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Viscoplasticity of Transversely Isotropic Clays

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Introduction

Most experimental investigations concerning the mechanical behavior of one-dimensionally consolidated clays indicate some degree of anisotropy. One cause of this phenomenon is the prevailing orientation of the clay platelets normal to the direction of the major principal consolidation stress. The magnitude of the principal stress difference necessary to develop an anisotropic fabric is a function of the clay type, pore fluid chemistry, and initial particle orientation. Depending on the degree of anisotropy of the clay structure, the mechanical properties are also anisotropic and nonlinear.

Constitutive relations for anisotropic clays can be approached by considering the micromechanics of elementary cells of clay particles (5), but this approach is complicated for three-dimensional behavior and has not yet been fully developed. In the present study, a phenomenological approach, supported by a qualitative understanding of the physical processes in the microstructure, is adopted. A constitutive relation for transversely isotropic clays is developed by extending previous concepts based on endochronic theory, and the capability of the model is demonstrated by its application to describe data from hollow cylinder, conventional triaxial, and plane strain tests. This work is an extension of previous studies on isotropic clays (1,10).

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Endochronic Theory as Viscoplasticity with Strain-Rate-Dependent Viscosity

The basic idea underlying the proposed theory is the characterization of inelastic strain increments in terms of a nondecreasing scalar variable, z, whose rate of increase depends on strain rates. The variable z, which has been originally called reduced time (15), is now generally known as intrinsic time, a term coined by Valanis (18), who was first to apply this concept successfully to complicated nonlinear behavior (particularly metals) and called the theory "endochronic." This theory is most appropriately treated as a special case of viscoplasticity (15,18). To set the background for the further treatment of clay, we will first indicate how endochronic theory may be obtained from certain quite reasonable hypotheses about viscoplasticity with strain-rate dependent viscosity (2).

The viscoplastic constitutive relation (for small strains) may be generally written as

$$d\epsilon_{ij} = D_{ijkl}\, d\sigma_{kl} + d\epsilon''_{ij} \quad \ldots \quad (1a)$$

$$d\epsilon''_{ij} = \frac{\partial \Phi}{\partial \sigma_{ij}}\, dz; \quad dz = \frac{dt}{a(\underset{\sim}{\sigma}, \underset{\sim}{\epsilon})} \quad \ldots \quad (1b)$$

in which ϵ_{ij} = components of the linearized strain tensor $\underset{\sim}{\epsilon}$, ϵ''_{ij} = inelastic part of the strain tensor; σ_{ij} = components of the stress tensor $\underset{\sim}{\sigma}$ (all in cartesian coordinates, x_i, $i = 1, 2, 3$); D_{ijkl} = flexibility coefficients; a = viscosity coefficient; $\Phi = \Phi(\sigma_{ij})$ = loading function, which must be convex and satisfy proper invariance requirements; and repeated indices imply summation. In classical viscoplasticity, a is a function of $\underset{\sim}{\sigma}$ and possibly also $\underset{\sim}{\epsilon}$. However, as pointed out by Schapery (15), the viscosity coefficient, a, must generally be considered to depend also on the strain rate, $\dot{\epsilon}_{ij}$. This dependence may be assumed in the form $a(\underset{\sim}{\sigma}, \underset{\sim}{\epsilon}) = a_1(\underset{\sim}{\sigma}, \underset{\sim}{\epsilon})\, a_2(\dot{\underset{\sim}{\epsilon}})$; superimposed dots are used to indicate derivatives with respect to time, t.

If the inelastic strain develops gradually, as in clays, the function $a_2(\dot{\underset{\sim}{\epsilon}})$ may be expected to be continuous and smooth; then, a Taylor series expansion is possible:

$$\left[\frac{1}{a_2(\dot{\underset{\sim}{\epsilon}})}\right]^r = p_o + p_{ij}\dot{\epsilon}_{ij} + p_{ijkl}\dot{\epsilon}_{ij}\dot{\epsilon}_{kl} + p_{ijklmn}\dot{\epsilon}_{ij}\dot{\epsilon}_{kl}\dot{\epsilon}_{mn} + \ldots \quad \ldots \quad (2)$$

in which r is some exponent to be determined later. Now, the following two requirements may be imposed on physical grounds: (1) The coefficient $1/a_2$ should increase as $||\dot{\epsilon}_{ij}||$ increases, in particular, $1/a_2$ for any $\dot{\underset{\sim}{\epsilon}} \neq 0$ should be larger than $1/a_2$ for $\dot{\underset{\sim}{\epsilon}} = 0$; and (2) the ratio of the inelastic to total increment magnitudes must be greater than zero and finite as the strain magnitude tends to infinity, i.e.

$$\infty > \lim_{||\dot{\underset{\sim}{\epsilon}}|| \to \infty} \frac{||d\underset{\sim}{\epsilon}''||}{||d\underset{\sim}{\epsilon}||} > 0$$

in which $||..||$ represents the magnitude of the tensor (defined as $||\epsilon_{ij}|| = (\epsilon_{ij}\epsilon_{ij})^{1/2}$ or as $[J_2(\underset{\sim}{\epsilon})]^{1/2}$, where J_2 = second invariant of the deviator).

If the linear terms, $p_{ij}\dot{\epsilon}_{ij}$, were present in Eq. 2, it would be impossible to satisfy the first condition because $\dot{\epsilon}_{ij}$ can be either positive or negative of any magnitude; thus, $p_{ij} = 0$. For the same reason the cubic terms must also vanish; therefore, $p_{ijklmn} = 0$. On the other hand, the quadratic form, $p_{ijkl}\dot{\epsilon}_{ij}\dot{\epsilon}_{kl}$, can be chosen to be positive for any $\dot{\epsilon}_{ij}$. Thus, by truncating the Taylor series after the cubic term, which must be sufficiently accurate for small enough values of $\dot{\epsilon}_{ij}$, we are left with

$$\left[\frac{1}{a_2(\dot{\underset{\sim}{\epsilon}})}\right]^r = p_o + p_{ijkl}\dot{\epsilon}_{ij}\dot{\epsilon}_{kl} \quad \ldots \quad (3)$$

In the limit for very rapid loading, we must expect time-independent behavior; thus, it must be possible to eliminate the time dependence. Dividing Eq. 1 by $||d\underset{\sim}{\epsilon}|| = (d\epsilon_{ij}\, d\epsilon_{ij})^{1/2}$, we obtain

$$\frac{||d\underset{\sim}{\epsilon}''||}{||d\underset{\sim}{\epsilon}||} = \left|\left|\frac{\partial \phi}{\partial \sigma_{ij}}\right|\right| \frac{1}{a_1(\underset{\sim}{\sigma}, \underset{\sim}{\epsilon})} \left(\frac{p_o}{||\dot{\underset{\sim}{\epsilon}}||^r} + \frac{p_{ijkl}\, d\epsilon_{ij}\, d\epsilon_{kl}}{d\epsilon_{pq}\, d\epsilon_{pq}}\, ||\dot{\underset{\sim}{\epsilon}}||^{2-r}\right)^{1/r} \quad \ldots \quad (4)$$

