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Structural stability

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Abstract

The paper attempts a broad overview of the vast field of stability of structures, including elastic and anelastic structures, static and dynamic response, linear and non-linear behavior, energy approach, thermodynamic aspects, creep stability and fracture or damage-induced instability. The importance of stability theory to various fields of engineering and applied science is pointed out and the history of the discipline is briefly sketched. The principal accomplishments are succinctly reviewed, and fruitful recent trends, particularly the stability analysis of damage localization and fracture, are emphasized. Only selected references are given.

1. Introduction

Stability represents a fundamental problem in solid mechanics, which must be mastered to ensure the safety of structures against collapse. The theory of stability is of crucial importance for structural engineering, aerospace engineering, nuclear engineering, offshore, ocean and arctic engineering. It plays an important role in certain problems of space structures, geotechnical structures, geophysics and materials science.

The importance of the subject is evident from the history of structural collapses caused by neglect or misunderstanding of the stability aspects of design. The most famous among these is perhaps the collapse of the Tacoma Narrows Bridge in 1940, due to aerodynamic instability, and the collapse of Quebec Bridge over St. Lawrence in 1907, but numerous other disasters provided important lessons; e.g. collapse of the space frame of Hartford Arena in 1978 and of the reticulated dome of Post College Theater in the same year, the collapse of steel box girder bridge in Melbourne several years earlier, the collapse of Ferrybridge cooling tower as well as an early history of loss of dynamic stability of aircraft

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wings, or control in rocket propulsion, instability failures of mountain slopes, open or underground excavations, ocean oil platforms, etc.

Stability analysis in solid mechanics began with Euler's solution of buckling of an elastic column (Euler, 1744). Most basic linear elastic problems of structural stability were solved by the end of the 19th century, although further solutions have been appearing as new structural types were being introduced. The twentieth century has witnessed a great expansion of the stability theory into nonlinear behavior, caused either by large deflections or by nonlinearity of the constitutive law of the material. In the second half of this century, dynamic stability, important especially for non-conservative systems, became reasonably well understood. Certain aspects, such as chaos attracted attention nevertheless only during the last few decades, and so did the intricate nonlinear aspects of post-critical behavior in static instabilities. The greatest emphasis is currently being placed on the analysis of instabilities and bifurcations caused by propagation of softening damage or fracture in materials, which is important not only from the physical and engineering viewpoint, but also from the viewpoint of computational modeling.

Most of the subjects covered in this brief overview are expounded in detail in the book by Bažant and Cedolin (1991), which is henceforth referenced as [BC].

2. Stability of columns, frames and arches

The concept of a critical load of an elastic structure at which the equilibrium bifurcates was introduced by Euler (1744) who also provided the solutions of critical loads of columns with various end restraints. Experiments, however, could not verify the calculated critical loads. This fact was explained by Young (1807), who realized that imperfections such as initial curvature, initial bending moments or load eccentricity play an important role and derived a formula for what is known today as the magnification factor for deflections and bending moments in columns due to axial load. Kirchhoff (1859) extended the theory to geometrically non-linear large deflections and provided an elegant solution of the deflection curve, called the elastica, in terms of elliptic integrals. The effect of shear, which is manifest in columns with a low effective shear stiffness, was clarified by Engesser (1889) (ignorance of his solution decades later was unfortunately the prime cause for the collapse of the Quebec Bridge in 1907, precipitated by buckling of one latticed diagonal of a truss having insufficient shear stiffness).

The flexibility method of analysis of frames was extended to critical load analysis by formulating the dependence of the flexibility matrix of the column on its axial force (von Mises and Ratzersdorfer, 1926; Chwalla, 1928), and the same was soon done for the stiffness matrix (James, 1935; Livesley and Chandler, 1956). The flexibility method, applied to the primary statically determinate structure of a redundant frame, can also be used but can be misleading if there are many statically indeterminate internal forces because the flexibility matrix of the primary structure, unlike the stiffness matrix of the original structure, can and typically does, lose positive definitiveness before the first critical load.

