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Stochastic Process for Extrapolating Concrete Creep

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Nature of Problem and Objective

In the design of concrete structures for nuclear reactors, it has become standard practice to carry out creep tests of the particular concrete to be used. These tests are inevitably of limited duration, such as 6 months–12 months, and an extrapolation to the end of lifespan, usually 40 yr, is necessary (for an example of such a problem, see Fig. 3 of Ref. 7). Because of safety considerations, the designer is interested not merely in the expected value of 40-yr creep, but mainly in the extreme values that have a certain specified small probability (such as 5%) of being exceeded. Up until now most of the research has been concerned with trying to predict creep using deterministic models that best fit the data available in the literature. This has its merits as far as predicting the average behavior over long periods of time, but an estimate of the expected statistical variation is lacking. Its qualitative estimate can be obtained only by statistical means and this is going to be the prime concern of this paper.

Literature on statistical treatment of creep of concrete is rather limited. Most of the work has dealt with creep in connection with long-term deflections of reinforced concrete beams (8,9,26,28). Certain statistical models have been suggested for deflection (26,28); however, the deflection problem is not equivalent to the problem of constitutive behavior of the material. Too many factors enter in the deflection problem and they are hard to isolate. This is in addition to the fact that the data analyzed pertain to beams tested in flexure without control

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of humidity conditions. As far as the prediction of long-term creep from short-time tests is concerned, Brooke and Neville (9) introduced statistical distribution of certain material parameters into an assumed deterministic law and looked for correlation between long-time and short-time creep, using regression analysis. They suggested a linear form for the relationship between long-time and short-time creep and they imposed both the mean and variance on the model. Properly, extrapolation of creep data should be treated by means of a stochastic process. The only work that dealt with creep as a stochastic process seems to be the pioneering paper by Benjamin, Cornell, and Gabrielson (8), whose innovative ideas the present study attempts to continue. In their model, however, deflection variation has been modeled as a Poisson process, which has certain limitations that will be indicated in the text. Consideration of some more general models, including nonhomogeneous process statistical analysis and general discrete-state Poisson processes, rather than just the Poisson process, was suggested in subsequent dissertations and reports (15,17,25).

Statistically tractable creep data are scarce in the literature and even those available do not follow a consistent statistical procedure, especially as far as reading times are concerned. This makes it hard to verify the choice of any stochastic model in the strict statistical sense. On the other hand, in recent years more light has been thrown on the physical mechanism of creep itself (1,4,5,6), and this will be used as basis for the choice of the stochastic model. At the same time, attention will be paid to the existing deterministic laws for concrete creep. For this purpose, a simple analytical formula is needed; deterministic creep functions that are given by a set of graphs are of little use for the statistical treatment. This is true, e.g., of the creep function which was recently proposed by Rüsch et al. (24) for the Comite European de Beton (C.E.B.) International Recommendations, even if its disagreement with most test data (2) were deemed not to be serious. The deterministic formulation that appears to give the best overall fits of creep curves at constant stress and at various ages of loading is the double power law (1,2,3):

$$J_s = \frac{1}{E_0} + \beta\xi; \quad \xi = s^n; \quad \beta = \frac{\phi_1}{E_0}(t'^{-m} + \alpha); \quad s = t - t' \quad \text{. . . . . (1)}$$

in which J_s = creep function = strain at time s due to constant unit normal stress applied at time $s = 0$; t = current age of concrete; t' = age of concrete when stress was applied; E_0, ϕ_1, m, n, α = parameters of double power law. According to Eq. 1, the unit creep rate $\partial J_s/\partial s$ is not stationary, and so a stochastic process that would be used to model creep also could not be stationary. However, instead of actual time s, it is possible to use ξ as the independent variable; then the rate $\partial J_s/\partial \xi$ is constant. Thus, the stochastic process, $J_{s_i} = Y_1 + Y_2 + \ldots Y_i$, in which Y_i are the increments over time intervals Δs_i, can be transformed to a stationary process. This fact, along with the hypothesis that increments Y_i are independent and gamma distributed, will be the crucial points of the stochastic model that follows. The distribution parameters will be estimated and, using simulation techniques, the model will be compared to the best data sets available in the literature.

Basic Assumptions.—A prismatic concrete specimen with the following idealized properties will be considered:

1. The length of the specimen is large compared to the maximum size of aggregate.
2. Concrete is macroscopically homogeneous, i.e., material properties do not change from one position to another.
3. The specimen is in a state of homogeneous uniaxial stress, σ. Actually, the creep strain exhibits statistical variation that must cause statistical differences in lateral strains throughout the specimen and must, therefore, induce local three-dimensional stress states; these effects are neglected.
4. The stress, σ, is sufficiently small, so that the expected deformation may be considered to be linearly dependent upon stress. This means that the compressive stress must be less than about 0.4 of the compression strength of the specimen.
5. Consideration is restricted to basic creep, i.e., creep that is not accompanied by moisture exchange, and to creep under time-constant stress and constant temperature.

In view of assumptions 1–4, creep may be characterized by the creep function, J_s, which is here defined as

$$J_s = \frac{l_s - l}{l\sigma} = \frac{w_s}{l\sigma}; \quad t \geq t' \quad \text{. . . . . (2)}$$

in which s = current time; σ = constant normal stress acting since time 0; l = initial length of specimen just before loading; $l_s = l + w_s$ = length observed at time s; and w_s = displacement of end cross section of initial coordinate $x = l$, the other end cross section $x = 0$ being fixed. Stress σ is negative for compression. The present notation differs from the usual notation $J_s = J(t,t')$, in which t' = age of concrete at the instant when stress is applied; and $t = t' + s$. Creep, per se, is usually understood as the difference $C(t,t') = J(t,t') - J(t',t')$, but this is better avoided because the definition of the instantaneous strain, $J(t',t')$, is rather ambiguous. Normally, the length change of a companion specimen without load would have to be also subtracted from w_s in Eq. 2, but this is not necessary here in view of assumption 5 and because the length change of sealed load-free specimens (called autogeneous shrinkage) is small and can be neglected.

Basic Hypothesis of Stochastic Model.—Creep of concrete represents a stochastic process, which is definitely not time-homogeneous. The present model rests mainly on two hypotheses:

Hypothesis A.—The stochastic process, J_s, is a pure jump increasing process with independent increments.

Hypothesis B.—Creep is a local gamma process.

