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INELASTICITY AND FAILURE OF CONCRETE: A SURVEY OF RECENT PROGRESS

by Z. P. BAŽANT

POLITECNICO DI MILANO

CORSO DI PERFEZIONAMENTO PER LE COSTRUZIONI IN CEMENTO ARMATO "FRATELLI PESENTI"

Cinquantenario della istituzione 1927/28 – 1977/78

ANALISI DELLE STRUTTURE IN CEMENTO ARMATO MEDIANTE IL METODO DEGLI ELEMENTI FINITI

seminario 20-23 giugno 1978

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Nella seconda metà del Febbraio 1928 iniziavano al Politecnico di Milano le prime lezioni del Corso di Perfezionamento per le Costruzioni in Cemento Armato, voluto e patrocinato dalla famiglia Pesenti con lo scopo di contribuire a formare degli "ingegneri specializzati in tutte le applicazioni del cemento, e specialmente nelle costruzioni in cemento armato". Ricorre così in questo anno accademico il cinquantenario di questa istituzione, la cui storia è riportata nel volume XIV (1977) degli Studi e Rendiconti del Corso.

Fra le molte iniziative possibili per una degna celebrazione di una data così importante si è scelta quella più corrispondente allo scopo istituzionale del corso, realizzando in maniera libera e pubblica un Seminario che descriva lo "stato dell'arte" nella analisi delle strutture in cemento armato mediante il metodo degli elementi finiti. Tale metodo, discretizzando la struttura in un insieme di elementi, permette di adottare per ciascuno di questi la descrizione analitica più appropriata. Questo è particolarmente importante nelle strutture in cemento armato, perchè consente di tenere conto, in un'unica analisi, sia di componenti diverse quanto a tipologia strutturale, sia del comportamento oltremodo complesso del calcestruzzo di cemento, caratterizzato da legame costitutivo non lineare, deformazioni di tipo viscoso e fessurazione.

Il Seminario è stato idealmente diviso in due parti. La prima riguarda l'esposizione delle ricerche relative alla formulazione dei legami costitutivi da introdurre nella analisi, campo in cui le possibilità analitiche del metodo in oggetto sopravanzano di gran lunga la conoscenza del materiale dal punto di vista fisico. La seconda riporta una descrizione della metodologia di analisi e delle sue applicazioni ai principali tipi di strutture della ingegneria civile, la quale mette in evidenza le condizioni cui deve soddisfare il modello analitico affinchè possa riprodurre il comportamento reale delle strutture. A tale proposito sento il dovere di ringraziare i professori Zdenek P. Bazant, William Schnobrich e Alex C. Scordelis, che sono stati dei pionieri in questo campo di ricerca, per l'entusiasmo con cui hanno accettato di tenere questo Seminario, e per la maniera così aderente allo spirito informatore con cui hanno preparato il testo delle loro lezioni. Ringrazio anche il mio collaboratore, ing. Luigi Cedolin, per avere svolto così efficacemente il lavoro di coordinamento dei diversi interventi.

Anche in questa occasione rinnovo alla famiglia Pesenti, e per essa alla Società Italcementi di Bergamo, l'espressione della gratitudine per il mecenatismo con cui ha da sempre sostenuto il Corso di Perfezionamento, che ha reso possibile anche questa iniziativa.

Il Direttore del Corso Prof. Sandro Dei Poli

Milano, 20 giugno 1978

Prof. Zdeněk P. Bažant

Il professor Zdeněk P. Bažant ha ricevuto il diploma di dottorato dalla Accademia delle Scienze Cecoslovacca nel 1963 e divenne docente al Politecnico di Praga nel 1967. Dopo aver ricoperto incarichi di insegnamento al C.E.B.T.P. di Parigi, all'Università di Toronto e all'Università di California a Berkeley, dal 1969 egli insegna presso la Northwestern University, dove nel 1973 divenne professore ordinario. E' consulente di Argonne National Laboratory, Oak Ridge National Laboratory e Sargent and Lundy, Engrs. Ha presieduto la commissione ASCE-EMD sulle proprietà dei materiali e fa parte della direzione editoriale dei giornali ASCE-EMD e Cement and Concrete Research. Il professor Bazant ricevette la medaglia RILEM del 1975, essendo citato per "brillanti sviluppi nella meccanica dei materiali, nella termodinamica dei fenomeni viscosi, e nella teoria della stabilità, fondendo ricerca sperimentale e formulazione teorica". Nel 1976 ricevette dalla ASCE il premio W.L. Huber per la ricerca nella ingegneria civile, essendo citato per "ricerca sugli effetti della viscosità, inelasticità e contenuto di umidità nel calcestruzzo, comportamento non lineare e dipendente dal tempo nelle strutture, teoria della stabilità e della frattura" e nel 1977 il premio T.Y. Lin per il cemento armato precompresso, per la memoria "Viscosità e ritiro nei gusci di contenimento di reattori nucleari". Egli ha appena ricevuto la prestigiosa borsa di studio Guggenheim per la ricerca per l'anno accademico 1978-79.