The matrix stiffness method in the form of finite elements of beams has been proven more suitable for computer analysis and has made the calculation of the critical loads of elastic frames a routine problem. For large regular frames, the critical loads can be obtained analytically by methods of difference calculus [BC, sec. 2.9]. Even simpler analytical solutions can be obtained by approximating the regular frame with a micropolar continuum [BC, sec. 2.10].

A considerably more difficult problem is the buckling of slender high arches or rings. Boussinesq's initial solution of a two-hinge arch was later corrected by Hurlbrink (1908), but a good understanding of arbitrary statically indeterminate arches was not reached until the 1970's.

3. Dynamic instabilities and chaos

A structure can lose stability while under accelerated motion. The treatment of dynamic instabilities necessitates a general stability definition which was contributed by Liapunov (1893). Roughly, it states that the motion of a structure is stable if any possible small change in the initial conditions can lead only to a small change in the response. This is important for non-conservative loads, for example, those produced by wind and generally by fluids or by jet propulsion. Solutions of instabilities of columns under various idealized non-conservative loads such as the follower forces occupied mechanicians in the middle of the century.

Dynamic instability, also called flutter, is an important consideration for aircraft wings, suspension bridges, tall chimneys, guyed masts and other structures (Simiu and Scanlan, 1986). Another type of instability, important for foundations of rotating machinery as well as bridge columns, is the parametric resonance, engendered by the fact that the axial displacement of a column has double the frequency of its lateral vibrations, permitting them to resonate with a load of that frequency. An idealized form of the problem leads to Mathieu differential equation, which was solved approximately by Rayleigh (1894).

An important special case are conservative systems, for which a theorem due to Lagrange (1788) and Dirichlet states that the system is stable if its potential energy is positive definite. This theorem makes it possible to forgo dynamic analysis and reduce the stability problem to an investigation of the shape (and topology) of the potential energy surface as a function of the generalized displacements of the structure. Only limited success has been obtained in a search for functions similar to potential energy, called Liapunov functions, which would decide the stability of non-conservative systems.

The Coriolis (gyroscopic) force, even though it does no work, was found to be the reason for stability of shafts rotating at supercritical speeds. An interesting phenomenon is that nonconservative systems such as rotating machinery stabilized by gyroscopic forces, fluid conveying pipes, aircraft wings and structures under follower forces can be destabilized by damping (Semler et al., 1998; Crandall, 1995; Nissim, 1965).

Recently, the problem of chaotic vibrations of strongly non-linear systems has attracted enormous attention. In such systems, the long-time response may appear completely unpredictable but its trajectory in the phase space exhibits a certain order, being attracted to the fractal basins (Thompson, 1982, 1989, 1986, 1986).

4. Energy methods, post-critical behavior and catastrophe theory

The Lagrange–Dirichlet theorem reduces stability analysis of conservative systems to a check of the positive definiteness of the tangential stiffness matrix of the structure. As a consequence of Liapunov's theorem, the critical loads can be determined from the stiffness matrix of the linearized system, for which the potential energy is quadratic.

The post-critical behavior is characterized by the higher than quadratic terms of the potential energy as a function of generalized displacements. The basic types of post-critical behavior can be classified as stable symmetric (which is always imperfection insensitive), unstable symmetric, and asymmetric (which are both imperfection sensitive, the latter more than the former). For all systems, the initial post-critical behavior is described by Koiter's (1945) power laws, a celebrated result of stability theory according to which, for every elastic structure, an imperfection causes a reduction of the maximum load proportional to either the 2/3 or the 1/2 power of the imperfection magnitude, for all elastic systems. Moderate reductions due to imperfections occur in some types of elastic frames, but in cylindrical shells subjected to axial compression or bending and in spherical domes the maximum load reduction due to inevitable imperfections is major, down to about 1/8 to 1/3 of the critical load.

Another important type of instability of elastic systems is the snapthrough. It occurs in nonlinear systems in which bifurcation with symmetric deflections does not exist, e.g., in flat arches or shallow shells.