According to the second condition, this ratio must be greater than zero and less than infinity. If $2 - r < 0$, then $||d\underset{\sim}{\epsilon}''||/||d\underset{\sim}{\epsilon}|| \to 0$ for $||\dot{\underset{\sim}{\epsilon}}|| \to \infty$, i.e., the material would become perfectly elastic when subjected to very rapid loading; this is unreasonable and violates the second condition. On the other hand, if $2 - r > 0$, then $||d\underset{\sim}{\epsilon}''||/||d\underset{\sim}{\epsilon}|| \to \infty$ for $||\dot{\underset{\sim}{\epsilon}}|| \to 0$, which is impossible. Therefore, the only remaining possibility is $2 - r = 0$ or $r = 2$; thus, under the assumptions stated, the intrinsic time increments must be expressed in the form

$$dz = \frac{dt}{a} = \frac{dt}{a_1(\underset{\sim}{\sigma}, \underset{\sim}{\epsilon})}\sqrt{p_o + p_{ijkl}\dot{\epsilon}_{ij}\dot{\epsilon}_{kl}}$$

$$= \frac{1}{a_1(\underset{\sim}{\sigma}, \underset{\sim}{\epsilon})}\sqrt{p_o\, dt^2 + p_{ijkl}\, d\epsilon_{ij}\, d\epsilon_{kl}} \quad \ldots \quad (5)$$

An alternative derivation of Eq. 5 has been given by Bazant and Bhat (3). Note that, for a certain choice of p_{ijkl}, the reduced time coefficient, a, is a positive function of the total octahedral shear strain rate, as suggested first by Schapery (Ref. 15, p. 279). The particular square root type form of dz in Eq. 5, deduced here from two reasonable hypotheses, was directly introduced by Valanis (18). Eq. 5 may be written as

$$dz = \sqrt{\left(\frac{d\zeta}{Z_1}\right)^2 + \left(\frac{dt}{\tau_1}\right)^2} \quad \ldots \quad (6a)$$

$$d\zeta = \sqrt{p_{ijkl}\, d\epsilon_{ij}\, d\epsilon_{kl}} \quad \ldots \quad (6b)$$

in which $1/\tau_1 = \sqrt{p_o}/a_1(\underset{\sim}{\sigma}, \underset{\sim}{\epsilon})$; $p_{ijkl} = Z_1^2 p_{ijkl}[a_1(\underset{\sim}{\sigma}, \underset{\sim}{\epsilon})]^{-2}$; and Z_1 = constant. The symbol τ_1 may be interpreted as a retardation time characteristic of the material within the desired time range. Note that τ_1 can depend on $\underset{\sim}{\sigma}$ and $\underset{\sim}{\epsilon}$, which would model classical viscoplastic behavior. The variable ζ, which will be called the modified path length, represents the length of the path traced by the states of the material in six-dimensional strain space (of variable metric p_{ijkl}). The special case, $d\zeta = \sqrt{de_{ij}\, de_{ij}/2}$, in which e_{ij} = the strain deviator, was used already in the 1950's and early 1960's by Il'yushin, Rivlin, and Pipkin (see for comparison Ref. 3). An important innovation proposed by Valanis (18) was to make p_{ijkl} dependent on a scalar function of ζ, which is called a hardening function and is particularly effective in modeling the hardening in cyclic loading. The theory was further extended by devising practical ways to model various particular features of geological materials, such as the existence of peak points on the stress-strain curves, strain-softening, inelastic dilatancy, hydrostatic stress sensitivity, long-term creep (3), inelastic strains due to hydrostatic stress (6), and other phenomena (2). It should also be noted that the basic stress-strain relation (Eq. 1) was put into the framework of thermodynamics by Schapery (15) and Valanis (18).

Setting $\tau_1/Z_1 = 0$ (i.e., dropping dt^2 from Eq. 6*a*), we obtain a time-independent theory. For lack of data on rate dependence, only this limiting time-independent case is considered in the sequel, but the extension to rate dependence is obvious.

The hardening and softening of the material in the course of inelastic straining is simplest to describe by means of scalar hardening and softening functions f and F:

$$d\zeta = \frac{d\eta}{f(\eta)}; \quad d\eta = F(\underset{\sim}{\sigma}, \underset{\sim}{\epsilon})d\xi; \quad d\xi = \sqrt{J_2(d\underset{\sim}{\epsilon})} = \sqrt{\frac{1}{2}\, de_{ij}de_{ij}} \quad \ldots\ldots (7)$$

in which $F(\underset{\sim}{\sigma}, \underset{\sim}{\epsilon})$ = a function of proper transversely isotropic invariants of the total strain tensor, $\underset{\sim}{\epsilon}$, and the total stress tensor, $\underset{\sim}{\sigma}$; $J_2(d\underset{\sim}{\epsilon})$ = a certain, properly invariant, quadratic form in $d\epsilon_{ij}$, which is assumed to have constant coefficients and to be an overall characteristic of shear strain increments, which give rise to inelastic phenomena; and ξ represents the path length in strain space (of constant metric). The functions f and F are both required to increase proportionally with increasing strain. The function $f(\eta)$ models chiefly the fact that the material hardens as a function of the path length, rather than the strain or stress level (this is corroborated by cyclic tests), and the function $F(\underset{\sim}{\epsilon}, \underset{\sim}{\sigma})$ reflects the weakening of the material at high strain and stress levels. Note that, upon integration over η and differentiation, Eq. 7 can be arranged to the form $d\zeta = F_1(\underset{\sim}{\sigma}, \underset{\sim}{\epsilon}, \zeta)\, d\xi$, in which F_1 is a combined hardening-softening function.

A very important property of geological materials is the inelastic volume dilatancy, λ, which is caused chiefly by shear straining; therefore, following Bažant and Bhat (3), it is logical to set

$$d\lambda = L(\underset{\sim}{\epsilon}, \underset{\sim}{\sigma}, \lambda)\, d\xi \quad \ldots\ldots (8)$$

A few comments on the preceding postulation of a loading function, $\phi(\sigma_{ij})$, are in order. First of all, one compelling reason for the use of a loading function is that it greatly simplifies the derivation of the expressions for inelastic strains, as will be seen in the sequel. On the other hand, in viscoplasticity and its endochronic version in particular, the loading function does not appear in the basic theory, as in plasticity, because Drucker's postulate is generally not satisfied and the constitutive relation is not incrementally linear. Nevertheless, based on a recent study of the basic structure of endochronic inelasticity (2), it appears that, at each loading stage, a loading function ought to exist, at least for a set of all load directions which are sufficiently close to one chosen reference direction (in strain space). This is due to the fact that, for all such directions, an incremental linearization of the endochronic formulation is possible (2), thus making the endochronic formulation equivalent to a classical plasticity formulation for which Drucker's postulate can be satisfied. In this light and since the loading function characterizes the tensorial aspects of inelastic strain for the foregoing set of loading directions, it is not unreasonable to adopt it for the complete constitutive relation. Finally, it should be pointed out that a loading function exists and has been determined for previous practical endochronic formulations for geological materials (1,2,3,4,7).