Hypothesis A is crucial, and three different justifications will be given for it. Hypothesis B is less important, and it will be based partly on a physical model, partly on practical need for simplicity and statistical tractability. The definition and rigorous mathematical theory for local gamma processes may be found in Ref. 13. The available data and experience support the hypotheses quite well and no evidence contrary to our model has been found. However, it must be pointed out that scarcity of good quality data prevents statistical tests on the hypotheses. At any rate, this is a first attempt at a consistent stochastic model; and since we give the considerations that lead us to this model in some detail, the other researchers will be able to form their own opinion and, should they differ, they will hopefully quickly realize what needs to be changed.

Hypothesis A means that J_s is a sum of independent random variables. This condition will now be justified by showing that $J_s - J_u$ must be a sum of identically distributed random variables, so that the distribution law of any increment $J_s - J_u$ must be infinitely divisible.

Infinite Divisibility of Distribution of Creep

Justification 1: Additivity of Deformations.—Let σ be 1 and let $w_s(x)$ denote the displacement of a cross section with coordinate x; then $J_s = w_s(1)$. Consider a specimen of length $l = 1$ subdivided at midlength in two halves. The deformations of the halves are $w_s(1/2)$ and $w_s(1) - w_s(1/2)$. According to assumption 3 both halves are subjected to the same stress, $\sigma = 1$; according to assumption 2 they have the same material properties; and, according to assumption 1, each half remains macroscopically homogeneous. This implies that $w_s(1/2)$ and $w_s(1) - w_s(1/2)$ have the same distribution function, and because $J_s = w_s(1/2) + [w_s(1) - w_s(1/2)]$, it follows that J_s is a sum of two stochastically independent and identically distributed random variables. The same arguments still apply when the specimen is divided into n equal parts instead of just two, provided that $n << n_0$, in which n_0 is the ratio of specimen length to average aggregate size. Thus, J_s is a sum of n independent identically distributed random variables. Therefore, if n_0 were infinite, J_s would be infinitely divisible by definition (16). Since n_0 is finite, an infinitely divisible distribution is only an approximation to the distribution of J_s. For large n_0, this approximation must be very good. Aside from that, it is practically impossible to find a distribution that is "n_0 divisible" but not infinitely divisible. Consequently, the infinite divisibility of J_s is a reasonable hypothesis. All of the foregoing arguments apply when J_s is replaced by the increment $J_s - J_u$, and so the time increments of J_s are also infinitely divisible.

Justification 2: Additivity of Stresses.—Let $w_s(\sigma)$ be the deformation of a specimen of unit length under stress σ; then $J_s = w_s(1)$. In deterministic terms, $w_s(\sigma_1 + \sigma_2) = w_s(\sigma_1) + w_s(\sigma_2)$, and this is possible only if creep shows no randomness, which is not true. In random process terms, the correct interpretation is that the random variable $w_s(\sigma_1 + \sigma_2)$ has the same distribution as the sum of two independent random variables whose distributions are those of $w_s(\sigma_1)$ and $w_s(\sigma_2)$. By induction, since $J_s = w_s(\sigma_1 + \ldots + \sigma_n)$ with $\sigma_1 = \ldots = \sigma_n = 1/n$, this implies that J_s has the same distribution as the sum of n independent and identically distributed random variables. Thus, by definition, J_s is infinitely divisible, and the same holds for any increment $J_s - J_u$.

The preceding arguments, together with the known results on Laplace transforms of infinitely divisible random variables, imply the following. If $w_s(l,\sigma)$ is the deformation for length l and stress σ, then (16):

$$E\{\exp[-\lambda w_s(l,\sigma)]\} = \exp\left\{-l\sigma\left[\lambda c_s + \int_0^\infty \nu_s(dy)(1 - e^{-\lambda y})\right]\right\} \quad . . . . . (3)$$

for any $\lambda \geq 0$; here c_s is a constant and ν_s is a measure on $(0,\infty)$ satisfying

$\int_0^\infty \nu_s(dy)\, y \leq 1$. The condition on ν_s is stronger than the usual ones because of the boundedness of J_s by 1. To completely specify the probability law of $\nu_s(l,\sigma)$, we need to specify the constants c_s and the measures of ν_s for all $s \geq 0$. Thus, without loss of generality, we may restrict ourselves to the stochastic process $J_s = \nu_s(1,1)$.

Micromechanism of Creep.—Further conclusions on the stochastic properties of concrete creep can be derived from consideration of its microscopic mechanism. Based on the present state of knowledge, the creep mechanism can be described as follows.

Concrete is made up of aggregate and sand embedded in a matrix of hardened cement paste. The main solid component of this matrix is the cement gel, which consists largely of sheets of colloidal dimensions with average thickness of 30Å and average gaps of 15Å between the sheets. These sheets are formed mostly of calcium silicate hydrates and are strongly hydrophylic. The hardened cement paste matrix contains interconnected pores of different sizes. The largest pores are called macropores, are of round shape, contain capillary water, and are interconnected by a system of thinner pores. The thinnest ones, called micropores or gel pores, represent the gaps between the sheets. They are essentially laminar and some of them possibly tubular in shape, and they are only one to several molecules in thickness. The micropores contain water strongly held by solid surfaces, which could be regarded as hindered absorbed water or interlayer water. The micropores also contain relatively weakly held and partially mobile particles of solids bridging the gap between the opposite surfaces of the pores. The water in laminar micropores can exert on the pore walls a significant transverse pressure, called disjoining pressure.

When the load is applied on concrete, most of the resulting compression across the laminar micropores is carried by the solid particles bridging the pores. Water in the micropores receives only a small portion of the applied load because it has undoubtedly much smaller stiffness than the solid particles. The stress across the micropores causes certain particles of solids, probably Ca-ions, to slowly migrate out of the compressed pores (1,5,6), in a direction normal to the compressive stress (i.e., along the pores). The solid particles that can possibly migrate under load are held by bonds of various strengths and are subjected to various stresses. Those that are held weakly or receive higher stress are most likely to lose their bond (i.e., jump over their activation energy barrier) and migrate to a stress-free location or one of lower stress. As the number of weakly held and highly stressed particles becomes exhausted, the rate of migrations diminishes, causing a decline of the creep rate. As the particles leave the micropores, the transverse pressure (disjoining pressure) is relaxed and the applied load is transferred partly onto the elastic aggregate, partly onto other micropores. This also causes a decline of the creep rate. In addition, simultaneous hydration, which fills available pores by additional cement paste opposing the deformation, causes further deceleration of creep.

