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Wölbkrafttorsion dünnwandiger Stäbe mit veränderlichem Querschnitt

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Introduction

Thin walled-bars constitute a highly efficient type of structural element. They are frequently subjected not only to tension, compression and bending, but also to torsion. An engineer's theory of torsion of thin walled bars with constant cross section has already been presented [3], [7]. However, the thin-walled bars, used especially in bridge and aircraft structures, are usually of variable cross section, i. e., of section with variable form and thickness of walls.

In order to satisfy the needs of the aircraft industry, attention has been, until recently, paid mainly to tapered and conical bars and shells used as wings. Their exact, as well as an engineer's solution has been given in papers [1, 7, 5], and others (for a detailed survey see [3]).

On the other hand, CYWIŃSKI [2] recently presented an engineer's solution of torsion of monosymmetrical thin-walled bars having a variable open cross section, which are cut out from a cylindrical or prismatical surface. The bars considered by him were of constant thickness and were loaded only by transverse moments. He also verified the solution experimentally.

In this paper an attempt will be made to derive the solution of non-uniform torsion for monosymmetrical closed profiles (Fig. 1b) and, at the same time, to extend CYWIŃSKI's solution of open profiles (Fig. 1a) for a general loading and, approximately, to a general variable monosymmetrical section and also to show other methods for deriving the equations. The analysis will be effected analogically to the VLASOV's theory of thin-walled bars with constant open section and UMANSKIY's theory relating to closed sections.

Assumptions

1. The walls of thin-walled bars are thin with respect to the dimensions of the cross section (greater than $\sim 1:10$). The dimensions of the section are small with respect to the length of bar as well as with respect to the radius of curvature of the bar (greater than $\sim 1:7$).

2. The material is linearly elastic.

3. The deformations are small.

For the engineer's solution the following deformation hypotheses will be adopted:

4. The form of the cross sections of a bar is indeformable (absolutely rigid in their plane) which is attained, e.g., by a sufficient number of cross walls (diaphragms).

5. The variation of the cross section along the bar is continuous and gradual. The distribution of the longitudinal normal strains and stresses in the section can be regarded as similar to the distribution for a bar with constant section.

6. For the deformations caused by bending moments and axial loads the Navier-Bernoulli hypothesis of maintaining the planarity and the perpendicularity of the cross sections is admissible.

7 a) In the case of the torsion of open profiles the shear strain in the central surface of the walls is zero (Wagner's assumption of a shear-resistant skim).

b) In closed profiles the non-uniform torsion causes a deplanation (warping) of the primarily plane cross sections which is proportional to the deplanation occurring in the case of pure (simple, Bredt's) torsion (Umanskiy's assumption).

Assumptions 4, 6 and 7 are the same as for bars with constant section where their admissibility has already been verified (e.g., [7]). With respect to the introduction of these approximate deformation hypotheses, the compatibility equations will not be fulfilled.

For the analysis we shall introduce the cartesian left-handed coordinate system x, y, z according to Fig. 2. The z axis is the chosen longitudinal axis of the bar. It is advantageous to choose this axis in such a way that it repre-

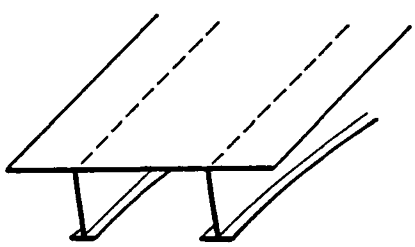

Fig. 1 a).

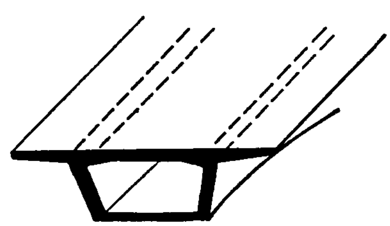

Fig. 1 b).

sents the average position of the shear centres of the sections. The z coordinate is the length of the bar. The plane (z, y) is the plane of symmetry of the bar. The cross sections are perpendicular to the z axis. Let us denote by $\bar{x}$ and $\bar{y}$ their principal centroidal axes of inertia. With respect to symmetry the axis $\bar{y}$ lies in the plane (y, z) and $\bar{x} \| x$. Let the position of the shear centre A of the section be determined by the coordinate $y = \alpha_y\,(x = 0)$. The line connecting all the shear centres will be termed the shear centre line. Generally it is neither straight, nor parallel with z. The same may be said regarding the gravity centre line.

Let us denote by ξ, η, ζ the displacement of the point of intersection of the z axis with the section which is caused by the loading in the directions x, y and z, and by ϑ the rotation of the section about the z axis (Fig. 2). Let the

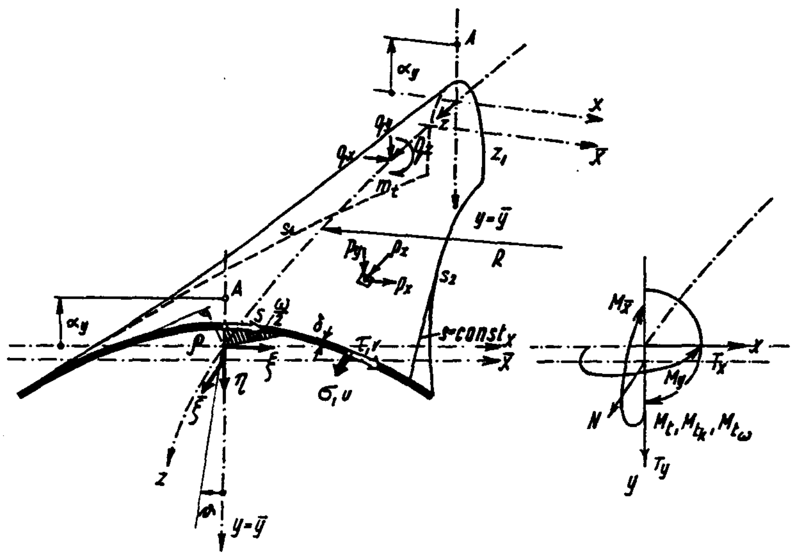

Fig. 2.

longitudinal displacement of the centre of gravity be denoted by $\bar{\zeta}$. For the displacement of a point on the central surface of the walls, caused by loading in the direction of the z axis or the directions of the centre line of the section (Fig. 2), the notation u and v will be used.

Thin-walled Bars of Open Profile

In each cross section we shall introduce, according to VLASOV [6], the sectorial coordinate $\omega(s) = \int_0^s \rho(s)\,ds$ where ρ is the distance of the tangent of the centre line of the section from the pole lying on the z axis and s is the length of the centre line from the zero point which lies in the intersection of the $\bar{y}$ axis with the wall: ω represents the doubled area of the sector formed by the arc $(0, s)$ to the pole. This definition of ω may also be extend to a section in which the wall bifurcates into several branches.

With the aid of the coordinates x, y and ω in each section the following expressions are defined

$$\begin{aligned} J_x &= \int_F y^2\,dF, \qquad J_y = \int_F x^2\,dF, \qquad J_{xy} = \int_F x\,y\,dF, \\ S_y &= \int_F x\,dF, \qquad S_x = \int_F y\,dF, \end{aligned} \tag{1}$$

where F is the area of the walls in section, $dF = \delta\,ds$, δ is the (variable) thickness of the wall, J_x, J_y are the moments of inertia. For the principal coordinates x, y: $J_{\bar{x}\bar{y}} = S_{\bar{y}} = S_{\bar{x}} = 0$.