At this point we have stated the basic structure of the constitutive relation (Eqs. 1*a*, 1*b*, 6*a*, 6*b*, 7, and 8). Although no particular attention has been paid to unloading, the nature of endochronic theory is such that unloading irreversibility, a salient feature of inelastic behavior, is automatically built in without imposing any inequality conditions. In view of previous experience (1,3,4,7), we may expect that the formulation would give roughly correct forms of unloading diagrams and large hysteresis loops, but no data are available to check this. However, based on recent work (2), it cannot be expected that the formulation as stated so far would also provide correct behavior for small unload-reload cycles and for small cyclic stresses superimposed on large static stresses. In particular, the positiveness of the hysteretic energy dissipation during such loading would not be assured. An extension of this endochronic theory, which consists of a special form of kinematic hardening and assures proper behavior under such loads has been proposed recently (2) and would have to be incorporated into the present formulation to achieve a general model for cyclic loading.

Transverse Isotropy and Tensorial Invariance

Recalling the well-known stress-strain relations for elastic transversely isotropic materials (12) and augmenting the elastic strain increments by inelastic strain increments, $d\epsilon''_{ij}$, the transversely isotropic form of Eq. 1 may be written as

$$d\epsilon_{11} = C_1\, d\bar{\sigma}_{11} + C_2\, d\bar{\sigma}_{22} + C_4\, d\bar{\sigma}_{33} + d\epsilon''_{11}; \quad d\epsilon_{22} = C_2\, d\bar{\sigma}_{11} + C_1\, d\bar{\sigma}_{22} + C_4\, d\bar{\sigma}_{33} + d\epsilon''_{22}; \quad d\epsilon_{33} = C_4\, d\bar{\sigma}_{11} + C_4\, d\bar{\sigma}_{22} + C_3\, d\bar{\sigma}_{33} + d\epsilon''_{33};$$
$$d\epsilon_{12} = (C_1 - C_2)\, d\sigma_{12} + d\epsilon''_{12}; \quad d\epsilon_{23} = C_5\, d\sigma_{23} + d\epsilon''_{23};$$
$$d\epsilon_{13} = C_5\, d\sigma_{13} + d\epsilon''_{13} \quad \ldots\ldots (9)$$

in which superimposed bars denote effective stress defined later (Eq. 18). Eqs. 9 are referred to cartesian coordinates x_1, x_2, and x_3 of which x_3 is normal to the plane of isotropy. Eqs. 9 are the most general possible relations which are invariant for the transversely isotropic group of transformations—any rotation about axis x_3 and reflections on planes (x_1x_2), (x_2x_3), and (x_1x_3) (13). The coefficient of $d\sigma_{12}$ must equal $C_1 - C_2$ in order to satisfy these invariance conditions. The symmetry conditions, $\partial\epsilon_{ij}/\partial\sigma_{kl} = \partial\sigma_{ij}/\partial\epsilon_{kl}$, which are required for the elastic parts of the strains to possess a potential, are also satisfied by Eq. 9.

The five independent elastic moduli C_1, ... C_5 may be expressed as

$$C_1 = \frac{1}{E}; \quad C_2 = -\frac{\nu}{E}; \quad C_3 = \frac{1}{E'}; \quad C_4 = -\frac{\nu'}{E'}; \quad C_5 = \frac{1}{2G'} \quad \ldots\ldots (10)$$

in which E and ν are Young's modulus and Poisson's ratio in the plane of isotropy; and E', ν', and G' are Young's modulus, Poisson's ratio, and the shear modulus in the planes normal to the plane of isotropy.

The inelastic strains may be expressed in terms of a loading function, $\Phi(\sigma_{ij})$. Similar to modeling the behavior of isotropic clays, it will be useful to divide the inelastic strains into two parts—those which involve a volume change and those which do not. Accordingly, we postulate that Φ is a sum of the function J_2^σ, which is independent of the hydrostatic effective stress, and the function Φ_1, which depends on the hydrostatic effective stress. This may be satisfied by choosing

$$\Phi(\sigma_{ij}) = \Phi_1 + J_2^\sigma; \quad \Phi_1 = D_3(\bar{\sigma}_{11} + \bar{\sigma}_{22}) + D_4\,\bar{\sigma}_{33} \qquad (11)$$

$$J_2^\sigma = \frac{1}{2}\,[D_1(\bar{\sigma}_{11} - \bar{\sigma}_{22})^2 + D_2(\bar{\sigma}_{11} - \bar{\sigma}_{33})^2 + D_2(\bar{\sigma}_{22} - \bar{\sigma}_{33})^2$$
$$+ 6D_1\,\sigma_{12}^2 + 2\,C_5(\sigma_{13}^2 + \sigma_{23}^2)] \qquad (12)$$

These are the most general possible linear and quadratic expressions which are invariant for the transversely isotropic group of transformations. Because dz involves an arbitrary constant, the function $\Phi(\sigma_{ij})$ may be adjusted by an arbitrary common multiplier. By virtue of this fact, one coefficient may by chosen, and in Eq. 12 the coefficient of σ_{13}^2 was chosen to be equal to $2C_5$. Note that J_2^σ is a special case of the yield function for orthotropic materials, as deduced by Hill (9) and Fischer (8). For isotropic materials J_2^σ reduces to the second invariant of the stress deviator.

The inelastic strain increments may now be expressed as $d\epsilon''_{ij} = (\partial\Phi/\partial\sigma_{ij})\,dz$. Thus, remembering that $6\sigma_{12}^2$ must be expressed in the symmetric form $3\sigma_{12}^2 + 3\sigma_{21}^2$ before taking the derivative $\partial/\partial\sigma_{12}$, we get

$$d\epsilon''_{11} = D_1(\bar{\sigma}_{11} - \bar{\sigma}_{22})\,dz + D_2(\bar{\sigma}_{11} - \bar{\sigma}_{33})\,dz + (1 - b_3)\,d\lambda;$$
$$d\epsilon''_{22} = D_1(\bar{\sigma}_{22} - \bar{\sigma}_{11})\,dz + D_2(\bar{\sigma}_{22} - \bar{\sigma}_{33})\,dz + (1 - b_3)\,d\lambda;$$
$$d\epsilon''_{33} = D_2(\bar{\sigma}_{33} - \bar{\sigma}_{11})\,dz + D_2(\bar{\sigma}_{33} - \bar{\sigma}_{22})\,dz + (1 + 2b_3)\,d\lambda;$$
$$d\epsilon''_{12} = 3\,D_1\,\sigma_{12}\,dz; \quad d\epsilon''_{23} = C_5\,\sigma_{23}\,dz; \quad d\epsilon''_{13} = C_5\,\sigma_{13}\,dz \qquad (13)$$

Here we have used the notations $1 - b_3 = D_3\,dz/d\lambda$ and $1 + 2b_3 = D_4\,dz/d\lambda$, in which b_3 appears in a form which guarantees that $d\epsilon''_{kk} = 3\,d\lambda$. The derivation of Eq. 13 attests to the usefulness of postulating an endochronic loading function, as proposed by Bazant (2).