The topology of the potential energy surface near the critical load can give rise to very intricate postcritical behavior. Similar behavior occurs in many problems of physics and other sciences. It has recently been intensely studied in the theory of catastrophes (Thompson, 1982, 1989, 1986). A famous result is Thom's (Thom, 1975) proof that for systems with no more than two generalized displacements and no more than four control parameters involving the loads and imperfection magnitudes, there exist no more than seven fundamental catastrophes, called the fold, cusp, swallowtail, butterfly, hyperbolic umbilic, elliptic umbilic, and parabolic umbilic.

The potential energy is also useful as the basis of approximate solutions of critical loads and postcritical behavior. A fundamental role is played by Rayleigh's quotient (Rayleigh, 1894) whose evaluation on the basis of an approximate deflection curve or surface is known to yield an upper bound on the exact critical load. When a column or beam structure is statically determinate, a closer upper bound is obtained from the Timoshenko quotient (Timoshenko and Gere, 1961), which is nothing but the Rayleigh quotient for a reduced-order (second-order) differential equation for beam-column. Minimizing the Rayleigh quotient calculated from the deflection surface expressed as a linear combination of a complete system of chosen linearly independent functions is equivalent to the Ritz direct variational method. The minimization approaches the critical load from above.

In view of structural safety, methods providing lower bounds would be preferable. However, this problem is much more difficult, and the available bounds are usually not close [BC sec. 5.8].

The potential energy expression is also useful for deriving the differential equation of the problem and the boundary conditions via the calculus of variations.

5. Thin-wall beams, plates and shells

Long thin-wall beams, such as metallic cold-formed profiles and steel or concrete girders for bridges or buildings represent long shells which can be approximately treated by a semi-variational approach (Kantorovich variational method), in which the basic deformation modes of the cross section are judiciously selected so that the potential energy, as well as the corresponding differential equation resulting by variational calculus, be one-dimensional. For open cross sections, the most important deformation mode is the warping of the cross section, with the bimoment being the associated force variable. The problem of warping torsion is amenable to simple formulas for lateral buckling and axial torsional buckling of beam columns. For more complicated deformation modes characterizing box cross sections, systems of ordinary differential equations have been solved numerically, both for linear analysis of the critical load and for large nonlinear post-critical deflections.

The critical loads of elastic rectangular or circular plates are easily solved by series expansions. An interesting point is that among the many critical loads of plates (as well as shells), the lowest one often does not correspond to the longest wavelength of the deflection profiles. Studies of postcritical behavior, beginning with von Kármán (1910) and Föppl (1907), have shown that plates are generally not imperfection sensitive and, in fact, possess normally a huge postcritical reserve, which is, however, mobilized only at very large deflections. The limit capacity in postcritical deflections is reached by plasticization of the plate, which develops ridge-shaped buckles causing the plate to act approximately as a truss. Using such a truss analogy, simple formulas have been developed for the maximum loads of rectangular plates (von Kármán et al., 1932), with the remarkable property that the maximum distributed load is independent of the plate dimensions.

The theory of shell buckling has had a fascinating history with a long gestation. After the critical loads of pressurized spherical and axially compressed cylindrical shells were solved early in the century (Lorenz, 1908; Timoshenko, 1910 and Southwell, 1914), experiments showed failure loads that were about 3 to 8 times smaller. This disagreement was much debated and was not explained until von Kármán and Tsien (1941) in their seminal paper found the explanation in highly non-linear postcritical behavior which causes the bifurcation at critical load to be strongly asymmetric. This explanation was later found to fit the general postcritical theory of Koiter (1945). It further took several decades to experimentally demonstrate that the theoretical loads of shells can indeed be closely approached provided the imperfections are extremely small (Almroth et al., 1964; Tennyson, 1969; Tennyson and Chan, 1990). Another reason for the extreme imperfection sensitivity of shells is that there exist many different buckling modes with critical loads so close that they interact. The calculation of failure loads of imperfect shells is a difficult problem, even with finite elements (Budiansky and Hutchinson, 1964, 1971; [BC ch. 7]). Therefore, the practice relies on an empirical correction of the critical load by an empirical 'knock-down' factor, whose values have been tabulated for many typical shell forms.