Prof. William C. Schnobrich

Il professor William C. Schnobrich ricevette i diplomi di Bachelor of Science nel 1953, Master of Science nel 1955 e Philosophy Doctor nel 1962, tutti dall'Università dell'Illinois. Dopo aver ricoperto incarichi di ricerca in varie agenzie governative, nel 1962 cominciò ad insegnare presso l'Università dell'Illinois, dove nel 1968 divenne professore ordinario. E' consulente di molte società di progettazione per l'analisi di strutture speciali in cemento armato. E' membro della commissione ASCE-EMD per i metodi matematici e delle commissioni ASCE-STD per le strutture speciali, la progettazione delle strutture a guscio di cemento armato e l'analisi per elementi finiti delle strutture in cemento armato. Molta parte della attività di ricerca del professor Schnobrich è stata dedicata allo sviluppo di metodi analitici per lo studio della risposta statica e dinamica delle strutture. Si è occupato in modo particolare dell'analisi per elementi finiti di strutture piane e a guscio, in cemento armato, per le quali ha sviluppato metodi di calcolo capaci di tenere conto di legami costitutivi non lineari e fessurazione ora universalmente adottati. Recentemente si è occupato della resistenza alle azioni sismiche di pareti di controvento e torri di refrigerazione ad iperboloide. Tra le molte opere di cui è stato consulente per la progettazione va ricordato il "Gymnasium and Natatorium" del George Williams College. Nel 1977 ha ricevuto il premio A. Von Humboldt per le Scienze.

Prof. Alex C. Scordelis

Il professor Alex C. Scordelis ricevette il diploma di Bachelor of Science dall'Università di California nel 1948 ed il diploma di Master of Science dal Massachusetts Institute of Technology nel 1949. Nello stesso anno cominciò ad insegnare presso l'Università di California, dove successivamente divenne professore ordinario. E' consulente di diverse società di progettazione ed agenzie governative per l'analisi di strutture a guscio e ponti in cemento armato. E' presidente della commissione ASCE per le strutture in cemento armato e membro delle commissioni: ASCE-ACI e IASS per i gusci in cemento armato, ASCE-STD per l'analisi delle strutture in cemento armato con il metodo degli elementi finiti, ACI per i ponti in cemento armato. La ricerca svolta dal professor Scordelis è stata sempre rivolta allo studio teorico e sperimentale delle strutture in cemento armato, ed in particolare dei gusci e delle travi da ponte a cassone. Fin dagli anni '50 ha riconosciuto le possibilità del metodo degli spostamenti, in associazione con lo sviluppo dei calcolatori elettronici, per l'analisi delle strutture a guscio ed è stato tra i primi ad applicare il metodo degli elementi finiti allo studio delle strutture in cemento armato. Fra le molte opere di cui è stato consulente per la progettazione va ricordata la copertura a paraboloide iperbolico della St. Mary's Cathedral San Francisco. Nel 1976 ha ricevuto dall'ASCE il premio Moisseiff per la memoria "Analisi non lineare di gusci di cemento armato di forma qualsiasi", e nel 1978 è stato eletto membro della Accademia Nazionale di Ingegneria degli Stati Uniti.

PART I

INELASTICITY AND FAILURE OF CONCRETE: A SURVEY OF RECENT PROGRESS

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INTRODUCTION

Emergence of powerful computational methods, such as the finite element methods, makes it today possible for the structural analyst to consider rather complicated mechanical properties of materials. While twenty years ago the knowledge of material behavior was ahead of the computational capabilities, recently it has fallen behind. This is especially true of concrete, a material with extremely complicated, one could say "messy," properties. With the powerful finite element codes available, we indeed could today analyze concrete structures more accurately if more realistic models of material behavior were available to us. In fact, this is not only a possibility, but a compelling need, as far as certain modern types of structures, e.g., the concrete reactor vessels or ocean structures are concerned. It will be characteristic for some time to come that in investigating the methods of structural analysis, i.e., the finite element methods, we will have to pay a great deal of attention to the models of material behavior to be used and the effective adaptation of the computational algorithms to these models.

Significant advances in the modeling of the inelastic behavior of concrete have been achieved during the last decade. The purpose of this work is to attempt to give a survey of the recent developments, serving as a background and summary of a series of twelve lectures delivered at the Politecnico di Milano as part of an international seminar commemorating the 50th anniversary of the Graduate School for Reinforced Concrete Structures (Fondazione Fratelli Pesenti). No claims for completeness of this survey are set; a certain bias emphasizing the contributions made at the writer's home institution is inevitable, especially since many of the results surveyed have not yet been published in periodicals and exist still only in report form. The reader should judge the following text in this light.

Emphasis will be placed on the fundamentals of the constitutive equations for nonlinear triaxial behavior. The coverage of creep and moisture effects will be disproportionately narrow and will be focused only on selected few very recent developments because a comprehensive survey of this subject was published not too long ago [39].