Presence of water in the micropores is essential for the migration to be possible. Thus, without water in the pores, as in predried concrete, there is almost no creep. At constant water content of concrete, movements of water along the pores are rather limited and play no significant role in creep. This is the case of basic creep. When water content varies, e.g., because of external drying, a large amount of water diffuses along the micropores. The movement of water endows the solid particles with greater mobility (1) and causes an acceleration of the migration of solid particles along the micropores, thereby accelerating creep. This explains the phenomenon of drying creep. In this study, however, only the basic creep will be considered, and so no attention need be paid to the movements of water.

When the applied stress is purely volumetric, it causes the solid particles to migrate along the micropores into the largest, round-shaped pores (macropores) whose walls do not receive any pressure as a result of applied load. When the applied stress is purely deviatoric (shear), it causes the particles to migrate from the laminar micropores normal to the direction of principal compressive stress, σ_1, into the laminar micropores normal to the direction of the principal tensile stress, $\sigma_2 = -\sigma_1$, and passage of the particles does not have to intersect any of the macropores. For a general state of applied stress, including the case of uniaxial stress, both types of migration happen simultaneously (1).

Justification 3: Consequences of Micromechanism of Creep.—It is clear from the foregoing exposition that creep involves local relaxations of pressure and transfer of the load on other macropores. Thus, while the transverse pressure averaged over all micropores decreases in time, due to transfer of load upon the aggregate, the pressure across any given micropore fluctuates randomly in time. A rigorous study should consider the pressure field throughout the cement paste as time varies, and relate the migration process to it. This, however, seems to be beyond the current capabilities. Therefore, only approximations that depend on the average pressure field will be made.

Considering the fluctuations of pressure at a fixed location, the time between two successive peaks is very large compared with the durations of the peak pressure. Therefore, all the migrations taking place during a high pressure period may be considered to be an instantaneous event. Accordingly, migrations at location x can be represented as a sequence of pairs (S_i^x, N_i^x), in which S_i^x is the time of the ith peak for pressure and N_i^x is the number of particles migrating during the ith peak period. The migrations taking place at two different points of the same micropore are certainly not independent. However, supposing that a particle migrating out of one micropore is quite unlikely to pass into a parallel micropore, it becomes reasonable to assume that migrations taking place at two different locations, x and y, are conditionally independent upon the transverse pressure at those locations ($x \neq y$). But the pressures at x and y should be almost independent if the distance $\|x - y\|$ is large enough. The number of micropores intersecting even a very small line segment is extremely large; approximately, the spacing of micropores equals the average thickness of sheet, 30Å plus a gap of 15Å in between (1Å = 10^{-7} mm). Assuming the macropores to occupy about 50% of volume, the number of micropores per 1 mm length is about $0.5\ [(30 + 15)10^{-7}]^{-1} \approx 100{,}000$. Similar conclusions can be deduced from the internal surface area of cement gel, which is about 2×10^6 cm²/cm³ to 6×10^6 cm²/cm³ of paste. Consequently, the dependence of pressures at locations x and y should be negligible when $\|x - y\|$ exceeds 1 mm.

Accordingly, let the hardened cement paste within the specimen be subdivided into a great number of regions $R_1, R_2, \ldots$, which are small compared with specimen size but large enough to insure independence between migration processes within R_j and R_k whenever $j \neq k$. For any rectangle $A \times B$ in

the plane $[0,\infty) \times [0,\infty)$, let $N_k(A \times B)$ be the number of all pairs (S_i^x, N_i^x) belonging to $A \times B$ as x varies over the region R_k. By the way R_k are chosen, the stochastic processes $N_k = \{N_k(A \times B): A \subset [0,\infty), B \subset [0,\infty)\}$, $k = 1, 2, \ldots$, are independent.

For fixed k, N_k is a random counting measure on $[0,\infty) \times [0,\infty)$, and the sum

$$N = \sum_{k=1}^{\infty} N_k \qquad (4)$$

is again a random counting measure. Compared with the points of N, those contributed by N_k for fixed k are uniformly sparse (13). It now follows from theorems on the superposition of uniformly sparse processes (13) that the process N is approximately a Poisson random measure. [A random measure M is Poisson with mean measure μ if $M(A_1), \ldots, M(A_n)$ are independent whenever the sets $A_1, \ldots, A_n$ are disjoint and if the random variable $M(A)$ has the Poisson distribution with parameter $\mu(A)$ for every set A.]

Let (S_i, X_i) be chosen such that

$$N(A \times B) = \sum_i 1_A(S_i)\, 1_B(X_i), \qquad (5)$$

in which $1_A(x) = 1$ or 0 according to whether $x \in A$ or $x \notin A$. Then, S_i are the times of migrations and X_i are the corresponding numbers of particles involved. Depending on the location at which migration takes place, the pressure and the local stiffness of microstructure, the contribution of X_i particles to creep strain will be some random variable Y_i. Supposing that Y_i is conditionally independent of all Y_j for $j \neq i$, the hypothesis that N is a Poisson random measure implies that the random measure

$$M(A \times B) = \Sigma\, 1_A(S_i)\, 1_B(Y_i) \qquad (6)$$

is again a Poisson random measure (on $R_+ \times R_+$). In terms of M, the total creep from time 0 to time s is

$$J_s - J_0 = \sum_i Y_i\, 1_{(0,s]}(S_i) = \int_{u=0^+}^{s} \int_{y=0^+}^{\infty} y M(du, dy) \qquad (7)$$

If the number of jump times, S_i, during a finite interval $[0,t]$ were finite, then Eq. 7 would represent a compound Poisson process, i.e., a process whose jump times form a Poisson process and the sizes of the jumps (instead of being all equal to 1 as in a Poisson process) are independent random variables (p. 91 of Ref. 12). However, and this will be the case herein, every interval $(t, t + \varepsilon)$ will in general contain infinitely many jump times S_i no matter how small $\varepsilon > 0$ may be. Under the hypothesis that M is a Poisson random measure, Eq. 7 shows that the creep process J_s (on $s \in R_+$) has independent increments. This provides the desired justification. Conversely, given the representation in Eq. 7, J_s has independent increments only if the random measure M is Poisson.