Usually the rise of the buckles on the shell surface can be considered as shallow. This is assumed in the shallow shell theory, in which the problem of critical load of a cylindrical shell can be reduced to one 8th order partial differential equation for the deflection (Donnell, 1934). For general shells, the problem leads to a system of eight first-order partial differential equations, known as the Donnell–Mushtari–Vlasov theory. In sandwich plates and shells, interaction of global and local buckling is important (Plantema, 1966)

6. Buckling of elasto-plastic structures

In 1889, Engesser (1889) suggested that the critical load of an inelastic column is obtained by simply replacing the elastic modulus with the tangent modulus for loading. But in 1895 he reversed himself (Engesser, 1895, 1899) by proposing that a certain geometry-dependent weighted average of the moduli for loading and unloading, called the reduced modulus, should be used. This theory was later refined and extended by von Kármán (1910). After blatant disagreements with measurements on aluminum alloys were detected in aeronautical industry, Shanley (1947), in an epoch-making paper, showed that Engesser's (Engesser, 1889) original proposal, namely the initial tangent modulus value, should be used because the column does not buckle at constant load, but at increasing load.

Shanley's theory, which was generalized by Hill (1958), is today generally accepted for calculating the first bifurcation of an elastoplastic structure. The fact that a structure must buckle at its first bifurcation load was later established by analysis of imperfections, and still later much more easily on the basis of entropy increment calculation ([BC sec. 10.2], Bažant, 1988). A salient feature of elastoplastic buckling is that the structure is not at the stability limit at the bifurcation state and that the deflected post-bifurcation states are stable.

The distinction between the critical loads of Engesser's reduced modulus theory and Shanley's tangent modulus theory is small for materials such as mild steel, which reach a horizontal yield plateau abruptly. However, hot-rolled steel profiles indicated a large difference between the tangent modulus load and the reduced modulus load. This fact had remained puzzling until it was discovered that the reason consists in large thermal stresses locked in after cooling (Osgood, 1951; Yang et al., 1952), which cause that the diagram of the axial force versus shortening of the column is smoothly curved, without a sudden transition to a yield plateau.

Measurement of bifurcation loads further provided a surprising result with important consequences for the theory of plastic constitutive equation. Tests of torsional buckling of cruciform columns (Gerard and Becker, 1957), in which the critical load depends on the initial tangent modulus for shear, revealed that Hencky's simple deformation theory, criticized in other respects, gives correct results while the incremental plasticity theories based on a single loading surface (e.g. von Mises or Tresca) give values much too high (Hutchinson, 1974). Very complicated behavior is observed in reinforced concrete columns, in which inelastic deformations are combined with tensile cracking and bond slip ([BC], Sec. 8.5; Bažant and Xiang, 1997a,b).

Large plastic deflections of columns need to be understood for predicting the energy absorption capability of the impact, blast, or earthquake. For very large deflections of very slender columns, one can assume formation of plastic yield hinges, which greatly simplifies the calculations. A difficult problem, attacked by finite-strain finite element solutions (e.g. Needleman, 1982; Tvergaard, 1982), has been the localization of plastic strain such as necking in tensioned bars. Plastic localization instabilities are important for the bursting of pipes and other shells due to internal pressure, for bending failure of tubes due to ovalization of the cross section, and for postcritical reserves in plates and thin-wall girders.

7. Thermodynamic analysis of structural stability

Although stability of an inelastic structure can be decided by analyzing the effects of all possible imperfections, it is much simpler and more general to use a thermodynamic approach. Since an inelastic structure is normally far from a state of thermodynamic equilibrium at which all the dissipative processes would come to a standstill, the use of irreversible thermodynamics would in principle be necessary.

The classical thermodynamics, which deals only with states infinitely close to thermodynamic equilibrium (and is much simpler), can nevertheless be used by introducing the hypothesis of a tangentially equivalent inelastic structure [BC, ch. 10]. The existence of such a structure is of course tacitly implied in finite element programs in which the loading increments are analyzed on the basis of tangential stiffness. Because various combinations of loading and unloading are possible, there is generally a number of tangentially equivalent elastic structures to consider.