CHAPTER A1

INCREMENTALLY LINEAR THEORIES FOR TRIAXIAL BEHAVIOR

A1.1 Simple Generalizations of Elasticity

The uniaxial stress-strain diagram of concrete is easily described by a simple formula, such as those due to Saenz, Sargin, Popovics, etc. Where the complexity comes from is the triaxial nature of the response and the fact that the response must be modeled without violating basic principles of continuum mechanics [1], which include the principle of objectivity and the associated conditions of invariance during a rotation of coordinate axes. Many of the recently proposed models for nonlinear triaxial behavior of concrete do not satisfy these invariance conditions under general loading. Consequently, these models are of limited usefulness and will not be described herein.

Recognizing that classical plasticity, as developed for metals, is inapplicable to concrete, Coon and Evans [2] adopted the far more general framework of hypoelasticity, in which the increments of stress tensor $\underset{\sim}{\sigma} = [\sigma_{ij}]$ and strain tensor $\underset{\sim}{\epsilon} = [\epsilon_{ij}]$ are assumed to be related linearly and the proportionality coefficients are assumed to be functions of σ_{ij} and ϵ_{ij} but not of their history, and not of their rates. Thus

$$d\sigma_{ij} = C_{ijkm}(\underset{\sim}{\sigma},\underset{\sim}{\epsilon})d\epsilon_{km} \quad \text{or} \quad d\epsilon_{ij} = D_{ijkm}(\underset{\sim}{\sigma},\underset{\sim}{\epsilon})d\sigma_{km} \quad \text{...........(A1.1)}$$

in which C_{ijkm} = tensor of tangent stiffness moduli and D_{ijkm} = tensor of tangent compliances (flexibilities). Lower case latin subscripts refer to cartesian coordinates x_i ($i = 1,2,3$) and repetition of a subscript implies summation. Only small (linearized) strains are considered in this work.

Assuming that the response diagrams are smooth at least up to the peak point (strength), which is true of concrete, the peak point of the response diagram is obtained by setting $d\sigma_{ij} = 0$; Eq. (A1.1) then forms a homogeneous linear equation system and a non-zero solution $d\epsilon_{ij}$ is possible only if the determinant of the (6×6) matrix of C_{ijkm} vanishes, which yields the strength criterion [2]. To satisfy the condition of isotropy, C_{ijkm} (or D_{ijkm}) must

be an isotropic tensor polynomial in σ_{ij} and ϵ_{ij}, the general forms of which have been determined by Truesdell. Polynomials of degree two and higher involve too many constants, and so Coon and Evans [2] limited attention to a linear tensor polynomial, which contains only seven material constants, two of which are related to Young's modulus E and Poisson's ratio ν in stress-free state. This formulation gave a reasonable description of the uniaxial and biaxial response diagrams up to about 75% of the peak point (strength). However, the failure condition resulting from this formulation was very poor and, especially, the strains that corresponded to the peak point were infinite. These shortcomings could undoubtedly be alleviated by using higher-order tensor polynomials, but these involve too many terms and too many constants which are not practically manageable. Therefore, physical concepts are needed to guide the selection of the functional dependence of C_{ijkm} or D_{ijkm} upon $\underset{\sim}{\sigma}$ and $\underset{\sim}{\epsilon}$. Incremental plasticity, which can be regarded as a special case of hypoelasticity, offers the necessary physical concepts.

An important property reflected in Eq. A1.1 is the stress-induced anisotropy (or strain-induced anisotropy, or a combination of both). The tangent moduli tensor C_{ijkm} has an anisotropic form unless the stress is zero or the stress tensor is isotropic (hydrostatic pressure). Indeed, if we admit that the tangent moduli depend on stress, and if the stress tensor is not isotropic, it would be entirely unreasonable to a priori assume an anisotropic form of C_{ijkm}. This is confirmed by the fact that non-isotropic stress states, e.g., the uniaxial compression or tension, produce microcracks that are not oriented randomly but exhibit a certain prevalent orientation depending on the stress. The property of stress-induced anisotropy is characteristic of all models for the behavior of concrete, especially the incremental plastic and fracturing models as well as the endochronic theory.

Various simplified forms of the hypoelastic formulation have been appearing in the literature and are widely used in finite element programs. In one form, the (6×6) tangent compliance matrix $\underset{\sim}{D}$ is expressed in terms of E and ν in the same (isotropic) form as for the stress-free state, and the tangent modulus is varied as a function of the maximum principal strain. Alternatively, $\underset{\sim}{D}$ is expressed in terms of bulk modulus K and shear modulus G in the same form as for the stress-free state and G and K are varied as functions of the stress and strain invariants. These formulations, of course, do not reflect the stress-induced anisotropy. Another model, called "orthotropic," assumes the tangent moduli matrix $\underset{\sim}{C}$ to be of the same form as for an orthotropic material and depending on the principal stresses, strains, and their ratios. This model cannot reflect, however, a general stress-induced anisotropy which is always produced if the increase of stress components or strain components is non-proportional, i.e., when the directions of principal stresses or strains rotate in the course of loading. The orthotropic formulations have been made to represent uniaxial and biaxial stress-strain curves for proportional loading up to the peak stress point, but they do not seem to be capable of representing a broader set of test data (especially on volume change and lateral strains).