Next it is necessary to consider the shape of the mean measure m of Poisson random measure M, i.e., $m(C) = E[M(C)]$ (with C is a Borel subset belonging to $R_+ \times R_+$), in which E is expected value. Once m is known, Eq. 7 implies that

$$E\{\exp[-\lambda(J_s - J_r)]\} = \exp\left[-\int_{u=r}^{s}\int_{y=0^+}^{\infty} m(du,dy)(1-e^{-\lambda y})\right] \qquad (8)$$

for any $\lambda \geq 0$ and $0 \leq r < s$. (Note that the left-hand side of Eq. 8 represents Laplace transformation.) Eq. 8 specifies the probability law of J_t completely in view of the hypothesis that J_t has independent increments. Comparing Eq. 8 with Eq. 3, it is seen that $c_s = 0$ and $\nu_s(dy) = \int_0^s m(du,dy)\,du$.

Creep as Local Gamma Process

According to Hypothesis B, creep is a local gamma process. This is a process that satisfies Eq. 8 with mean measure of the form (see Ref. 14 for rigorous definition)

$$m(du,dy) = a'_u\,du\,e^{-b_u y}\frac{dy}{y} \qquad (9)$$

in which $b_u = b(u)$ represents the scale function ($b > 0$); $a'_u = da_u/du$; and $a_u = a(u)$ represents the shape function of the local gamma process ($a'_u > 0$). Incidentally, both a and b must depend on the fixed time t' of loading. Hypothesis B (i.e., Eq. 9) can be justified as follows.

Further Consequences of Micromechanism of Creep.—First, note that measure m must be continuous. Therefore

$$m(du,dy) = g(u,y)\,du\,dy \qquad (10)$$

for some positive measurable function g (on $R_0 \times R_0$), in which $R_0 = (0,\infty)$. Now $g(u,y)$ may be interpreted as the *expected rate of migrations whose contribution to creep rate* (strain per unit of time) *is* y. Next, consider this migration rate $g(u,y)$ for fixed time u and fixed contribution y to creep rate. The important relevant factors are the average transverse pressure, p, the length, q, of the passages contributing to creep rate by the amount y, the average micropore thickness, h, and the expected number, $n(u,y)$, of micropores contributing to creep by the amount y at time u.

It is reasonable to assume that the effect of transverse pressure is linear on the migration rate, i.e., $g(u,y)$ is proportional to p. The effect of thickness h is to cause a sharp increase in the migration rate, perhaps of the order of h^3 as in viscous flow. So, $g(u,y)$ is proportional to ph^3.

The effect of the length q of micropore passages is a little more involved. First, the longer the passage, the greater the number of particles having the potential to migrate, and so this particular effect is proportional to q. On the other hand, the longer the passage, the smaller is the number of particles that accomplish moving out of it during a fixed time. Supposing that the motions of particles out of a micropore can be approximated by a one-dimensional random walk, it takes an average particle $c_1 q^2$ steps to move out of a passage of length q, c_1 being some proportionality constant. Putting all the factors together, one has

$$g(u,y) = k_0 ph^3\left(\frac{q}{q^2}\right)n(u,y) \qquad (11)$$

in which the proportionality constant, k_0, may depend on u.

Furthermore, p must depend on h, and to estimate this dependence, one may assume p to be proportional to E_p/h, E_p being the stiffness (elastic modulus) of the solid particle bridges across the micropore per unit segment of micropore thickness and per unit area of wall. From measurements as well as a statistical argument based on a discrete model of microstructure and joint probability (19), the elastic modulus of a porous material is known to be approximately proportional to $(1 - n_p)^3$, in which n_p = porosity; and $1 - n_p$ = fraction of solids in the porous material. Here, $1 - n_p$ may be associated with the fraction of solid particles in the micropore, and since the number of particles sticking out into the micropore should be constant for a given area of micropore wall, $1 - n_p$ should be proportional to $1/h$. Thus, $E_p \sim 1/h^3$; $E_p/h \sim 1/h^4$, and so $g(u,y) \sim ph^3 \sim 1/h$. This result agrees with the fact that $g(u,y)$ must tend to 0 as $h \to \infty$, since no migrations can be induced by load in very thick pores.

Finally, y should be proportional to the number of particles that potentially migrate, and this number should be proportional to the volume of micropore along which migration takes place, i.e., $y = hq/c_0$, in which c_0 is some constant and hq represents passage volume assuming a unit width of passage. Inserting these into Eq. 11 in the preceding, it follows that

$$g(u,y) = k(u)\frac{n(u,y)}{y} \qquad (12)$$

in which $k(u)$ is a proportionality constant equal to k_0/c_0; and $n(u,y)$ is the expected number of micropore passages of volume $c_0 y$ at time u. Coefficient $k(u)$ must decrease with time u because the number of migrating particles decreases with time as the local stress peaks within the micropore are getting exhausted, and also because p diminishes with time as the stress is being transferred on the aggregate.

The number $n(u,y)$ decreases as volume $c_0 y$ increases. The total micropore volume, V, is a sum of micropore volumes $c_0 y$ times their number, which is proportional to y. Thus, the expected value of V is

$$\bar{v} = \int_0^{\infty} k_1\,n(u,y)\,dy \qquad (13)$$

in which k_1 is some constant. Volume V is a bounded random variable, with a bound V_0 that is less than the volume of specimen. The simplest way to ensure that the integral in Eq. 13 be bounded is to choose

$$n(u,y) = a_0 e^{-by} \qquad (14)$$

in which b and a_0 depend on u but not on y. Because y is an indicator of the contribution of a micropore to creep rate, and because the creep rate decreases with time, $n(u,y)$ must also decrease. So, $b = b_u = b(u)$ should be an increasing function and a_0 should be a decreasing function of time u. Furthermore, noting that the number of migrations in a given micropore decreases with time as stress peaks in the microstructure become exhausted and the load becomes more uniformly distributed, coefficient a_0 must be a decreasing function of time. Consequently

$$g(u,y) = \frac{a'_u}{y} e^{-b_u y} \qquad (15)$$

in which a'_u is a decreasing function of time u; and b_u is an increasing function of time. This justifies the hypothesis made in Eq. 9.

Essential Characteristics of Model.—The foregoing justification uses a number of assumptions concerning the physical processes involved, some of which might be far fetched. For example, the assumption of a random walk of particles along micropore is certainly a simplification, because the migration rate of particles must also depend on the gradient (along the micropore) of the pressure acting across the micropore (1,4,5), as in a diffusion process. Nevertheless, if the migration of particles were treated as diffusion, the time for a particle to diffuse out of a given micropore of length q under the same initial pressure p would be also proportional to q^2, same as in the random walk model.

