

PHENOMENOLOGICAL THEORIES FOR CREEP OF CONCRETE BASED ON RHEOLOGICAL MODELS

By

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The influence of age is a basic characteristic of creep of concrete differing from other structural materials. The creep law of concrete is therefore time-variant. The laws of creep derived in the viscoelasticity theory from various rheological models are, however, time-invariant. As far as these laws have been used for concrete to interpret its stress-strain relation, either the ageing of concrete has been approximately neglected or these laws were adapted in a pure formal way — the constant coefficients in the differential equation for the creep law have been changed to time functions. This is, however, incorrect.

In the present paper there has been shown that, correctly, the time variation of the mechanical parameters themselves, in particular elements of model, i.e. the time variation of spring constants of springs and viscosity constants of dashpots, has to be considered. (For a spring, e.g., then the relation $\dot{\sigma}_x = E\dot{\epsilon}_x$ and not the relation $\sigma_x = E\epsilon_x$ applies.) In general, every linear (additive) creep law for which the principle of superposition applies may be then interpreted by a model consisting of springs and dashpots. Equivalently, one may also take an electric circuit consisting of time variable resistances and self-inductions or capacities as a model.

In particular, there has been shown that the well-known approximate creep law of Dischinger-Whitney, can be interpreted by the Maxwell's model (Fig. 4a) and the Arutyunyan-Maslov's law by the Boltzmann's (standard) model (Fig. 4b). On the basis of models there are further discussed suitable and physically admissible forms of functions in these laws at which there has been shown, e.g., that the ageing term $\varphi_\infty/E = C_0 + A/\tau$ used frequently in the Arutyunyan-Maslov's function (7.14), gives nonsense results for a rather young concrete and therefore rather the expression (7.18) has to be used for which the integral appearing in relaxation problems has been tabulated (Tab. 1) and a chart of corresponding relaxation functions constructed. Also more accurate forms of creep law based on models consisting of three or more elements have been discussed. They enable an analytical solution by transforming the Volterra's integral equation into differential equation.

Further some simplifying assumptions suitable for creep of concrete — the reduction of time with respect to environment (moisture) conditions, the creep affinity in non homogeneous structures — are introduced and also equations for multiaxial stress are formulated.

As a generalization of the well-known elastic-viscoelastic analogy for time-invariant creep, and analogy of time-variant creep with elasticity has been presented. It makes possible to obtain the equations for creep problems directly from the equations of problem in elasticity theory by a mere interchange of the elasticity constants with the differential (or integral) operators for creep. To demonstrate this, the equations for a nonhomogeneous redundant frame, for a homogeneous body and for bending of a composite beam at various creep laws have been derived.

The use of Laplace transform is of advantage only if the Maxwell's model for creep is considered.

For nonlinear effects in creep of concrete at low stresses a new rheological model with a ratched pawl according to Fig. 4d has been suggested. Its ratched pawl gets closed only if the long-time part of strain becomes to decrease, so that this model can interpret the fact that the reversibility of creep after a release of stress is considerably smaller than according to the principle of superposition in time.

Finally there has been mentioned that, with respect to a great influence of moisture and dimensions of cross section, there will be necessary in future to set up a space-time theory of creep of concrete, considering the drying process. Only then it will be possible to determine the real time distribution of stress in cross sections.

For creep analysis of concrete structures many different creep laws have been suggested till now. The most important of them are the Dischinger-Whitney's law (theory of aging), the Maslov-Arutyunyan's law and the law of ideal deformation modulus (time-invariant theory). The aim of the present paper is to make a critical mathematical discussion of these theories and to suggest a new approach in using the rheological models for a time-variant (i.e. aging) material, as well as to derive some more suitable forms of creep law and a new rheological model for irreversible effects in concrete creep. Next there will be generally presented an efficient aid for the structural creep analysis — the analogy of time-variant creep with elasticity, expressed by creep operators.

The creep laws based on mechanical rheological models, which are rather an intuitive help for interpretation of stress-strain relation, are widely used in the theory of viscoelasticity of materials which are not subjected to ageing. The mechanical parameters are considered time invariable. With view to the advantages of this approach, efforts have been made to introduce these models into the creep analysis of concrete. The most significant contribution here were perhaps the papers of Freudenthal [14, 15] including the study of nonlinear creep, the book of Hansen [17], next also the papers of Skudra [28] and others. If taking the mechanical parameters the creep equation in the analysis as time invariable, sufficiently good interpretation cannot be attained. The only possible thing to do is to adjust the parameters according to the age of concrete in time of load application [e.g. 26]. To change the parameters in final stress-strain equation for time-invariant creep to time functions is a purely formal and thus not suitable approach, corresponding yet to behaviour of no actual model.

In the present paper, it will be suggested for concrete to use time-invariant rheological models, in which the parameters of each element yet are variable with the age of concrete. Particularly, it will be shown that, e.g., the time-variant Maxwell's model (Fig. 3a) gives the most usual Dischinger-Whitney's creep law, the time-variant Boltzmann's (i.e. standard) model (Fig. 3b) interpretes the Arutyunyan-Maslov's law, etc. Let us note that the time-variant models have their counterparts in electric analogue. — electric circuits (Fig. 3A, B) with time variable resistances (designated as R), self-inductions (L) and capacities (C), which have been derived in the paper [7].

The principal results in this problem have already been presented by the author in [4], [7], [8]. (It should be also noted here that properly the plastics under time variable temperature are also a time-variant material. However, it is possible to attain invariant models and equations by certain reduction of time scale according to temperature.)

1. REDUCTION OF TIME ACCORDING TO ENVIRONMENT CONDITIONS

It is known that the greatest difference between creep observed on actual structures in nature and the experiments in laboratory under invariable conditions is the large influence of variation of environment moisture and, to a smaller extent, of temperature. In order to express this circumstance, the assumption of reduction

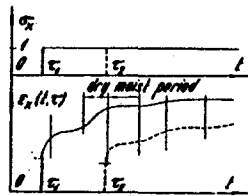


Fig. 1a.

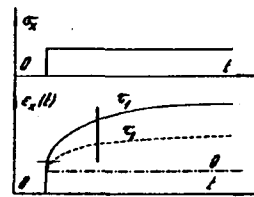


Fig. 1b.

of time according to environment conditions has been suggested by the author [5], [6]. It expresses the acceleration of creep by dry environment and its moderation by moist environment (fig. 1a) sufficiently well, mainly during periodic changes between summer and winter, by an apparent moderation or acceleration of time course. Thus, instead of real time A the reduced time t is introduced by the relation

$$(1.1) \quad dt = \varrho(t) dt \quad \text{or} \quad t = \int_0^t \varrho(t) dt$$

$\varrho(t)$ — time reduction coefficient, depending on environment conditions (i.e. for summer $\varrho \approx 1.7$, for winter $\varrho \approx 0.3$, for spring and autumn $\varrho \approx 1.0$). Thanks to this assumption, all creep equations can be considered as for steady environment conditions and all our further considerations will apply for the reduced time t .

2. ASSUMPTION OF CREEP AFFINITY IN A NONHOMOGENEOUS STRUCTURE

In analysing the creep of statically indeterminate concrete structures, it is advantageous to express the stress-strain relation of all parts of structure by equations of the same type in one time t common for all structural parts. The physical non-homogeneity of concrete structures in creep is caused mainly by cooperation of

concrete and steel reinforcement, steel parts, and eventually other materials (elastic or consolidating foundations), or by cooperation of differently aged concrete (bridges cast by cantilever method) of differently thick concrete parts, of different kind of concrete of the parts under water and in air, etc. For nonhomogeneous creep of concrete structures, a simplifying assumption of affinity of creep curves (fig. 1b) under constant stress can be introduced. Hence, the assumption of creep affinity will be introduced in the form:

$$(2.1) \quad C_{(\kappa)}(t, \tau) = \kappa C(t, \tau)$$

$C(t, \tau)$ — the chosen basic creep function (for the average age of concrete) which expresses the creep strain in time t caused by the action of the constant unit stress applied in time τ (fig. 2a), κ — the creep affinity coefficient of the part of structure, $C_{(\kappa)}(t, \tau)$ — the creep function for the part with a creep affinity coefficient κ . The coefficient κ should be determined from the measured values of creep function for extreme times, i.e.

$$(2.2) \quad \kappa = \frac{C_{(\kappa)}(\infty, t_0)}{C(\infty, t_0)}$$

t_0 — the instant of applying the loading on the structure. The assumption of creep affinity is exactly valid for steel parts (or rock, sand, gravel foundation), for which $\kappa \equiv 0$ and hence elasticity is obtained. For a homogeneous concrete structure (of equal concrete age, thickness, etc.), there is $\kappa \equiv 1$. (Particularly for Dischinger-Whitney's creep law, this assumption was introduced by the author already in 1961 [5], [6]. For Arutyunyan-Maslov's creep law, the creep affinity for differently aged concrete is already included in its known formulation [21], [3], [1], [4], [24], etc.)

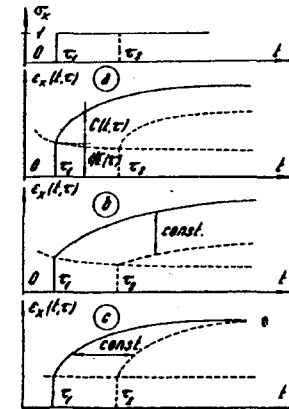


Fig. 2.

3. GENERAL RELATIONS FOR LINEAR CREEP OF CONCRETE

Now, let us briefly formulate the — principally known — general integral relations for the linear creep of concrete which will be needed further. Basic assumption is the principle of superposition [3], [13], [4], [3a] stating that the mutually corresponding curves of stress and strain are additive, e.g. particularly that the responses of the strain or stress to steps of stress or strain in different time may be added. This is valid for stresses lower than about 0.5 of compressive strength and 0.9 of tensile strength and in time for increasing stress or only slightly decreasing stress. Otherwise, nonlinear effects take place.