To define $d\xi$, we might be tempted to adopt for $J_2(d\epsilon)$ an expression analogous to J_2^σ; however, this would be incorrect. The crucial condition, from which the proper form must be derived, is that the hydrostatic stress increments must result in a purely elastic deformation. Indeed, when an array of particles dispersed within a fluid is subjected to hydrostatic pressure, every microelement of the particles, as well as the fluid, will be subjected to the same hydrostatic pressure and no forces between particles will be produced. This cannot cause any inelastic strain. Thus, substituting $d\epsilon''_{ij} = 0$ and $d\bar{\sigma}_{11} = d\bar{\sigma}_{22} = d\bar{\sigma}_{33} = -dp$ into Eq. 9, we obtain $d\epsilon_{11} = d\epsilon_{22} = -(C_1 + C_2 + C_4)\,dp$ and $d\epsilon_{33} = -(C_3 + 2C_4)\,dp$. Thus, a zero inelastic strain is obtained for the ratios $d\epsilon_{11}/d\epsilon_{33} = d\epsilon_{22}/d\epsilon_{33} = b_1$ such that

$$b_1 = \frac{C_1 + C_2 + C_4}{C_3 + 2C_4} \qquad (14)$$

The condition that $d\xi$ vanish for this ratio of normal strain increments is satisfied by the quadratic expression

$$d\xi^2 = J_2(d\epsilon) = \frac{3}{2 + b_1}\left\{\frac{1}{6}\,[(d\epsilon_{11} - d\epsilon_{22})^2 + (d\epsilon_{11} - b_1\,d\epsilon_{33})^2\right.$$
$$\left. + (d\epsilon_{22} - b_1\,d\epsilon_{33})^2] + d\epsilon_{12}^2 + b_2(d\epsilon_{23}^2 + d\epsilon_{13}^2)\right\} \qquad (15)$$

It may be verified that this expression is the most general one that is invariant with regard to the transversely isotropic group of transformations. While the coefficient b_1 is fixed by Eq. 14, the coefficient b_2 is arbitrary. For isotropic materials, $b_1 = b_2 = 1$ and Eq. 15 reduces to the second invariant of the deviator of $d\epsilon$. Note that uniform dilation $d\epsilon_{11} = d\epsilon_{22} = d\epsilon_{33}$ does affect the value of $d\xi$ and produces inelastic strain. This is in agreement with the fact this dilation causes elastic shear stresses, according to Eq. 9.

The deformation increment which does not produce inelastic strain increments is such that $d\epsilon_{11}:d\epsilon_{22}:d\epsilon_{33} = 1:1:b_1$; thus, it is logical to define

$$I_1^\epsilon = \epsilon_{11} + \epsilon_{22} + b_1\,\epsilon_{33} \qquad (16)$$

By analogy with Eq. 15, we may also introduce

$$J_2^\epsilon = \frac{3}{2 + b_1}\left\{\frac{1}{6}\,[(\epsilon_{11} - \epsilon_{22})^2 + (\epsilon_{22} - b_1\,\epsilon_{33})^2\right.$$
$$\left. + (\epsilon_{11} - b_1\,\epsilon_{33})^2] + \epsilon_{12}^2 + b_2(\epsilon_{23}^2 + \epsilon_{13}^2)\right\} \qquad (17)$$

which reduces to the second invariant of the strain deviator if the material is isotropic ($b_1 = 1$). It may again be checked that Eqs. 15, 16, and 17 are invariant with respect to the transversely isotropic group of transformations.

Bore Pressure and Effective Stress

Because of the two-phase nature of clay, the stress, $\bar{\sigma}_{ij}$, in all preceding equations (Eqs. 1, 5, 7, 8, 9, 11, 12, and 13) must be considered as the effective stress, which characterizes the forces between the particles of the solid skeleton. In terms of the total stress, σ_{ij}, of the two-phase medium, we have

$$\bar{\sigma}_{ij} = \sigma_{ij} + \delta_{ij}u \qquad (18)$$

in which δ_{ij} = Kronecker delta; and u = pore pressure.

The pore pressure increments in the undrained condition can be calculated in terms of the volumetric strain (1):

$$du = \frac{C_w}{n}\,(d\epsilon_{11} + d\epsilon_{22} + d\epsilon_{33}) \qquad (19)$$

in which n = porosity of the clay; and C_w = compressibility of water. If Eqs.

9 and Eqs. 13 are substituted for $d\epsilon_{ij}$ and if the effective stresses $\bar{\sigma}_{ij}$ are expressed in terms of σ_{ij} from Eq. 18, it follows that

$$du = \frac{C_w}{H_1}(C_1 + C_2 + C_4)(d\sigma_{11} + d\sigma_{22}) + (C_3 + 2C_4)\, d\sigma_{33} + 3\, d\lambda \quad \ldots\ldots \quad (20)$$

in which $H_1 = n + C_w(2C_1 + 2C_2 + 4C_4 + C_3)$ . . . . (21)

Step-by-Step Integration of Stress-Strain Relations

To calculate the response to a prescribed loading history, central difference approximations of the stress-strain relations are rather effective. The approximations in the rth step, $\Delta t = t_{r+1} - t_r$, may be obtained from Eqs. 9 and 13 if the differential signs, d, are replaced by the increment signs, Δ, and the values of $\bar{\sigma}_{ij}$ are replaced by the midstep values $\bar{\sigma}_{ij_{r-1/2}} = \bar{\sigma}_{ij_r} + \Delta\bar{\sigma}_{ij_r}/2$. If the effective stresses, $\bar{\sigma}_{ij}$, are at the same time expressed in terms of the total stresses, σ_{ij}, the following equations for the strain increments ensue:

$$\Delta\epsilon_{11_r} = H_2\Delta\sigma_{11_r} + H_3\Delta\sigma_{22_r} + H_4\Delta\sigma_{33_r} + [D_1(\sigma_{11} - \sigma_{22})_r$$