The creep mechanism has been described in considerable detail, so as to have some reasonable, specific picture in mind. Yet, much of this description consists of logical conjectures that might be revised at a later time. Therefore, it is appropriate to list those characteristics of the present model of the creep mechanism that have been essential for obtaining the mathematical result (Eq. 15). These consist merely in the following:

1. The creep is the sum of a very large number of small contributions originating sparsely over time and space.
2. The mean rate of contributions of size y is proportional to a/y as $y \to 0$ and to ae^{-by} as $y \to \infty$ (see, for comparison, Eq. 15). Coefficients a and b are functions of time.

There may be different mechanisms leading to these essential characteristics 1 and 2. In the present work, characteristics 1 and 2 are shown to be justified merely on the basis that creep is due to migrations (diffusion) of some particles forming the solid microstructure along some sort of micropore passages from loaded to unloaded locations.

Practical Considerations.—In addition to the preceding physical justifications, it is worth adding some pragmatic ones in favor of Eq. 9. Namely, by Hypothesis A, the stochastic process is increasing, has independent increments, and is bounded. The only statistically tractable processes with independent increments are the Gaussian, stable, Poisson, and gamma processes.

Gaussian processes are excluded because creep is increasing. Stable processes are excluded because the only increasing ones have index less than 1 and stable processes with index less than 1 have infinite expectations.

Poisson processes must be excluded because creep should have a continuous distribution; if a Poisson distribution is to be the approximation, then its parameter must be very large. But a Poisson distribution with a large parameter λ is approximated closely by the normal distribution with mean λ and variance λ, so that its practical range is $\lambda \pm 3\sqrt{\lambda}$. But for large λ, the value of $\sqrt{\lambda}$ is insignificant compared with λ (e.g., for $\lambda = 10^6$, $\sqrt{\lambda} = 0.001\ \lambda$), which means that the creep rate woule be essentially constant and equal to λ. This is of course not true.

Incidentally, this argument also shows that the model from Ref. 8, which assumes creep, viewed to be caused by viscous flow of water, to be a constant multiple of a (nonstationary) Poisson process, cannot really explain the experimentally observed magnitude of the variation in creep, which can be as high as 30% of the mean. [However, in subsequent works (15,17), which were not devoted specifically to concrete, consideration of stochastic models that are not limited to Poisson processes has been suggested.]

So, this elimination process leaves us with gamma related processes. Of course, an ordinary gamma process is excluded because of the obvious nonstationarity of creep, but happily, local gamma processes are capable of handling nonstationarity and yet are statistically tractable (14).

Also, among continuous distributions, after excluding the Gaussian and exponential distributions, the gamma distributions are the easiest to deal with and are also known to fit a wide range of experimental distributions. Thus, a priori, there are good reasons to try and reduce the problem to a form involving the gamma family of distributions.

Transformation of NonStationary Creep Process into Stationary Process

Substitution of Eq. 9 into Eq. 8 provides

$$-\ln E(e^{-\lambda(J_s - J_r)}) = \int_r^s a'_u\,du \int_0^\infty \frac{1}{y} e^{-b_u y}(1 - e^{-\lambda y})\,dy$$

$$= \int_{u=r}^{s} \ln\left[1 + \frac{\lambda}{b(u)}\right] da(u) \qquad (16)$$

Eq. 8 implies that J_s has independent non-negative increments and that the increments $J_s - J_r$ $(s > r)$ has the infinitely divisible distribution (Eq. 8 or 3) with the exponent given by Eq. 16. In the special case that $b(u) = \beta =$ constant, Eq. 16 yields exponent $(a_s - a_r) \ln (1 + \lambda/\beta)$, and so the right-hand side of Eq. 8 becomes $[\beta/(\lambda + \beta)]^{a_s - a_u}$. This may be recognized to be the Laplace transform of the gamma distribution with shape parameter β and scale parameter $(a_s - a_u)$, i.e.:

$$\left(\frac{\beta}{\lambda + \beta}\right)^{a_s - a_u} = \int_0^\infty e^{-\lambda x} \frac{\beta e^{-\beta x}(\beta x)^{a_s - a_r - 1}}{\Gamma(a_s - a_u)}\,dx \qquad (17)$$

Therefore, if $b(u) = \beta =$ constant:

$$P\{J_s - J_r \le z\} = \int_0^z \frac{\beta e^{-\beta x}(\beta x)^{a_s - a_r - 1}}{\Gamma(a_s - a_u)}\,dx \qquad (18)$$

in which P denotes probability of the event in $\{\ldots\}$. If further $a'_u = \alpha =$ constant, then $a_s - a_u = \alpha(s - u)$, and one has the ordinary gamma process. However, if b_u is not a constant, then the distribution of the increments $J_r - J_s$ is not of any recognizable form. Therefore, it is of interest to find transformation of process J_s into some recognizable simple process.

The mean and the variance of any increment of the local gamma process can be ex[illegible]

$$E[J_s - J_r] = \int_{u=r}^{s}\int_{y=0^+}^{\infty} m(du,dy)\,y = \int_r^s a'_u\,du \int_0^\infty e^{-b_u y}\frac{dy}{y}\,y$$

$$= \int_r^s \frac{da(u)}{b(u)} \qquad (19)$$

$$\mathrm{Var}\,[J_t - J_s] = \int_{u=r}^{s}\int_{y=0^+}^{\infty} m(du,dy)\,y^2$$

$$= \int_r^s a'_u\,du \int_0^\infty e^{-b_u y}\frac{dy}{y}\,y^2 = \int_r^s \frac{da(u)}{b(u)^2} \qquad (20)$$

In the special case of ordinary gamma process (constant a'_u and b_u), the mean and variance reduce to $(s-r)a'_u/b$ and $(s-r)a'_u/b^2$. The local gamma process can be shown to have infinitely many jumps in any finite interval, and so the jumps must be rather small to add up to a finite value.

The local gamma process can be transformed into a simple gamma process by changes of scale and of time, and this transformation is invertible. According to the results of Ref. 14, the following corollary may be stated. Let J_s represent a local gamma process with shape parameter $a(u)$ and scale parameter $b(u)$, and let $a(u)$ be increasing and continuous. Define $\alpha(u) = \sup\{s: a(s) < u\}$ = value of s when $a(s) = u$; and $\beta(u) = 1/b(u)$ [for $s \in R_0 = (0,\infty)$]. Then

$$X_u = T_{\alpha\beta} J_s, \quad \text{with} \quad T_{\alpha\beta} J_s = \int_{0^+}^{\alpha(u)} \frac{dJ_s}{b(s)} \qquad (21)$$

in which $T_{\alpha\beta}$ denotes the transformation; and X_u is an ordinary gamma process with shape parameter $\beta = 1$ and scale parameter $\alpha = 1$, i.e., the distribution of $X_{s+r} - X_s$ has the derivative $F'(x) = e^{-x}x^{r-1}/\Gamma(r)$. The inverse transformation is $J_s = T_{ab} X_u$, i.e., it is of the same form except that the roles of a, b and α, β are interchanged.

