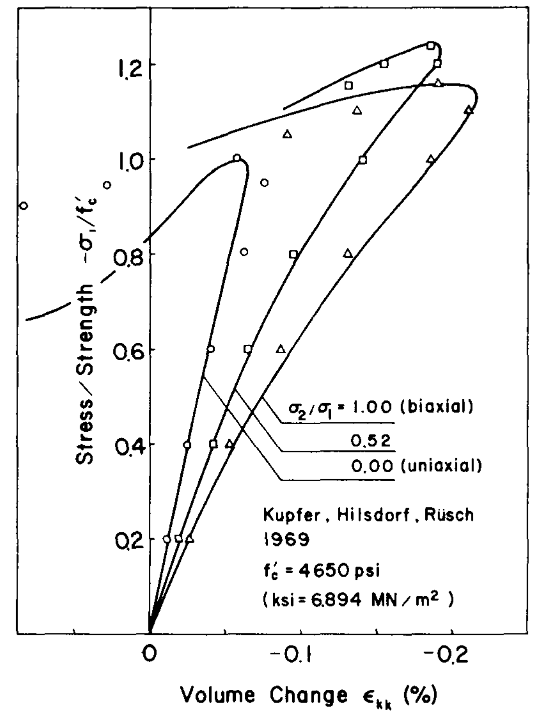

FIG. 6.—Fits of Uniaxial and Biaxial Data; Volume Change

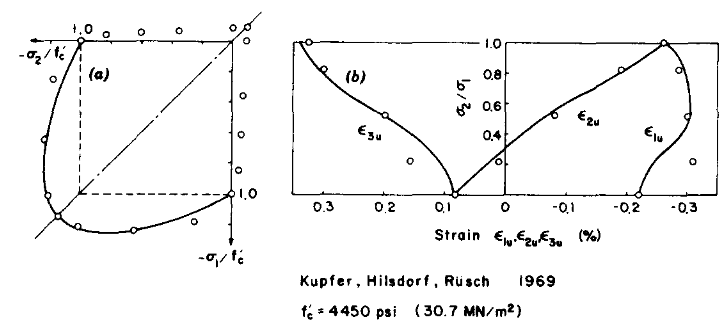

FIG. 7.—Fits of Uniaxial and Biaxial Data: (*a*) Failure Envelope; (*b*) Failure Strains

shown in Figs. 7(*a*) and 8. Eqs. 5 and 8 yield

$$\tau = 2\, G_0\, \gamma_p \frac{(n-1)q}{q^n + rq + n - 1} \quad \text{(14)}$$

and noting that $\gamma = \gamma_p$ (or $q = 1$) gives $\partial\tau/\partial q = 0$, we see that γ_p is the strain at peak stress, $\tau = \tau_p$. Substituting $q = 1$ we get

$$\tau_p = 2\, G_0 \gamma_p \frac{n-1}{n+r} \quad \ldots \quad (15)$$

If we use the principal stress components, set $\sigma_2/\sigma_1 = k_2$ and $\sigma_3/\sigma_1 = k_3$, and use the expression for the second deviator invariant to express τ_p, we get

$$\tau_p = -\sigma_{1_p} \phi(k_2, k_3) \quad \ldots \quad (16)$$

in which $$\phi(k_2,k_3) = \frac{1}{\sqrt{3}}\left(k_2^2 + k_3^2 - k_2 k_3 - k_2 - k_3 + 1\right)^{1/2} \quad \ldots \quad (17)$$

and σ_{1_p} represents the value of σ_1 at the peak stress state. Substituting Eq. 15 into Eq. 16 we finally obtain

$$-\sigma_{1_p} = \frac{2\, G_0}{\phi(k_2,k_3)} \frac{n-1}{n+r} \gamma_p \quad \ldots \quad (18)$$

which gives the stress at the points of the failure envelope as a function of

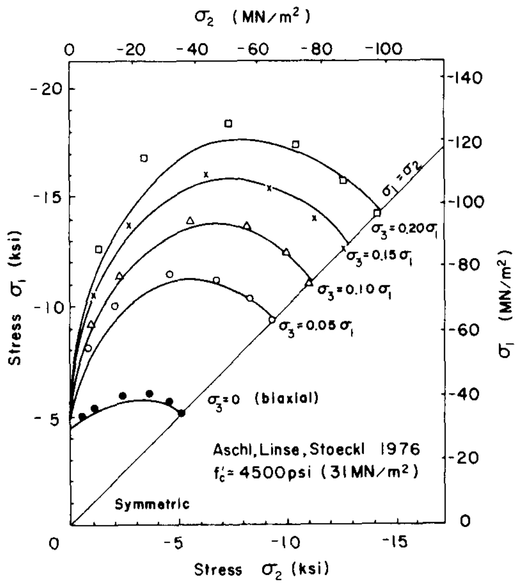

FIG. 8.—Fit of Triaxial Failure Envelopes for Proportional Triaxial Loading

ratios k_2 and k_3. Fitting Eq. 16 to the failure data [Figs. 7(*a*) and 8], the functions that define γ_p and r in Eq. 8 have been determined.

A similar procedure was used to determine from Eq. 6 and Eqs. 8 and 9 the equation for λ and λ' at the points of the failure envelope, which was exploited for determining the functions for λ and λ' given in Eqs. 9 and 10. The possibility of obtaining algebraic expressions for the failure states considerably facilitates fitting of the test data with the total strain theory.

For the final tuning of the fits of the stress-strain diagrams in Figs. 1-8, a computer program that solves the stress-strain relations and automatically plots the response curves was written. By trying various adjustments of the functions obtained by direct analysis of certain response curves and failure envelopes, and by optimizing the material parameters, the expressions in Eqs. 8-10 have finally been obtained.