For uniaxial compression or tension, we obtain, by adding the strain responses to steps of stress $d\sigma_x = [d\sigma_x(\tau)/(d\tau)] d\tau$, the following relation between normal stress σ_x and normal strain ε_x :

$$(3.1) \quad \varepsilon_x(t) = \sigma_x(t_0) \left[\frac{1}{E(t_0)} + \kappa C(t, t_0) \right] + \int_{t_0}^t \left[\frac{1}{E(\tau)} + \kappa C(t, \tau) \right] \frac{d\sigma_x(\tau)}{d\tau} d\tau$$

$E(t)$ — modulus of elasticity in time t . This above equation can be written also in an operator form:

$$(3.2) \quad \varepsilon_x = E^{-1} \sigma_x$$

in which E^{-1} is the integral operator defined by the right side of eq. (3.1). Using integration by parts in the eq. (3.1), it follows

$$(3.3) \quad \varepsilon_x(t) = \frac{\sigma_x(t)}{E(t)} + \int_{t_0}^t \frac{\sigma_x(\tau)}{E(\tau)} L(t, \tau) d\tau,$$

where

$$\frac{L(t, \tau)}{E(\tau)} = - \frac{\partial}{\partial \tau} \left[\frac{1}{E(\tau)} + \kappa C(t, \tau) \right] > 0$$

By this equation the operator E^{-1} is defined as a Volterra's integral operator. If the strain is prescribed, the eq. (3.3) represents a Volterra's integral equation for stress. The function $L(t, \tau)$ may be called strain memory function (or hereditary function, influence function) and represents the strain response in time t to the unity stress impulse ($\Delta t \sigma_x / E = 1$, $\Delta t \rightarrow 0$, $\sigma_x \rightarrow \infty$; Dirac's function) in time τ (or to the stress impulse of magnitude $E(\tau)$). The following relations are valid:

$$(3.4) \quad \lim_{t \rightarrow \infty} E(t) = E(\infty) < \infty, \quad \partial E / \partial t > 0;$$

$$C(t, \tau) = 0, \quad C(t, \tau) > 0, \quad \partial C / \partial t > 0, \quad \partial C / \partial \tau < 0$$

$$\lim_{t \rightarrow \infty} C(t, \tau) < \infty \text{ and } > 0, \quad \lim_{\tau \rightarrow \infty} C(\infty, \tau) > 0;$$

On the other hand, the creep laws may be also expressed on the basis of the relaxation function $R_{(\kappa)}(t, \tau)$ denoting the release of stress in time t (in the part (κ) of the structure) if a constant unit strain enforced in time τ is prescribed. From the principle of superposition, we get

$$(3.5) \quad \sigma_x(t) = \varepsilon_x(t_0) [E(t_0) - R_{(\kappa)}(t, t_0)] + \int_{t_0}^t [E(\tau) - R_{(\kappa)}(t, \tau)] \frac{\partial \varepsilon_x(\tau)}{\partial \tau} d\tau$$

or in an operator form

$$(3.6) \quad E \varepsilon_x = \sigma_x$$

in which the operator E is defined by the eq. (3.5). Using integration by parts, a Volterra's integral equation for strain is obtained:

$$(3.7) \quad \sigma_x(t) = E(t) \varepsilon_x(t) - \int_{t_0}^t E(\tau) \varepsilon_x(\tau) K(t, \tau) d\tau$$

where

$$(3.8) \quad E(\tau) K(t, \tau) = \frac{\partial}{\partial t} [E(\tau) - R_{(\kappa)}(t, \tau)] > 0$$

The eq. (3.7) defines E as a Volterra's integral operator. $K(t, \tau)$ is the stress memory function and represents the stress response in time t to a unity strain impulse ($E \varepsilon_x \Delta t = 1$, $\Delta t \rightarrow 0$; $\varepsilon_x \rightarrow \infty$; Dirac's function) in time τ . There are valid the relations:

$$R(\tau, \tau) = 0, \quad R(t, \tau) > 0; \quad \partial R / \partial t > 0, \quad \partial R / \partial \tau < 0,$$

$$\lim_{t \rightarrow \infty} R(t, \tau) < \infty \text{ and } > 0, \quad \lim_{\tau \rightarrow \infty} R(\infty, \tau) > 0,$$

$$K(\tau, \tau) = 0, \quad K(t, \tau) > 0, \quad \partial K / \partial t > 0, \quad \partial K / \partial \tau < 0,$$

$$\lim_{t \rightarrow \infty} K(t, \tau) < \infty \text{ and } > 0, \quad \lim_{\tau \rightarrow \infty} K(\infty, \tau) > 0 \quad (\partial^2 R / \partial t \partial \tau < 0).$$

The creep and relaxation functions are mutually related. $R_{(\kappa)}(t, \tau)$ is given by the solution of (3.1) or (3.3) for stress if $\varepsilon_x \equiv 1$ and vice versa for $C(t, \tau)$. Further it can be seen that $K(t, \tau)$ is the resolving kernel of the integral equation (3.3) and vice versa for $C(t, \tau)$. The memory functions $L(t, \tau)$, $K(t, \tau)$ are related by the Volterra's integral equation [16], [2]

$$(3.9) \quad L(t, \tau) - K(t, \tau) = \int_{\tau}^t L(\vartheta, \tau) K(t, \vartheta) d\vartheta = \int_{\tau}^t L(t, \vartheta) K(\vartheta, \tau) d\vartheta$$

from which one memory function may be determined if the other memory function is given. The eq. (3.9) is derived by adding the eq. (3.3) and (3.7), then substituting the stress $\sigma_x \equiv 1$ and the strain $\varepsilon_x = 1/E(t_0) + \kappa C(t, t_0)$ or $\varepsilon_x \equiv 1$ and $E(t_0) - R_{(\kappa)}(t, t_0)$, differentiating the equation with respect to t_0 , and finally substituting left sides of the eq. (3.4) and (3.8) and replacing the letters t_0 and τ by τ and ϑ in turn.

The equations for the relation of shear stress τ_{xy} and shear strain γ_{xy} and, in general, the equations for the multiaxial plane or space state of stress are analogic to the foregoing equations for uniaxial state of stress. Besides the shear modulus G and Poisson's ratio μ , it is necessary to introduce the creep function $C_1(t, \tau)$ for transverse creep normal strain and the creep function $C_2(t, \tau)$ for creep shear strain caused

by unit shear stress. With respect to the conditions of isotropy (quasi-isotropy), a relation: $C(t, \tau) + C_1(t, \tau) = \frac{1}{2}C_2(t, \tau)$ between creep functions results as an analogue to the relation between elastic constants: $E^{-1} + \mu E^{-1} = \frac{1}{2}G^{-1}$. In the place of the eq. (3.2), we have then generally

$$(3.10) \quad \begin{aligned} \epsilon_x &= E^{-1}\sigma_x - \mu E^{-1}(\sigma_x + \sigma_y), \dots \\ \gamma_{xy} &= G^{-1}\tau_{xy}, \dots \end{aligned}$$

where μE^{-1} and G^{-1} are Volterra's integral operators which have a similar form as according to (3.3), while in the place of $C(t, \tau)$ it stays $C_1(t, \tau)$ or $C_2(t, \tau)$. With respect to isotropy, it holds that

$$(2.11) \quad E^{-1} + \mu E^{-1} = \frac{1}{2}G^{-1}.$$

Similarly as for uniaxial stress, it is also possible to use the relaxation function as basic, in which case rather than equations (2.10), it is better to use the equations analogous to the Lamé's equations in elasticity (with corresponding relaxation functions and operators). The corresponding strain memory functions are related with stress memory functions by equations analogous to (3.9).

4. RHEOLOGICAL MODELS FOR LINEAR CREEP

The creep equations (3.2), (3.6), (3.10) are sufficient to solve the problems of structural mechanics. They are resulting in Volterra's integral equations in time which are not convenient for an analytical solution, although they are rather useful for a numerical solution [29], [25]. In order to find an analytical solution, it is necessary to introduce certain special types of creep function $C(t, \tau)$ which make possible to transform the integral equations or directly to express the creep law with help of a differential equation rather than integral equation.

For introducing convenient special analytical forms of creep function $C(t, \tau)$, the rheological models are of advantage. The load of model is identified with the stress σ_x (or τ_{xy} etc.) and its deformation with the strain ϵ_x (or γ_{xy} etc.). The linear creep can be modelled by a set of linear springs and linear dashpots coupled mutually in series or parallel. (In series coupling the forces are equal and deformations are added whereas in the parallel coupling the deformations are equal and the forces are added.) In contradistinction to the deformation equation for a time variable spring $\sigma_x = E\epsilon_x$ at time-variant material, the equation $\dot{\sigma}_x = E\dot{\epsilon}_x$ must be considered. For a dashpot $\sigma_x = \eta\dot{\epsilon}_x$ applies (which is the same as for time-variant material).

5. TIME-VARIANT MAXWELL'S MODEL

The Maxwell's model with time-variant mechanical parameters (fig. 3a) is the most simple model which can approximately interpret the creep of concrete.