From fitting of extensive test data (1,9,10) it appeared that the average creep curves are quite satisfactorily described by the deterministic double power law in Eq. 1. Because of this fact, and in view of the scarcity of data usable for statistical analysis, functions $a(u)$ and $b(u)$ should be chosen so as to yield the expected value in the form of power law, s^n. One possible choice is

$$a(s) = \int_0^s \frac{a_0}{u^{1-n}}\,du = \frac{a_0}{n} s^n; \quad b(s) = b_0 \qquad (22)$$

in which a_0, b_0, and n are constants. Indeed, Eqs. 19 and 20 yield

$$E[J_s] = \frac{a_0}{b_0} z; \quad \mathrm{var}\,[J_s] = \frac{a_0}{b_0^2} z; \quad z = \frac{1}{n} s^n \qquad (23)$$

According to Eq. 21 it follows that, if J has a jump of size y at $s = \alpha(u)$, then X has a jump of size $y/b(x)$ at u. Thus, $X_u = J_{\alpha(u)}$, in which $\alpha(u)$ = value of s when $a(s) = u$ or $a_0 s^n/n = u$, which yields

$$s = \alpha(u) = \left(\frac{nu}{a_0}\right)^{1/n} \qquad (24)$$

This will transform J_s into an ordinary gamma process with shape parameter a_0 and scale parameter b_0.

Estimation of Model Parameters

Considering the ordinary gamma process X obtained by the transformation, we now describe the estimation of its two parameters, a_0 and b_0, which we simply denote by a and b. Supposing the transformed times of observations were $u_1, u_2, \ldots, u_n$, we let $u_0 = 0$ and define

$$v_i = u_i - u_{i-1}; \quad Y_i = X_{u_i} - X_{u_{i-1}}; \quad Z_i = Y_i - \frac{a v_i}{b} \qquad (25)$$

Then, the distribution of Y_i has the density $be^{-by}(by)^{av_i - 1}/\Gamma(av_i)$, and the Y_i are independent.

Method of Moments.—For each i, we have

$$E[Z_i] = 0; \quad E[Z_i^2] = \frac{a v_i}{b^2} \qquad (26)$$

Furthermore, introducing the means $\bar Y = \Sigma Y_i/\Sigma v_i$ and $\bar Z = \Sigma Z_i/\Sigma v_i$, and $\bar Z = \bar Y - a/b$, it follows that

$$E[\bar Z] = 0, E[\bar Z^2] = \frac{1}{(\Sigma v_i)^2}\sum_i E[Z_i^2] = \frac{1}{(\Sigma v_i)^2}\left(\frac{a}{b^2}\sum_i v_i\right) \qquad (27)$$

Now one may calculate $E[\bar Y] = a/b$ and

$$\sum_i (Y_i - \bar Y v_i)^2 = \sum_i \left[Y_i - \frac{a v_i}{b} - \left(Y - \frac{a}{b}\right)v_i\right]^2$$

$$= \sum_i (Z_i - \bar Z v_i)^2 = \sum_i (Z_i^2 - 2Z_i \bar Z v_i + v_i^2 \bar Z^2) \qquad (28)$$

Considering the second term in the sum, one has

$$E[Z_i \bar Z] = \frac{E(Z_i \Sigma Z_j)}{\Sigma v_j} = \frac{1}{\Sigma v_j}\left[Z_i^2 + Z_i \sum_{i \neq j} Z_i\right] = \frac{E(Z_i^2)}{\Sigma v_j} \qquad (29)$$

since $E[Z_i] = 0$. From Eqs. 28 and 29 and the preceding results it follows that

$$E\left[\sum_i (Y_i - \bar Y v_i)^2\right] = \frac{a}{b^2}\left(\Sigma v_i - \frac{\Sigma v_i^2}{\Sigma v_i}\right) \qquad (30)$$

Finally, to get the first estimates for a and b by the method of moments, one obtains

$$\frac{\hat a}{\hat b} = \bar Y; \quad \frac{\hat a}{\hat b^2}\left(\Sigma v_i - \frac{\Sigma v_i^2}{\Sigma v_i}\right) = \sum_i (Y_i - \bar Y v_i)^2 \qquad (31)$$

Method of Maximum Likelihood.—The parameters pertaining to Y_i are a and b, and since the increments Y_i are assumed to be independent, the likelihood function is given by $L(a,b) = \Pi_i f_i(Y_i)$, in which f_i is the distribution of Y_i. Estimates are obtained by solving the equations

$$\frac{\partial}{\partial a} L(a,b) = 0; \quad \frac{\partial}{\partial b} L(a,b) = 0 \quad \ldots \quad (32)$$

for a and b. The solution will give two values $\hat{a}$ and $\hat{b}$ in terms of y_i and v_i. For gamma distribution, $L(a,b) = \Pi_i\, b e^{-by_i}(by_i)^{av_i - 1}/\Gamma(av_i)$, and because maximizing L is the same as maximizing log L, one may first take logarithms and then derivatives. Thus, $\ln L = \Sigma_i\, [-by_i + av_i \ln b + (av_i - 1) \ln y_i = \ln \Gamma(av_i)]$. The partial derivatives are

$$\frac{\partial}{\partial a} \ln L = \sum_i \left[v_i \ln b + v_i \ln y_i - \frac{u_i \Gamma'(av_i)}{\Gamma(av_i)} \right] \quad \ldots \quad (33)$$

$$\frac{\partial}{\partial b} \ln L = \sum_i \left[-y_i + \frac{av_i}{b} \right] \quad \ldots \quad (34)$$

in which $\Gamma(x)$ is the known gamma function and $\Gamma'(x) = d\Gamma(x)/dx$. Equating these to zero, one obtains the following equations for the optimal values $\hat{a}$, $\hat{b}$ (maximum likelihood estimates) for a and b:

$$\frac{\hat{a}}{\hat{b}} \Sigma v_i = \sum_i Y_i; \quad \log \hat{b} \sum_i v_i + \sum_i v_i \log y_i = \sum_i v_i \frac{\Gamma'(\hat{a} v_i)}{\Gamma(\hat{a} v_i)} \quad \ldots \quad (35)$$

Note that the first of these equations is the same as the first equation in the method of moment estimation.