Furthermore, we introduce

$$\begin{aligned} J_\omega &= \int_F \omega^2\,dF, \qquad J_{\omega x} = \int_F \omega\,x\,dF, \\ J_{\omega y} &= \int_F \omega\,y\,dF, \qquad S_\omega = \int_F \omega\,dF, \end{aligned} \tag{2}$$

where J_ω is termed the sectorial moment of inertia (or bimoment of inertia). With a view to symmetry $S_\omega = J_{\omega y} = 0$. The principal sectorial coordinate $\bar{\omega}(s) = \int_0^s \bar{\rho}(s)\,ds$ is the sectorial coordinate for such a principal pole A, for which $J_{\bar{\omega}\bar{x}} = 0$ $(J_{\bar{\omega}\bar{y}} = S_{\bar{\omega}} = 0)$. This condition determines the position of the pole A which is

$$y = \alpha_y = -\frac{J_{\omega\bar{x}}}{J_{\bar{y}}} = -\frac{J_{\omega x}}{J_y}, \qquad x = 0. \tag{3}$$

For the principal coordinate $\bar{\omega}$ we have

$$\bar{\omega} = \omega + \alpha_y x, \qquad J_{\bar{\omega}} = J_\omega + \alpha_y J_{\omega x}. \tag{4}$$

The solution for the given loading and for the given boundary conditions can be found variationally in the form of an infinite series

$$u = \sum_{i=1}^{\infty} U_i(z) f_i(s), \qquad v = \sum_{i=1}^{\infty} V_i(z) g_i(s), \tag{5}$$

in which $f_1(s), f_2(s), \ldots$ and $g_1(s), g_2(s), \ldots$ are certain chosen systems of linearly independent functions and the functions $U_1(z), U_2(z), \ldots$ $V_1(z)$, $V_2(z), \ldots$ are to be determined. In accordance with the approximate assumptions 4, 5, 6, and 7 for the engineer's solution, our further analysis will be restricted to the first four terms for u and to the first three terms for v. From these assumptions it follows that

$$u = \bar{\zeta} - \xi'\,\bar{x} - \eta'\,\bar{y} - \vartheta'\,\omega, \tag{6}$$

$$v = \xi\cos(x, s) + \eta\sin(x, s) + \vartheta\,\rho, \tag{7}$$

where (x, s) is the angle between the x axis and the tangent of the centre line of the section; $\eta', \ldots$ here denotes, as well as in the following relations the derivatives with respect to z. Eq. (7) is an obvious consequence of assumption 4. Eq. (6) results from assumption 7a. stating that the shear strain $\gamma = \partial u / \partial s + \partial v / \partial z = 0$. Substituting here the Eq. (7) and carrying out the integration, Eq. (6) is obtained.

Differentiating the Eq. (6) with respect to z and neglecting, in accordance with the assumption 5a, the derivatives $\partial x / \partial z$, $\partial y / \partial z$, $\partial \omega / \partial z$ along the longitudinal fibres s = const., we obtain

$$\epsilon = \bar{\zeta}' - \xi'' \bar{x} - \eta'' \bar{y} - \vartheta'' \omega . \tag{8}$$

Here $\epsilon = \partial u / \partial z$ is the normal strain perpendicular to the section.

The simplest derivation of the equations for the unknown functions $\xi(z)$, $\eta(z)$, $\bar{\zeta}(z)$, $\vartheta(z)$ can be effected by a variational method with the aid of the Euler's equation for the minimum value of the potential energy of the bar. This energy is equal to the sum of the potential energy of the internal forces (strain energy) and of the potential energy of external forces. The specific strain energy per length z in cross section is:

$$W = \int_F \tfrac{1}{2} E \epsilon^2 dF + \int_F \tfrac{1}{2} G \gamma^2 dF , \tag{9}$$

where E is the tensile (Young's) modulus and G the shear modulus of elasticity. According to assumptions 6 and 7a the second term for open profiles corresponds only to a non-uniform shear strain distribution throughout the thickness of the wall which is caused solely by pure torsion. Thus it assumes the value $\frac{1}{2} G J_k \vartheta'^2$ where $J_k = \frac{1}{3} \int_F \delta^3 ds$ denotes the moment of inertia in pure (simple) torsion.

The first term in Eq. (9) is determined by substituting Eq. (8) and performing the integration. With respect to the Eqs. (1) and (2), here is obtained

$$W = \tfrac{1}{2} (E F \bar{\zeta}'^2 + E J_y \xi''^2 + E J_{\bar{x}} \eta''^2 + E J_\omega \vartheta''^2 + 2 E J_{\omega x} \xi'' \vartheta'' + G J_k \vartheta'^2) . \tag{10}$$

The potential energy of the whole bar is (taking into consideration Eq. (6))

$$\begin{aligned} U = \int_{z_1}^{z_2} \{ W - q_x \xi - q_y \eta - m_t \vartheta - \int_{s_1}^{s_2} p_z (\bar{\zeta} - \xi' x \\ - \eta' \bar{y} - \vartheta' \omega) - [q_z (\bar{\zeta} - \xi' x - \eta' \bar{y} + \vartheta' \omega)]_{s_1}^{s_2} \} dz \\ - [N \bar{\zeta} + T_x \xi + T_y \eta + M_x \xi' - M_y \eta' + M_t \vartheta]_{z_1}^{z_2} . \end{aligned} \tag{11}$$

Here $s_1(z)$ and $s_2(z)$ $(s_1 < s_2)$ are the two longitudinal edges of the open profile; q_x, q_y are the x- and y-components of the specific load per length z acting on the section; m_t is the specific traverse moment per length z of the specific loads acting on the section to the axis of rotation ϑ, i.e., to the z axis; p_z is the specific surface load per area of wall, parallel with the axis z; q_{z_1} and q_{z_2} are the specific loads per length z at the edges of the open profile in the direction of the shear stress τ. The term outside integral (11) is the energy of the concentrated loads and the reactions of the elastic supports.

The Euler's equations minimizing the expression (11). are

$$\begin{aligned} &\frac{\partial \phi}{\partial \xi} - \left(\frac{\partial \phi}{\partial \xi'}\right)' + \left(\frac{\partial \phi}{\partial \xi''}\right)'' = 0, \quad \ldots, \\ &\frac{\partial \phi}{\partial \vartheta} - \left(\frac{\partial \phi}{\partial \vartheta'}\right)' + \left(\frac{\partial \phi}{\partial \vartheta''}\right)'' = 0, \end{aligned} \tag{12}$$

where ϕ is the integrand in expression (11).

From the first and the last terms of Eq. (12) it follows that

$$(E J_y \xi'')'' + (E J_{\omega x} \vartheta'')'' = q_x + (\int_{s_1}^{s_2} p_z x \, ds)' + [(q_z x)']_{s_1}^{s_2}, \tag{13}$$

$$(E J_{\omega x} \xi'')'' + (E J_\omega \vartheta'')'' - (G J_k \vartheta')' = m_t + (\int_{s_1}^{s_2} p_z \omega \, ds)' + [(q_z \omega)']_{s_1}^{s_2}. \tag{14}$$

The other two Eqs. (12) for η and ζ give

$$(E J_{\bar{x}} \eta'')'' = q_y + (\int_{s_1}^{s_2} p_z \bar{y} \, ds)' + [(q_z \bar{y})']_{s_1}^{s_2}. \qquad (E F \zeta')' = -\int_{s_1}^{s_2} p_z ds - [q_z]_{s_1}^{s_2}. \tag{15}$$

Eq. (15) for the axial tension and for bending in the vertical plane (y, z) are mutually independent, as well as being independent of (13) and (14). They are well known and are commonly used in design practice.

Eqs. (13) and (14) constitute a system of two simultanous differential equations, namely for bending in the horizontal plane (x, z) and for non-uniform torsion. If the longitudinal specific loads are symmetrical with respect to the y axis, they assume the simpler form

$$\boxed{\begin{aligned} &(E J_y \xi'')'' + (E J_{\omega x} \vartheta'')'' = q_x, &\text{(13a)} \\ &(E J_{\omega x} \xi'')'' + (E J_\omega \vartheta'')'' - (G J_k \vartheta')' = m_t. &\text{(14a)} \end{aligned}}$$

2. It is also possible to derive the equations for torsion from equilibrium conditions. This can be done in the following manner.

By means of a system of mutually orthogonal functions $1, \bar{x}, \bar{y}, \bar{\omega}$ in the section we define the internal forces

$$N = \int_F \sigma \, dF, \qquad M_{\bar{x}} = \int_F \sigma \bar{y} \, dF, \qquad M_y = -\int_F \sigma x \, dF, \tag{16}$$

$$\bar{B} = \int_F \sigma \bar{\omega} \, dF. \tag{17}$$

N is the normal force, $M_{\bar{x}}$ and $M_{\bar{y}}$ are the bending moments to the principal $\bar{x}$ and $\bar{y}$ axes and $\bar{B}$ is termed the bimoment to the shear centre. The static quantities N, $M_{\bar{x}}$, $M_{\bar{y}}$ are necessarily equal to the resultants of all the external forces acting on the bar at one side of the section to its centre of gravity and principal $\bar{x}$, $\bar{y}$ axes.

Furthermore it will be necessary to define the bimoment B to the z axis, i.e., with respect to Eq. (4)

$$B = \int_F \sigma \omega \, dF = \bar{B} + \alpha_y M_y. \tag{18}$$

Writing $\sigma = E\epsilon$, let us substitute Eq. (8) in Eqs. (16), (17) and (18). In this way (taking into consideration that $S_y = S_\omega = J_{xy} = J_{\omega y} = S_{\bar{x}} = J_{\bar{\omega} x} = 0$), the following equations are obtained

$$N = E F \zeta', \qquad M_{\bar{x}} = -E J_{\bar{x}} \eta'', \tag{19}$$

$$M_y = E J_y \xi'' + E J_{\omega x} \vartheta'' = E J_y (\xi'' - \alpha_y \vartheta''), \tag{20}$$

$$\bar{B} = -E (J_{\bar{\omega}} - \alpha_y J_{\bar{\omega} x}) \vartheta'' = -E J_{\bar{\omega}} \vartheta'', \tag{21}$$

$$B = -E J_{\omega x} \xi'' - E J_\omega \vartheta''. \tag{22}$$

where in (20) the Eq. (4) has been used. Applying now the differential equations of equilibrium for N, M_x, M_y, i.e., the equations

$$M_y'' = q_x + \int_F (p_z x)' \, dF + [(q_z x)']_{s_1}^{s_2},$$

the Eq. (15) and the first of the Eqs. (13) are obtained.