$$+ D_2(\sigma_{11} - \sigma_{33})_r]\,\Delta z_r + \left(1 - b_3 - \frac{3C_wH_8}{H_1}\right)\Delta\lambda_r; \quad \Delta\epsilon_{22_r} = H_3\Delta\sigma_{11_r}$$

$$+ H_2\Delta\sigma_{22_r} + H_4\Delta\sigma_{33_r} + [D_1(\sigma_{22} - \sigma_{11})_r + D_2(\sigma_{22} - \sigma_{33})]\,\Delta z_r + \left(1 - b_3 - \frac{3C_wH_8}{H_1}\right)\Delta\lambda_r; \quad \Delta\epsilon_{33_r} = H_4\Delta\sigma_{11_r} + H_4\Delta\sigma_{22_r} + H_5\Delta\sigma_{33_r}$$

$$+ [D_2(\sigma_{33} - \sigma_{11}) + D_2(\sigma_{33} - \sigma_{22})_r]\,\Delta z_r + \left(1 + 2b_3 - \frac{3C_wH_9}{H_1}\right)\Delta\lambda_r;$$

$$\Delta\epsilon_{12_r} = H_6\Delta\sigma_{12_r} + (C_1 - C_2)\sigma_{12_r}\Delta z_r; \quad \Delta\epsilon_{13_r} = H_7\Delta\sigma_{13_r} + C_5\sigma_{13_r}\Delta z_r;$$

$$\Delta\epsilon_{23_r} = H_7\Delta\sigma_{23_r} + C_5\sigma_{23_r}\Delta z_r \quad \ldots\ldots \quad (22)$$

in which
$$H_2 = \left(C_1 + \frac{\Delta z_r}{6}\, C_1 - C_2 + C_3 - C_4\right) - \frac{C_w(C_1 + C_2 + C_4)^2}{n + C_w(2C_1 + 2C_2 + 4C_4 + C_3)};$$

$$H_3 = C_2 - \frac{\Delta z_r}{6}(C_1 - C_2) - \frac{C_w(C_1 + C_2 + C_4)^2}{n + C_w(2C_1 + 2C_2 + 4C_4 + C_3)};$$

$$H_4 = C_4 - \frac{\Delta z_r}{6}(C_3 - C_4) - \frac{C_w(C_1 + C_2 + C_4)(2C_4 + C_3)}{n + C_w(2C_1 + 2C_2 + 4C_4 + C_3)};$$

$$H_5 = C_3 + \frac{\Delta z_r}{3}(C_3 - C_4) - \frac{C_w(2C_4 + C_3)^2}{n + C_w(2C_1 + 2C_2 + 4C_4 + C_3)};$$

$$H_6 = (C_1 - C_2)\left(1 + \frac{\Delta z_r}{2}\right); \quad H_7 = C_5\left(1 + \frac{\Delta z_r}{2}\right);$$

$$H_8 = C_1 + C_2 + C_4; \quad H_9 = 2C_4 + C_3 \quad \ldots\ldots \quad (23)$$

The coefficients needed to calculate dz and $d\lambda$ in these equations also depend on $\bar{\sigma}_{ij_{r-1/2}}$, and so iterations in each loading step are required to achieve a second-order approximation accuracy. In the first iteration, all coefficients are evaluated on the basis of $\sigma_{ij_{r-1}}$.

The complete formulation of the incremental stress-strain-pore pressure relations is provided by Eqs. 6*a*, 7, 8, 15, 20, and 22 (with d replaced by Δ). These equations involve five elasticity coefficients (E, ν, E', ν', and G') and five coefficients for inelastic anisotropy (b_1, b_2, b_3, D_1 and D_2). This is eight parameters more than required for isotropic clays. A number of further material parameters are needed to define the hardening and softening functions and the dilatancy function. However, these parameters are the same as for isotropic clays. So, by virtue of the preceding rational arguments, the anisotropy of clays is completely described by only eight parameters. This is a remarkably small number of parameters to characterize both elastic and inelastic anisotropy.

Material Parameters and Their Variation

For the special case of isotropic clays a comparison with the constitutive equation from a previous study (1) indicated that the following relations hold:

$$D_1 = \frac{C_1 - C_2}{3}; \quad D_2 = \frac{C_3 - C_4}{3} \quad \ldots\ldots \quad (24)$$

To reduce the number of unknowns it has been assumed, somewhat arbitrarily, that these relations apply also for anisotropic clays. This assumption is justified by the fact that close fits of test data have been achieved (although from the theoretical point of view D_1 and D_2 do not have to be related to C_1, ... C_4).

Two different sets of tests, involving a total of seven undrained shear tests on two different clays, were selected for the purpose of verifying the proposed constitutive model. One of these data sets includes torsional hollow cylinder shear tests on one-dimensionally consolidated kaolinite samples (14); the inclination of the major principal stress with respect to the axis of symmetry is varied to investigate the directional behavior. The other data set includes triaxial and plane strain tests on undisturbed natural Haney clay (17); these samples were anisotropically consolidated under triaxial and plane strain conditions, and tests were performed along different paths. The engineering properties of these two clays are summarized in Table 1, in which w_L = liquid limit, w_P = plastic limit, G_s = specific gravity, μ_{xy} = Poisson's ratio, σ_{vc} = vertical consolidation pressure, K_o = stress ratio, and e_o = void ratio.

The form of the hardening and softening functions, the dilatancy function, and the variation of elastic moduli are assumed to be the same as for isotropic clays, as determined in a previous study (1). Anisotropy is accounted for by replacing the stress and strain invariants in these functions with the appropriate transversely isotropic invariants, as determined before. The numerical coefficients in these functions are, however, determined individually for each particular data set. The hardening and softening functions, appearing in Eq. 7, are given as

$$d\zeta = \frac{d\eta}{1 + \dfrac{\beta_1 \eta}{1 + \beta_2 \eta}}; \quad d\eta = \left[4 + \frac{|1 - 500\, I_1^\epsilon|(1 + a_d J_2^\epsilon)}{0.01 + \dfrac{0.75\, I_1^{\bar\sigma}}{p_a}} \right] d\xi \quad \ldots\ldots \quad (25)$$

and the densification-dilatancy function is

$$d\lambda = \frac{C_d |1 + 2{,}500\, I_1^\epsilon|}{(1 + 3{,}000\, J_2^\epsilon)\left(\dfrac{1 + 0.25\, I_1^{\bar\sigma}}{p_a}\right)(1 + C_\lambda \lambda)} d\xi \quad \ldots\ldots \quad (26)$$

in which a_d, β_1, β_2, C_d and C_λ are material parameters that must be determined for each clay; and $I_1^{\bar\sigma} = \bar\sigma_{11} + \bar\sigma_{22} + \bar\sigma_{33} = 3p$ = first invariant of effective stress.

The term $\beta_1 \eta$ yields hardening with regard to the length of the path traced in strain space, and the term $\beta_2 \eta$ stops this hardening at a large value of η.