One approach for solving the stress-strain relations in the program is to differentiate them to get incremental stress-strain relations and then integrate those using small loading steps. This is, however, inconvenient when the expressions for material functions such as γ'' and ϵ'' are not yet known because the analytical expressions for the derivatives needed for the incremental stress-strain relations are different for various choices of functions. Therefore, a different approach was used in the computer program. The prescribed stresses or displacements have been increased in small loading steps and the response for the end of the step was determined directly from the total stress-strain relations by iterations, in which the values of γ'' and ϵ'' in Eqs. 5 and 6 were evaluated on the basis of the previous iteration (and in the first iteration on the basis of the previous loading step). Taking small enough loading steps this process converged satisfactorily. It should be also noted that at least one strain variable was prescribed instead of a stress variable; e.g., ϵ_1 was prescribed instead of σ_1 in the uniaxial test. This appears to be necessary to achieve convergence near and beyond the peak stress point.

Using the foregoing expressions for γ'' and ϵ'', one can calculate the secant moduli as follows

$$\frac{1}{G} = \frac{1}{G_0} + \frac{2\gamma''}{\tau}; \quad \frac{1}{K} = \frac{1}{K_0} + \frac{3\epsilon''}{\sigma} \quad \ldots \quad (19)$$

Since ϵ'' may reach, due to dilatancy λ, large positive values while σ is negative (compression), the value of $1/K$ changes from positive to negative when the deviator strains are large. This is manifested by the fact that the volume change diagrams in Figs. 5, 6, and 7 cross to the left of the stress axis. At this crossing K approaches $+\infty$, jumps to $-\infty$ and then decreases into finite negative values. The previous total stress-strain models have stopped short of this behavior and were limited to positive finite K. It was for this reason why they were unable to model the inelastic dilatancy and had to be limited to states well before the peak stress point (failure) is reached.

The deformation due to pure hydrostatic stress (i.e., at $\tau = \gamma = 0$, $I_3 = \sigma^3$) may be described by $\epsilon = \sigma/3K_1$ in which

$$\frac{1}{K_1} = \frac{1}{K_0} + \frac{3\epsilon_0''}{\sigma} \quad \ldots \quad (20)$$

Here ϵ_0'' is the expression for ϵ'' obtained from Eq. 9 substituting $\tau = \gamma = 0$ and $I_3 = \sigma^3$; and K_1 is the secant modulus for pure hydrostatic stress. In the previous total stress-strain models, K_1 was used to describe the volume change in general, which of course neglects the significant effect of τ (or γ) upon K. To correct this deficiency, Kupfer and Gerstle (31) proposed to write the volumetric strain under a general stress state in the form

$$\epsilon = \frac{\sigma}{3K_1} + \frac{\tau_o}{H} \qquad (21)$$

and called H the coupling modulus, with $\tau_o = \sqrt{2/3}\,\tau$ being the octahedral shear stress. Our formulation of course includes this coupling effect and H may be calculated from Eqs. 9–10 as follows:

$$\frac{1}{H} = \sqrt{\frac{2}{3}}\,\frac{1}{\tau}\left[\frac{\sigma}{3}\left(\frac{1}{K_0} - \frac{1}{K_1}\right) + \epsilon''\right] \qquad (22)$$

It should be realized, however, that the coupling modulus represents only one correction to K_1. Further corrections are necessary to take into account the effects of γ and I_3. In the same spirit, one could define a coupling modulus between ϵ and $I_3^{1/3}$, or between γ and σ. For this reason we prefer to express all inelastic behavior in terms of ϵ'' and γ'' or K and G and avoid introducing further coupling moduli.

Path-Dependent Generalization

In standard triaxial tests, the hydrostatic pressure is applied first and the axial stress is increased subsequently while the lateral stress is kept constant [Fig. 9(*a*)]. This stress path appears to produce stiffer response and less damage in a concrete specimen than the proportional triaxial loading test. This is because the hydrostatic pressure applied at first prevents the growth of microcracks due to shear, particularly in the early stage of the axial loading which follows the hydrostatic loading. This trend is only partially offset by the fact that the hydrostatic loading itself also causes some collapse of microstructure in concrete [(20) Fig. 1].

The experimental information on the differences in response between standard triaxial tests and proportional triaxial tests (carried out on cubic specimens) is not quite consistent. In a recent private communication (1979), H. Aschl of Munich indicates that in his tests there is only a small difference between these peak stresses. On the other hand, comparison of his proportional tests with standard triaxial tests of some others reveals a very large difference. This is shown in Fig. 9(*b*), in which a comparison between the peak stress values (strength) obtained for these loading paths is made using the limited data available in the literature. Despite the scatter, it appears that there exists a significant systematic difference between the two loading paths. Consequently, the present model cannot be expected to give good predictions for a general nonproportional loading path, including the standard triaxial loading path, even though it probably would give the correct incremental stiffness for the first nonproportional loading increment, i.e., the increment to the side mentioned before. In fact, when the present model was applied to a nonproportional loading path, the deviation of the peak stress predicted by the model and that of test data was normally about 10%, and in some cases, up to 40%.

The basic philosophy in developing any refined theory of constitutive behavior is to begin with a simple model describing well some basic broad class of response and to adjust then the model by additional terms. In plasticity, e.g., the underlying simple concept is that of the loading surface and the normality rule. We refine it then by violating the normality rule, e.g., by introducing vertex hardening.