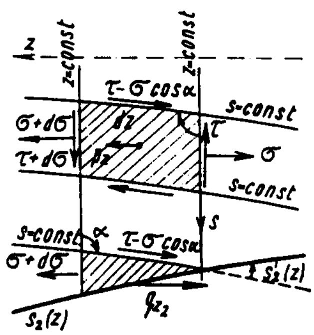

Fig. 3.

The equilibrium conditions will now be used in order to transform Eq. (21). The equilibrium conditions of the wall and of the longitudinal edges in the coordinate lines z = const. and s = const. which are generally non-orthogonal (Fig. 3), are

$$\frac{\partial}{\partial z}(\sigma \delta) = -p_z - \frac{\partial}{\partial s}(\tau_0 \delta) + \frac{\partial}{\partial s}[\sigma \delta \cos \alpha), \tag{23a}$$

$$\left.\begin{aligned} \sigma(s_1)\,\delta_1 \frac{\partial s_1(z)}{\partial z} &= \tau_0(s_1)\,\delta_1 - \sigma(s_1)\,\delta_1 \cos\alpha(s_1) - q_{z_1}, \\ \sigma(s_2)\,\delta_2 \frac{\partial s_2(z)}{\partial z} &= \tau_0(s_2)\,\delta_2 - \sigma(s_2)\,\delta_2 \cos\alpha(s_2) - q_{z_2}. \end{aligned}\right. \tag{23 b}$$

Here τ_0 denotes the shear stress in the centre line of the sections $s_1(z)$ and $s_2(z)$ $(s_1 < s_2)$ the longitudinal edges; α is the angle from the centre line z = const. to the longitudinal fibre s = const. There applies $\cos\alpha \doteq x' \frac{\partial x}{\partial s} + y' \frac{\partial y}{\partial s}$ where

$x' = \partial x / \partial z$ and $y' = \partial y / \partial z$ are the derivatives along the fibre s = const. For a cylindrical surface the coordinate lines are orthogonal. $\cos\alpha = 0$. For other bars $\cos\alpha \neq 0$, but, according to assumption 5, $\cos\alpha$ is small compared with unity.

It should be noted that a non-zero value for the shear stress τ_0 is, of course, contrary to the initial assumptions 6 and 7. This is, however, inevitable in a simplified engineer's solution, as it is likewise in the solution for bending.

If the integral (18), with limits depending on z is differentiated with respect to z, and then the Eq. (23b) is substituted and finally, having substituted the Eq. (23a), the integration by parts is applied, there follows successively:

$$\begin{aligned} B' &= \int_{s_1}^{s_2} (\sigma\,\omega\,\delta)'\,ds + \sigma(s_2)\,\delta_2\,s_2'(z)\,\omega(s_2) - \sigma(s_1)\,\delta_1\,s_1'(z)\,\omega(s_1) = \\ &\int_{s_1}^{s_2} (\sigma\,\delta)'\,\omega\,ds + \int_{s_1}^{s_2} \sigma\,\delta\,\omega'\,ds + [(\tau_0\,\delta - \sigma\,\delta\cos\alpha - q_z)\,\omega]_{s_1}^{s_2} = \\ &\int_{s_1}^{s_2} \tau_0\,\delta\,d\omega - \int_{s_1}^{s_2} p_z\,\omega\,ds - [q_z\,\omega]_{s_1}^{s_2} + \int_{s_1}^{s_2} \sigma\,\delta\,(\omega' - \rho\cos\alpha)\,ds \doteq \\ &\int_F \tau_0\,\delta\,d\omega - \int_{s_1}^{s_2} p_z\,\omega\,ds - [q_z\,\omega]_{s_1}^{s_2}. \end{aligned} \tag{24}$$

The expression $(\omega' - \rho\cos\alpha)$ was here regarded as being approximately as zero. The reason for this is the approximate geometric relation $d\omega \doteq \rho\,(dz\cos\alpha)$.

The first term in Eq. (24) represents the torsional moment about the z axis which corresponds to the mean value τ_0 of the shear stress τ in the thickness of the wall, resulting from equilibrium with normal stresses. The torsional moment about the z axis of all stresses which correspond, according to the equilibrium conditions of the wall, to the normal stresses σ will be termed the warping-torsional (bending-torsional) moment M_{t_ω}. As in the wall, with regard to the stress equilibrium perpendicularly to the surface of wall, the shear stresses also act perpendicularly to the surface of wall. The resultant x and y components of these stresses are $\sigma\,\delta\,x'$ and $\sigma\,\delta\,y'$, and the expression for M_{t_ω} is

$$M_{t_\omega} = \int_F \tau_0\,\delta\,d\omega + \int_F \sigma\,(-x'\,y + y'\,x)\,dF. \tag{25}$$

Substituting it in (24) it follows that

$$B' = M_{t_\omega} - \int_{s_1}^{s_2} p_z\,\omega\,ds - [q_z\,\omega]_{s_1}^{s_2}. \tag{24a}$$

The torsional moment acting in the section about the z axis is

$$M_t = \int_F \tau\,\rho_\tau\,dF + \int_F \sigma\,(-x'\,y + y'\,x)\,dF, \tag{26}$$

where ρ_τ is the distance of the shear stress vector τ from the z axis. It is equal to the resulting moment of the external loads acting on the bar at one side of the section about the z axis. The shear stress τ is composed of the mean value τ_0 in the thickness of the wall ($\rho_\tau = \rho = \partial\,\omega / \partial\,s$) and of the component τ_k

which is non-uniform in the thickness of wall and is due to St.-Venant's pure (simple) torsion: $\tau = \tau_0 + \tau_k$. Thus the following relation results

$$M_t = M_{t_\omega} + M_{t_k}. \tag{27}$$

in which

$$M_{t_k} = \int_F \tau_k \rho_\tau \, dF = G J_k \vartheta', \tag{28}$$

i.e.. the moment corresponding to the pure (simple) torsion.

Substituting Eqs. (28), (24a) and (22) in (27), the final equation for non-uniform torsion is obtained, i.e..

$$-(E J_{\omega x} \xi'')' - (E J_\omega \vartheta'')' + G J_k \vartheta' = M_t - \int_{s_1}^{s_2} p_z \omega \, ds - [q_z \omega]_{s_1}^{s_2}. \tag{29}$$

Differentiating this equation and considering that equilibrium requires that $M_t' = -m_t$. Eq. (14) is obtained.

If in Eq. (24a) B is expressed in terms of (18) and then Eqs. (28). (24a) and (21) are substituted in (27). it follows that

$$-(E J_{\bar\omega} \vartheta'' - \alpha_y M_y)' + G J_k \vartheta' = M_t - \int_{s_1}^{s_2} p_z \omega \, ds - [q_z \omega]_{s_1}^{s_2}. \tag{30}$$

This equation may be used for computation. if M_y and M_t are known. By differentiation of (30) the following equation results

$$(E J_{\bar\omega} \vartheta'' - \alpha_y M_y)'' - (G J_k \vartheta')' = m_t + \left(\int_{s_1}^{s_2} p_z \omega \, ds\right)' - [(q_z \omega)']_{s_1}^{s_2} \tag{30a}$$

and it may be used if M_y is known.

It is also possible to derive an equation relating B, M_t and M_y. For this purpose it is necessary to substitute Eq. (22) in (29), then divide it by GJ_k. and thereafter derivate it with respect to z and finally substitute the Eq. (21) and (18). Thus it follows (for the sake of brevity let $p_z = 0$. $q_{z_1} = q_{z_2} = 0$) that

$$\left(\frac{B'}{GJ_k}\right)' - \frac{B - \alpha_y M_y}{E J_{\bar\omega}} = \left(\frac{M_t}{GJ_k}\right)' \tag{31}$$

or with respect to (18)

$$\left(\frac{\bar{B}' + (\alpha_y M_y)'}{G J_k}\right)' - \frac{\bar{B}}{E J_{\bar\omega}} = \left(\frac{M_t}{G J_k}\right)'. \tag{31a}$$

This equation is useful if two boundary conditions for B are known in addition to M_t and M_y.