Another generalization of elasticity, originally introduced by Hencky and Nádai to describe plasticity of metals, is the deformation theory (or total-strain theory) [3-5,1], in which a unique relation is assumed to exist between the total stresses and the total strains; for an isotropic material:

$$s_{ij} = 2G(\underset{\sim}{\sigma},\underset{\sim}{\epsilon})e_{ij} \quad \text{(A1.2)}$$

$$\sigma_{kk} = 3K(\underset{\sim}{\sigma},\underset{\sim}{\epsilon})\epsilon_{kk} \quad \text{(A1.3)}$$

where s_{ij}, e_{ij} = deviators of σ_{ij} and ϵ_{ij}; G and K are now the secant shear and bulk moduli. This formulation has the advantage that one does not need to worry about the stress-induced anisotropy, for if the stresses and strains are referred to the initial stress-free state which is isotropic, the stress-strain relation must also be of isotropic form. So, the secant moduli G and K may be assumed to depend only on the invariants of stress and strain. Such a formulation has been developed for concrete, for example, by Kupfer and Gerstle [6] and by Kotsovos and Newman [7], and it has been shown to fit reasonably well proportional biaxial and triaxial tests, but not the strain-softening branches and the peak loads. The chief shortcoming of the deformation theory is that strains do not depend on the stress and strain paths prior to the current state, which is not true. Nevertheless, theories of this type generally give a better prediction of the stiffness of the material in the direction tangent to the current loading surface than does incremental plasticity with the normality rule (which was for metals pointed out by Budianski [8]). Differentiating Eq.(A1.1)

$$ds_{ij} = 2G\,de_{ij} + 2e_{ij}\left(\frac{\partial G}{\partial \sigma_{km}} d\sigma_{km} + \frac{\partial G}{\partial \epsilon_{km}} d\epsilon_{km}\right) \quad \text{(A1.4)}$$

$$d\sigma_{kk} = 3K\,d\epsilon_{kk} + 3\epsilon_{kk}\left(\frac{\partial K}{\partial \sigma_{km}} d\sigma_{km} + \frac{\partial K}{\partial \epsilon_{km}} d\epsilon_{km}\right) \quad \text{(A1.5)}$$

we see that the relation between ds_{ij} and de_{ij} is linear but not isotropic. So, the deformation theory exhibits a sort of stress-induced anisotropy.

A1.2 Incremental Plasticity

Incremental plasticity results also in a linear relation between stress and strain increments, same as the theories discussed so far. However, the difficult question of stress-induced anisotropy is not attacked in a direct, phenomenological manner, but physical concepts are employed to answer this question indirectly, as a by-product. These physical concepts center on the notion of yield surface (yield function) $F(\sigma_{ij}) = 0$, defined as the envelope of all states that can be reached from the current stress state elastically. Obviously this property should in general depend on the current state, and so the yield surface is generally variable in the loading process, which is called hardening. (This feature is neglected in ideal plasticity.) The existence of a current yield surface is an assumption, since for concrete practically no adjacent state can be reached by purely elastic deformation (as is actually reflected in the endochronic theory); this is to a lesser extent true even of metals [9,10].

The change of the loading surface in the process of loading may be characterized by hardening parameters H_k, writing the current loading surface as

$$F(\sigma_{ij}, H_k) = 0 \quad \text{(A1.6)}$$

Choosing $(\partial F/\partial H_k)dH_k > 0$ for loading (inelastic straining), we obtain

$$\frac{\partial F}{\partial \sigma_{km}} d\sigma_{km} > 0 \quad \text{(A1.7)}$$

as the loading criterion (or the condition for inelastic straining). Hence, $d\epsilon''_{ij}$ must be a function of this expression. Assuming a linear dependence of inelastic strains $d\epsilon''_{ij}$ upon $d\sigma_{ij}$ we further obtain

$$d\epsilon''_{ij} = g_{ij}(\partial F/\partial \sigma_{km})d\sigma_{km} \quad \text{.......... (A1.8)}$$

where g_{ij} are some proportionality constants.

Note that it would be incorrect to introduce a loading criterion separately for individual strain components, saying, e.g., that ϵ_{11} loads if $d\epsilon_{11} > 0$ and ϵ_{22} unloads if $d\epsilon_{22} < 0$; the fallacy in this approach is that upon rotation of coordinates the signs of individual strain increments may become different, making it impossible to make the loading criterion invariant.