Having reduced the problem to elastic structures, one can introduce the incremental internally produced entropy of the structure-load system as well as thermodynamic potentials such as the incremental total energy, Gibbs' or Helmholtz's free energy or enthalpy, which represent the thermodynamic potentials under adiabatic and isothermal conditions, and with either the generalized displacements or the associated forces as the variables. Simple thermodynamic criteria for stability of inelastic structures in the pre-peak and post-peak deflections and for snapback behavior under various types of load control, displacement control, and mixed control have been established [BC, sec. 10.1–10.2]. Generally it appears that stability is decided by the positive definiteness of the second variation of these potentials, which represents the second-order work based on the tangential stiffness matrix and is equivalent to the negative of the entropy increment of the structure-load system.

Thermodynamic analysis makes it also possible to determine which branch is followed by the structure after a bifurcation. To this end, one may consider a deviation from equilibrium at constant values of independent variables and changing controls (loads), and subsequent approach at constant controls to a new equilibrium on one or another branch of the post-bifurcation equilibrium path. The path that is followed is that for which the second-order increment of entropy on approach to the new equilibrium state is maximized.

The bifurcation state itself is indicated by singularity of the tangential stiffness matrix [BC, ch. 10, Hill, 1958, 1962; Maier et al., 1973; Petryk, 1985b; Nguyen, 1987]. The tangential stiffness matrix that decides the first bifurcation is determined under the assumption that the tangential modulus for loading applies at all points of the structure. When the lowest eigenvalue of this matrix becomes negative, the bifurcation point has been passed. But the structure is not necessarily unstable, and for this reason, a bifurcation state can be missed in a finite element program if the tangential stiffness matrix is not calculated and checked (de Borst, 1987, 1988a,b).

More complicated behavior is encountered when phenomena such as friction or damage cause the tangential stiffness matrix to be non-symmetric. Singularity of that matrix decides bifurcation, however, stability is decided strictly by the symmetric part of the stiffness matrix, whose eigenvalue is known to be smaller or equal to the lowest eigenvalue of the non-symmetric matrix (Bromwich theorem; [BC] Sec. 10.4).

Stability analysis of load cycles in an elasto-plastic material provides important restrictions for the constitutive laws. It turns out that, in frictional materials, a non-associated flow rule does not necessarily cause instability. This can be analyzed on the basis of the so-called frictionally-blocked second-order energy density [BC, sec. 10.7], which generalizes previous Mandel's (1964) example.

8. Damage localization instabilities

Localization of damage is a favored mechanism of failure of inelastic structures whose material exhibits strain-softening damage. Such damage is described by stress-strain relations that exhibit a postpeak descent of stress at increasing strain (Bažant 1986, 1994; Bažant and Chen 1997; Bažant and Planas, 1998), and in general a loss of positive definiteness of the tangential stiffness matrix of the material. Stress-strain relations of this kind have been used empirically for concrete since the 1950's. However, mechanicians who understood the implications for stability regarded all studies of strain-softening damage for several decades with contempt until it was realized that the concept of strain softening can be put on a sound basis by introducing a characteristic length of the material. Curiously, the continuum damage mechanics escaped such contempt despite the absence of material length, perhaps because its strain-softening features were hidden in a separate damage variable while the 'true' stress exhibited no strain softening. In absence of a characteristic length, the material cannot propagate waves (loading waves, not unloading ones), the dynamic boundary value problem becomes ill-posed, and the partial differential equation changes its type from hyperbolic to elliptic (Hadamard 1903). There were intense polemics on these questions until about 1985.

Strictly mathematically, the concept of strain softening does make sense even without the material length. Unique exact solutions to some wave propagation problems have been given [BC, sec. 13.1]. However, they exhibit physically unacceptable features. The dynamic problem is ill-posed, and as soon as postpeak strain softening is triggered, the damage instantly localizes into a zone of measure 0 (a point, a line, a surface, with zero volume). Thus, the structure is indicated to fail with a zero energy dissipation.