The normal stress σ can be obtained from Eq. (8), writing $\sigma = E\epsilon$. If the first relation (4) is substituted in (8) and then Eq. (19) and (20) are used, it follows that

$$\sigma = \frac{N}{F} - \frac{M_y}{J_y} x + \frac{M_{\bar x}}{J_{\bar x}} \bar y + \frac{\bar B}{J_{\bar\omega}} \bar\omega. \tag{32}$$

The expression for shear stresses τ in the centre line of the section can be established according to the equilibrium condition (23a) of the wall. For the sake of simplicity let $p_{z_1} = q_{z_2} = q_z = 0$. Integrating this equation, we obtain

$$\tau\,\delta = \tau(s_1)\,\delta_1 - \int_{s_1}^{s} \frac{\partial}{\partial z}(\sigma\,\delta)\,ds + \sigma\,\delta\cos\alpha - \sigma(s_1)\,\delta_1\cos\alpha(s_1) - \int_{s_1}^{s} p_z\,ds\,. \tag{33}$$

Using the equilibrium condition (23b) for the edge and substituting (32), the following expression is obtained

$$\begin{aligned}\tau\,\delta &= -\frac{\partial}{\partial z}\int_{s_1}^{s}\sigma\,\delta\,ds + \sigma\,\delta\cos\alpha - \int_{s_1}^{s} p_z\,ds - q_{z_1} = \\ &-\left(\frac{M_{\bar y}}{J_{\bar y}}\,S_{\bar y(s)}\right)' + \left(\frac{M_{\bar x}}{J_{\bar x}}\,S_{\bar x(s)}\right)' + \left(\frac{B}{J_{\bar\omega}}\,S_{\bar\omega(s)}\right)' + \sigma\,\delta\cos\alpha - \int_{s_1}^{s} p_z\,ds - q_{z_1}\,.\end{aligned} \tag{34}$$

in which

$$S_{\bar y(s)} = \int_{s_1}^{s}\bar x\,dF\,,\qquad S_{\bar x(s)} = \int_{s_1}^{s}\bar y\,dF\,,\qquad S_{\bar\omega(s)} = \int_{s_1}^{s}\bar\omega\,dF\,. \tag{35}$$

The shear stress in the longitudinal fibre s = const. is $\tau\,\delta - \sigma\,\delta\cos\alpha$.

The Eqs. (13), (14), (13a), (14a), (29), (30), (30a), (31), (31a) become independent equations for torsion and for bending if either $M_y = 0$ or $\alpha_y \equiv J_{\omega x} \equiv 0$.

The case $M_y \equiv 0$ occurs if the supports cannot produce horizontal forces and moments and $q_x = 0$. The second case arises if the shear centre line is straight so that the z axis can be situated coincidentally. A special case of this is the bar of constant section.

For bars of cylindrical surface which are loaded only by traverse moments m_t the Eqs. (13a), (14a), (19), (20), (21), (30), (30a), (32), (34) have already been derived by Cywiński [2].

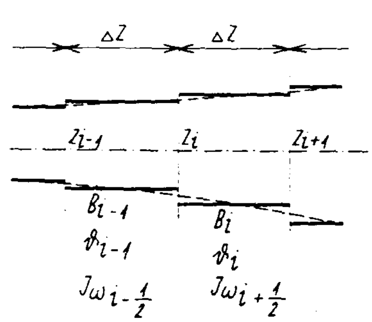

Fig. 4.

3. We shall also show a third possibility for the derivation of Eq. (13) and (14) which is based on the known equations for bars of constant section. Let us divide the bar by sections $z_1, z_2, \ldots, z_i, \ldots$ into short equal lengths Δz (Fig. 4).

The bar of variable section can be approximately solved as a bar, the section of which varies stepwise, i.e., discontinuously in the sections z_i, so that in a portion Δz the cross-section is constant. Let the loads be acting only in the sections z_i and for the sake of brevity let $q_z = q_{z_1} = q_{z_2} = 0$. The constants of the section in a portion (z_i, z_{i+1}) will be denoted as $EJ_{\omega_{i+\frac{1}{2}}}$, $GJ_{k_{i+\frac{1}{2}}}, \ldots$ and the constant moment by $M_{t_{i+\frac{1}{2}}}$; they represent the mean values of EJ_ω, $GI_k, \ldots$ in this portion of the given bar.

Denoting the rotation in the portion (z_i, z_{i-1}) by $\vartheta_i(z)$ the expansion of ϑ_i' and ϑ_{i-1}' into a Taylor's series gives

$$\begin{aligned} \vartheta_i'(z_{i-1}) &= \vartheta_i'(z_i) + \vartheta_i''(z_i)\Delta z + \tfrac{1}{2}\vartheta_i'''(z_i)\Delta z^2 + \ldots, \\ \vartheta_{i-1}'(z_{i-1}) &= \vartheta_{i-1}'(z_i) - \vartheta_{i-1}''(z_i)\Delta z + \tfrac{1}{2}\vartheta_{i-1}'''(z_i)\Delta z^2 \ldots, \end{aligned} \tag{36}$$

$$\begin{aligned} \xi_i''(z_{i-1}) &= \xi_i''(z_i) + \xi_i'''(z_i)\Delta z + \ldots, \\ \xi_{i-1}''(z_{i-1}) &= \xi_{i-1}''(z_i) - \xi_{i-1}'''(z_i)\Delta z + \ldots \end{aligned} \tag{37}$$

In the portion Δz the solution is governed by the well-known special case of Eq. (29) for constant section [7]. Hence we have

$$\begin{aligned} -\tfrac{1}{2}(EJ_{\omega x_i} + EJ_{\omega x_{i+1}})\xi_i'''(z_i) - EJ_{\omega_{i+\frac{1}{2}}}\vartheta_i'''(z_i) + GJ_{k_{i+\frac{1}{2}}}\vartheta_i'(z_i) &= M_{t_{i-\frac{1}{2}}}, \\ -\tfrac{1}{2}(EJ_{\omega x_{i-1}} + EJ_{\omega x_i})\xi_{i-1}'''(z_i) - EJ_{\omega_{i-\frac{1}{2}}}\vartheta_{i-1}'''(z_i) + GJ_{k_{i-\frac{1}{2}}}\vartheta_{i-1}'(z_i) &= M_{t_{i-\frac{1}{2}}}. \end{aligned} \tag{38}$$

The continuity conditions for torsion in the section z_i, except the condition $\vartheta_{i-1}(z_i) = \vartheta_i(z_i)$, are problematical. This is due to the discontinuity of the wall, i.e., to the discontinuity of u and σ and of x, y, ω. For bending (Eq. (13), (15)) this does not matter, however, because the integral condition of equilibrium and the equality of rotations and displacements of entire sections are sufficient. It is probably appropriate to require that there should be equality of the deplanation parameters in Eq. (6) and of the bimoments B to the z axis. With respect to the Eq. (22) these conditions are

$$\begin{aligned} &\vartheta_{i-1}'(z_i) = \vartheta_i'(z_i), \\ &EJ_{\omega x_{i-\frac{1}{2}}}\xi_{i-1}''(z_i) + EJ_{\omega_{i-\frac{1}{2}}}\vartheta_{i-1}''(z_i) = EJ_{\omega x_{i+\frac{1}{2}}}\xi_i''(z_i) + EJ_{\omega_{i-\frac{1}{2}}}\vartheta_i''(z_i). \end{aligned} \tag{39}$$

If the terms $\xi_i'''(z_i)$ and $\xi_{i-1}'''(z_i)$ from (37) are substituted in (38), and then the terms $\vartheta_{i-1}'''(z_i)$, $\vartheta_i'''(z_i)$, $\vartheta_{i-1}'(z_i)$, $\vartheta_i'(z_i)$ are calculated from Eqs. (38) and (39) and substituted in (36), the following difference equation (with a residuum of higher order) is obtained

$$\begin{aligned} &-\frac{1}{2\Delta z}[EJ_{\omega x_{i+1}}\xi_i''(z_{i+1}) - EJ_{\omega x_{i-1}}\xi_{i-1}''(z_{i-1})] \\ &-\frac{1}{2\Delta z^2}\{EJ_{\omega_{i+\frac{1}{2}}}[\vartheta_i'(z_{i+1} - \vartheta_i'(z_i)] - EJ_{\omega_{i-\frac{1}{2}}}[\vartheta_{i-1}'(z_i) - \vartheta_i'(z_{i-1})]\} + GJ_{k_i}\vartheta_i'(z_i) = M_{t_i}, \end{aligned} \tag{40}$$

in which the substitutions $\frac{1}{2}(GJ_{k_{i-\frac{1}{2}}} + GJ_{k_{i+\frac{1}{2}}}) = GJ_{k_i}$ and $\frac{1}{2}(M_{t_{i-\frac{1}{2}}} + M_{t_{i-\frac{1}{2}}}) = M_{t_i}$ were used. Proceeding to the limit for $\Delta z \to 0$ this difference equation transforms to the differential Eq. (29) or (14a). Eq. (13) can be obtained analogically.