In endochronic theory, the underlying simple concept is the intrinsic time and the lack of unloading criteria, and improvement is then obtained by introducing unloading criteria. We may regard our theory in the same light. The fact that the stress-strain relation is algebraic is the underlying simple form, and we

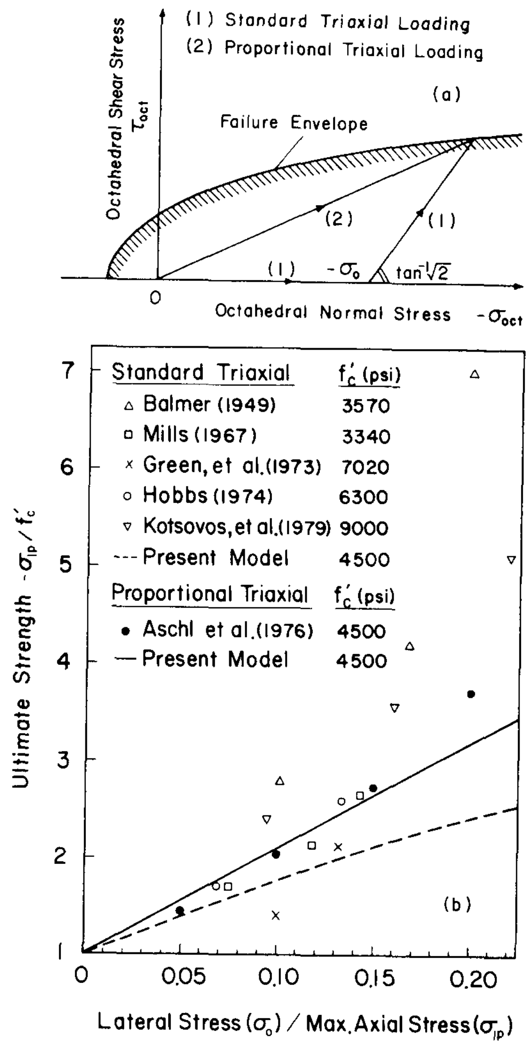

FIG. 9.—Standard Triaxial Loading and Proportional Triaxial Loading: (*a*) Stress Paths; (*b*) Comparison of Ultimate Strength

will now consider necessary refinement to represent the response for nonproportional loading paths.

The proportional loading paths are defined by the relations $ds_{ij} = s_{ij}\,d\kappa$ and $d\sigma = \sigma\,d\kappa$ in which $d\kappa$ = a common scalar parameter. Thus, we see that deviations from the proportional path are measured by the terms, $ds_{ij} - s_{ij}d\kappa$, and $d\sigma - \sigma d\kappa$, which vanish for proportional loading path. Therefore, the additional deviatoric and volumetric inelastic strains due to the nonproportionality of loading may be considered in the form:

$$e^*_{ij} = \int_L \frac{\phi_1(\epsilon^*)}{H_1(\sigma, \epsilon)}(ds_{ij} - s_{ij}d\kappa); \quad \epsilon^* = \int_L \frac{\phi_2(\epsilon^*)}{H_2(\sigma, \epsilon)}(d\sigma - \sigma\, d\kappa) \quad \dots \quad (23)$$

Squaring the relation $ds_{ij} = s_{ij}d\kappa$, we get $ds_{ij}ds_{ij} = s_{ij}s_{ij}d\kappa^2$, and by multiplying by 1/2 and taking square root we obtain $d\tau = \tau d\kappa$. Subtracting from this the relation $\mu d\sigma = \mu\sigma d\kappa$, in which μ = some constant, we obtain $d\tau - \mu d\sigma = (\tau - \mu\sigma)d\kappa$ from which

$$d\kappa = \frac{d\tau - \mu d\sigma}{\tau - \mu\sigma} \quad \dots \quad (24)$$

For all numerical calculations the value $\mu = 1$ was used. This value would give $d\kappa = 0/0$ when $\sigma = \tau$, but this case can never arise because $\tau \geq 0$ and tensile stress states ($\sigma > 0$) are excluded from our theory. The generalized path-dependent total stress-strain relation now becomes

$$e_{ij} = \frac{s_{ij}}{2G_0} + e''_{ij} + e^*_{ij}; \quad \epsilon = \frac{\sigma}{3K_0} + \epsilon'' + \epsilon^* \quad \dots \quad (25)$$

In numerical calculations, the stress increments in Eqs. 23 and 24 must be evaluated at each load increment and in each of its iterations. Then, the path-dependent increments Δe^*_{ij} and $\Delta\epsilon^*$ at a point on the stress path are accumulated into e^*_{ij} and ϵ^*.

Variables e^*_{ij} and ϵ^* add another degree of freedom to the present theory; e^*_{ij} is effective to control the peak of the stress-strain curve in standard triaxial tests, while ϵ^* is effective for adjusting the inelastic volume change in these tests. By analyzing the available test data for proportional and standard triaxial tests, the following expressions for moduli H_1 and H_2 have been identified:

$$\frac{1}{H_1(\sigma, \epsilon)} = h_1\gamma \frac{1 - h_0F_6}{1 + F_6}; \quad \frac{1}{H_2(\sigma, \epsilon)} = h_2\gamma\,(h_3 + F_7);$$

$$\phi_1(\epsilon^*) = \phi_2(\epsilon^*) = 1 \quad \dots \quad (26)$$

$$F_6 = \frac{-\left(\sigma + \dfrac{\tau}{\sqrt{3}}\right)}{(1\text{ psi})}; \quad F_7 = 2.2^s; \quad s = 0.01\,F_6 - 20;$$

$$h_0 = 1.963 \times 10^{-4}; \quad h_1 = 0.4317\,(\text{psi})^{-1} \times \left(\frac{f'_c}{6{,}500\text{ psi}}\right)^6;$$

$$h_2 = 10^{-4}\,(\text{psi})^{-1}; \quad h_3 = \frac{f'_c}{6{,}000\text{ psi}} \quad \dots \quad (27)$$

Variables γ and F_6 are useful to characterize the effects of shear deformation

and the confining pressure for standard triaxial loading paths, in contrast to the proportional ones.