The strain rate in the spring is $\dot{\sigma}_x/E(t)$ and the strain rate in the dashpot $\sigma_x/\eta(t)$ so that the stress-strain equation for creep has the form:

$$(5.1) \quad \dot{\epsilon}_x = \frac{\dot{\sigma}_x}{E(t)} + \frac{\sigma_x}{\eta(t)}.$$

Instead of time t let us introduce in this equation a new independent variable $\varphi(t)$ according to the relation:

$$(5.1) \quad \begin{aligned} \eta(t) &= \frac{E_0}{\kappa \varphi(t)} \\ \text{or } \kappa \varphi(t) &= E_0 \int_0^t \frac{d\tau}{\eta(\tau)} \end{aligned}$$

in which E_0 is the mean (or initial) value of $E(t)$. This substitution is correct, since $\varphi(t)$ is a still increasing continuous function of time. This equation takes then the form

$$(5.2) \quad \frac{\partial \epsilon_x}{\partial \varphi} = \frac{1}{E(t)} \frac{\partial \sigma_x}{\partial \varphi} + \frac{\kappa \sigma_x}{E_0}$$

in which the coefficient at σ_x turns to a constant. This equation can also be written in operator form (3.2), in which E^{-1} is the following differential operator:

$$(5.3) \quad E^{-1} = \frac{\frac{1}{E(t)} \frac{\partial}{\partial \varphi} + \frac{\kappa}{E_0}}{\frac{\partial}{\partial \varphi}}.$$

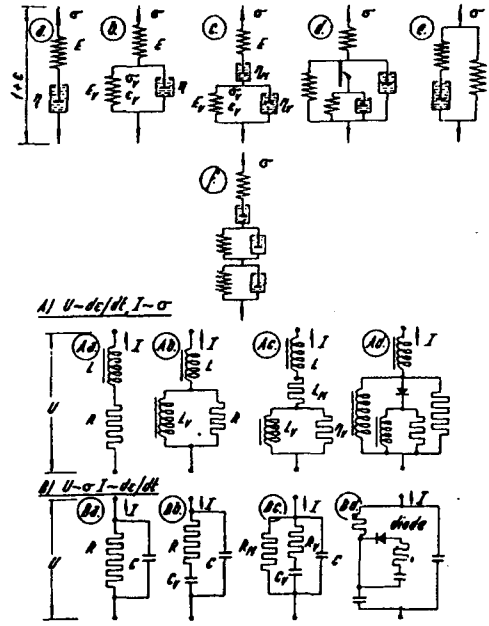


Fig. 3.

Solving the strain under constant stress from eq. (5.2), it is found (with respect to (2.2)) that the creep function has the form

$$(5.4) \quad C(t, \tau) = \frac{\varphi(t) - \varphi(\tau)}{E_0}.$$

On the other hand, if this form of $C(t, \tau)$ would be primarily introduced, the eq. (5.3) is obtained by substituting it in (3.3) and differentiating with respect to $\varphi(t)$. Hence, the creep function (5.4) is equivalent to a time-variant Maxwell's model.

The direct introduction of the expression (5.4) can be based on the assumption that the curves $C(t, \tau)$ for various τ are identical, parallel and mutually translated

perpendicularly to the axis t (fig. 2b). Otherwise, the eq. (5.4) is also obtained as an easy consequence of the assumption that, at releases of stress, all creep deformations are irreversible (plastic), i.e. that, after taking the load off, the strain remains constant which is possible only if the curves $C(t, \tau)$ are parallel.

It is seen that the creep function in form of (5.4) represents the well-known Dischinger-Whitney's creep law [11], [4], [30], [28] (for $E = \text{const.}$ firstly suggested by Whitney [32] and for structural analysis basically developed by Dischinger, which is the most simple creep law for concrete and is widely used in structural analysis. (In Russian it is called theory of ageing.) The function $\varphi(t)$ is called creep factor. It has a finite limit $\varphi(\infty) < \infty$.

For analytical expression of $\varphi(t)$, various formulae have been suggested. Dischinger [11] used $\varphi(t) = \varphi(\infty)(1 - e^{-\gamma t})$ where γ should be taken by 1 to 2 years⁻¹, Mörsch introduced $\varphi(\infty) \cdot \sqrt{[1 - e^{-\gamma t}]}$ where $\gamma \approx \text{year}^{-1}$ (prescribed till now by ČSN-standard); also it has been used $\varphi(\infty) t^b / (a + t^b)$ (where $b = 0.6$, $a = 0.28$ · √(year) etc. The first of these formulae is the best one, because it satisfies exactly the assumption of creep affinity (2.1) for differently aged concrete, requiring that $\kappa = [\varphi_{(\kappa)}(\infty) - \varphi_{(\kappa)}(t_0)] / [\varphi(\infty) - \varphi(t_0)]$ is a constant which is true as $\kappa = e^{-\gamma \Delta t}$ · $\varphi_{(\kappa)}(\infty) / \varphi(\infty)$ (part (κ) is older). However, all the formulae for creep factor have an essential short-coming in the fact that, for loads applied on very old concrete, a zero creep would result.

Therefore, it is more accurate to introduce the creep factor according to the actual (measured) creep function for the (average) instant t_0 of application of steady load on structure, i.e. to define it by the relation

$$(5.5) \quad \varphi(t) = \varphi(t_0) + E_0 C(t, t_0)$$

where $\varphi(t_0)$ is chosen arbitrarily. If the load is applied on an infinitely old concrete ($t_0 \rightarrow \infty$), a non-zero creep is then obtained.

For changes of stress after the instant t_0 of load application, however, the additional creep produced by them tends to zero if $t \rightarrow \infty$. This remains the principal deficiency of this creep law. In consequence, the values of stress relaxation, due to enforced deformations in structures according to this creep law, are obtained greater than according to a more exact law, because actually the relative creep (negative), corresponding to releases of stress after long time, tends to a constant non-zero value. For the same reason, the deflections obtained for buckling of columns are smaller than the exact values are. Second deficiency of this creep law lies in the fact that the reversible part of creep, i.e. delayed elastic deformations (representing about 10 to 30% [20]), are not expressed.

In (5.5) the rate of creep for different t_0 may approximately be considered the same, i.e.

$$(5.6) \quad \varphi(t) - \varphi(t_0) = \varphi_\infty(t_0) \beta(t - t_0) = E_0 C(t, t_0)$$

where β is function of $t - t_0$ ($\beta(0) = 0$, $\beta(\infty) = 1$, $\beta = C(t, t_0) / C(\infty, t_0)$) and

$\varphi_\infty(t_0)$ is the (actual) creep factor from t_0 to infinity, $\varphi_\infty(t_0) = E_0 C(\infty, t_0)$. The analytical expressions used for β are, e.g., $\beta = 3/[3 + a(t - t_0)]$ ($a \approx 100$ days) or $\beta = \text{tgh}[a(t - t_0)]^b$ ($a \approx 0.5 \text{ year}^{-1}$, $b \approx 0.5$). Analytically most suitable is, however, $\beta = 1 - e^{-\gamma(t-t_0)}$ ($\gamma \approx 40$ to 60 days^{-1}), since this form eventually allows also to use the more accurate creep law (7.10). The dependence of $\varphi_\infty(t_0)$ on the composition and properties of concrete itself, on the moisture (and temperature) of environment, on thickness of concrete (dimensions, form of cross section) and on the age t_0 of concrete at instant of load application can be expressed by the parameters φ_c , k_1 , k_2 , k_3 in turn, in form of product $\varphi_\infty(t_0) = \varphi_c k_1 k_2 k_3$ [30]. The eq. (5.6) exactly satisfies the assumption (2.1) of creep affinity for differently aged concrete, $\kappa = \varphi_{\infty(\kappa)}(t_0) / \varphi_\infty(t_0)$.

The relaxation function $R_{(\kappa)}(t, \tau)$ corresponding to (5.4) is obtained by solving the equation (5.2) for constant ε_x :

$$(5.7) \quad R(t, \tau) = E_0 \left[1 - \exp \left(- \int_{\varphi(\tau)}^{\varphi(t)} \frac{E(\tau)}{E_0} \kappa \, d\varphi(\tau) \right) \right].$$

Assuming approximately $E(t) = E_0(1 + \delta \kappa \varphi(t))$ [30], the integral equals $\kappa \bar{\varphi} (1 + \frac{1}{2} \delta \kappa \bar{\varphi})$ where $\bar{\varphi} = \varphi(t) - \varphi(\tau)$.

Approximately, the modulus E may be considered as constant.

6. TIME INVARIANT BOLTZMANN'S MODEL

For a structure under steady load, a time-invariant Boltzmann's (or standard) model (Fig. 3b) may also be approximately used for modelling the relation of ε_x and σ_x at creep. For the elements of model it holds that $\sigma_x = E(\varepsilon_x - \varepsilon_{xV})$, $\sigma_{xV} = E_V \varepsilon_{xV}$, $(\sigma_x - \sigma_{xV}) = \eta \dot{\varepsilon}_{xV}$. Eliminating ε_{xV} and σ_{xV} , the equation:

$$(6.1) \quad \dot{\varepsilon}_x + \gamma \varepsilon_x = \frac{\dot{\sigma}_x}{E} + \frac{\gamma}{E_{id}} \sigma_x$$

is obtained, if the notation $E_{id}^{-1} = E^{-1} + E_V^{-1}$, $\gamma = E_V / \eta$ is introduced. Solving this equation for constant ε_x , the following creep function and relaxation function results:

$$(6.2) \quad C(t, \tau) = \frac{\varphi_\infty}{E} (1 - e^{-\gamma(t-\tau)}), \quad \frac{\kappa \varphi_\infty}{E} = \frac{1}{E_V} = \frac{1}{E_{id}} - \frac{1}{E},$$

$$R(t, \tau) = (E - E_{id}) (1 - e^{-\gamma(t-\tau)E/E_{id}})$$

where φ_∞ denotes the (actual) creep factor up to infinity.

As it is seen from (6.1), for a steady load (or a load tending to a constant value for $t \rightarrow \infty$), under which the state of stress and strain tends also to a steady value for $t \rightarrow \infty$ ($\dot{\sigma}_x = \dot{\varepsilon}_x = 0$), we have $\sigma_x(\infty) = E_{id} \varepsilon_x(\infty)$. Hence the limit values of

stresses and strains in the structure (which are most important) may be obtained according to elasticity theory, the structure having moduli E_{id} instead of E . This is the well-known law of ideal deformation modulus (which is represented by E_{id}) — a theory widely used in approximate engineering analysis — and the time-invariant Boltzmann's model is therefore its rheological model.

The relatively best approach is attained if φ_∞ is ascertained according to the age of concrete at instant t_0 of the first load application, i.e. if we define

$$(6.3) \quad E_{id} = \frac{E_0}{1 + \kappa \varphi_\infty(t_0)} = \frac{E_0}{1 + E_0 \kappa C(\infty, t_0)}$$

The eq. (6.2) satisfies the assumption (2.1) of creep affinity for differently aged concrete.