Thin-walled Bars with Closed Profiles

Let us now consider a thin walled bar with a closed single-cell, variable monosymmetrical cross-section. For sake of simplicity let its specific loads be solely transversal. On the basis of assumptions 4, 5, 6 and 7, it is possible to derive ([4], [6]) that

$$u = \bar{\zeta} - \xi' x - \eta' \bar{y} + \kappa \omega_c, \tag{41}$$

$$v = \xi \cos(x, s) + \eta \sin(x, s) + \vartheta \rho, \tag{42}$$

where the warping function ω_c for a closed section is

$$\omega_c(s) = \omega(s) - \frac{J_k}{\Omega} \int_0^s \frac{ds}{\delta}. \tag{43}$$

Here Ω denotes the doubled area, enclosed by the centre line of the section, $\Omega = \oint \rho\, ds$, J_k is the Bredt's moment of inertia in pure (simple) torsion [4], $J_k = \Omega^2 \Big/ \oint \frac{ds}{\delta}$. If an open branch is joined to the closed contour, then the second term in (43) must be taken with its value in the node of bifurcation. The function $\kappa = \kappa(z)$ will be termed the deplanation parameter. The function $\omega(s)$ for $s \geqq 0$ denotes the sectorial coordinate as for an open section (obtained by cutting the closed section in $s = 0$) with the pole on the z axis of the bar and the zero point $s = 0$ on the axis of symmetry of the section. In place of Eq. (2) we shall define for a closed section, the sectorial moment of inertia (or bimoment of inertia) etc., i. e.,

$$J_\omega = \int_F \omega_c^2\, dF, \qquad J_{\omega x} = \int_F \omega_c x\, dF. \quad \ldots \tag{44}$$

With respect to the symmetry of the section it is $S_\omega = J_{\omega y} = 0$. The principal warping function $\bar{\omega}_c(s)$ which has the pole of the corresponding $\omega(s)$ in the shear centre, is defined by the condition $J_{\bar{\omega} x} = 0$. The y-coordinate of the shear centre is [4]

$$\alpha_y = -\frac{J_{\omega x}}{J_y} = -\frac{1}{J_y}\left(\int_F \omega x\, dF + \frac{J_k}{\Omega} \oint S_{y(s)} \frac{ds}{\delta}\right). \tag{45}$$

Analogically to the Eq. (4) the following expression is valid

$$\bar{\omega}_c = \omega_c + \alpha_y x, \qquad J_{\bar{\omega}} = J_\omega + \alpha_y J_{\omega x}. \tag{45a}$$

From the geometric equation $\gamma = \partial u / \partial s + \partial v / \partial s$, if the Eq. (41) and (42) have been substituted in this equation, it follows that

$$\gamma = \kappa \frac{\partial \omega_c}{\partial s} + \vartheta' \rho = \kappa \left(\rho - \frac{J_k}{\delta \Omega}\right) + \vartheta' \rho. \tag{46}$$

Here ρ' was neglected in accordance with assumption 5. Substituting $\tau = G\gamma$ in Eq. (26), defining the torsional moment, we obtain

$$\boxed{\kappa = -\frac{\vartheta'}{\nu} + \frac{M_t}{\nu G J_p},} \qquad \nu = \frac{1}{J_p}\int_F \rho\, d\omega_c = 1 - \frac{J_k}{J_p}, \tag{47}$$

where $J_p = \int_F \rho^2 dF$ is the tangential-polar moment of inertia.

The simplest method of deriving the equation for torsion is by a variational method. The specific strain energy in the section is given by Eq. (9). According to assumption 5 the normal strain is

$$\epsilon = \bar{\zeta}' - \xi'' x - \eta'' \bar{y} + \kappa' \omega_c. \tag{48}$$

Substituting Eq. (46) and (48) in Eq. (9), it is possible to obtain successively

$$\begin{aligned} W &= \tfrac{1}{2}(EF\bar{\xi}'^2 + EJ_y \xi''^2 + EJ_{\bar{x}} \eta''^2 - 2EJ_{\omega x}\xi''\kappa' + EJ_\omega \kappa'^2 \\ &\quad + GJ_p \vartheta'^2 + 2G(J_p - J_k)\kappa\vartheta' + G(J_p - J_k)\kappa^2) = \\ &\quad \tfrac{1}{2}\left(EF\bar{\xi}'^2 + EJ_y\xi''^2 + EJ_{\bar{x}}\eta''^2 - 2EJ_{\omega x}\xi''\kappa' + EJ_\omega\kappa'^2 - \nu GJ_k\kappa^2 + \frac{M_t^2}{GJ_p}\right). \end{aligned} \tag{49}$$

The specific potential energy, corresponding to the transverse moments, is $-m_t\vartheta$. It will be convenient to write it, with respect to the equilibrium equation $m_t = -M_t'$ in the form $-M_t\vartheta' + (M_t\vartheta)'$. To maintain uniformity, this will also be done with $-q_x\xi$ and $q_y\eta$. Then the sum of the strain energy and of the potential energy of the external forces is obtained in the form

$$U = \int_{z_1}^{z_2} (W - M_y\xi'' + M_x\eta'' - M_t\vartheta')\,dz, \tag{50}$$

where for the sake of brevity we have omitted the terms corresponding to p_z, q_{z_1}, q_{z_2}, N. Denoting the integrand in this equation by ϕ, the minimizing conditions for this expression are the Euler equations

$$\frac{\partial\phi}{\partial\xi''} = 0, \qquad \dots\dots \qquad \frac{\partial\phi}{\partial\kappa} + \frac{\partial\phi}{\partial\vartheta'}\frac{\partial\vartheta'}{\partial\kappa} - \left(\frac{\partial\phi}{\partial\kappa'}\right)' = 0. \tag{51}$$

The first and the last of these equations give, respectively,

$$\boxed{EJ_y\xi'' - EJ_{\omega x}\kappa' = M_y, \qquad \frac{1}{\nu}(-EJ_{\omega x}\xi'' + EJ_\omega\kappa')' - GJ_k\kappa = M_t.} \tag{52}$$

In the case of longitudinal loads the same terms as in (29) remain on the right-hand side of the second equation.

The two remaining Eqs. (51) yield the same equations as (15).

In a manner similar to that in the preceding paragraph Eq. (52) represents a system of two simultanous differential equations for $\kappa(z)$ and $\xi(z)$. They can be used in this form if M_t and M_y are known. Otherwise, it is necessary to differentiate them and substitute the equilibrium relations, e.g.,

$$E J_y \xi'' - E J_{\omega x} \kappa' = M_y,$$
$$\frac{1}{\nu}(-E J_{\omega x} \xi'' + E J_\omega \kappa')' - \frac{G}{\nu}(J_c + \nu^2 J_p)\kappa = M_t. \tag{52a}$$

Having solved κ, it is possible to calculate ϑ by solving the differential Eq. (47). Otherwise, if Eq. (47) is substituted in (52), a system based directly on the unknowns ξ and ϑ is obtained.

Calculating ξ'' from the first Eq. (52) and substituting it in the second one, the following expression is obtained

$$\frac{1}{\nu}(E J_{\bar{\omega}} \kappa' + \alpha_y M_y)' - G J_k \kappa = M_t. \tag{53}$$

It can easily be seen that the system (52) becomes converted into an independent equation for torsion if either $M_y = 0$, or $J_{\omega x} = 0$ is valid. The second of these can only be attained, if the shear centre-line is straight and can thus be made coincident with the z axis.

The bimoment for a closed section to the shear centre-line or to the z axis will be defined as

$$\bar{B} = \int_F \sigma \bar{\omega}_c \, dF, \qquad B = \int_F \sigma \omega_c \, dF = \bar{B} + \alpha_y M_y, \tag{54}$$

where the relation (45a) was used. Substituting (48) we obtain ($J_{\bar{\omega} x} = 0$)

$$B = -E J_{\omega x} \xi'' + E J_\omega \kappa', \qquad \bar{B} = E J_{\bar{\omega}} \kappa'. \tag{55}$$

If the first equation is substituted in Eq. (52), and then this equation is divided by $G J_k$ and differentiated, and finally the Eqs. (55) and (54) are once more substituted, it follows that

$$\left(\frac{B'}{\nu G J_k}\right)' - \frac{B - \alpha_y M_y}{E J_{\bar{\omega}}} = \left(\frac{M_t}{G J_k}\right)' \tag{56}$$

or

$$\left(\frac{\bar{B}' + (\alpha_y M_y)'}{\nu G J_k}\right)' - \frac{\bar{B}}{E J_{\bar{\omega}}} = \left(\frac{M_t}{G J_k}\right)'. \tag{56a}$$

These equations can be used if M_t, M_y and two boundary conditions for $\bar{B}$ are known.