The onset of strain softening, however, can generally be analyzed without introducing the material length. Such analysis was pioneered by Rudnicki and Rice (1975) and Rice (1976), who solved the effect of the geometrically nonlinear features of finite strain on localization of (nonsoftening) plastic strain into an infinite layer of arbitrary thickness within an infinite body. A similar approach to the onset of localization (or bifurcation) has later been pursued for softening materials (in which case the finite strain features become unimportant if the softening is steep). The bifurcation is indicated by the singularity of the so-called acoustic tensor of the material when the orientation of the localization layer is fixed, or the singularity of the tangential stiffness tensor of the material when the orientation is arbitrary (e.g. Rizzi et al., 1995). Localizations can also be triggered by a lack of normality of plastic flow in the case of non-associated flow rule (de Borst, 1988a; Leroy and Ortiz, 1989). In the case of infinite body, the bifurcation condition represents also the stability limit, but for a finite body the thickness of the localization layer becomes important for stability, and stability is lost later. This was shown first for a strain softening bar (Bažant, 1976) and later for a layer of finite thickness in a finite body. The conditions for bifurcation and for the loss of stability have also been analytically formulated for ellipsoidal localization domains, on the basis of Eshelby's theorem [BC, sec. 13.4]. Dynamic bifurcations with localization have been shown to occur under seismic loading in concrete structures (Bažant and Jirasek, 1996)

Strain-softening constitutive relations as an approximation to distributed cracking in reinforced concrete structures have been introduced into finite element analysis by Rashid (1968), Murray and others. Such computational approaches have, however, been shown unobjective with respect to the choice of the mesh size and geometry (exhibiting incorrect convergence on mesh refinement) [BC, ch. 10, Bažant, 1976].

The realization that objectivity of finite element calculations necessitates introducing a characteristic length of the material led first to the formulation of the crack band model (Bažant, 1976; Bažant and Cedolin, 1979; Bažant and Oh, 1983), and then to transplanting the non-local concept from elasticity (Eringen, 1965) into the analysis of strain-softening damage (Bažant et al., 1984). The crack band model, and to a lesser extent the non-local damage model, have undergone many refinements (e.g. Cervenka, 1998) and have found wide practical applications, especially in the analysis of concrete structures and geotechnical excavations. However, the nonlocal model often requires inconveniently small finite elements, and to deal with such cases finite elements with embedded discontinuities (either an embedded band with strain discontinuity or an embedded line with displacement discontinuity) were introduced (Ortiz et al., 1987; Belytschko et al., 1988; see the comparative study by Jirásek, 1998). As an approximation to the nonlocal averaging integral, which seems to have some computational advantages, a second-gradient model for strain softening has been proposed [BC, eq. 13.10.25]. Its effective form solves the nonlocal strain from the local strain from a system of separate Helmholtz partial differential equation (Peerlings et al., 1996). Intricate slip localization instabilities have been found in velocity dependent friction (Rice and Ruina, 1982)

9. Stability problems of fracture propagation

Fracture mechanics presents numerous stability problems, especially when different crack tips interact. In the case of a single crack tip, the limit of stability of crack propagation is reached when the curve of the energy release rate at constant load versus crack length becomes tangent to the R-curve of the material. For some fracture geometries, and for a sufficiently large structure size, crack propagation can lead to snapback instability of the structure. In the case of the cohesive crack model, the stability limit is given in terms of a certain integral equation over the length of the cohesive zone (Bažant and Planas, 1998; Bažant and Li, 1995). Simultaneous growth of many cracks typically leads to bifurcations as well as stability loss, which can be analyzed on the basis of the tangential stiffness matrix expressed in terms of the partial derivatives of the stress intensity factor of each crack tip with respect to the length of every crack in the structure.

Often it is found that, in a homogeneous body, simultaneous propagation of several crack tips does not represent the stable path of the system. Rather, the fracture growth localizes into a single crack and the other cracks stop growing or start unloading. An important example of such behavior are parallel cracks caused by cooling or drying shrinkage in porous materials. The result is that when the parallel cracks reach a certain depth, every other crack stops growing and the intermediate ones propagate further until again every other crack stops growing, etc. In this manner, stability considerations govern the spacing of open parallel cracks. The problem is of interest for the pavements of runways and highways (Li et al., 1995), in geology for the interpretation of drying cracks in mud or cooling cracks in ancient lava flows, etc. [BC] Ch. 12; (Parker, 1999).