The greatest shortcoming of this law is that φ_∞ is a constant, in consequence of which the relative creep for additional changes of stress produced by creep in very old concrete is the same as on the beginning (at instant t_0 of first stress). For this reason, the value of stress relaxation computed according to this law are smaller than the actual values, the deflections at column buckling are obtained greater than the correct values. Second deficiency lies in the fact that, according to this creep law, all creep strains and stresses are totally reversible (delayed elastic deformations only). On the other hand, if the reversibility of creep is here taken as a primary assumption, the creep curves $C(t, \tau)$ for various τ must be identical, parallel and mutually translated in direction of the time axis t (fig. 2c), which gives in special case the expression (6.2).

For a young concrete, this law is less accurate than the Dischinger-Whitn law, for an old concrete (more than one year), it corresponds better to actual behaviour.

7. TIME-VARIANT BOLTZMANN'S MODEL

The time-variant Boltzmann's model (or standard) (Fig. 3b) can interpret the creep of concrete more accurately, although in a mathematically more involved manner, than the preceding two models. The differential equations for the elements of model are $\dot{\sigma}_x = E(t)(\dot{\epsilon}_x - \dot{\epsilon}_{xv})$, $\dot{\sigma}_{xv} = E_v(t)\dot{\epsilon}_{xv}$, $\sigma_x - \sigma_{xv} = \eta(t)\dot{\epsilon}_{xv}$.

Eliminating ϵ_{xv} and σ_{xv} from this system of three differential equations, we receive the following second order differential equation [4], [7]:

$$(7.1) \quad E\eta\ddot{\epsilon}_x + E(E_v + \dot{\eta})\dot{\epsilon}_x = \eta\ddot{\sigma}_x + [E + E_v + \dot{\eta} - \eta\dot{E}/E]\dot{\sigma}_x$$

The initial equations at instant t_0 of first stress application are following:

$$(7.2) \quad \epsilon_x(t_0) = \frac{\sigma_x(t_0)}{E(t_0)}, \quad \dot{\epsilon}_x(t_0) = \frac{\dot{\sigma}_x(t_0)}{E(t_0)} + \frac{\sigma_x(t_0)}{\eta(t_0)}$$

as it results from $\dot{\epsilon}_{xv}(t_0) = \sigma_{xv}(t_0) = 0$. Integrating the eq. (7.1) (or directly the equations for model elements) for constant stress $\sigma_x(\tau)$ applied in time τ , we receive the creep function:

$$(7.3) \quad C(t, \tau) = g(\tau) [h(t) - h(\tau)]$$

in which

$$(7.4) \quad g(t) = e^{F(t)}, \quad h(t) = \int_{t_0}^t \frac{1}{\eta(\vartheta)} e^{-F(\vartheta)} d\vartheta, \quad F(t) = \int_{t_0}^t \frac{E_v(\vartheta)}{\eta(\vartheta)} d\vartheta$$

On the other hand, let us now consider the creep function (7.3) as primarily chosen and introduce it in the Volterra's differential equation (3.3), (3.4). Writing then the first and the second derivative of (3.3) with respect to t :

$$(7.5) \quad \dot{\epsilon}_x = \frac{\dot{\sigma}_x}{E} + gh\sigma_x - h \int_{t_0}^t \sigma_x(\tau) \frac{dg(\tau)}{d\tau} d\tau$$

$$(7.6) \quad \ddot{\epsilon}_x = \frac{\ddot{\sigma}_x}{E} + \left[gh + \left(\frac{1}{E} \right) \right] \dot{\sigma}_x + g\dot{h}\sigma_x - \dot{h} \int_{t_0}^t \sigma_x(\tau) \frac{dg(\tau)}{d\tau} d\tau$$

we can see that the same integral stays in both equations. Hence, by its elimination, combining these equations, the integral equation (3.3) is transformed into the following differential equation:

$$(7.7) \quad \ddot{\epsilon}_x - \frac{\dot{h}}{h}\dot{\epsilon}_x = \frac{\ddot{\sigma}_x}{E} + \left[gh - \frac{\dot{h}}{Eh} + \left(\frac{1}{E} \right) \right] \dot{\sigma}_x$$

with initial conditions

$$(7.8) \quad \sigma_x(t_0) = E(t_0)\epsilon_x(t_0), \quad \dot{\epsilon}_x(t_0) = \frac{\dot{\sigma}_x(t_0)}{E(t_0)} + g(t_0)h(t_0)\sigma_x(t_0)$$

resulting from the eq. (3.3) and (7.5) for $t = t_0$.

The equations (7.7) with (7.8) and (7.1) with (7.2) are identical if the relations

$$(7.9) \quad \eta = \frac{1}{gh}, \quad E_v = -\eta \frac{\dot{h}}{h} - \dot{\eta}$$

apply. These relations are reciprocal to (7.4). The correspondence of model parameters $\eta(t)$ and $E_v(t)$ and functions $g(t)$, $h(t)$ is unique and therefore also the creep function (7.3) has the rheological model in time-variant Boltzmann's model.

The Maslov-Arutyunyan's creep law [21], [3], [24], [4], [23]:

$$(7.10) \quad C(t, \tau) = \frac{\varphi_\infty(\tau)}{E(\tau)} (1 - e^{-\gamma(t-\tau)})$$

is evidently a special case of (7.3) for

$$g(\tau) = \frac{\varphi_{\infty}(\tau)}{E(\tau)} e^{\gamma\tau}, \quad h(t) = 1 - e^{-\gamma t}.$$

In this creep function, the rate of creep, expressed by the coefficient γ , is considered as independent of t_0 . $\varphi_{\infty}(\tau)$ is the creep factor from instant τ of stress application to infinity ($\partial\varphi_{\infty}/\partial\tau < 0$, $\varphi_{\infty}(\infty) < \infty$ and > 0). The creep function (7.10) satisfies obviously the assumption of creep affinity (2.1) for differently aged concrete, $\kappa = \varphi_{\infty(\kappa)}(t_0)/\varphi_{\infty}(t_0)$.

By substitution in (7.7) and (7.8), we receive, with respect to (2.1), a differential equation between stress and strain corresponding to Maslov-Arutyunyan's creep law:

$$(7.11) \quad E\ddot{\varepsilon}_x + E\gamma\dot{\varepsilon}_x = \ddot{\sigma}_x + \gamma(1 + \kappa\varphi_{\infty} - \dot{E}/\gamma E)\dot{\sigma}_x$$

with initial conditions

$$(7.12) \quad \varepsilon_x(t_0) = \frac{\sigma_x(t_0)}{E(t_0)}, \quad \dot{\varepsilon}_x(t_0) = \frac{\dot{\sigma}_x(t_0) + \gamma\kappa\varphi_{\infty}(t_0)\sigma_x(t_0)}{E(t_0)}.$$

According to (7.9) and (7.4) the relations between $\varphi_{\infty}(t)$, γ and the parameters E_V , η of Boltzmann's model, which interprets the Maslov-Arutyunyan's creep law, are:

$$(7.13) \quad \eta(t) = \frac{E(t)}{\gamma\kappa\varphi_{\infty}(t)}, \quad E_V(t) = \frac{E(t)}{\kappa\varphi_{\infty}(t)} - \eta(t)$$

or

$$(7.14) \quad \kappa\varphi_{\infty}(\tau) = \frac{1}{1 + \eta(\tau)/E_V(\tau)}, \quad \gamma = \frac{E(\tau)}{\kappa\varphi_{\infty}(\tau)\eta(\tau)}.$$

The eq. (7.11) can be simplified by introducing a new independent variable $\xi = 1 - e^{-(t-t_0)}$ (a continuous still increasing function). Substituting

$$\frac{d}{dt} = \gamma(1 - \xi) \frac{d}{d\xi}, \quad \frac{d^2}{dt^2} = -\gamma^2(1 - \xi) \frac{d}{d\xi} + \gamma^2(1 - \xi)^2 \frac{d^2}{d\xi^2}$$

in (7.11), it is thus obtained after rearrangements:

$$(7.15) \quad E \frac{d^2 \varepsilon_x}{d\xi^2} = \frac{d^2 \sigma_x}{d\xi^2} + \left[\frac{\kappa\varphi_{\infty}}{1 - \xi} - \frac{dE/d\xi}{E} \right] \frac{d\sigma_x}{d\xi}.$$

The second initial condition (7.12) then is

$$(7.16) \quad E(t_0) \frac{d\varepsilon_x(t_0)}{d\xi} = \frac{d\sigma_x(t_0)}{d\xi} + \kappa\varphi_{\infty}(t_0)\sigma_x(t_0).$$

The equations (7.1), (7.7), (7.11), (7.15), as well as the initial conditions (7.2), (7.8), (7.12), (7.16), can also be written in operator form (3.2). Thus, for instance, for the eq. (7.11) and the second initial condition (7.12), the following creep operators apply:

$$(7.17) \quad E^{-1} = \frac{\frac{\partial^2}{\partial t^2} + \gamma(1 + \kappa\varphi_{\infty} - \dot{E}/\gamma E) \frac{\partial}{\partial t}}{E \frac{\partial^2}{\partial t^2} + \gamma E \frac{\partial}{\partial t}}, \quad E^{-1}(t_0) = \left[\frac{\frac{\partial}{\partial t} + \gamma\kappa\varphi_{\infty}}{E \frac{\partial}{\partial t}} \right]_{t=t_0}.$$

According to the model representation, we can make sure if the analytical expression chosen for $C(t, \tau)$ is basically possible or not. For instance, Arutyunyan [3] introduced $\varphi_{\infty}(\tau)/E(\tau) = C_0 + A/\tau$ where C_0 and A are constants. Substituting it in (7.13), it can be ascertained that η varies from 0 for $\tau = 0$ up to $1/\gamma C_0$ for $\tau \rightarrow \infty$ and E_V varies from the value $-1/A\gamma$ for $\tau = 0$ up to the value $1/C_0$ for $\tau \rightarrow \infty$; $E_V = (C_0\tau^2 + A\tau - A/\gamma)(C_0\tau + A)^{-2}$. Here, for very young concrete, exactly for $\tau < -A/2C_0 + \sqrt{[(A/4C_0 - 1/\gamma)A/C_0]}$, negative values of the spring constant E_V in Boltzmann's model are resulting, which is not physically possible, of course, and the foregoing expression for $\varphi_{\infty}(\tau)$ can be used only for older concrete. The same would also appear for $\varphi_{\infty}(\tau)/E(\tau) = C_0/(1 - e^{-b\tau})$, in which case η varies from 0 to η_{∞} , but E_V varies from $-1/C_0$ to $1/C_0$.