A standard library subroutine was used to determine $\hat{a}$ and $\hat{b}$ from the preceding equations. This subroutine uses Brown's method, which is at least quadratically convergent, and a root is accepted if two successive approximations to a given root agree in the first five significant digits. For more details see Refs. 10 and 11. The gamma function was approximated by a fifth-degree polynomial and the error in the approximation was less than 5×10^{-5}. Initial estimates for $\hat{a}$ and $\hat{b}$ were provided by the method of moments and from this the final estimate was obtained.

Analysis of Data and Results

From the basic laws of thermodynamics it is known that creep is an ever increasing process in time, which implies that creep increments are strictly positive. The available data, however, do not show this trend all the time. This is more pronounced at short times from loading, which suggests the experimental error to be the reason. However, trying to take into account that error would yield a model that would be far from simple. On the other hand, some of the data sets exhibit positive increments only. This does not necessarily rule out experimental error. Nevertheless, on the overall, such data should be more reliable. In the present study all data points that represent negative creep increments have been omitted. This introduces, of course, an added error.

Such adjustments have been made in the data of the Bureau of Reclamation (18) and the data for Dworshak Dam (23).

A large number of data points is very desirable since it allows more reliable parameter estimates. For instance, in the case of Dworshak Dam an average

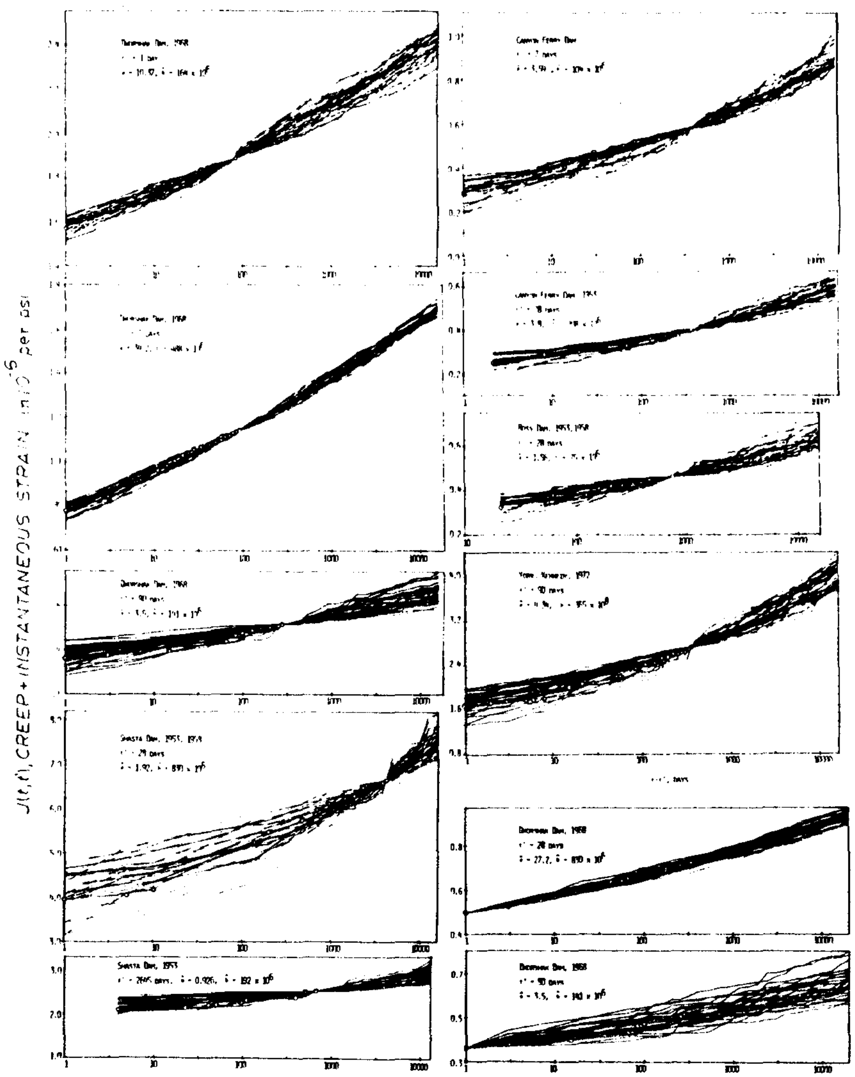

FIG. 1.—Monte Carlo Simulation of Proposed Nonstationary Gamma Process, Compared with Test Data (1 psi = 6.89 kN/m²)

of 18 data points were reported for each age at loading. The data for Shasta Dam and [illegible] Dam are [illegible] Ross Dam, [illegible] for [illegible] at loading)

All the data used here (Fig. 1) are restricted to specimens that are loaded uniaxially, although some statistically usable data for multiaxially loaded specimens (27) are available. These data have not been considered, since converting them to a uniaxial case would require knowledge of Poisson's ratio, whose value is uncertain. The best data set available appeared to be that for Dowrshak Dam (23).

A computer program has been written for the data analysis. For each age at loading, the values of $J_s = J(t,t')$ are read and the corresponding values of s_i, J_{s_i} are noted. Then, the transformed values $u_i = (1/n)(s_i)$, $v_i = u_i - u_{i-1}$, and the increments $Y_i = J_{s_i} - J_{s_{i-1}}$ are computed. Using the method of moments estimates for $\hat{a}$ and $\hat{b}$ are obtained according to Eq. 31. A subprogram is written to evaluate the functions $F(\hat{a},\hat{b}) = 0$ according to Eqs. 33 and 34,

TABLE 1.—Dependence of Statistical Parameters on Age at Loading

t', in days (1)	$\hat{a}$ (2)	$\hat{b}/10^8$ psi (3)	$(\hat{a}/\hat{b}) \times 10^8$ psi (4)	$(\hat{a}/\hat{b}^2) \times (10^8\ \text{psi})^2$ (5)
(a) Canyon Ferry Dam				
2	4.27	1.11	3.86	3.47
7	3.59	1.04	3.45	3.32
28	3.80	2.03	1.88	0.93
90	4.08	3.48	1.17	0.34
365	2.97	3.08	0.96	0.31
(b) Dworshak Dam				
1	10.37	1.64	6.33	3.86
3	41.49	4.90	8.35	1.73
7	34.24	4.88	7.02	1.44
28	27.20	8.30	3.27	0.39
90	3.50	1.41	2.48	1.76
(c) Shasta Dam				
28	1.92	8.33	0.23	0.03
91	8.41	0.29	29.00	100.00
2,645	0.93	1.92	0.48	0.25

Note: 1 psi = 6.89 kN/m².

and using a and b obtained from the method of moments as the initial estimate, the final values for $\hat{a}$ and $\hat{b}$ are obtained.