An essential further concept is the Drucker's stability postulate [11,12,3-5], which requires that

$$\Delta W_2 = \frac{1}{2}\, d\sigma_{km}\, d\epsilon''_{km} \geq 0 \quad \text{.......... (A1.9)}$$

This condition guarantees that the work done on the material during a cycle of applying and removing $d\sigma_{ij}$ be non-negative. (This is only the second-order part of the work, since the first-order work, $\oint \sigma_{ij} d\epsilon_{ij}$, cancels with the work of applied loads according to the principle of virtual work.) Under a wide range of situations, the non-negativeness of ΔW_2 is equivalent to a material stability condition. However, this is not true when friction is involved, as has been shown by J. Mandel [13] and G. Maier [14]. In fact, a simple example of a spring-loaded friction block which violates inequality (A1.9) yet is stable in the Liapunov sense can be given. It has been a surprise when an example was discovered [15] for an opposite situation; namely, a case when the material satisfies Drucker's postulate (A1.9) yet is unstable in Liapunov sense. (This may be observed in a material which has no elastic response, i.e., is inelastic also for unloading, as is true of concrete.) Consequently, the connection of Drucker's postulate with stability is not unambiguous. Nevertheless, violation of Drucker's postulate should be admitted only when there is a clear physical reason for doing so, such as the effect of internal friction, which is absent in metals but important for concrete and soils.

Adopting Drucker's postulate (A1.9), comparing it to inequality (A1.7), and noting that both inequalities must hold for any $d\sigma_{km}$, it follows that

$$d\epsilon''_{ij} = \frac{\partial F}{\partial \sigma_{ij}}\, d\mu \quad \text{.......... (A1.10)}$$

where the proportionality constant $d\mu$ is, by comparison with Eq. (A1.8)

$$d\mu = \frac{1}{h}\left(\frac{\partial F}{\partial \sigma_{km}}\, d\sigma_{km}\right) \quad \text{.......... (A1.11)}$$

where h is a parameter depending on σ_{ij} and ϵ_{ij}. Eq. (A1.10) is the well-known normality rule, which tremendously simplifies the treatment of inelastic phenomena. According to the previous discussion, however, the use of normality rule is not a necessity. In fact we could again give examples of various frictional materials where violation of the normality rule must be expected. E.g., one may consider a pile of sheets (a book) subjected to shear. According to the normality rule, the pile would have to dilate upon shearing, but it does not. Or one may consider the sliding of two plates with toothed surfaces; if the surface teeth are steep enough, the dilatancy (movement apart) enforced by the sliding may exceed that indicated by the normality rule.

Noting that $d\sigma_{ij}$ is involved in Eq. (A1.11) linearly, it is clear that the stress-strain relations of incremental plasticity may always be put in hypoelastic form $d\epsilon_{ij} = D_{ijkm}(\underset{\sim}{\sigma},\underset{\sim}{\epsilon})d\sigma_{km}$. This form may further be inverted to yield the tangential moduli $C_{ijkm}(\underset{\sim}{\sigma},\underset{\sim}{\epsilon})$. Although the inversion of a (6×6) matrix in general leads to prohibitively complex expressions, a rather simple expression for the inverse matrix of C_{ijkm} is possible here, as was first recognized by Prager [16]. An analogous inverted matrix was introduced by Yamada and Zienkiewicz [65,66] when they formulated the finite element method for incremental plasticity.

The yield function for concrete must reflect the internal friction. The simplest form is then the loading surface of Drucker-Prager type [64]:

$$F(\sigma_{ij}, H_k) = \bar{\tau} + g(\sigma) - H_1 = 0 \quad \text{.......... (A1.12)}$$

where $\sigma = \sigma_{kk}/3$ = mean stress, $\bar{\tau} = (\frac{1}{2}\, s_{km} s_{km})^{\frac{1}{2}}$ = stress intensity. The incremental stress-strain relations based on this loading surface can represent reasonably well uniaxial, biaxial and triaxial stress-strain diagrams. However, lateral strain and volume dilatancy data, and especially the strain-softening branches, cannot be described well, not even with more complicated forms of the loading function. The problem is not with the shape of the loading surface; abandoning the von Mises-type deviatoric loading surface, based on $\bar{\tau}$, and introducing more complex forms that involve I_3, the third invariant of stress, brings only minor improvements which are inconclusive in view of the scarcity and scatter of test data on triaxial behavior of one and the same type of concrete. The problem is rather with the inherent limitations of the incremental-plastic formalism and its unsuitability to represent strain-softening. This limitation can be surmounted by a new concept, the theory of fracturing material.

A1.3 Plastic-Fracturing Material with Friction and Dilatancy

Plastic slip does not lead to softening of the material. The only physical mechanism which can explain strain-softening is the microcracking. This was realized by Dougill [17], who proceeded to formulate a theory for a purely fracturing material (no plasticity) which is completely analogous to incremental plasticity. He begins by postulating the existence of a fracturing surface

$$\phi(\varepsilon_{ij}, H_k) = 0 \qquad \text{(A1.13)}$$

which envelops all states that can be reached without further fracturing. Choosing $(\partial\phi/\partial H_k)dH_k$ to be negative when fracturing occurs (H_k are the fracturing parameters), the condition of fracturing is obtained as

$$\frac{\partial\phi}{\partial\varepsilon_{km}} d\varepsilon_{km} \geq 0 \qquad \text{(A1.14)}$$