It is therefore necessary to use other analytical expressions for $\varphi_{\infty}(\tau)$, for which this discrepancy does not appear. For instance, for $\varphi_{\infty}(\tau) = C_0 + A/\tau^n$, where $n > 1$, η varies from 0 for $\tau = 0$ up to $1/\gamma C_0$ for $t \rightarrow \infty$ and E_V varies from 0 for $\tau = 0$ up to $1/C_0$ for $t \rightarrow \infty$; $E_V = (C_0 + A\tau^n - nA\tau^{n-1}/\gamma)(C_0 + A\tau^n)^{-2}$.

The mostly convenient expression is the exponential:

$$(7.18) \quad \frac{\varphi_{\infty}(\tau)}{E(\tau)} = C_0 + Ae^{-\gamma_1\tau}.$$

Then η varies from $1/\gamma(C_0 + A)$ for $\tau = 0$ up to $1/\gamma_1 C$ for $\tau \rightarrow \infty$ and E_V varies from the value $(C_0 + A - \gamma_1 A/\gamma)(C_0 + A)^{-2}$ for $\tau \rightarrow \infty$ which is positive if $\gamma_1 < \gamma(1 + C_0/A)$ up to the value $1/C_0$ for $\tau \rightarrow \infty$;

$$E_V = [C_0 + (1 - \gamma_1/\gamma) Ae^{-\gamma_1\tau}] [C_0 + Ae^{-\gamma_1\tau}]^{-2}.$$

However, from the standpoint of analytical solution, the Arutyunyan's function $\varphi_{\infty}(\tau)/E(\tau) = C_0 + A/\tau$ seems to be mostly convenient, because the relaxation function [3] and the problems of stress variation in statically indeterminate structures, as well as the deflections at column buckling (ref. in [8]), may be solved with help of the non-complete gamma-function, while in other cases, the obtained integrals are neither elementary nor special functions, except for (7.18) at column buckling [8], where also a non complete gamma function is obtained. Using (7.18) to solve the

Table 1. Integral $\int_0^u x^u e^x dx$

$u \backslash v$	0.00	0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00
0.00	0	0	0	0	0	0	0	0	0
0.25	0.284	0.158	0.096	0.059	0.037	0.030	0.017	0.010	0.006
0.50	0.649	0.443	0.320	0.235	0.176	0.140	0.103	0.079	0.061
0.75	1.117	0.860	0.691	0.565	0.471	0.403	0.339	0.289	0.249
1.00	1.718	1.449	1.292	1.133	1.000	0.907	0.780	0.735	0.718
1.25	2.490	2.245	2.112	1.979	1.873	1.807	1.709	1.694	1.708
1.50	3.482	3.319	3.277	3.241	3.241	3.291	3.319	3.440	3.602
1.75	4.755	4.757	4.902	5.077	5.316	5.636	5.970	6.252	6.991
2.00	6.389	6.671	7.142	7.700	8.389	9.236	10.19	11.38	12.78
2.25	8.487	9.206	10.20	11.40	12.86	14.64	16.72	19.27	22.33
2.50	11.18	12.55	14.36	16.56	19.27	22.60	26.61	31.56	37.59
2.75	14.64	16.96	19.97	23.71	28.37	34.20	41.38	50.37	61.55
3.00	19.09	22.75	27.51	33.53	41.17	50.87	63.10	78.67	98.43

relaxation function and the stresses in statically indeterminate structures, the integral $\int_0^u u^u e^x dx$ is obtained. To enable the use of (7.18), we have computed a numerical table of its values — table 1. We can demonstrate the appearance of this integral by determining the relaxation function corresponding to Arutyunyan-Maslov's law. Solving the differential equation (7.11) (first order differential equation for $d\sigma_x/dt$, by separation of variables), we obtain:

$$(7.19) \quad R_{(x)}(t, \tau) = \gamma \kappa \varphi_\infty(\tau) \int_t^E E(\vartheta) e^{-F(\vartheta)} d\vartheta$$

where

$$(7.20) \quad F(\vartheta) = \gamma \int_t^\vartheta [1 + \kappa \varphi_\infty(t)] dt.$$

Substituting the function (7.18) into (7.20) and introducing $\gamma_1 \doteq \gamma$, $E = \text{const.}$, we obtain the expression:

$$(7.21) \quad R_{(x)}(t, \tau) = E \kappa \varphi_\infty(\tau) e^{-x(t) - [1 + \kappa \varphi_\infty(\infty)] \int_{x(t)}^{x(\tau)} x^{\kappa \varphi_\infty(\infty)} e^x dx}$$

where $x(t) = \kappa \varphi_\infty(t) - \kappa \varphi_\infty(\infty)$. We note that, for $\gamma_1 \doteq \gamma$ in (7.18), the limit value $R_{(x)}(\infty, \tau)$ do not depend on γ , i.e. on the creep rate. This relaxation function is represented in Fig. 4.

The time-variant Maxwell's model (Dischinger-Whitney's law) as well as the time-invariant Boltzmann's model (ideal modulus) are special cases of time-variant Boltzmann's model for $E_V = 0$ ($g(\tau) = \text{const.}$) or for invariable parameters. However,

in general, this is not the case for the Arutyunyan-Maslov's creep law — only if using the function (7.18) in (7.10), we obtain the Dischinger-Whitney's law as a special case for $\varphi_\infty(\infty) = C_0 = 0$, $\gamma_1 \doteq \gamma(\varphi(t) = 1 - e^{-\gamma t})$ and the law of ideal modulus for $\varphi_\infty(\tau) = \varphi_\infty(\infty) = \text{const.}$ ($A_2 = 0$), $E = \text{const.}$ This is a further important advantage of the form (7.18).

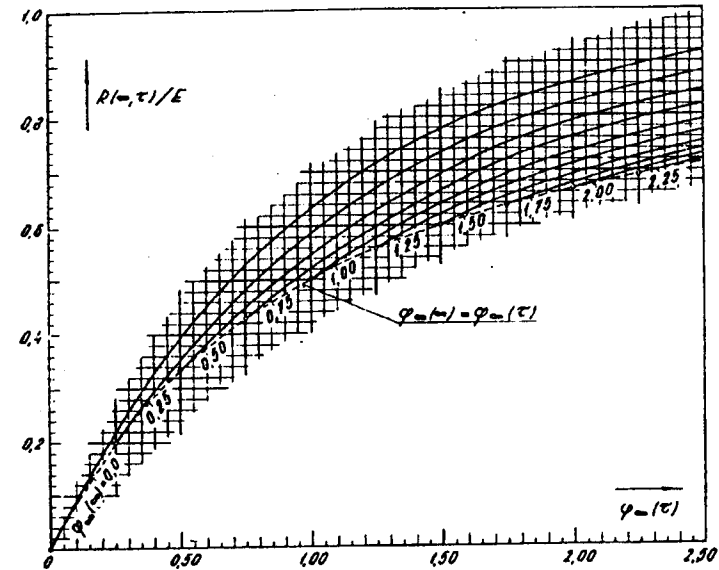


Fig. 4.

The relaxation function for concrete according to (7.21) has the form $R_{(x)}(t, \tau) = g_r(\tau) [h_r(t) - h_r(\tau)]$. Let us note that it can be also represented with elementary analytical functions for $g_r(t)$ or $h_r(t)$ which could be advantageous in certain problems. Similarly as we evolved the eq. (7.7) from (7.3), we can find, according to the principle of superposition, the differential equation for creep:

$$(7.21) \quad E \dot{\epsilon}_x - (g_r h_r + E \dot{h}_r / h_r + \dot{E}) \dot{\epsilon}_x = \ddot{\sigma}_x - \frac{\dot{h}_r}{h_r} \dot{\sigma}_x$$

with initial conditions for $t = t_0$:

$$(7.22) \quad \sigma_x(t_0) = E(t_0) \epsilon_x(t_0), \quad \dot{\sigma}_x(t_0) = E(t_0) \dot{\epsilon}_x(t_0) - g_r(t_0) h_r(t_0) \epsilon_x(t_0).$$

Comparing these equations with (7.1) and (7.2), the correspondence with the parameters of Boltzmann's model can be ascertained in the form:

$$(7.23) \quad \eta = \frac{E^2}{g_r h_r}, \quad E_V = -\dot{\eta} - E - \left(\frac{\dot{h}_r}{h_r} - \frac{\dot{E}}{E} \right) \eta$$

or reciprocally

$$(7.24) \quad g_r = \frac{E^2}{\eta} e^{-\int A dt}, \quad h_r = \int \left[\exp \int A dt \right] dt$$

where

$$A = -E/\eta - (E_V + \dot{\eta})/\eta + \dot{E}/E.$$

Instead of Boltzmann's model, it would be also possible to interpret the creep equivalently by a time-variant three-element model according to the Fig. 3e, for which the differential equation for creep can be evolved in the form:

$$(7.25) \quad E\dot{\epsilon}_x + \left\{ E_2 \frac{\dot{\eta}}{\eta} + \dot{E}_1 + \frac{E_1 E_2}{\eta} \left[1 + \left(\frac{\eta}{E_2} \right) \right] \right\} \dot{\epsilon}_x = \ddot{\sigma}_x + \frac{E_2}{\eta} \left[1 + \left(\frac{\eta}{E_2} \right) \right] \dot{\sigma}_x.$$

However, the correspondence between the model parameters and the function (7.3) or (7.10) leads to much more involved formulae and therefore this interpretation is not convenient.