Having substituted the Eq. (47) in the second term of Eq. (53) and taking into account that $M_{t_k} = G J_k \vartheta'$, it is found that Eq. (53) can be written in the form of Eq. (27), in which the warping-torsional moment is $M_{t_\omega}^{\cdots} = B' = \bar{B}' + (\alpha_y M_y)'$ (Eq. (55)). — *Note:* If instead of the Eq. (43) an arbitrarily chosen warping function ω_c is assumed, then the Eq. (45), (45a), (49), (52), (53), (56), (57) and the last expressions in the Eq. (46), (47), (54) and (55) would not be valid. In (49) it would be necessary to replace the terms containing G by $\frac{1}{2}(G J_c \kappa^2 + 2\nu G J_p \kappa \vartheta' + G J_p \vartheta'^2)$ where $J_c = \int_F \left(\frac{\partial \omega_c}{\partial s}\right)^2 dF$. Instead of (52) we should obtain

$$E J_y \xi'' - E J_{\omega x} \kappa' = M_y,$$
$$\frac{1}{\nu}(-E J_{\omega x} \xi'' + E J_\omega \kappa')' - \frac{G}{\nu}(J_c + \nu^2 J_p)\kappa = M_t. \tag{52a}$$

Sometimes it is also possible to introduce $\omega_c = -x\,y$ which is identical with (43) for a rectangular section and δ = const. in each plate in the section. The Eq. (45a), (54) and the Eq. (45) except for the last expression are then valid and, in addition the equation analogous to (53), (56) and (56a) can easily be derived.

The normal stresses are given by Eq. (48), $\sigma = E\,\epsilon$. Substituting this in Eq. (55) and the first Eq. (52), there follows the formula:

$$\sigma = \frac{N}{F} - \frac{M_y}{J_y}\,x + \frac{M_{\bar{x}}}{J_{\bar{x}}}\,\bar{y} + \frac{\bar{B}}{J_{\bar{\omega}}}\,\bar{\omega}_c, \tag{57}$$

which is similar to (32). The shear stresses τ can be derived from σ according to the differential equilibrium Eq. (23a), in a similar manner to that for a non-variable section.

In an analogous manner to that for open profiles, the solution may also be derived by statical considerations, just as for a non-variable section [4], which would, however, be much more laborious, or by determining the limit of the solution for a bar with a stepwise variable section under suitable continuity conditions (continuity of κ and of B).

Bars with a multi-cell profile will not be treated here, but can be solved in a similar manner. The foregoing equations may also be directly applied to symmetrical two-cell profiles because in them the wall located in the plane of symmetry is not stressed by torques.

Curved Bars

Let us now consider a bar with a circularly curved z axis, the radius of curvature of which is R, and with sections symmetrical to the $\bar{y}$ axis, which is perpendicular to the plane of curvature (x, z); z denotes the length of the axis (Fig. 2).

The changes in the curvature of the axis of the bar caused by bending will be denoted by k_x and k_y, the torsional curvature, i.e., the specific angle of twist by k_t and the extension in the axis of the bar by ϵ_R. Summing the influences of the changes in curvature along the portion (z_0, z_1) of the circular axis of the bar, the displacements ξ, η, ζ and rotation ϑ can be expressed by the equations

$$\left.\begin{aligned}
\xi_1 &= \xi_{01}(z_1) + R\int_{z_0}^{z_1} k_y \sin\frac{z_1 - z}{R}\,dz - \int_{z_0}^{z_1} \epsilon_R \sin\frac{z_1 - z}{R}\,dz,\\
\eta_1 &= \eta_{01}(z_1) - R\int_{z_0}^{z_1} k_x \sin\frac{z_1 - z}{R}\,dz + R\int_{z_0}^{z_1} k_t\left(1 - \cos\frac{z_1 - z}{R}\right)dz,\\
\zeta_1 &= \zeta_{01}(z_1) + R\int_{z_0}^{z_1} k_y\left(1 - \cos\frac{z_1 - z}{R}\right)dz + \int_{z_0}^{z_1} \epsilon_R \cos\frac{z_1 - z}{R}\,dz,\\
\vartheta_1 &= \vartheta_{01}(z_1) + \int_{z_0}^{z_1} k_t \cos\frac{z_1 - z}{R}\,dz + \int_{z_0}^{z_1} k_x \sin\frac{z_1 - z}{R}\,dz,
\end{aligned}\right\} \tag{58}$$

in which ξ_{01}, η_{01}, ζ_{01}, ϑ_{01} represent the influence of the displacements and rotations in the section z_0, i.e., the displacements and rotations of the bar as a rigid body. The differential equations for k_x, k_y, k_t and ϵ_R can be derived from the system (58) of integral equations. This is done by elimination of the integrals from Eqs. (58) and their derivatives with respect to z_1. The following equations are obtained in this way:

$$k_x = -\eta'' + \frac{\vartheta}{R}, \qquad k_y = \xi'' + \frac{\zeta'}{R}, \qquad k_t = \vartheta' + \frac{\eta'}{R}, \qquad \epsilon_R = \zeta' - \frac{\xi}{R}. \tag{59}$$

Their validity may be also proved by substituting Eq. (58) in (59). On the other hand, the Eqs. (58) represent a solution of the linear differential Eqs. (59) which is expressed with the aid of the Green's functions corresponding to Eq. (59).

The equilibrium of the portion (z_0, z_1) of the curved bar requires that

$$\begin{aligned}
M_{x_1} &= M_{x_{01}} - R\int_{z_0}^{z_1} q_y \sin\frac{z_1 - z}{R}\,dz + \int_{z_0}^{z_1} m_t \sin\frac{z_1 - z}{R}\,dz,\\
M_{y_1} &= M_{y_{01}} + R\int_{z_0}^{z_1} q_x \sin\frac{z_1 - z}{R}\,dz - R\int_{z_0}^{z_1} q_z\left(1 - \cos\frac{z_1 - z}{R}\right)dz,\\
M_{t_1} &= M_{t_{01}} - R\int_{z_0}^{z_1} q_y\left(1 - \cos\frac{z_1 - z}{R}\right)dz - \int_{z_0}^{z_1} m_t \cos\frac{z_1 - z}{R}\,dz,\\
N_1 &= N_{01} - \int_{z_0}^{z_1} q_z \cos\frac{z_1 - z}{R}\,dz - \int_{z_0}^{z_1} q_x \sin\frac{z_1 - z}{R}\,dz.
\end{aligned} \tag{60}$$

Here $M_{x_{01}}$, $M_{y_{01}}$, $M_{t_{01}}$, N_{01} are the internal forces produced in the section z_1 by the forces acting on the section z_0: $q_x dz$, $q_y dz$, $q_z dz$ and $m_t dz$ denote the resultants of all the loads acting on the portion dz of the bar which for curved bars is limited by non-parallel sections, the planes of which form an angle dz/R. By means of the specific area loads $p_x \ldots$ they can be expressed as

$$q_x = \int_{s_1}^{s_2} p_x\left(1 - \frac{x}{R}\right)ds, \quad \ldots, \quad m_t = \int_{s_1}^{s_2} (p_y x - p_x y)\left(1 - \frac{x}{R}\right)ds.$$

Eliminating the integrals from the Eqs. (60) and their derivatives, the following differential equations of equilibrium are obtained:

$$q_x = M_y'' - \frac{N}{R}, \quad q_y = -M_x'' - \frac{M_t'}{R}, \quad q_z = -N' - \frac{M_y'}{R}, \quad m_t = -M_t' + \frac{M_x}{R}. \tag{61}$$

We note that the Eq. (58)—(59) are formally analogic to the Eq. (60)—(61), q_x, q_y, q_z, m_t, M_x, M_y, M_t, N corresponding to k_x, k_y, k_t, ϵ_t, ξ, η, ζ, ϑ.

Starting from the same assumptions as for straight bars, the normal stress distribution can also be approximately adopted in the form of Eqs. (8) or (48) and (47), in which ζ', ξ'', η'' and ϑ' are replaced by ϵ_R, k_y, $-k_x$ and k_t in turn, although strictly speaking the right-hand sides of these equations should have been divided by $(1 - x/R)$. Substituting the stress distribution $\sigma = E\epsilon$ in the Eq. (16), the equations for bending and extension are obtained in the form of (19) and (20) or the first Eq. (52), in which ζ', ξ'', η'' are replaced by ϵ_R, k_y and $-k_x$. However, it would be rather difficult to obtain the equation for torsion in this way. For this reason we shall employ the energy method.