Since the matrix of the purely fracturing material is perfectly elastic, the relation $\sigma_{ij} = C_{ijkm}\varepsilon_{km}$ must hold, indicating that upon unloading the material always returns to the initial strain-free state. By differentiating, we then have

$$d\sigma_{ij} = C_{ijkm} d\varepsilon_{km} - d\sigma_{ij}^{fr} \qquad \text{(A1.15)}$$

$$d\sigma_{ij}^{fr} = -dC_{ijkm}\varepsilon_{km} \qquad \text{(A1.16)}$$

where $d\sigma_{ij}^{fr}$ are called fracturing stress increments (actually decrements). By comparison of Eq. (A1.16) with Eq. (A1.14) we acquire

$$d\sigma_{ij}^{fr} = -g_{ij}\left(\frac{\partial\phi}{\partial\varepsilon_{km}} d\varepsilon_{km}\right) \qquad \text{(A1.17)}$$

where g_{ij} are some proportionality constants.

To gain a simple, manageable form of the theory we further need some work inequality analogous to Drucker's postulate. This is provided by the Il'yushin's postulate (cf. Ref. 18):

$$\Delta H_2 = \frac{1}{2} d\sigma_{ij}^{fr} d\varepsilon_{ij} \geq 0 \qquad \text{(A1.18)}$$

where ΔH_2 represents the second-order complementary work done on the material during the application and removal of the strain increments $d\varepsilon_{ij}$. Comparison with inequality (A1.14) and further comparison with Eq. (A1.17) then yield

$$d\sigma_{ij}^{fr} = -\frac{\partial\phi}{\partial\varepsilon_{ij}} 2d\kappa \qquad \text{(A1.19)}$$

$$2\,d\kappa = \phi\frac{\partial\phi}{\partial\varepsilon_{km}} d\varepsilon_{km} \qquad \text{(A1.20)}$$

in analogy with plasticity. Eq. (A1.19) represents the normality rule.

Similarly to the Drucker-Prager yield function in plasticity, the fracturing surface may be introduced as

$$\phi(\varepsilon_{ij}, H_k) = \bar{\gamma} + k(\varepsilon) - H_1 = 0 \qquad \text{(A1.21)}$$

where $\varepsilon = \varepsilon_{kk}/3$, $\bar{\gamma} = (\frac{1}{2} e_{ij}\ e_{ij})^{\frac{1}{2}}$ = strain intensity. Applying Eqs. (A1.19) and (A1.20), it follows that [19]

$$ds_{ij} = 2G\,de_{ij} - ds_{ij}^{fr}, \quad ds_{ij}^{fr} = e_{ij}\frac{d\kappa}{\bar{\gamma}} \qquad \text{(A1.22)}$$

$$d\sigma = 2K d\varepsilon - d\sigma^{fr}, \quad d\sigma^{fr} = \frac{2}{3}\alpha\, d\kappa \qquad \text{(A1.23)}$$

$$d\kappa = \frac{1}{2}\phi(d\bar{\gamma} + \alpha' d\varepsilon) \text{ if } d\kappa \geq 0; \text{ else } d\kappa = 0 \qquad \text{(A1.24)}$$

in which $\alpha' = dk(\varepsilon)/d\varepsilon$. Full normality, as indicated by Eq. (A1.19), is obtained for $\alpha = \alpha'$; but this restriction is deliberately relaxed, for the same reasons as those regarding internal friction.

With regard to plasticity, there is, however, one conspicuous difference in that the elastic properties, C_{ijkm}, are not constant but variable. Dougill [17] has explored the consequences of the normality rule (Eq. A1.19) for the degradation of material constants C_{ijkm} (Eq. A1.16) and for the admissible shape of the loading surface. It appeared that a fracturing elastic material which exhibits full path-independence is described by a fracturing surface that is linear in ϵ_{ij}. This corresponds [17] to a material that consists of a random system of bonded straight fibers, which is not a realistic model for concrete. Therefore, when modeling the actual behavior of concrete, one cannot follow the concepts of fracturing surface, path independence and normality rule completely.

While the stress-induced anisotropy of the relation between the total strain and stress increments is essential, their elastic parts may approximately be related by a tangent moduli tensor that is isotropic (except for purely tensile cracking, for which the microcracks are highly oriented). This "elastic" isotropy has in fact been assumed in Eqs. (A1.22) and (A1.23); thus,

$$ds_{ij}^{fr} = -2\, e_{ij}\, dG, \quad d\sigma^{fr} = -3\,\epsilon\, dK \qquad \text{(A1.25)}$$

Squaring and summing of the first relation furnishes the approximation [19]:

$$dG = -\frac{d\kappa}{2\bar{\gamma}}, \quad dK = -\frac{2\alpha}{9}\frac{d\kappa}{\epsilon} \qquad \text{(A1.26)}$$

from which the fracturing dilatancy factor α may be expressed as [19]:

$$\alpha = \frac{9\epsilon}{4\bar{\gamma}}\frac{dK}{dG} \qquad \text{(A1.27)}$$

This relation allows one to utilize a theoretical relationship between K and G for an elastic material containing a random array of cracks of varying area. Such a relationship has recently been derived by Budianski and O'Connell [20] using Hill's self-consistent method for composites. This development allows us to theoretically determine the function [19]:

$$\frac{dK}{dG} = f\left(\frac{G}{G_0}\right) \qquad \text{(A1.28)}$$

where G_0 = value of G in the stress-free state. The dilatancy factor then follows from Eq. (A1.28), and clearly it does not obey normality except by accident. Eq. (A1.28) gave a reasonable agreement with test data for compression softening of concrete [19].