The solid lines in Figs. 10–12 demonstrate that satisfactory fits of standard triaxial tests at low and moderately high confining pressure have been achieved. For the tests at high confining pressure, e.g., Balmer's tests (4) for p = 10 ksi–25 ksi (69 MN/m^2–172 MN/m^2), the present expression gives response below test data, although improvement might be possible with more refined expressions for H_1 and H_2.

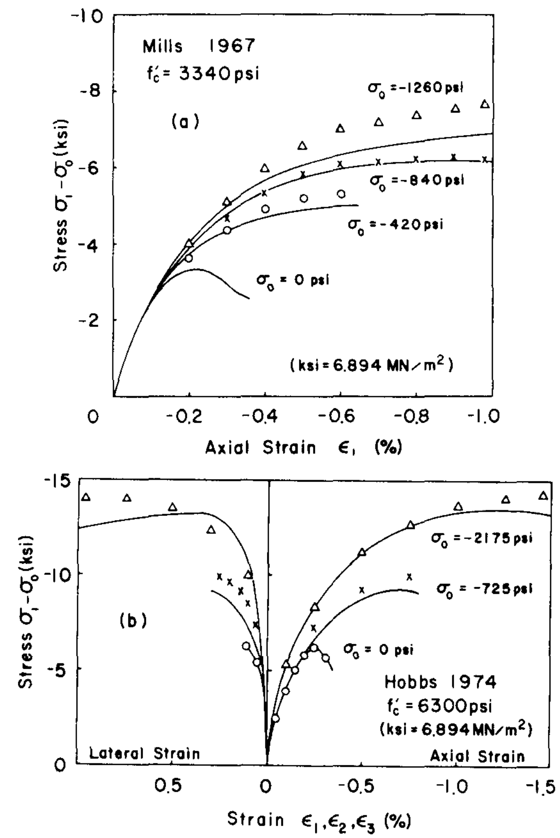

FIG. 10.—Fits of Standard Triaxial Test Data at Low Confining Pressure: *(a)* Low Strength Concrete; *(b)* Higher Strength Concretes (Stress Versus Strain)

One important consequence of introducing path-dependent terms e^*_{ij} is that the principal directions of stress and strain cease to coincide. This is typical of other theories, such as plastic or endochronic ones and is supported by experiments.

The pure total strain theory can be defined for only one set of loading path, but it does not necessarily have to be the set of proportional paths in the stress space. For example, if we had proper test data, we could define it for

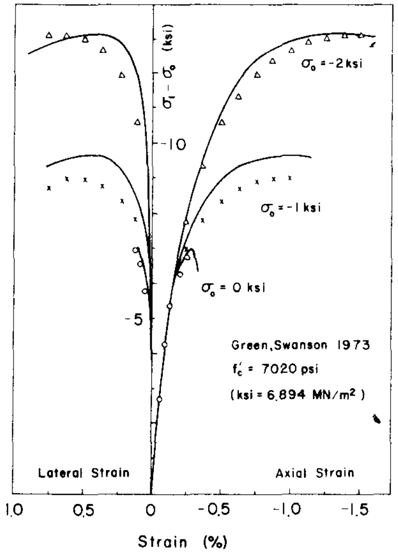

FIG. 11.—Fits of Standard Triaxial Test Data at Low Confining Pressure of Higher Strength Concretes: Strain as Percentage

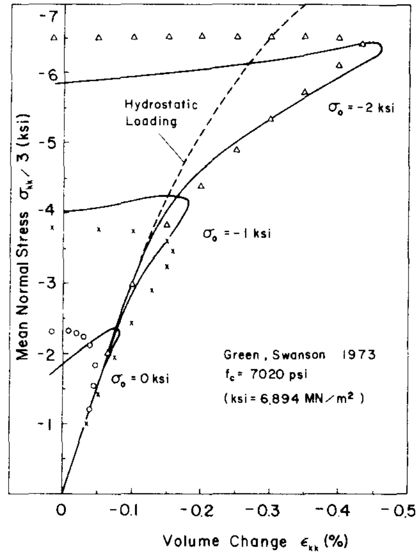

FIG. 12.—Fits of Standard Triaxial Test Data at Low Confining Pressure of Higher Strength Concretes: Mean Normal Stress Versus Volume Change

the set of proportional paths in the strain space. In that case, an extension to an arbitrary loading path could be achieved by terms analogous to Eq. 23 in which s_{ij}, σ, and τ are replaced by e_{ij}, ϵ, and γ.

One may also envisage a more general mixed formulation, such that mixed terms of stress and strain are combined with the previous expressions for e^*_{ij} and e^*. In such a formulation, however, there would be no loading path for which the response could be described by algebraic equations.

Initial Stiffness to Side

We may pay closer attention to the incremental stiffness for the first load increment to the side following a proportional stress path. Eq. 23 with $\phi_1 = \phi_2 = 1$ alter this stiffness, which seems undesirable because, as we said before, the total strain theory as such probably gives roughly correct values for this stiffness. This means that $1/H_1$ and $1/H_2$ should be made very small for the first to-the-side loading.