TABLE 1.—Properties of the Selected Clays

Clay type (1)	w_L (2)	w_P (3)	G_s (4)	μ_{xy} (5)	$\sigma_{vc'}$ in kilonewtons per square meter (6)	K_o (7)	e_o (8)
Kaolinite	63	29	2.65	0.28	414	0.465	0.96
Haney Clay	44	18	2.80	0.24	588	0.550	0.95

The invariant J_2^ϵ is used in Eq. 25 to achieve a weakening of the response for large distortions, and the invariant I_1^ϵ is used to model the stiffening of the response after densification has occurred. The invariant $I_1^{\bar\sigma}$ in Eqs. 25 and 26 introduces the stiffening of the inelastic response as the effective hydrostatic stress increases. The term J_2^ϵ in Eq. 26 reflects the fact that dilatancy increments are less at larger distortions, and the term $C_\lambda \lambda$ in Eq. 26 causes the dilatancy rate to drop after much dilatancy has occurred. Apart from logically reflecting such properties, the functions in Eqs. 25 and 26 are, of course, empirical. Compared to other endochronic models (3,6), these functions are relatively simple. This is, however, due to the fact that the existing test data are comparatively limited. It must be expected that much more sophisticated functions will be required to fit more comprehensive test data when they become available.

Our formulation can also account for the variation of elastic moduli along the stress path. This is modeled by means of two factors: (1) Effective normal stress appearing as the first invariant of the effective stress; and (2) void ratio appearing indirectly in terms of the accumulated densification. The anisotropy is characterized by the ratio $r_a = E'/E$. This parameter is considered to be a function of the ratio of the previous consolidation stresses, i.e.

$$r_a = \frac{E'}{E} = \left(\frac{\sigma_{3c}}{\sigma_{1c}}\right) \quad \ldots\ldots \quad (27)$$

The values of Poisson's ratio in the planes parallel and perpendicular to the plane of isotropy are assumed to be equal and are determined from the relation given by Lade and Musante (11). Together with the values for Poisson's ratio and Young's modulus, this enables the number of the independent elastic

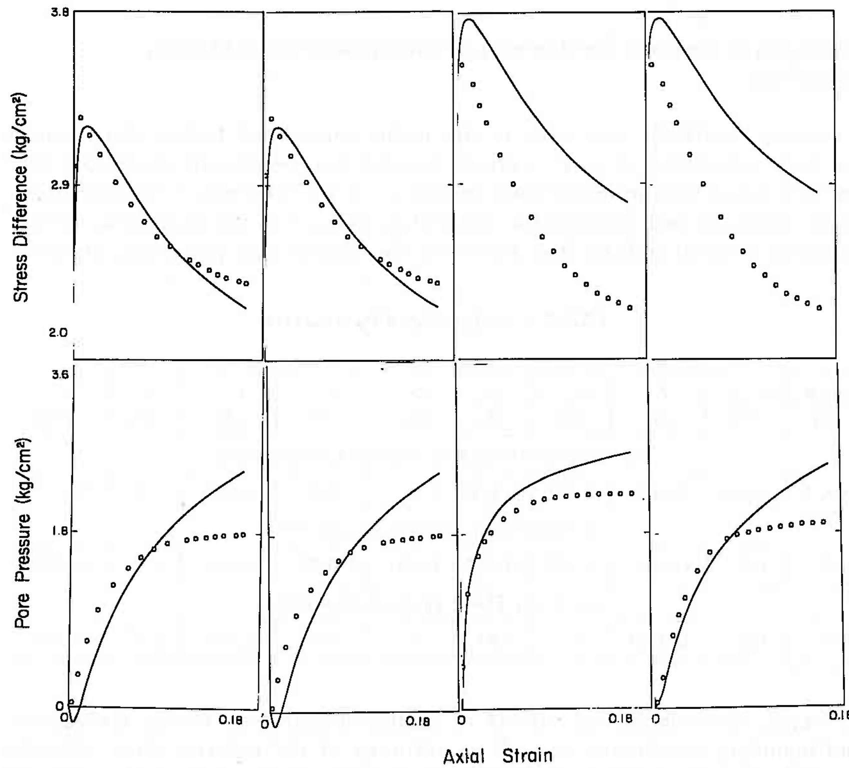

FIG. 1.—Stress-Strain Pore Pressure Data for Anisotropically Consolidated Haney Clay [after Vaid (16)] and Theoretical Response Obtained by Fitting First Test and Predicting Others Based on This One Fit

coefficients needed to be reduced from five to three, two of which are Young's modulus and the shear modulus in the planes perpendicular to the plane of isotropy. Both of these moduli are treated as variable functions of the inelastic volumetric strain and the effective confining stress, and they may be written as

$$E = E_o\left(1 + \frac{I_1^{\bar\sigma} - I_1^{\bar\sigma_o}}{10\, I_1^{\bar\sigma_o}} + \frac{3\lambda}{10 n}\right); \quad G' = G'_o\left(1 + \frac{I_1^{\bar\sigma} - I_1^{\bar\sigma_o}}{10\, I_1^{\bar\sigma_o}} + \frac{3\lambda}{10 n}\right) \quad \ldots \quad (28)$$

in which E_o and G'_o are the initial values of the moduli. Since the anisotropy ratio and Poisson's ratio are determined initially, the rest of the moduli can be calculated with respect to E and G' at each increment. Because the proposed relations are based on the data selected, they are limited to a certain extent. The major difficulty is the lack of reliable and representative indicies to quantify the properties of the different clay types and the different degrees of anisotropy. The relations given are time- and rate-independent; for conditions where these effects become significant, a similar logic can be followed in the formulation to include time-dependent behavior.

Application of Proposed Constitutive Law and Identification of Material Parameters

Due to significant variations in the many interrelated factors that control the basic properties of clays, certain material parameters are introduced into the formulation to achieve increased generality. At this stage empirical correlations might be established between the material parameters of the model and various practical material indices that represent the engineering properties of clays.

TABLE 2.—Optimized Parameters

E/p_a (1)	G'/p_a (2)	Z_1 (3)	a_d (4)	β_1 (5)	β_2 (6)	C_d (7)	C_λ (8)	b_2 (9)	b_3 (10)
(*a*) From Conventional Triaxial Compression									
800.0	195.0	0.0136	0.173	0.172	1.0	0.062	9,000	1.0	0.0573
(*b*) From Four Tests on Haney Clay									
795	195	0.012	0.173	0.172	0.904	0.0587	9,800	1.0	0.0005
(*c*) From Three Tests on Kaolinite									
712	120	0.0222	5.0	3.84	5.0	0.0083	2,000	0.57	1.288

However, the complicated effects of sample preparation, testing techniques, and boundary conditions, as well as accuracy of the reported data, virtually preclude the development of reliable correlations of this type, and it appears more suitable to determine the material parameters in the constitutive law by performing simple stress-strain tests.