The mechanism of inelastic deformation of concrete consists of both microcracking and plastic slip. The former prevails at low hydrostatic pressure and in the later stages of the uniaxial compression test. The latter dominates at high hydrostatic pressures and is also important in the early stages of the uniaxial compression test. An impression of the nature of inelastic strain may be gained from the slopes of the unloading branches; as long as they are roughly parallel to the initial slope at the first loading, fracturing cannot constitute a significant part of inelastic strain, and only when the unloading slope becomes much less steep than the initial loading slope, as is characteristic of the states at the falling branch of the uniaxial compression diagram, the fracturing behavior becomes important.

Accordingly, a realistic model for concrete must combine both the plastic and fracturing strains. Because the plastic stress-strain relations express the stress increment as the sum of the elastic and inelastic strain increments, while in the fracturing model it is the elastic and inelastic stresses rather than strains which are summed, it is necessary to first invert the incremental plastic stress-strain relations based on Eq. (A1.12). After doing this, the plastic and fracturing stress increments may be superimposed. This procedure leads to the following relations [19]:

$$ds_{ij} = 2G\, de_{ij} - 2G\, s_{ij}\frac{d\mu}{\bar{\tau}} - e_{ij}\frac{d\kappa}{\bar{\gamma}} \qquad \text{(A1.29)}$$

$$d\sigma = 3K d\epsilon - 2K\beta\, d\mu - \frac{2}{3}\alpha\, d\kappa \qquad \text{(A1.30)}$$

in which

$$d\mu = \frac{G\, s_{km}\, d\epsilon_{km} + \bar{\tau} K\beta' d\epsilon_{kk}}{2\bar{\tau}(G + h + K\beta\beta')} \quad \text{if } d\mu \geq 0\text{; else } d\mu = 0 \qquad \text{(A1.31)}$$

$$d\kappa = \frac{\phi}{2}\,(d\bar{\gamma} + \alpha' d\epsilon) \quad \text{if } d\kappa \geq 0\text{; else } d\kappa = 0 \qquad \text{(A1.32)}$$

while the variation of G and K is determined from Eq. (A1.26). The coefficients h, ϕ, β, β', α, α' are certain functions of stress and strain invariants (as well as the characteristics of concrete, such as strength f'_c). With the help of various intuitive considerations the proper form of these functions has been deduced and material parameters have been identified by fitting various test data available in the literature [19]. A very good agreement with test data, comparable to that previously achieved with the endochronic theory (and exemplified in Figs. A2.2-6 in the sequel), has been attained for tests under monotonically increasing load [19]. However, the behavior at unloading and cyclic loading is not predicted by Eqs. (A1.29)-(A1.32), and further developments would be necessary for this purpose.

CHAPTER A2

INCREMENTALLY NONLINEAR THEORIES FOR TRIAXIAL BEHAVIOR

A2.1 Endochronic Theory and Viscoplasticity

Endochronic theory is a novel form of constitutive relations for time-independent as well as time-dependent inelastic behavior. The theory is characterized by the use of reduced or intrinsic time, a non-decreasing scalar variable whose increments depend on strain increments as well as actual time increments. The most important difference with respect to incremental plasticity and all other theories discussed so far is that the relation between $d\sigma_{ij}$ and $d\epsilon_{ij}$ is not linear, i.e., endochronic theory is not a special case of hypoelasticity. The endochronic theory for concrete will be explained with less detail than the preceding one because a thorough explanation is available in a recent paper [21] and report [22].

Endochronic theory is best approached as a special case of viscoplasticity [15]. To elucidate it, consider the classical viscoplastic constitutive relation:

$$\dot{\epsilon}_{ij} = D^{el}_{ijkm}\,\dot{\sigma}_{km} + \frac{\partial F}{\partial \sigma_{ij}}\,\phi(\underset{\sim}{\sigma},\underset{\sim}{\epsilon}) \quad \ldots\ldots \quad \text{(A2.1)}$$

where superimposed dots refer to derivatives with respect to time t, $F(\underset{\sim}{\sigma})$ is a loading surface and $1/\phi$ may be regarded as a viscosity parameter. Schapery [23] (1969) proposed a generalization such that the viscosity associated with the inelastic strain rate is also dependent on $\dot{\epsilon}_{ij}$:

$$\dot{\epsilon}_{ij} = D^{el}_{ijkm}\,\dot{\sigma}_{km} + \frac{\partial F}{\partial \sigma_{ij}}\,\phi_1(\underset{\sim}{\sigma},\underset{\sim}{\epsilon})\,\phi_2(\dot{\underset{\sim}{\epsilon}}) \quad \ldots\ldots \quad \text{(A2.2)}$$