We could achieve this by proper functions $\phi_1(e^*)$ and $\phi_2(\epsilon^*)$ where $e^* = [e^*_{ij}e^*_{ij}/2]^{1/2}$ = intensity of e^*_{ij}, choosing ϕ_1 and ϕ_2 to be very small when e^* and ϵ^* tend to zero, and equal to 1.0 when e^* and ϵ^* are large. This has not been done for lack of good measurements of the initial to-the-side stiffness, but it is something that should be kept in mind for the future refinement.

Special Path-Dependent Form when Normality Rule is Satisfied

Without getting into an examination of the applicability of the normality rule,

it is of interest to observe that our path-dependent corrective terms allow us to obtain an incremental stress-strain relation which satisfies the normality rule yet still gives the same response to proportional loading as our total strain theory. Choosing now $d\kappa = d\tau/\tau$ and differentiating Eqs. 25 and 23, we obtain

$$de_{ij} = \frac{ds_{ij}}{2G_0} + de^{pl}_{ij} \qquad (28)$$

in which $$de^{pl}_{ij} = -s_{ij}\frac{dG}{2G^2} - \left(\frac{1}{2G_0} - \frac{1}{2G}\right) ds_{ij} + \frac{\phi_1}{H_1}\left(ds_{ij} - s_{ij}\frac{d\tau}{\tau}\right) \qquad (29)$$

For proportional loading the last term vanishes, and so the integral of these equations for proportional loading is identical to the total strain theory regardless of the definitions of ϕ_1 and H_1. If we choose von Mises loading surface in the stress space, then the terms with s_{ij} satisfy the normality rule whereas the terms with ds_{ij} in general do not, unless the loading is proportional. Therefore the normality rule is satisfied when the coefficient at ds_{ij} in Eq. 29 vanishes, i.e., when we choose ϕ_1 and H_1 such that

$$\frac{\phi_1}{H_1} = \frac{1}{2G_0} - \frac{1}{2G} \qquad (30)$$

Furthermore, in classical plasticity the magnitude of de^{pl}_{ij} is required to vanish when the loading direction is parallel to the loading surface, i.e., normal to s_{ij} (or $s_{km}ds_{km} = 0$), which is known as the continuity condition. To achieve this, we must further have $dG/2G^2 + \phi_1 d\tau/H_1\tau = 0$ when $d\tau = 0$, or $dG = 0$ when $d\tau = 0$. This latter condition can, in general, be attained only if G depends solely on τ and is independent of other invariants of stress and strain, which is not a very realistic model.

Incremental (Tangential) Form of Proposed Total Stress-Strain Relation

For structural analysis, we must convert the stress-strain relation to an incremental form. Differentiating Eq. 25 we have the following incremental form:

$$ds_{ij} = 2G_0(de_{ij} - de''_{ij} - de^*_{ij}); \quad d\sigma = 3K_0(d\epsilon - d\epsilon'' - d\epsilon^*) \qquad (31)$$

By definition: $$d\sigma_{ij} = ds_{ij} + d\sigma\delta_{ij} = 2G_0\,d\epsilon_{ij} + \left(K_0 - \frac{2}{3}G_0\right)d\epsilon_{kk}\,\delta_{ij}$$

$$- [2G_0(de''_{ij} + de^*_{ij}) + 3K_0(d\epsilon'' + d\epsilon^*)\delta_{ij}] \qquad (32)$$

in which $de''_{ij} = d\left(s_{ij}\frac{\gamma''}{\tau}\right) = \frac{\gamma''}{\tau}ds_{ij} + \frac{s_{ij}}{\tau}d\gamma'' - \frac{s_{ij}\gamma''}{\tau^2}d\tau$;

$$d\epsilon'' = \frac{\partial\epsilon''}{\partial\sigma_{km}}d\sigma_{km} + \frac{\partial\epsilon''}{\partial\epsilon_{km}}d\epsilon_{km} \qquad (33)$$

If we substitute

$$d\gamma'' = \frac{\partial\gamma''}{\partial\sigma_{km}}d\sigma_{km} + \frac{\partial\gamma''}{\partial\epsilon_{km}}d\epsilon_{km}; \quad d\tau = \frac{s_{km}d\sigma_{km}}{2\tau}; \quad d\sigma = \frac{d\sigma_{kk}}{3} \qquad (34)$$

into Eq. 32, then we obtain the tangential stress-strain relation:

$$d\sigma_{ij} = D_{ijkm}\, d\epsilon_{km} \qquad (35)$$

in which $D_{ijkm} = A^{-1}_{pqij}(D^{el}_{pqkm} - B_{pqkm})$;

$$A_{pqij} = \left[\frac{1}{2} + G_0\left(\frac{\gamma''}{\tau} + \frac{1}{H_1}\right)\right](\delta_{pi}\delta_{qj} + \delta_{qi}\delta_{pj}) + B'_{pqij} - D^*_{pqij};$$

$$B_{pqkm} = 2G_0 \frac{s_{pq}}{\tau}\frac{\partial\gamma''}{\partial\epsilon_{km}} + 3K_0\delta_{pq}\frac{\partial\epsilon''}{\partial\epsilon_{km}}; \quad B'_{pqij} = 2G_0\frac{s_{pq}}{\tau}\left(\frac{\partial\gamma''}{\partial\sigma_{ij}} - \frac{\gamma''}{2\tau^2}s_{ij}\right)$$

$$+ 3K_0\delta_{pq}\frac{\partial\epsilon''}{\partial\sigma_{ij}} - \frac{2\gamma''}{3\tau}G_0\delta_{pq}\delta_{ij}; \quad D^*_{pqij} = \left(\frac{2G_0}{3H_1} - \frac{K_0}{H_2}\right)\delta_{pq}\delta_{ij}$$

$$+ \frac{1}{\tau - \mu\sigma}\left(\frac{2G_0}{H_1}s_{pq} + \frac{3K_0}{H_2}\sigma\,\delta_{pq}\right)\left(\frac{s_{ij}}{2\tau} - \frac{\mu}{3}\delta_{ij}\right);$$

$$D^{el}_{pqkm} = G_0(\delta_{pk}\delta_{qm} + \delta_{qk}\delta_{pm}) + \left(K_0 - \frac{2}{3}G_0\right)\delta_{pq}\delta_{km} \qquad (36)$$

In these expressions, the partial derivatives of γ'' and ϵ'' with respect to stresses and strains may be obtained either analytically or numerically by finite difference approximations. Eq. 35 may then serve as basis for structural analysis with small loading increments.