For transversely isotropic clays with the proposed relations among the elasticity coefficients, two moduli (E and G') and six material coefficients (a_d, β_1, β_2, Z_1, C_d, and C_λ) in the intrinsic functions and two anisotropy coefficients (b_2 and b_3) from the stress-strain relations must be determined. Accordingly, several tests with various stress-strain paths should be performed to obtain the values of these 10 coefficients. Once determined (e.g., with the use of a mathematical optimization procedure), it should be theoretically possible to predict the stress-strain-pore pressure dependence for all realistic stress-strain paths.

Most testing in soil mechanics practice has been restricted to triaxial compression tests, oedometer tests, and direct shear tests. The triaxial and oedometer techniques impose conditions where the minor and intermediate principal stresses are equal; however, this condition rarely exists in field situations. On the contrary,

there are many field problems in which conditions approximate more closely those of plane strain. Thus it is desirable to develop a constitutive law that can predict plane strain response from data obtained from triaxial tests. Such an attempt is made in this study for Haney clay. Fig. 1 (first column of figures) shows the optimized fit for data obtained from undrained triaxial compression tests in which the specimen is sheared by increasing the axial stress under a constant confining stress. Using the value of the optimized parameters given in Table 2(*a*), the response is predicted for (1) Triaxial compression tests where

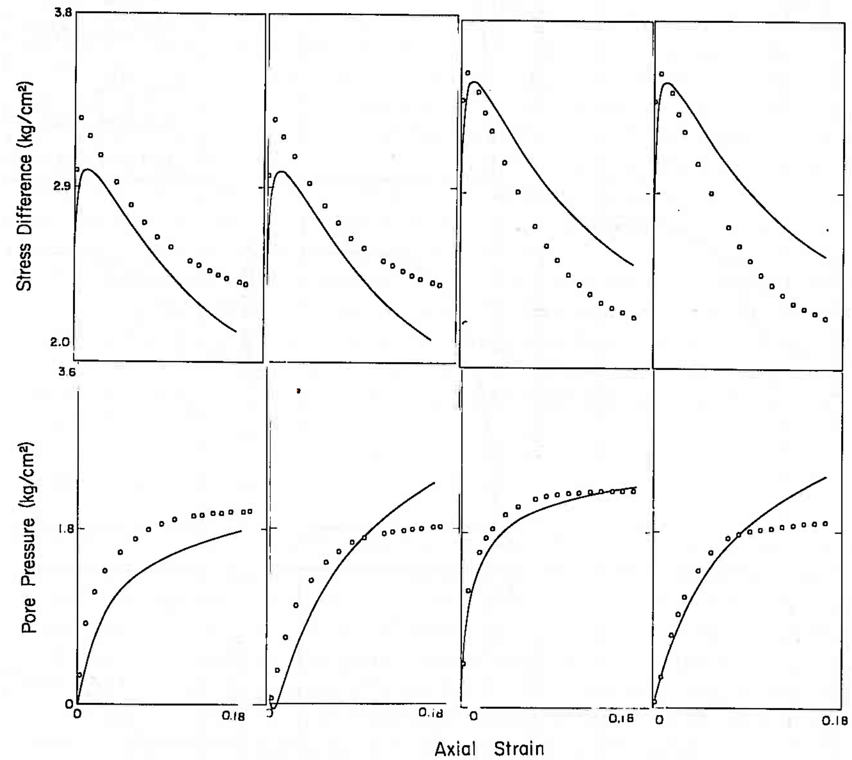

FIG. 2.—Stress-Strain Pore Pressure Data for Anisotropically Consolidated Haney Clay [after Vaid (16)] and Theoretical Response Obtained by Fitting All Four Tests at Same Time

the axial stress is maintained constant and the confining stress is decreased (Fig. 1, second column of figures); (2) plane strain tests with an increasing axial stress (Fig. 1, third column of figures); and (3) plane strain tests in which the horizontal stress is decreased (Fig. 1, fourth column of figures). These predictions are reasonably accurate in most cases, yielding a slight overprediction of the stress difference in the case of the plane strain tests.

Another alternative, which is a more suitable approach, involves optimizing the data from all four tests on Haney clay simultaneously to obtain more representative values for the material parameters. To prevent mathematical difficulties due to interdependence between the adopted parameters, six of the 10 parameters are optimized while the other four are chosen by a trial-and-error

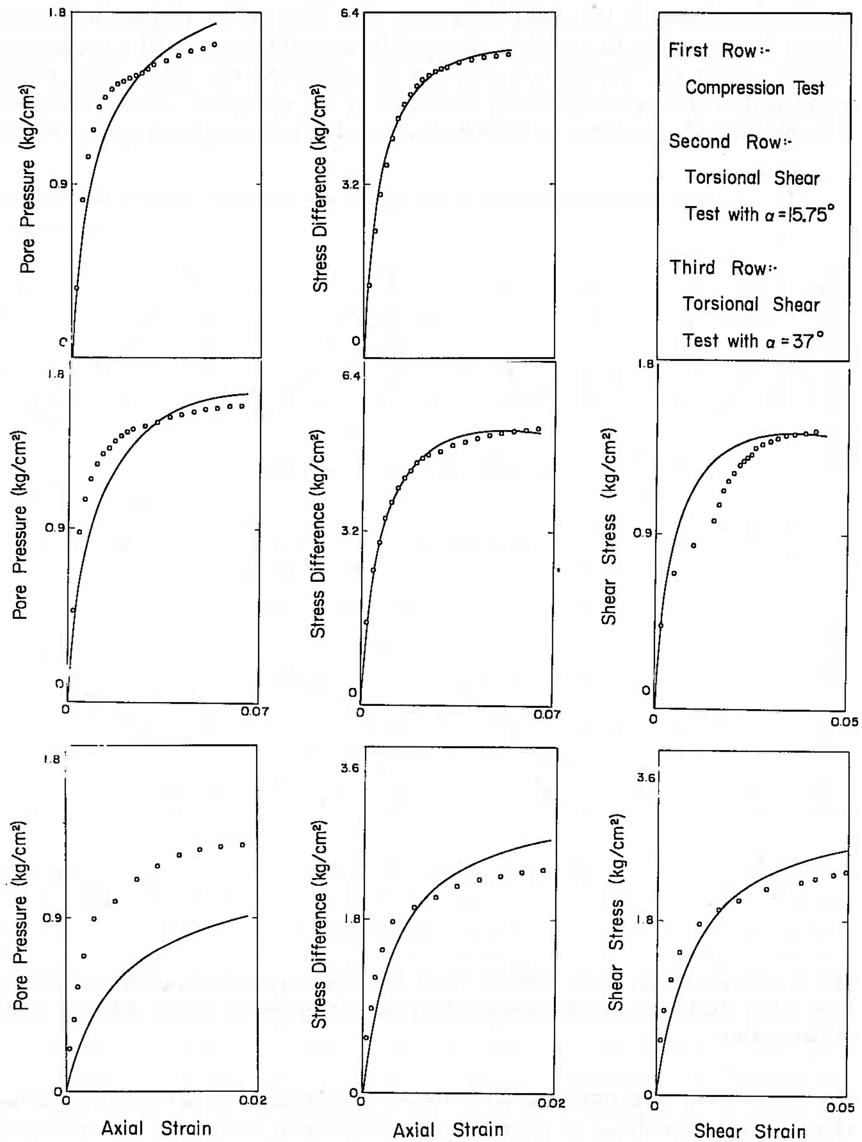

FIG. 3.—Stress-Strain Pore Pressure Data for Anisotropically Consolidated Kaolin from Torsional Hollow Cylinder Compression Tests [after Ou (14)] and Theoretical Response Obtained by Fitting All Three Tests at Same Time