The specific strain energy of a curved bar is

$$W_R = \tfrac{1}{2}\int_F (E\,\epsilon^2 + G\,\gamma^2)\left(1-\frac{x}{R}\right) dF. \tag{62}$$

If Eqs. (8) or (48) are substituted in (62) the equation

$$W_R \doteq W + (E\,J_y\,k_y + E\,J_{\omega x}\,k_t')\frac{\epsilon_R}{R} + E\left(\frac{1}{2}k_t'\int_F \omega^2 x\,dF + k_y\int_F \omega\,x^2\,dF\right)\frac{k_t'}{R} \tag{63}$$

is obtained.

W denotes here the expression (10) or (49) and (47) for straight bars, in which ζ', ξ'', η'' and ϑ' are replaced by ϵ_R, k_y, $-k_x$ and k_t in turn. Approximately, for large values of R, it can be considered that $W_R \doteq W$.

The expression for the potential energy of the bar can be transformed, substituting, firstly, the Eq. (61), then using integration by parts twice and finally substituting the Eq. (59) and the equation for bending in the plane $(x\,z)$, in the following manner:

$$\begin{aligned} U &= \int_{z_1}^{z_2}(W_R - q_x\,\xi - q_y\,\eta - q_z\,\zeta - m_t\,\vartheta)\,dz + [\ldots]_{z_1}^{z_2} = \\ &\int_{z_1}^{z_2}\left[W - M_y\left(\xi'' + \frac{\zeta'}{R}\right) + M_x\left(\eta'' - \frac{\vartheta}{R}\right) - M_t\left(\vartheta' + \frac{\eta'}{R}\right) - N\left(\zeta' - \frac{\xi}{R}\right)\right] dz + [\ldots]_{z_1}^{z_2} = \\ &\int_{z_1}^{z_2}[W - M_y\,k_y - M_x\,k_x - M_t\,k_t - N\,\epsilon_R]\,dz + [\ldots]_{z_1}^{z_2}. \end{aligned} \tag{64}$$

It can be seen that this expression is identical with the expression (50) for straight bars, if the terms ζ', ξ'', η'' and ϑ' are replaced by ϵ_R, k_y, $-k_x$ and k_t in turn. The Euler's equations are then identical with the Eq. (12) or (51) if the same replacement is effected in them.

In consequence of this, all the foregoing equations for straigth bars may be used for curved bars. In these equations ζ', ξ'', η'' and ϑ' can be replaced by ϵ_R, k_y, $-k_x$, k_t if they contain only the internal forces $M_x, \ldots$ and not the loads, i.e. the Eqs. (19)—(22), (24a), (27), (29); (30), (31), (32), (34), (47), (52), (52a), (53), (57), but not the Eqs. (13)–(15), (30a), (52a), in which this replacement is not possible. The equation containing the loads and not the internal forces may be derived from them by substituting the equilibrium conditions. It is, however, more appropriate to compute directly the curvatures k_x, k_y, k_t and ϵ_R from the equations containing internal forces and to determine ξ, η, ϑ and ζ, if it is necessary at all, according to Eq. (58) (or by means of the force deformation analogy [8]).

The foregoing results for circularly transversely curved bars can be generalized for arbitrarily curved bars. Instead of Eq. (58) we can write the integral equation of the deflection line of the bar in the vectorial form

$$\vec{u}_1 = \vec{u}_0 + \vec{\varphi}_0 \times (\vec{r}_1 - \vec{r}_0) + \int_{z_0}^{z_1} [\vec{k}\, dz \times (\vec{r} - \vec{r}_0)] + \int_{z_0}^{z_1} (\vec{\epsilon}_z\, dz + \vec{\gamma}_z\, dz), \tag{58a}$$

where x denotes the vectorial product, u is the displacement vector and φ the (axial) rotation vector for the sections of the bar. $\vec{r} = \vec{r}(z)$ is the radius vector determining the axis of the bar. $\vec{k}\,dz$ is the (axial) vector of relative rotation of two neighbouring sections at a distance dz and $\epsilon_z dz$, $\gamma_z dz$ are the vectors of their relative displacement; $\vec{k}$ is termed the vector of curvature change (with bending and torsional components k_x, k_y, k_t). Similarly, the vectorial equation of equilibrium of the portion (z_0, z_1) of the bar is

$$\vec{M}_1 = \vec{M}_0 + \vec{T}_0 \times (\vec{r}_1 - \vec{r}_0) + \int_{z_0}^{z_1} [\vec{q}\, dz \times (\vec{r}_1 - \vec{r}_0)] + \int_{z_0}^{z_1} \vec{m}\, dz, \tag{60a}$$

where $\vec{M}$ is the moment vector acting in the section to the point of the z axis of the bar (with bending and torsional components M_x, M_y, M_t): $\vec{T}$ is the internal force vector in the section (with normal and shear components). $\vec{q}\,dz$ is the resulting vector of the loads acting on the portion dz and $\vec{m}\,dz$ their moment vector with respect to the axis of the bar. Because of the formal analogy of Eq. (58a) and (60a) the expression for the potential energy can be transformed in the same way as in Eq. (64).

Therefore all the equations mentioned which are valid for circular curved bars can also be used for arbitrarily curved bars. However, the differential relations (59) and (61) for $k_x, k_t, \ldots, M_t$ must be replaced by more general equations, resulting, e.g., from (58a) and (60a).

Note: Assuming the z axis of the bar to be arbitrarily curved, the shear centre-line can be taken as an axis of the bar. Then we shall have $J_{\omega x} = \alpha_y = 0$ and the equations for torsion and for horizontal bending will be independent, as in the case of a constant section. However, this would only rarely be advantageous, even if the difficulties caused by the curved axis could be disregarded, because the curvature would often be much too great and the supports, e.g. the restraint, would often be non-perpendicular (skew) to the z axis.

Method of Solution of Continuous Bars and Frames

A general solution of bars having a variable section can only be effected in a few special cases. It is therefore necessary to use approximate numerical methods, the most convenient of which is to replace the differential equations for torsion by algebraic difference equations, as for instance the Eq. (29) by (40).

A direct method of solution is to use the equations relating the deformations and the loads and containing the equilibrium conditions. However, this procedure is not advantageous in the case of point loads and for continuous beams, frames etc. In this case it is better to use a method similar to the force method or, if necessary, similar to the displacement method, for the bending and the simple torsion in frames, continuous beams and beam grillages. The fact that the problem of non-uniform (pure and warping-) torsion is infinitely statically indeterminate, renders it impossible to introduce a statically determinate primary system.

In the force method it is necessary to choose certain internal forces (torsional moments, bending moments, bimoments or shear and normal forces) as unknowns X_i (statically indeterminate quantities) in such a manner that for $X_i = 0$ (primary system) the distribution of M_x, M_y and M_t is statically determinate and that the boundary conditions for all the bars in the structure are given, so that it will be possible to solve the differential Eqs. (30) or (31a) or (53) or (56a) for torsion. Their solution for specified external loads and the corresponding values $M_{t(0)}, M_{y(0)}, \ldots$ we shall denote by $\vartheta'_{(0)}$, or $k'_{t(0)}$, or $B_{(0)}$, for $X_j = 1$ by $\vartheta'_{(j)}$, etc. The compatibility conditions are expressed by the equations $\sum_i \delta_{ik} X_i + \delta_k = 0$ in which the deformation δ_{ik} caused in the primary system by $X_i = 1$ in the direction of X_k is determined by the principle of virtual works

$$\delta_{ik} = \int_z dz \int_F \left(\frac{\sigma_{(i)}\sigma_{(k)}}{E} + \frac{\tau_{(i)}\tau_{(j)}}{G}\right) dF. \tag{66}$$

The shear stresses relate, to the first approximation, only to the pure torsion.