The time-variant Maxwell's model gives an equation of first order without the term ϵ_x , whereas the time-variant Boltzmann's model gives yet an equation of second order. Thus it can arise a question if the general first order equation:

$$a_1(t) \dot{\epsilon}_x + a_0(t) \epsilon_x = b_1(t) \dot{\sigma}_x + b_0(t)$$

would not give a better approximation of creep of concrete than the time-variant Maxwell's model. Solving this equation for $\sigma_x = \text{const.}$, the corresponding creep function would be obtained in the form

$$C(t, \tau) = f_1(t) - f_1(\tau) + [1/E(t) - f_1(\tau)] f_2(\tau) \left[\frac{1}{f_2(t)} - \frac{1}{f_2(\tau)} \right]$$

As $C(\infty, \tau) < \infty$, it must be also $\lim_{t \rightarrow \infty} f_2(t) < \infty$ and for $t \rightarrow \infty$, $f_2(t)$ would tend to the function of the type $e^{\eta t}$. Then, however, $C(\infty, \tau)$ would have the same value for all τ which is not true for concrete. Therefore this approach is not possible. (The equation mentioned above is modelled by the Burgers's model with special interdependent time-variable parameters.)

MULTI-ELEMENT TIME-VARIANT LINEAR MODELS

Still a better approximation of creep of concrete can be attained by models consisting of more elements. Firstly this is the Burgers' model (Fig. 3c). The equations for its Maxwell's unit and for the spring and dashpot in its Voigt's unit are

$$\dot{\epsilon}_x - \dot{\epsilon}_{xV} = \dot{\sigma}_x/E + \sigma_x/\eta_M, \quad \dot{\sigma}_{xV} = E_V \dot{\epsilon}_{xV}, \quad \sigma_x - \sigma_{xV} = \eta_V \dot{\epsilon}_{xV}.$$

Eliminating σ_{xV} and ϵ_{xV} from these equations, we receive:

$$(8.1) \quad \dot{\epsilon}_x + \frac{E_V + \dot{\eta}_V}{\eta_V} \dot{\epsilon}_x = \frac{\ddot{\sigma}_x}{E} + \left[\frac{1}{\eta_V} + \frac{E_V + \dot{\eta}_V}{E\eta_V} + \left(\frac{1}{E} \right) + \frac{1}{\eta_M} \right] \dot{\sigma}_x + \left[\frac{E_V + \dot{\eta}_V}{\eta_M \eta_V} + \left(\frac{1}{\eta_M} \right) \right] \sigma_x$$

with initial conditions ($\sigma_{xV} = \epsilon_{xV} = 0$):

$$(8.2) \quad \sigma_x(t_0) = E(t_0) \epsilon_x(t_0), \quad \dot{\epsilon}_x(t_0) = \frac{\dot{\sigma}_x(t_0)}{E} + \left(\frac{1}{\eta_M(t_0)} + \frac{1}{\eta_V(t_0)} \right) \sigma_x(t_0).$$

Integrating (8.1) (or directly the equations for elements) for $\sigma_x = \text{const.}$, the creep function corresponding to this model is obtained in the form:

$$(8.3) \quad C(t, \tau) = f(t) - f(\tau) + g(\tau) [h(t) - h(\tau)]$$

where $f(t) = \int_0^t d\theta/\eta_M(\theta)$ or $\eta_M(t) = 1/f(t)$ and $g(t)$, $h(t)$ are the same as in (7.4) or (7.9) if η is interchanged with η_V . Similarly as for (7.7), we can solve the differential equation for creep on the basis of creep function. Approximately, it can be considered in this model that $g(\tau) \doteq \text{const.}$, in which case the last term of (8.3) has the time-variant form (6.2).

The five- and more-elements model (Fig. 3f) give differential creep equations of the third and higher order. The corresponding creep functions have obviously the form:

$$(8.5) \quad C(t, \tau) = \sum_{i=1}^n \psi_i(\tau) [1 - e^{-\tau(t-\tau)}]$$

or, according to (7.10), the special form (generalized Maxwell-Arutunyan's creep law [3]):

$$(8.6) \quad C(t, \tau) = \sum_{i=1}^n \psi_i(\tau) [1 - e^{-\tau(t-\tau)}].$$

The corresponding creep equations do not contain the terms with ϵ_x and, if $g_1 = \text{const.}$, nor the term with σ_x .

MULTIAXIAL STRESS

The foregoing models were considered only for an uniaxial state of stress. The creep function $C_1(t, \tau)$ for transverse strains can be assumed to be proportional to $C_1(t, \tau)$ [3], [4], i.e.

$$(9.1) \quad C_1(t, \tau) = \mu_1 C(t, \tau)$$

μ_1 — Poisson's ratio for creep. According to isotropy, the creep function for shear then is

$$(9.2) \quad C_2(t, \tau) = 2(1 + \mu_1) C(t, \tau)$$

and is also proportional to $C(t, \tau)$. Therefore the models for relation between ε_x and σ_x or γ_{xy} and τ_{xy} are the same as the foregoing models for the relation between ε_x and σ_x . The operators $\underline{\mu E}^{-1}$, \underline{G}^{-1} according to the eq. (3.10) have the same form as \underline{E}^{-1} with the only difference that E and the other model parameters E_ν , η_ν etc. must be interchanged with E/μ , E_ν/μ_1 , η/μ_1 , ... or with G , $\frac{1}{2}E_\nu/(1 + \mu_1)$, $\frac{1}{2}\eta/(1 + \mu_1)$, respectively. For instance, the differential operator, corresponding to the time-variant Boltzmann's model according to (7.1) is

$$(9.3) \quad \underline{G}^{-1} = \left\{ \frac{1}{G} \frac{\partial^2}{\partial t^2} + \left[\frac{2(1 + \mu_1)}{\eta} + \frac{E_\nu + \dot{\eta}}{G\eta} + \left(\frac{1}{G} \right) \right] \frac{\partial}{\partial t} \right\} \left[\frac{\partial^2}{\partial t^2} + \frac{E_\nu + \dot{\eta}}{\eta} \frac{\partial}{\partial t} \right]^{-1}$$

Expressing the operators with help of creep function, it is necessary to make in the operator \underline{E}^{-1} the same replacement by which $C(t, \tau)$ interchanges to $C_1(t, \tau)$ or to $C_2(t, \tau)$. For the Dischinger-Whitney's theory, e.g.,

$$C_1(t, \tau) = \frac{\mu_1}{E_0} [\varphi(t) - \varphi(\tau)], \quad C_2(t, \tau) = \frac{1 + \mu_1}{1 + \mu} \frac{1}{G_0} [\varphi(t) - \varphi(\tau)],$$

hence, according to (5.3), it results:

$$(9.4) \quad \underline{\mu E}^{-1} = \frac{\frac{\mu}{E} \frac{\partial}{\partial \varphi} + \frac{\mu_1 \kappa}{E_0}}{\frac{\partial}{\partial \varphi}}, \quad \underline{G}^{-1} = \frac{\frac{1}{G} \frac{\partial}{\partial \varphi} + \frac{1 + \mu_1}{1 + \mu} \frac{1}{G_0}}{\frac{\partial}{\partial \varphi}}$$

For the Arutyunyan-Maslov's creep law, according to (7.17), it follows:

$$(9.5) \quad \underline{\mu E}^{-1} = \frac{\mu \frac{\partial^2}{\partial t^2} + \gamma \left(\mu + \mu_1 \varphi_\infty - \mu \frac{E}{E} \right) \frac{\partial}{\partial t}}{E \frac{\partial^2}{\partial t^2} + E\gamma \frac{\partial}{\partial t}},$$

$$\underline{G}^{-1} = \frac{\frac{\partial^2}{\partial t^2} + \gamma \left[1 + 2(1 + \mu_1) \varphi_\infty - \frac{G}{G} \right] \frac{\partial}{\partial t}}{\frac{\partial^2}{\partial t^2} + \gamma \frac{\partial}{\partial t}},$$

etc. For simplification of analysis, it can be approximately assumed that $\underline{\mu} = \mu_1 = \mu$ in which case it holds that $\underline{\mu E}^{-1} = \underline{\mu E}^{-1}$, $\underline{G}^{-1} = 2(1 + \mu) \underline{E}^{-1}$.

It can be easily shown that, for linear Volterra's integral operators, as well as for the differential operators of type $f(t) \partial/\partial t$, the same principal rules as in algebra of numbers are valid (i.e. the associative and commutative rules for addition and multiplication and the distributive rule for multiplication). Let us note that this is yet a well known statement for time-invariant ($f(t) = \text{const.}$) differential operators and is derived in operational calculus, e.g. on the basis of Laplace transform [27]. (Division by this operator is introduced as an inverse operation to multiplication.)

With respect to this fact, the ageing derivation of equations for certain problem with creep at presence of ageing, in general, e.g. that of statically indeterminate structure, of beam bending, etc., would formally be the same as in the theory of elasticity, because the equations (3.10) are formally analogous to the Hook's law. Therefore this derivation needs not to be repeated for the case of creep and we can obtain the equations for creep only by replacing the elastic constants E , μ , G by the creep operators \underline{E} , $\underline{\mu}$, \underline{G} . Applying these operators, one must take care that they must be applied to quantities of the same type as in the creep law, for instance that the integral operator \underline{E}^{-1} must be applied only to forces, stresses, moments or their parameters, whereas the integral operator \underline{E} must be applied only to deformations, strains, displacements, rotations, etc. The derivatives of the numerator of differential operator \underline{E}^{-1} must be applied only to forces, moments, etc. and the derivatives of its nominator must be applied only to displacements, rotations, etc. Moreover, one must take care if, in the derivation in the theory of elasticity, the elastic constants were not eliminated as e.g. in the biharmonic equation for the stress function in plane problem in which case the introduction of creep operators is not possible, of course. (This is a consequence of the fact that completely all the rules of algebra are not valid, e.g. that $[(\partial/\partial t)/(\partial/\partial t)]y$ does not equal to y but more generally to $y + \text{const.}$ etc.)