Summary and Conclusions

The time-independent (short-time) nonlinear triaxial behavior of plain concrete that is free of continuous cracks and is subjected to monotonic loading is described in the form of an algebraic relation between total strains and stresses, analogous to the deformation theory of plasticity. The quadratic tensorial terms of stress and strain components are neglected. Good agreement with characteristic test data is achieved. The advantages of the formulation are:

1. The stress-strain relation is an algebraic rather than a differential or integral equation.
2. This fact lends the model conceptual simplicity and allows dispensing with more sophisticated concepts such as the loading surface or the intrinsic time used in other existing theories.
3. In contrast to the incremental theories, the present one yields an algebraic expression for failure envelopes. It also yields algebraic expressions for certain directly observable response curves, such as the hydrostatic loading test or the dilatancy curve.
4. The preceding facts simplify determination of material functions from test data. The process of data fitting can be split in separate identifications of various functions.
5. The theory probably represents the most simple theory that gives the correct incremental stiffness for the first to-the-side load increment following a proportional stress path.

The disadvantages of the present formulation are:

1. In contrast to some of the existing total stress-strain formulation, the present stress-strain relation is not explicit. Explicitness appears to be impossible if a broad range of experimental behavior should be represented well.
2. Consequently, the stress-strain relation must be differentiated and used as an incremental stress-strain relation in structural analysis. This is no simpler than the analysis by the plastic, plastic-fracturing or endochronic formulations.
3. Compared to these existing constitutive models, the present theory does not lead to fewer material constants.

Since the total stress-strain formulation is path-independent, it is limited to proportional or nearly proportional loading. It is shown, however, that by adding certain path-dependent tensorial terms that vanish for proportional loading, the total strain formulation can be extended to nonproportional loading and gives then path-dependent behavior. The addition of path-dependent terms causes that the principal directions of stress and strain cease to coincide for nonproportional loading, as indicated by experiment as well as the plastic, plastic-fracturing and endochronic theories. In the path-dependent form, the theory seems to be almost as general, and fit test data almost as equally well, as the endochronic and the plastic-fracturing theories for monotonic loading.

In contrast to the previous total strain models, the present one has a much broader applicability. While the previous models apply only up about 0.8 of strength, the present one gives the peak stress point as well as the subsequent strain softening. It also gives the triaxial failure envelopes, the corresponding strains at peak stress (failure) states, the inelastic dilatancy in uniaxial and triaxial tests, the stiffening which follows initial softening in purely hydrostatic loading, and it describes triaxial response curves, including lateral strains, up to much higher strains than the previous theories.

Acknowledgment

Support of the National Science Foundation under Grant ENG75-14848-A01 to Northwestern University is gratefully acknowledged. The first writer also thanks for partial support under Guggenheim Fellowship awarded to him for 1978–79.

Appendix I.—References

1. Argyris, J. H., Faust, G., Szimmat, J., Warnke, E. P., and Willam, K. J., "Recent Development in the Finite Element Analysis of Prestressed Concrete Reactor Vessels," *Nuclear Engineering and Design*, Vol. 28, 1974, pp. 42–75; see also *Report No. 151*, Institut für Statik und Dynamik, University of Stuttgart, Stuttgart, Germany, 1973.
2. Argyris, J. H., Faust, G., and Willam, K. J., "Limit Load Analysis of Thick-Walled Concrete Structures—A Finite Element Approach to Fracture," *Computer Methods in Applied Mechanics and Engineering*, Vol. 8, 1976, pp. 215–243.
3. Aschl, H., Linse, D., and Stoeckl, S., "Strength and Stress-Strain Behavior of Concrete under Multiaxial Compression and Tension Loading," *Proceedings of the 2nd International Conference on Mechanical Behavior of Materials*, Aug., 1976.
4. Balmer, G. G., "Shearing Strength of Concrete under High Triaxial Stress—Computation of Mohr's Envelope as a Curve," *Structural Research Laboratory Report No.*