Substituting the Eq. (57) or (32) in (66), we obtain for open as well as for closed profiles

$$\delta_{ik} = \int_z \left\{\frac{N_{(i)} N_{(k)}}{EF} + \frac{M_{y(i)} M_{y(k)}}{E J_y} + \frac{M_{\bar{x}(i)} M_{\bar{x}(k)}}{E J_{\bar{x}}} + \frac{\bar{B}_{(i)} \bar{B}_{(k)}}{E J_{\bar{\omega}}} + \frac{M_{t_k(i)} M_{t_k(k)}}{G J_k}\right\} dz. \tag{67}$$

z denotes the integration over the whole of the structure. The expression (67) or (55) can be transformed into another simple form, if the Eq. (21) is substituted in (67), then integration by parts is applied and the Eq. (24a) and (27) are used. In this manner it follows that in the case of zero longitudinal loads

$$\begin{aligned}\delta_{ik} = \int_F \Bigg\{&\frac{N_{(i)} N_{(k)}}{EF} + \frac{M_{y(i)} M_{y(k)}}{E J_y} + \frac{M_{\bar{x}(i)} M_{\bar{x}(k)}}{E J_{\bar{x}}} \\ &+ [M_{t(i)} - (\alpha_y M_{y(i)})'] \vartheta'_{(k)}\Bigg\} dz - [B_{(i)} \vartheta'_{(k)}]_{z_1}^{z_2},\end{aligned} \tag{68}$$

where the last term outside of the integral is generally found to be zero. Thus, instead of B and M_{t_k} it is necessary to calculate ϑ' and M_t which is often more advantageous, because M_t is known from equilibrium conditions.

For the deformation δ_k in the direction of X_k caused in the primary system by external loads, the equations analogic to (67) and (68) are valid, in which instead of $N_{(i)}, M_{y(i)}, \ldots$ the terms $N_{(0)}, M_{y(0)}$ are retained. If longitudinal loads are acting, then further terms in (68) according to (25) must be added. In the case of curved bars it is necessary to multiply the integrand of (66) by $(1 - x/R)$. Approximately, the Eq. (67) and (68) are also valid ($\vartheta' \to k_t$).

Conclusions

1. The theory presented makes it possible to solve the non-uniform (pure and warping-) torsion and the bending of thin-walled bars with an open or closed monosymmetrical variable section and a straight or planely curved axis. The main difference with respect to bars with constant section is, in addition to the influence of the derivatives of the sectional constants, the fact that it is not possible to use the shear centre line as an axis of the bar and the principal sectorial coordinate for warping-torsion. Thus, the torsion and the horizontal bending are not independent, but are connected by a system of two simultaneous differential equations.

2. With respect to the starting assumptions the solution presented is approximate. Since an exact solution or the evaluation of error would be enormously complicated, the theory needs an experimental verification including the determination of the limits of usability. The reason for which it can be accepted as admissible, is that it represents an analogous generalization of the solution for bars with constant section, which has already been experimentally verified (see [7]). In the special case of a monosymmetrical bar of open profile which is cut from a cylindrical surface and is loaded only by transverse specific moments, the solution presented is coincident with the solution of CYWIŃSKI [2] which has been verified by test.

3. In civil engineering the solution presented will be mainly of importance for bridge beams and frames of box section or of open section and, especially for horizontally curved bridges, because in these structures great torsional moments arise. This type of structure is widely used in modern highway design.

4. Further work has now to be done, e.g., on the solution of bars with unstiffened and deformable cross-sections, such as occur, for instance, in the case of bridges with traffic travelling inside the box beam.

Basic Notations

$(\ )' = \frac{\partial}{\partial z}$, $(\ )'' = \frac{\partial^2}{\partial z^2}$.	
$z;\ z_1, z_2$	length coordinate of the axis of the bar; the ends of the bar.
R	radius of curvature of the axis of the bar.
$x, y;\ \bar{x}, \bar{y}$	cartesian coordinates of the points of section: coordinates to the principal centroidal axis of inertia.
$s;\ s_1, s_2$	length of the centre line of the cross section; edges of the section.
$\omega, \bar{\omega}$	sectorial coordinate to the axis z of the points of the centre line; principal sectorial coordinate (to the shear centre).
$\omega_c, \bar{\omega}_c$	distribution of (unity) deplanation at the closed section.
δ	thickness of the wall.
ρ	distance of the tangent of the centre line from the z axis or from the shear centre.
F	area of the section.
Ω	area enclosed, by the centre line of the closed section.
$J_{\bar{x}}, J_{\bar{y}} = J_y$	principal moments of inertia of the section.
$J_{\omega}, J_{\bar{\omega}}, J_{\omega y}$	sectorial moment of inertia, principal sectorial moment of inertia, bending-sectorial moment of inertia.
J_k, J_ρ	moment of inertia in simple torsion, tangential-polar moment of inertia.
$\nu = 1 - J_k/J_\rho$	
α_y	y-coordinate of the shear centre (principal pole).
u, v	displacements of the point of the centre line perpendicularly to the section and tangentially to the centre line.
ϵ, γ	normal strain, perpendicularly to the section, and shear strain in the centre line.
σ, τ	normal stress, perpendicularly to the section, and shear stress in the centre line.
$\xi, \eta, \bar{\zeta}$	x- and y-displacement of the section in the z axis and the z-displacement in the centroid.
ϑ	transverse rotation of the section (around z).
κ	parameter of deplanation at the closed section.
k_x, k_y, k_t	changes of the curvature of the axis of the bar caused by loading in the planes yz and zx and the torsional curvature (specific angle of twist).
ϵ_R	extension corresponding to the z axis.
$q_x, q_y;\ m_t$	resultant x- and y-components and resultant moment to the z axis of the specific loads in the section.
$p_z;\ q_{z_1}, q_{z_2};\ q_z$	z-component of the specific area load on the wall; z-component of specific loads on the longitudinal edges; resultant z-component for the section.

N; $M_{\bar{x}}$, $M_{\bar{y}} = M_y$	normal force and bending moments to the principal centroidal axis of inertia.
M_t: M_{t_ω}, M_{t_k}	torsional moment to the z axis and its parts: bending-torsional moment and moment in simple torsion.
B, $\bar{B}$	bimoment to the z axis and to the shear centre.
T_x, T_y	shear forces.
E, G	moduli of elasticity.
W, W_R	specific strain energy in the section of the strain bar or of the curved bar.
X_i	statically indeterminate quantity.

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Summary

In this paper an engineer's solution of non-uniform (pure and warping) torsion, with simultaneous bending of thin-walled bars which have a monosymmetrical, slowly variable cross section, is presented. Solutions are provided for straight or curved bars of open as well as of closed profile. The initial simplifying assumptions of the solution are similar to those for bars with constant section, i. e., the non-deformability of the cross section and for open profiles Wagner's assumption of a shear-resistant skin, for closed profiles Umanskiy's assumption of similar deplanation (warping) to that for pure torsion. Unlike the solution of bars with constant section it is not possible to use the shear centre line as an axis of the bar and therefore the torsion and the transverse bending are mutually dependent and are given by two ordinary

simultaneous differential equations. The solution presented makes it possible to solve mainly bridge beams and frames of box section or of open section and especially horizontally curved bridges, where considerable torque occurs.

Résumé

L'auteur présente une solution approximative au problème de la torsion non uniforme (torsion pure et gauchissement) et de la flexion simultanée des barres à parois minces dont la section, lentement variable, présente un axe de symétrie. Les solutions présentées concernent des barres droites ou courbes de section ouverte ou fermée. Les hypothèses simplificatrices de base sont les mêmes que celles adoptées pour les barres à section constante: indéformabilité des sections transversales et, pour les profils ouverts, l'hypothèse de Wagner d'une membrane résistant au cisaillement, pour les profils fermés, celle d'Umanskij admettant un gauchissement semblable à celui de la torsion uniforme. Contrairement au cas des barres à section constante, il est ici impossible d'utiliser la ligne des centres de cisaillement comme axe de la barre; la torsion et la flexion latérale ne sont dès lors plus indépendantes et elles obéissent à deux équations différentielles ordinaires simultanées. La solution proposée permet de traiter les poutres de ponts et les ossatures à section ouverte ou en caisson: elle s'applique spécialement aux ponts courbes, où se produisent d'importants moments de torsion.

Zusammenfassung

Für das Problem der Wölbkrafttorsion unter gleichzeitiger Biegung von dünnwandigen Stäben mit einfach symmetrischem, langsam veränderlichem Querschnitt wird eine Näherungslösung entwickelt. Die Lösungen betreffen gerade und gekrümmte Stäbe mit offenem oder geschlossenem Querschnitt. Es gelten ähnliche vereinfachende Voraussetzungen wie bei Stäben mit konstantem Querschnitt: Starre Querschnittsform und, für offene Querschnitte, die Wagnersche Annahme einer schubfesten Wand, für geschlossene Querschnitte die Hypothese von Umanskij, daß die Verwölbungen jenen bei reiner Torsion ähnlich sind. Anders als bei geraden Stäben ist es hier nicht möglich, die Schubmittelpunktlinie als Stabachse zu wählen, so daß Torsion und Querbiegung voneinander abhängig und durch zwei gewöhnliche simultane Differentialgleichungen gegeben sind. Mit der vorgeschlagenen Lösung können Brückenträger und Rahmen mit kastenförmigem oder offenem Querschnitt, insbesondere gekrümmte Brücken, bei denen beträchtliche Drehmomente auftreten, behandelt werden.