To demonstrate the use of this analogy of creep with elasticity, let us derive the equations for redundant quantities X_i in n -times statically indeterminate nonhomogeneous structure. The elastic solution is given by the system of n algebraic equations

$$\sum_{j=1}^n \delta_{ij} X_j + \delta_{oi} + \delta_i = 0,$$

in which δ_i are the enforced deformations in sense of X_i on the primary system (shrinkage, temperature changes, movements of supports, prestressing) and

$$\delta_{kj} = \int S^{(k)} S^{(j)} \frac{dx}{EJ}$$

are the flexibility and load coefficients. Here $S^{(k)}$, for $k = 1, \dots, n$, or $S^{(0)}$ denotes the internal forces (bending and torsional moments, normal and shear forces) on the primary system caused by $X_k = 1$ or by prescribed external loads; EJ are the cor-

onding rigidities, x the length of bars. Replacing now $1/E$ by the operator \underline{E}^{-1} , equations for creep in operator form are:

$$1) \quad \sum_{j=1}^n X_j \int \underline{E} \underline{E}^{-1} S^{(j)} S^{(j)} \frac{dx}{EJ} + \int \underline{E} \underline{E}^{-1} S^{(0)} S^{(j)} \frac{dx}{EJ} + \delta_i = 0.$$

general they represent a system of Volterra's integral equations which is convenient a numerical solution. Using the Dischinger-Whitney's creep law, eq. (5.3), receive a system of n ordinary linear differential equations of first order [4], [6]:

$$2) \quad \sum_{j=1}^n \delta_{ij} \frac{dX_j}{d\varphi} + \sum_{j=1}^n \delta'_{ij} X_j + \delta'_{0i} + \delta_{0i,\varphi} + \frac{d\delta_i}{d\varphi} = 0 \quad (i = 1, \dots, n)$$

which δ'_{ij} and δ'_{0i} are the flexibility and load coefficients for the structure with duli $E' = E_0/x$ instead of $E(t)$ and

$$\delta_{0i,\varphi} = \int \frac{dS^{(0)}}{d\varphi} S^{(j)} \frac{dx}{EJ}.$$

ime variation of $E(t)$ is neglected, this system has constant coefficients and $\delta_{0i,\varphi} = d/d\varphi \delta_{0i}$. Introducing the creep operator according to the equations (7.16) and (7) for Arutyunyan-Maslov's creep law, we obtain a system of ordinary differential equations of first order for $dX_i/d\xi$:

$$3) \quad \sum_{j=1}^n \delta_{ij} \frac{d^2 X_j}{d\xi^2} + \sum_{j=1}^n \delta^*_{ij} \frac{dX_j}{d\xi} + \delta_{0i,\xi} + \delta^*_{0i,\xi} + \frac{d^2 \delta_i}{d\xi^2} = 0 \quad (i = 1, \dots, n)$$

h initial conditions:

$$4) \quad \sum_{j=1}^n \delta_{ij} \frac{dX_j(t_0)}{d\xi} + \sum_{j=1}^n \delta^*_{ij} X_j(t_0) + \delta_{0i,\xi}(t_0) + \delta^*_{0i,\xi}(t_0) + \frac{d\delta_i(t_0)}{d\xi} = 0.$$

he δ^*_{ij} and δ^*_{0i} are the flexibility and load coefficients for a structure with moduli

$$E^* = E \left[\frac{x\varphi_\infty}{1-\xi} - \frac{dE/d\xi}{E} \right]^{-1}$$

ead of $E(t)$; δ^*_{ij} , δ^*_{0i} are the flexibility and load coefficients for a structure with duli $E x \varphi_\infty(t_0)$ and

$$\delta_{0i,\xi} = \int \frac{d^2 S^{(0)}}{d\xi^2} S^{(j)} \frac{dx}{EJ}, \quad \delta_{0i,\xi} = \int \frac{dS^{(0)}}{d\xi} S^{(j)} \frac{dx}{EJ}, \quad \delta^*_{0i,\xi} = \int \frac{dS^{(0)}}{d\xi} S^{(j)} \frac{dx}{E^* J}.$$

urther, let us consider a homogeneous body ($x \equiv 1$) with $\mu = \mu_1 \equiv \mu$ which subjected to enforced deformations which would produce the stresses $\sigma_{xx}(t)$ if the

body would be elastic. Owing to the fact that the static, compatibility and geometric equations are the same as for elastic case, we obtain directly by comparison of the equations (3.10) for creep with the Hooke's law:

$$(10.5) \quad \underline{E} \underline{E}^{-1} \sigma_x = \sigma_{xx}, \quad \underline{E} \underline{E}^{-1} \tau_{xy} = \tau_{xy}, \dots$$

Finally, let us consider the bending of steel-reinforced concrete bar (column) with a symmetric section. For elastic small deflections y it holds that $(EJ + E_s J_s) y'' = M$ where y'' is the curvature, M the total bending moment (assumed that plane sections remain plane), EJ is the rigidity of the concrete part of cross section, $E_s J_s$ is the rigidity corresponding to reinforcement. Dividing this equation by E and interchanging $1/E$ with \underline{E}^{-1} , we obtain $EJ y'' = \underline{E} \underline{E}^{-1} (M - E_s J_s y'')$. Hence, for Dischinger-Whitney's law (eq. (5.3)), we have then [8]:

$$(10.6) \quad (EJ + E_s J_s) \frac{\partial y''}{\partial \varphi} = \frac{\partial M}{\partial \varphi} + \frac{x E}{E_0} (M - E_s J_s y'')$$

for Arutyunyan-Maslov's law (eq. (7.17)), we receive [8]:

$$(10.7) \quad (EJ + E_s J_s) \frac{\partial^2 y''}{\partial t^2} + \gamma \left[EJ + E_s J_s \left(1 + x\varphi_\infty - \frac{\dot{E}}{\gamma E} \right) \right] \frac{\partial y''}{\partial t} = \\ = \frac{\partial^2 M}{\partial t^2} + \gamma \left(1 + x\varphi_\infty - \frac{\dot{E}}{\gamma E} \right) \frac{\partial M}{\partial t}$$

with the initial condition:

$$(10.8) \quad [E(t_0) J + E_s J_s] \frac{\partial y''(t_0)}{\partial t} + \gamma x \varphi_\infty(t_0) E_s J_s y''(t_0) = \frac{\partial M(t_0)}{\partial t} + \gamma x \varphi_\infty(t_0) M(t_0).$$

Starting from these equations, it is possible to solve, e.g., the buckling of columns [8].

Analogically, we could have derived the differential equations for bending of beams with nonsymmetric section, for composite beams, for beams on elastic or viscoelastic foundation, for bending of plates, etc.

We see that, by use of analogy of creep with elasticity, the derivation of pertinent equations is essentially shortened. The detailed long derivation procedures which have appeared hitherto in literature are needless, since they are, in fact, nothing else than a repetition of their derivation in the theory of elasticity.

The presented analogy represents properly a generalization of the well-known elastic-viscoelastic analogy [13], [22], [32] used in time-invariant theory of viscoelasticity which is usually formulated with help of Laplace transform. However, the use of this transform for a time-variant creep is not advantageous, because the derivatives in creep operators are not multiplied by constants, but by time functions.

The only case, in which the Laplace transform is useful for concrete creep problems, is in case of Dischinger-Whitney's law if E is taken approximately as constant, since the equations (5.2) as well as (10.2), (10.6) have then constant coefficients. To obtain the relation between transforms of forces and deformations, it is then necessary to replace the elastic constants E , μ , G by (see [4], particularly for E [12]):

$$(10.9) \quad E = \frac{Ep}{p + \alpha}, \quad \bar{E} = \frac{E\bar{p}}{\mu p + \mu_1 \alpha}, \quad G = \frac{Gp}{p + \frac{1 + \mu_1}{1 + \mu} \alpha}$$

where p is the parameter of Laplace transform:

$$f(p) = \int_0^{\infty} f(\varphi) e^{-p\varphi} d\varphi.$$

For non-zero initial conditions in the obtained equation in each term with p , it is still necessary to subtract the initial values.

11. NONLINEAR EFFECTS AT REVERSIBLE PROCESSES

Even for stresses smaller than about 0.5 of compressive strength or 0.9 of tensile strength, the principle of superposition is not valid after great release of stress, because according to it the strain recovery would result much greater than it is in reality.

To interpret simply this phenomenon, the author suggested (originally for glass reinforced plastics) a model in Fig. 3d using a new nonlinear rheologic element — ratchet pawl — which enables free deformation in one direction, but prevents deformation in the opposite directions. (Let us note that the ratchet pawl could also be used to interpret the irreversibility of short-time deformations.) As long as the creep deformation is increasing, i.e. as far as the condition

$$(11.1) \quad \left| \dot{\epsilon}_x - \frac{\dot{\sigma}_x}{E} \right| \geq 0$$

is fulfilled, the model in Fig. 3d behaves still as a Boltzmann's model and the equations of paragraph 7 hold true. If the condition (11.1), beginning with time t_b , is not valid, the ratchet pawl gets closed and the model behaves again as a Boltzmann's model but with other values of mechanical parameters $E_{\nu b}$ and γ_b instead of E_{ν} and γ . The differential equations (7.1), (7.11) for creep have then the same form, only φ_{∞} and ϵ_x , σ_x is in them interchanged with $\varphi_{\infty b}$, γ_b and ϵ_{xb} , ϵ_{xb} , $\sigma_x - \sigma_{xb}$ in turn.