SP-23, Structural Research Laboratory, Denver, Colo., Oct., 1949.
5. Batdorf, S. B., and Budiansky, B., "A Mathematical Theory of Plasticity Based on the Concept of Slip," *Technicál Note 1871,* National Advisory Committee on Aeronautics, 1949.
6. Bathe, K. J., and Ramaswamy, S., "On Three-Dimensional Nonlinear Analysis on Concrete Structures," *Nuclear Engineering and Design,* Vol. 52, No. 3, May, 1979, pp. 385–409.
7. Bažant, Z. P., and Bhat, P., "Endochronic Theory of Inelasticity and Failure of Concrete," *Journal of the Engineering Mechanics Division,* ASCE, Vol. 102, No. EM4, Proc. Paper 12360, Aug., 1976, pp. 331–344.
8. Bažant, Z. P., "Endochronic Inelasticity and Incremental Plasticity," *International Journal of Solids and Structures,* Vol. 14, 1978, pp. 691–714.
9. Bažant, Z. P., and Shieh, C.-L., "Endochronic Model for Nonlinear Triaxial Behavior of Concrete," *Nuclear Engineering and Design,* Vol. 47, 1978, pp. 305–315.
10. Bažant, Z. P., and Shieh, C.-L., "Hysteretic Fracturing Endochronic Theory for Concrete," *Structural Engineering Report 78-9/640h,* Northwestern University, Evanston, Ill., Sept., 1978; *Journal of the Engineering Mechanics Division,* ASCE, Vol. 106, No. EM5, Proc. Paper 15781, Oct., 1980, pp. 929–950.
11. Bažant, Z. P., and Kim, S.-S., "Plastic-Fracturing Theory for Concrete," *Journal of the Engineering Mechanics Division,* ASCE, Vol. 105, No. EM3, Proc. Paper 14653, June, 1979, pp. 407–428.
12. Budiansky, B., "A Reassessment of Deformation Theories of Plasticity," *Journal of Applied Mechanics, Transactions,* American Society of Mechanical Engineers, Vol. 26, 1959, pp. 259–264.
13. Cedolin, L., Crutzen, Y. R. J., and Dei Poli, S., "Triaxial Stress-Strain Relationship for Concrete," *Journal of the Engineering Mechanics Division,* ASCE, Vol. 103, No. EM3, Proc. Paper 12969, June, 1977, pp. 423–439.
14. Chen, A. C. T., and Chen, W. F., "Constitutive Relations for Concrete," *Journal of the Engineering Mechanics Division,* ASCE, Vol. 101, No. EM4, Proc. Paper 11529, Aug., 1975, pp. 465–481.
15. Chen, W. F., and Suzuki, H., "Constitutive Models for Concrete," paper presented at the October 16–20, 1978, ASCE Annual Convention, held at, Chicago, Ill.
16. Christoffersen, J., and Hutchinson, J. W., "A Class of Phenomenological Corner Theories of Plasticity," *Report MECH-8,* Division of Applied Sciences, Harvard University, Cambridge, Mass., Jan., 1979.
17. Evans, R. J., and Pister, K. S., "Constitutive Equations for a Class of Nonlinear Elastic Solids," *International Journal of Solids and Structures,* Vol. 2, 1966, pp. 427–445.
18. Fung, Y. C., "Foundations of Solid Mechanics," Prentice Hall, Inc., Englewood Cliffs, N.J., 1965, pp. 477, 488.
19. Gardner, N., "Triaxial Behavior of Concrete," *American Concrete Institute Journal,* Vol. 66, No. 2, Feb., 1969, pp. 136–146.
20. Green, S. J., and Swanson, S. R., "Static Constitutive Relations for Concrete," *Technical Report No. AFWL-TR-72-2,* Terra-Tek, Inc., Salt Lake City, Utah, Apr., 1973.
21. Handelmann, G. H., Lin, C. C., and Prager, W., "On the Mechanical Behavior of Metals in the Strain-Hardening Range," *Quarterly of Applied Mathematics,* Vol. 4, 1947, pp. 397–407.
22. Hecker, S. S., "Experimental Studies of Yield Phenomena in Biaxially Loaded Metals," Constitutive Equations in Viscoplasticity: Computational and Engineering Aspects, presented at the Winter Annual Meeting of the ASME, New York, N.Y., Dec., 1976; Applied Mechanics Division, Vol. 20, pp. 1–33.
23. Hencky, H., "Zur Theorie Plastischer Deformationen und des hien durch in Material Lervogerufenen Dachspannungen," *Zeitschrift der Angewandten Mathematik und Mechanik,* Vol. 4, 1924, pp. 323–334.
24. Hobbs, D. W., "Strength and Deformation Properties of Plain Concrete Subject to Combined Stress, Part 3: Results Obtained on a Range of Flint Gravel Aggregate Concrete," *Technical Report,* Cement and Concrete Association, London, England, July, 1974.
25. Hognestad, E., Hanson, E. W., and McHenry, D., "Concrete Stress Distribution