A standard library subroutine (21,22) is then called to generate the moment vectors of gamma (A,B) deviates, which are distributed as $x^{A-1} \exp(-x/B)/B^A\Gamma(A)$, in which x, A, and B are all positive. In the present case $A = \hat{a}v$ and $B = 1/\hat{b}$. The subroutine uses the rejection technique. This subroutine requires more machine time than other subroutines, but the test results seem to allow more reliance on the deviates distributional form.

The simulated values for x_i are again transformed back to y_i and $\bar{y}_i = \bar{x}_i$ corresponding to each increment w_i are evaluated. Starting from the last value of $J_s = J_{s_N} = J(t_N,t')$, the process y_i can be plotted as shown in Fig. 1, using the output values $J(t_i,t') - \Sigma_{i=N}^{M} \bar{y}_i (M = N - i + 1)$; and $J(t_i,t') = 1/\bar{t}_0 + C(t_i,t')$. These values have been plotted for each vector value of

the generated deviates for as many simulations as desired (30 in Fig. 1). It can be seen from Fig. 1 that the deviates are concentrated within a band that looks reasonable when compared visually with the test data.

To extrapolate the creep and determine the creep values over long periods of time, estimates for $\hat{a}$ and $\hat{b}$ can be used. To exemplify it, Monte Carlo simulated extrapolations have been obtained (see Fig. 1). They were chosen from the last observation point, which is proper from the point of view of extrapolation. Backward simulations from the last point are also shown, in order to visualize the scatter in comparison to measured data. The bands within which creep values are expected to lie are shown for various ages at loading. For each data set and each age at loading the values of $\hat{a}$ and $\hat{b}$ are given in the figure.

Statistically tractable sets of data on concrete creep are rather scarce. However, by developing a stochastic process model, one can also extract statistical information from measurements on one single specimen. This information is provided by the statistical nature of subsequent creep increments and is useful even if statistical comparisons with tests on other specimens are lacking. Thus, the present model actually reduces the need for the scope of statistical data and thereby extends the feasibility of statistical analysis, even to the data pertaining to a single specimen. In fact, the data points in Fig. 1 each refer to a single specimen. When the time series for more than one specimen is available, which is highly desirable, the knowledge of the statistical parameters is, of course, greatly improved. In such a case, one can determine the stochastic process parameters $\hat{a}$ and $\hat{b}$ for each specimen (each time series) individually.

Furthermore, for practical application the statistical treatment of the dependence of creep on the age at loading, t', is needed. A two-dimensional stochastic process in age t', in addition to that in creep duration $t - t'$, can be postulated, but the test data on the effect of t' do not permit such a model because in every data test only a few ages at loading are included. Therefore, one has to be contented with considering the dependence on t' as deterministic.

Table 1 shows the dependence of $\hat{a}$ and $\hat{b}$ upon age at loading t'. This dependence does not seem to follow any recognizable form and is very much dependent on each particular data and the amount of scatter within the data. However, the value of $\hat{a}/\hat{b}$ seems to be decreasing as expected although the data for Dworshak Dam for $t' = 1$ day seems to give smaller value than for $t' = 3$ days as in Fig. 1. The dependence of $\hat{a}/\hat{b}^2$ upon t' indicated how the variation of the data varies with age.

Many of the test data in the literature cannot be analyzed as a stochastic process in time because the creep values have not been measured in sufficiently close time intervals and in properly spaced intervals. It would be beneficial if the experimentalists were taking readings that are optimal for statistical analysis. This can be achieved in such a way that the creep increments are identically distributed, i.e., by taking creep readings at times such that the intervals of transformed time z (Eq. 23) are constant. For this purpose one needs to have an estimate for the value of n, which can be taken as $n = 1/8$ on the average.

Conclusions

1. Creep of concrete can be considered as a nonstationary stochastic process

with independent increments of [illegible] distribution. This process can be transformed to a stationary gamma process. The transformation accounts for the deceleration of creep rate with creep deviation. The mean prediction agrees with the deterministic creep law in the form of a power law, amply justified previously.

2. Stochastic independence of increments and infinite divisibility of their distribution is assumed. This is justified by the requirement of distribution preservation when a homogeneous specimen is split into smaller ones and when the stress is considered as a sum of stresses. An even stronger justification is provided by considering the micromechanism of creep.

3. By treating creep as a stochastic process, some statistical information can be extracted merely from one specimen, by analyzing the time series of measured values. This is very useful because statistically tractable data including a number of specimens are scarce. (However, it is of course always better to use as many specimens as possible.)

4. With regard to considering creep at the same time as a stochastic process in the age at loading, t', no adequate time series of measurement in t' is available. For the time being, one has to be contented with identifying or even assuming a deterministic (mean) dependence on t' for the statistical parameters of the stochastic process in load duration at fixed t'.

5. To make statistical analysis of the time series of creep measurements more accurate, the readings must be taken in sufficiently close time intervals of z, which give equal spacing when creep duration s is transformed to $z = s^n/n$.

6. The main use of the present model is in extrapolating short-time creep measurements (e.g., of 6-month duration) to long times (such as 40 yr).

7. The model allows predicting for long-time creep the confidence limits of given probability cut-off, i.e., limits that have a specified small probability of being exceeded. In many design considerations, these maximum creep values should be used.

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13447 STOCHASTIC EXTRAPOLATING CONCRETE CREEP

KEY WORDS: **Concrete; Creep; Extrapolation;** Measurement; Predictions; Random processes; **Statistical analysis;** Stochastic processes; Viscoelasticity

ABSTRACT: Creep of concrete is modeled as a process with independent increments of locally gamma distribution. The process is transformed to a stationary gamma process. The mean prediction agrees with the deterministic double power law established previously. Infinite divisibility of the increment distribution is assumed. This is justified by additivity of deformations and of stresses, and also by considerations of the microscopic mechanism of creep, assuming creep to be due to migrations of widely spaced solid particles along micropore passages whose length is statistically distributed. The treatment of creep as a stochastic process allows extracting considerable information from measurements even on one specimen, although a greater number of specimens is preferable. The main use of the model is in extrapolation of short time creep data into long times, and calculation of confidence limits. Methods of determining process parameters from creep test data are given. Monte Carlo simulations demonstrate reasonable agreement with test data.

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