As an example, let us determine the strain in case of constant stress σ_{xa} applied at time t_a which in t_b is reduced to a smaller constant value σ_{xb} . Considering the

Arutyunyan-Maslov's creep law (7.10), the principle of superposition gives for $t \geq t_b$:

$$(11.2) \quad \epsilon_x(t) = \frac{\sigma_{xa}}{E(t_a)} [1 + \varphi_{\infty}(t_a)(1 - e^{-\gamma(t-t_a)})] - \frac{\sigma_{xa} - \sigma_{xb}}{E(t_b)} [1 + \varphi_{\infty}(t_b)(1 - e^{-\gamma(t-t_b)})].$$

The condition (11.1) requires that

$$(11.3) \quad \sigma_{xb} \geq \sigma_{xa} \left[1 - \frac{\varphi_{\infty}(t_b)}{\varphi_{\infty}(t_a)} e^{-\gamma(t_b-t_a)} \right] = \sigma_{xa}.$$

If the release of stress is too large, $\sigma_{xb} < \sigma_{xa}$, then for the stress $\sigma_{xb} - \sigma_{xc}$ the model with changed coefficients applies. Hence,

$$(11.4) \quad \epsilon_x(t) = \sigma_{xc} - \frac{\sigma_{xc} - \sigma_{xb}}{E(t_b)} [1 + \varphi_{\infty b}(t_b)(1 - e^{-\gamma_b(t-t_b)})]$$

and the strain is obviously less reversible than according to (11.2).

The mathematical advantage of this model lies in the greatest possible simplicity. For major cases of steady loaded structures, the linear theory is still valid, inclusive of the stress relaxation at constant stress (as in it the creep strain $\epsilon_x - \sigma_x/E$ is constantly growing). At the same time, we have thus exact limits for the applicability of linear theory.

For a comparison let us quote another similar approach suggested by Gvozdev [17]. He divides the elastic deformation in two parts: the part $\sigma_{x1}(t)/E(t)$ which is constantly increasing in time, $(\partial/\partial t)(\sigma_{x1}/E) \geq 0$, and the part $\sigma_{x2}(t)/E(t)$ which is constantly decreasing, $(\partial/\partial t)(\sigma_{x2}/E) \leq 0$; $\sigma_x(t)/E(t) = \sigma_{x1}(t)/E(t) + \sigma_{x2}(t)/E(t)$.

For each part he considers a different memory function L_1 or L_2 , $|L_1| < |L_2|$ and instead of (3.3), he introduces

$$(11.5) \quad \epsilon_x(t) = \frac{\sigma_x(t)}{E(t)} - \int_{t_0}^t \frac{\sigma_{x1}(\tau)}{E(\tau)} L_1(t, \tau) d\tau + \int_{t_0}^t \frac{\sigma_{x2}(\tau)}{E(\tau)} L_2(t, \tau) d\tau.$$

This approach seems to be useful also more generally for a repeated and cyclic loading and can interpret the fact (vibrocreep) that at cyclic loading, the creep can be greater than for a constant load equal to maximums of cyclic load (although, according to the principle of superposition, it should be always smaller).

12. REMARKS ON NONLINEAR EFFECTS AT HIGH COMPRESSIVE STRESSES

If the compressive stresses are greater than about 0.5 of strength, the relative creep is greater and quicker and the creep function depends on σ_x , that is $C = C(t, \tau, \sigma_x)$. At the same time, also the short-time deformations become nonlinear, $E = E(t, \sigma_x)$. Here it is possible to introduce $E(t, \sigma_x) = E(t)f(\sigma_x)$ and [31] $C(t, \tau, \sigma_x) = F(\sigma_x) \cdot C(t, \tau)$ or better [31]

$$(12.1) \quad C(t, \tau, \sigma_x) = C(t, \tau) + F(\sigma_x) C_n(t, \tau)$$

where $C_n(t, \tau)$ interprets the increased creep values and rates. Special forms of these functions can be introduced according to the models mentioned above, the elements of which are considered as nonlinear. For the time-variant Maxwell's model (generalization of Dischinger-Whitney's law), it may be introduced [4], [30]

$$(12.2) \quad \frac{\partial \epsilon_x}{\partial \varphi} = \frac{f(\sigma_x)}{E(t)} \frac{\partial \sigma_x}{\partial \varphi} + \frac{F(\sigma_x)}{E_0}$$

Neglecting the nonlinearity of elastic deformations ($f(\sigma_x) \equiv 1$), the integration in certain simple problems is easy. More involved is the solution for nonlinear Arutyunyan-Maslov's creep law [3].

13. INFLUENCE OF DIMENSIONS

There has been observed (e.g. [24], [18]) that the creep of concrete depends not only on the value of stress, but also on the thickness and form of the specimen. The creep of more thick elements is smaller and slower (about 3.5-times for thicknesses 100 cm and 5 cm [30]), the creep of plates is smaller (about 1.7-times [24]) than the creep of equally thick square prisms. This phenomenon is probably caused by drying of concrete, by loss of water [2].

In the creep law, the following interpretation of this fact is possible. The creep must depend not only on time t and on age τ , but also on the position of the respective point, i.e. on space coordinates, and on the geometry of surface. Then the creep of each structure is in fact nonhomogeneous. For instance, the creep in the middle of plate must be smaller than near the surface, in consequence of which, e.g., for axial compression the stress distribution becomes nonlinear, the stresses in the middle become greater and near the surface smaller. (For this reason also, the experimental determination of stresses according to measured strains in long-time loaded structures is complicated and so far rather problematic.)

In the above theory, the influence of dimensions is involved only integrally for the entire cross section at bending and compression of bars and plates and gives correctly only the total internal forces in sections. For determining the actual stress distribution in the sections, the above theory is rather approximative and, rigorously, it will be necessary to set up a time-space theory of concrete creep.

14. CONCLUSIONS

At present time, only the linear theory of creep is sufficiently developed for wide use in engineering analysis. The time-variant Maxwell's model (Dischinger-Whitney's law) is the most simple one and in many cases, it can sufficiently describe the actual behaviour. However, the time-variant Boltzmann's model (Arutyunyan-Maslov's law) should be used for simpler problems, at which the mathematical solution is still not too much complicated (e.g. stress relaxation in homogeneous structures, losses of prestressing, etc.) and for problems rather sensitive to creep (column buckling, suspension concrete bridges). Most accurate, of course, is to introduce the actual measured creep function according to the principle of superposition, in which case a numerical solution of the resulting Volterra's integral equations must be used (which is necessary, e.g., for computing the stress from a history of measured strain).

Further research is needed especially for the influence of dimensions, for moisture and drying effects, for nonlinear effects and to determine also all the experimental data.

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ЗАКОН ПОЛУЗУЧЕСТИ, ОСНОВАННЫЙ В ТЕОРИИ ВЯЗКОУПРУГОСТИ НА РАЗНЫХ РЕОЛОГИЧЕСКИХ МОДЕЛЯХ

Основной характеристикой ползучести бетона, отличающей его от других строительных материалов, является влияние старения вследствие чего закон ползучести вариантный во времени. Но законы ползучести, основанные в теории вязкоупругости на разных реологических

моделях в времени инвариантны и поэтому они не применимы для бетона. Насколько тоже использованы для бетона, старением бетона или пренебрежностью или использованием формальной ползучести — в дифференциальном законе ползучести отдельные постоянные функции за менялись функциями времени, что не правильно.

В настоящей работе показано, что правильно переменность механических параметров учесть уже в отдельных элементах модели (т. е. переменность в времени постоянных в пружин и вязкости демферов. Потом пружина, например, определяется уравнением $\dot{\epsilon} = \sigma_x / E_x$). Вообще каждый линейный закон ползучести для которого принцип $\dot{\epsilon}$ во времени имеет место (выследственная теория старения) может быть изображен составленной из пружин и демферов. Заметим, что эквивалентно для модели можно принимать электрическую цепь составленную из сопротивлений, индуктивностей и е переменных в времени.

В работе показано, что приближенный закон Динингера-Уитней (теория с изображен моделью Максвелла, Фиг. 4а, и закон Арутюняна-Маслова, моделью Бол Фиг. 4б. На основе моделей дискутированы полезные и физикально допустимые формы в законе ползучести, при чем например показано, что выражение $\dot{\epsilon} = C$ завышено принимаемое в функции Арутюняна-Маслова (7.14) дает невозможные результаты для очень молодого бетона. Поэтому лучше принимать выражение (7.18) для которого боте табулирована (Таб. 1) интеграл приходящий в релаксационных задачах и построения соответствующих релаксационных функций. Приведены тоже более точные формы ползучести основанные на моделях из трех или более элементов. Они позволяют анализ решения при помощи сведения интегрального уравнения Вольтерра к дифференциальному уравнению.

Далее приведено несколько упрощающих гипотез пригодных для ползучести без редуцирования времени по условиям окружающей среды (влажность) и аффинности кривых части для неоднородной конструкции — и формулированы уравнения для трех напряженного состояния.

Как обобщение известной упругой — вязкоупругой аналогии для ползучести инвариантно во времени выведения аналогия между ползучестью вариантной во времени и упругой. Она позволяет определять уравнение для задачи при ползучести прямо из уравнения в теории упругости только заменой упругих постоянных дифференциальными (или интегральными) операторами для ползучести. Это показано на выведении уравнений для неоднородно статически неопределенной рамы, для общего однородного тела и для отгиба сложной при различных законах ползучести. Примененные изображения Лапласа былого только ползучести по модели Максвелла.

Для нелинейных эффектов в ползучести бетона при малых напряжениях предложена численная модель с собакой изображенная в фиг. 4д. Собака заметает только если долгой часть деформация начинает уменьшаться, вследствие чего эта модель может изображать факт, что обратимость ползучести при разгрузке значительно меньше чем требовало по принципу наложения во времени.

Наконец замечено что в будущем — уважая очень сильное влияние влажностных условий и размеров сечений — необходимо построить пространственно-временную теорию ползучести бетона, уважающую эффекты высыхания. Только тогда будет возможно определить настоящее распределение напряжений в сечениях во времени.

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