10. Finite strain aspects of stability in three dimensions

To determine the tangential stiffness matrix and the critical state, the potential energy must be expressed correctly up to the quadratic terms in displacements. This means that the finite strain tensor must be expressed correctly up to the second-order terms. However, there are many types of finite strain measure which have different second-order terms (e.g. Green's Lagragian, Biot's, Hencky's, etc.). For each of them, the incremental equations of equilibrium, the tangential stiffness matrix and the critical loads are given by differently looking expressions. This fact has caused a long lasting confusion in the stability theory of three-dimensional bodies. A number of different theories which apparently were giving very different results were proposed by Southwell (1914), Biezeno and Hencky (1928), Trefftz (1933), Pearson, Hill (1958), Haringx (1942), Neuber (1943) and others.

This zoo of formulations led to controversies. It transpired, however, that all these theories are equivalent (Bažant, 1971) [BC, ch. 11] because the constitutive relations, and thus the tangential stiffness moduli of the material, are different, with different moduli values for theories associated with different quadratic components of the finite strain tensor. Simple relations between the tangential elastic moduli for different theories have been established. In this manner, it was for example shown that Engesser's and Haringx's formulae for shear buckling of columns are not in contradiction but in fact identical (although the shear modulus must of course be measured for each formula in a different manner).

The aforementioned problem does not arise for beams, plates and shells without shear because in those cases the second-order part of the finite strain tensor depends only on the rotations of the cross sections, in which there is no ambiguity.

Finite strain theory is needed to solve three-dimensional internal buckling of solids, surface buckling, buckling of thick columns and thick tubes, bulging of compressed bars, etc. The critical loads for these buckling modes are generally of the same order of magnitude as the tangential stiffness moduli of the material. This means that such instabilities can occur only in materials that exhibit a high degree of orthotropy, with a low shear stiffness, a condition that can arise as a result of material damage (especially oriented cracking), fibrous microstructure or latticed microstructure (e.g. built-up latticed columns). In compressed fiber composites, the three-dimensional instability gives rise to propagation of kink bands, which control compression strength (Rosen, 1965; Budiansky, 1983; Budiansky et al., 1997; Fleck, 1997; Bažant et al., 1999).

11. Buckling of viscoelastic and viscoplastic structures

Time dependence of material behavior, that is viscoelasticity or viscoplasticity, may lead to instabilities that develop not suddenly but over a long period of time. If the material is linearly viscoelastic and is a solid, a slender structure possesses a long-time critical load, which is the load that must be reached or surpassed for an infinitesimal imperfection to lead to a finite deflection in infinite time. The long-time buckling problem may be solved by replacing the elastic moduli in the elastic formulation by the corresponding viscoelastic operator, which may be of rate type or integral type (Freudenthal, 1950, 1952). The long-time critical load, however, need not be important for design. Important is the time to reach, for given imperfections, the maximum tolerable deflection or stress due to buckling. This time must exceed the design life time.

When the material is viscoplastic, there exists, in contrast to viscoelastic materials, a finite critical time at which the deflection triggered by an infinitely small imperfection becomes finite (Hoff, 1958). The critical time for viscoplastic buckling of metals controls the temperature to which various mechanical parts of heat engines can be exposed, determine the demands for insulation of steel structures against fire, etc. Considerable complications arise in the analysis of long-time buckling of concrete structures, due to chemically induced long-time aging of the material as well as distributed cracking and its gradual spread with time [BC, ch. 9]. Long time buckling is particularly important for thin shell concrete roofs.

12. Concluding comments

Stability of elastic structures appears to be reasonably well understood at present although many refinements are still needed and some basic advances may still be expected. The greatest challenge and opportunity probably lies in stability analysis of damage and fracture, and its interaction with geometrical nonlinearity of deformation. Coupled problems, in which structural stability analysis interfaces with chemical processes in materials, hygrothermal effects and various types of long-time degradation will no doubt play an increasingly important role. So will the probabilistic treatment of safety against the loss of stability or excessive deflection (Bolotin, 1969) — a subject that has also seen considerable advances but lies beyond the scope of this survey.

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