in Ultimate Strength," *American Concrete Institute Journal,* Vol. 52, No. 4, Dec., 1955, pp. 455–477.
26. Hutchinson, J. W., "Elastic Plastic Behavior of Polycrystalline Metals and Composite," *Proceedings of the Royal Society of London,* Series A, Vol. 319, No. 1537, Oct., 1970, pp. 247–272.
27. Hutchinson, J. W., "Plastic Buckling," *Advances in Applied Mechanics,* Yih, C.-S., ed., Vol. 14, Academic Press, New York, N.Y., 1974, pp. 67–144.
28. Kotsovos, M. D., and Newman, J. B., "Generalized Stress-Strain Relations for Concrete," *Journal of the Engineering Mechanics Division,* ASCE, Vol. 104, No. EM4, Proc. Paper 13922, Aug., 1978, pp. 845–856.
29. Kotsovos, M. D., and Newman, J. B., "A Mathematical Description of the Deformational Behavior of Concrete under Complex Loading," *Magazine of Concrete Research,* Vol. 31, No. 107, June, 1979, pp. 77–90.
30. Kupfer, H., Hilsdorf, H. K., and Rüsch, H., "Behavior of Concrete Under Biaxial Stresses," *American Concrete Institute Journal,* Vol. 66, Aug., 1969, pp. 656–666.
31. Kupfer, H. B., and Gerstle, K. H., "Behavior of Concrete Under Biaxial Stresses," *Journal of the Engineering Mechanics Division,* ASCE, Vol. 99, No. EM4, Proc. Paper 9917, Aug., 1973, pp. 853–866.
32. McDonald, J. E., "Time-Dependent Deformation of Concrete Under Multiaxial Stress Conditions," *Technical Report C-75-4,* Concrete Laboratory, Army Engineer Waterways Experiment Station, Vicksburg, Miss., Oct., 1975, (prepared for Oak Ridge National Laboratory, *ORNL-TM-5052/UC-57*).
33. Mills, L. L., "A Study of the Strength of Concrete Under Combined Compressive Loads," dissertation presented to New Mexico State University, at Las Cruces, N.M., in 1967, in partial fulfillment of the requirements for the degree of Doctor of Philosophy.
34. Mills, L. L., and Zimmerman, R. M., "Compressive Strength of Plain Concrete Under Multiaxial Loading Conditions," *American Concrete Institute Journal,* Vol. 67, No. 10, Oct., 1970, pp. 802–807.
35. Newmark, N. M., "Failure Hypothesis for Soils," *Proceedings of ASCE Research Conference on Shear Strength of Cohesive Soils,* Boulder, Colo., pp. 17–32, June, 1963.
36. Ottosen, N. S., "Constitutive Model for Short-Time Loading of Concrete," *Journal of the Engineering Mechanics Division,* ASCE, Vol. 105, No. EM1, Proc. Paper 14375, Feb., 1979, pp. 127–141.
37. Palaniswamy, R., and Shah, S. P., "Fracture and Stress-Strain Relationship of Concrete Under Triaxial Compression," *Journal of the Structural Division,* Vol. 100, No. ST5, Proc. Paper 10547, May, 1974, pp. 901–916.
38. Popovics, S., "A Numerical Approach to the Complete Stress-Strain Curve of Concrete," *Cement and Concrete Research,* Vol. 3, No. 5, Sept., 1973, pp. 583–599.
39. Prager, W., "Strain Hardening Under Combined Stress," *Journal of Applied Physics,* Vol. 16, 1945, pp. 837–840.
40. Rivlin, R. S., "Further Remarks on the Stress-Deformation Relations for Isotropic Materials," *Journal of Rational Mechanics and Analysis,* Vol. 4, 1955, pp. 681–702, (renamed *Journal of Mathematics and Mechanics*).
41. Rudnicki, J. W., and Rice, J. R., "Conditions of the Localization of Deformation in Pressure-Sensitive Dilatant Materials," *Journal of the Mechanics of Physics and Solids,* Vol. 23, 1975, pp. 371–394.
42. Saenz, L. P., "Equation for Stress-Strain Curve of Concrete," (Discussion) *American Concrete Institute Journal,* Vol. 61, No. 9, Sept., 1964, pp. 1229–1236.
43. Saugy, B., "Contribution a l'etude Theorique du Comportement Non Linearie des Structures Massive en Beton Arme sans Charges Rapides," *Bulletin Technique de la Suisse Romande,* Vol. 95, No. 22, Nov., 1969.
44. Saugy, B., Zimmerman, T., and Hussain Khan, M., "Three Dimensional Rupture Analysis of a Prestressed Concrete Pressure Vessel Including Creep Effects," *Paper H 2/5,* in the Sept., 1973, Second International Conference on Structural Mechanics in Reactor Technology, held at Berlin, Germany.
45. Schickert, G., and Winkler, H., "Results of Tests Concerning Strength and Strain of Concrete Subjected to Multiaxial Compressive Stresses," *Bundesanstalt für Materialprüfung (BAM),* Berlin, Germany, Bericht, No. 46, May, 1977.
46. Schimmelpfennig, K., "Ultimate Load Analysis of Prestressed Concrete Reactor

Pressure Vessels Considering a General Material Law," *Paper H 4/6, Transactions of the 3rd International Conference on Structural Mechanics in Reactor Technology*, London, England, Sept., 1975.

47. Willam, K. J., and Warnke, E. P., "Constitutive Model for the Triaxial Behavior of Concrete," presented at the May, 1974 International Association for Bridge and Structural Engineering Seminar on Concrete Structures Subjected to Triaxial Stresses, held at Bergamo, Italy.
48. Zienkiewicz, O. C., Phillips, D. V., and Owen, D. R. J., "Finite Element Analysis of Some Concrete Non-Linearities, Theory and Examples, *Paper III-2*, ISMES, presented at the May, 1974 International Association for Bridge and Structural Engineering Seminar on Concrete Structures Subjected to Triaxial Stresses, held at Bergamo, Italy.

15911 TOTAL STRAIN, PATH-DEPENDENCE OF CONCRETE

KEY WORDS: **Concrete; Concrete structures; Constitutive equations;** Deformation methods; Finite elements; **Inelastic action; Mathematical models;** Plasticity; Programs; **Strains; Stress strain relations;** Structural analysis; Triaxial tests

ABSTRACT: The nonlinear triaxial behavior of plane concrete that is free of continuous cracks is modeled by an algebraic relation between total strains and stresses, analogous to deformation theory of plasticity. Unloading is not modeled. Good agreement with numerous test data is achieved. The formulation implies algebraic expressions for failure envelopes and some stress-strain curves, which make data fitting easier. The model is enhanced by corrective path-dependent terms that vanish for proportional loading so as extend it for highly nonproportional loading. The principal directions of stress and strain then cease to coincide. In contrast to previous models, the present one applies to much largest atrains, gives the peak stress points, failure envelopes, strain softening, inelastic dilatancy, etc. A tangential incremental form for structural analysis is also indicated.

REFERENCE: Bazant, Zdenek P., and Tsubaki, Tatsuya, "Total Strain Theory and Path-Dependence of Concrete," *Journal of the Engineering Mechanics Division*, ASCE, Vol. 106, No. EM6, **Proc. Paper 15911**, December, 1980, pp. 1151-1173