process. The corresponding stress-strain-pore pressure relations for the optimized values, given in Table 2(*b*), are shown in Fig. 2. Considering the possible testing

errors, differences in the samples, and sample disturbance, the agreement between the test data and the optimized relations is satisfactory for all practical purposes.

In the case of hollow cylinder torsion-compression tests on one-dimensionally consolidated kaolinite (14), the latter approach was followed and all three selected tests were optimized simultaneously. The optimized stress-strain-pore pressure relations for the parameters given in Table 2(*c*) are shown in Fig. 3. The fits obtained are satisfactory; only the pore pressure response for the test with the principal stresses at an inclination of 37° to the vertical is significantly underestimated. If the stress differences are compared for all three tests and if proportionality between mean stresses is assumed, the calculated pore pressure response is more reasonable with respect to the measured pore pressures.

The fits were obtained with the help of a computer program that had been developed to integrate the constitutive relation in small steps. The program is similar to that described by Bažant and Bhat (3). Trial-and-error changes in material parameters were first needed to obtain qualitative agreement with the desired response; then, optimization techniques, based on a standard library subroutine (Levenberg-Marquardt algorithm for nonlinear least-square approximation) were used to obtain the best possible fits.

The present constitutive relation can be used with the finite element method to solve practical problems. Various suitable algorithms for endochronic theory have been described in the literature (3) and can be employed in conjunction with the present model. Large endochronic finite element programs have been developed and used successfully in a number of laboratories throughout the world.

Summary and Conclusions

A viscoplastic constitutive relation of the endochronic type (i.e., one in which the inelastic strain increments are characterized by intrinsic time) is formulated to describe the behavior of transversely isotropic clays produced by one-dimensional consolidation. The tensorial character of the inelastic strain increments is derived by postulating a suitable loading function which exhibits transversely isotropic invariance. The proper quadratic form defining the intrinsic time increments in terms of strain increments and the proper linear and quadratic strain invariants characterizing material states are derived from the hypothesis that hydrostatic stress must produce no inelastic strain, whereas volumetric strain must involve inelastic strain. Altogether, the formulation involves eight coefficients in addition to those needed in the previously published model for isotropic clays. The hardening and softening functions and the densification-dilatancy function are assumed to be given by the same expressions previously found for isotropic clays, but the invariants involved in these expressions are replaced by proper transversely isotropic invariants. Pore pressure is determined from the volume change and the compressibility of water, and the constitutive relation is written in terms of effective stresses. Elastic moduli are assumed to be functions of hydrostatic stress and inelastic dilatancy, and they are correlated with the consolidation stress. Experimental curves of stress difference, shear stress, and pore pressure versus axial strain for various anisotropically consolidated clays are fitted and good agreement is achieved. Based on this work, the following conclusions may be advanced: (1) The hypothesis that hydrostatic stress produces no inelastic strain greatly simplifies the formulation and gives reasonable results; (2) the proposed transversely isotropic invariants of strain and stress and the loading function give a logical and adequate description of the inelastic behavior of the soils studied; (3) pore pressures developed for undrained conditions can be calculated from volume change data and the compressibility of water and the stress-strain relation can be formulated in terms of effective stresses; and (4) the functions defining hardening, softening, inelastic dilatancy, and variation of elastic moduli for isotropic clays may be also used for anisotropic clays if the invariants appearing in these functions are replaced by proper transversely isotropic invariants.

Acknowledgment

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Appendix II.—Notation

The following symbols are used in this paper:

a_d = distortion coefficient;
b_1, b_2, b_3 = material parameters of anisotropy;
C_d = densification coefficient;
C_w = bulk modulus of pore water;
C_1, C_2, C_3, C_4, C_5 = elastic coefficients of anisotropic material matrix;
E', E = Young's moduli for transversely isotropic case;
e = void ratio;
G', G'_o = shear modulus and its initial value for an anisotropic material;
G_s = specific gravity;
I_1^σ = first stress invariant;
I_1^ϵ = first strain invariant;
J_2^ϵ = second deviatoric strain invariant;
K_o = ratio of horizontal to vertical stresses;
n = porosity;
p_a = atmospheric pressure;
r_a = anisotropy ratio;
t = time;
u = pore pressure;
w_p = plastic limit;
w_L = liquid limit;
Z_1 = material parameter;
z = intrinsic time measure;
β_1, β_2 = material parameters;
$\epsilon_{ij}, \underset{\sim}{\epsilon}$ = strain tensor;
ϵ = volumetric strain;
ζ = rearrangement measure;
η = continuous rearrangement;
λ = accumulated inelastic strain;
μ, μ' = Poisson's ratio for transversely isotropic case;
ξ = distortion measure; and
$\sigma_{ij}, \underset{\sim}{\sigma}$ = stress tensor.

14738 TRANSVERSELY ISOTROPIC CLAYS

KEY WORDS: **Anisotropy; Clays;** Constitutive equations; **Inelastic action; Materials;** Nonlinear systems; Plasticity; Pore pressure; **Viscoplasticity; Volume change**

ABSTRACT: A viscoplastic constitutive relation of the endochronic type (i.e., the inelastic strain increments are characterized by an intrinsic time) is formulated to describe the behavior of transversely isotropic clays produced by one-dimensional consolidation. The formulation contains eight material parameters in addition to those needed for isotropic clays. The hardening and softening functions and the densification-dilatancy function are assumed to be given by the same expressions previously found for isotropic clays, but the invariants involved in these expressions are replaced by the proper transversely isotropic invariants. The pore pressure is determined from the volume change and the compressibility of the water, and the constitutive relation is written in terms of the effective stresses. The elastic moduli are assumed to be functions of hydrostatic stress and inelastic dilatancy, and they are correlated with the consolidation stress. Experimental curves of axial strain for various anisotropically consolidated clays have been fit by a time-independent version of the theory, and a satisfatory agreement has been achieved.

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