If the inelastic strain develops gradually, as in concrete, function $\phi_2(\dot{\underset{\sim}{\epsilon}})$ may be expected to be continuous and smooth, and so it must allow Taylor series expansion [21,15]:

$$\phi_2(\dot{\underset{\sim}{\epsilon}}) = (p_0 + p_{ij}\dot{\epsilon}_{ij} + p_{ijkm}\dot{\epsilon}_{ij}\dot{\epsilon}_{km} + p_{ijkmpq}\dot{\epsilon}_{ij}\dot{\epsilon}_{km}\dot{\epsilon}_{pq})^m \quad \ldots\ldots \quad \text{(A2.3)}$$

where n is an exponent to be determined later. The series is truncated after the cubic terms.

On physical grounds, two restrictions may now be imposed:

(a) ϕ_2 must increase as $|\dot{\epsilon}_{ij}|$ increases.

(b) As the deformation becomes very rapid, the ratio of inelastic strain increment to the total strain increment must remain finite ($< \infty$), and it must also be > 0 because otherwise the instantaneous deformation would be incorrectly obtained as perfectly elastic; so,

$$\lim_{|\dot{\underline{\epsilon}}|\to\infty} \left(|d\underline{\epsilon}''|/|d\underline{\epsilon}|\right) > 0 \quad \text{and} \quad < \infty \qquad \text{(A2.4)}$$

Here $|\underline{\epsilon}|$ denotes the magnitude or norm of tensor $\underline{\epsilon}$, e.g. $(\epsilon_{ij}\,\epsilon_{ij})^{\frac{1}{2}}$.

It can now be shown that these two restrictions can be satisfied only if $P_{ij} = 0$, $P_{ijkmpq} = 0$ and $n = 1/2$. Eq. (A2.4) may now be rewritten in the form

$$d\epsilon_{ij} = D^{el}_{ijkm}\, d\sigma_{km} + d\epsilon''_{ij}, \qquad d\epsilon''_{ij} = \frac{\partial F}{\partial \sigma_{ij}}\, dz \qquad \text{(A2.5)}$$

$$dz = \sqrt{\left(\frac{d\zeta}{z_1}\right)^2 + \left(\frac{dt}{\tau_1}\right)^2}, \qquad d\zeta = \sqrt{\bar{P}_{ijkm}\, d\epsilon_{ij}\, d\epsilon_{km}} \qquad \text{(A2.6)}$$

where τ_1 may be regarded as a characteristic relaxation time.

Furthermore, the conditions of isotropy must be satisfied. Neglecting the possibility of stress-induced anisotropy of $\bar{P}_{ijkm}$, the tensor of $\bar{P}_{ijkm}$ must be isotropic. Noting that at low hydrostatic pressure the change of volume of concrete is associated with little inelastic deformation, the quadratic form $\bar{P}_{ijkm}\, d\epsilon_{ij}\, d\epsilon_{km}$ should involve only the second invariant of deviator increment de_{ij}, i.e.,

$$d\zeta = F(\underline{\sigma}, \underline{\epsilon}, \zeta)\sqrt{\tfrac{1}{2}\, de_{ij}\, de_{ij}} \qquad \text{(A2.7)}$$

where the dependence of F on the invariance of $\underline{\sigma}$ and $\underline{\epsilon}$ and on ζ reflects the fact that P_{ijkm} must in general be functions of $\underline{\sigma}$, $\underline{\epsilon}$ and the strain history.

At higher hydrostatic pressure the volume change is also significantly inelastic, but the mechanism of inelastic behavior is pore collapse rather than microcracking or plastic slip. For this reason the volumetric inelasticity is better described in terms of a separate variable

$$d\zeta' = F'(\underline{\sigma}, \underline{\epsilon}, \zeta')\sqrt{(d\epsilon_{kk})^2} \qquad \text{(A2.8)}$$

Variable z has been initially called reduced time [23] and is now generally known as intrinsic time. This term was introduced by Valanis [24] who was first to develop a successful endochronic concept for complicated nonlinear behavior, particularly cyclic loading and work hardening of metals. He also coined the Greek term "endochronic." The preceding derivation of the endochronic formulation parallels that from Ref. 21. The intrinsic time z is equivalent to the reduced time of Schapery [23], who suggested in 1968 that the ratio $a = dt/dz$ or coefficient ϕ in Eq. (A2.2) be considered as a non-negative function of the total octahedral strain rate, which is equivalent to Eq. (A2.7). The particular square-root expression for dz in Eq. (A2.6) was first postulated, as the starting assumption, by Valanis [24], who also placed the theory within the framework of continuum thermodynamics with internal variables. The intrinsic time may be geometrically interpreted as the length of the path traced by the states of the material in a strain space of suitable metric. Variables of this type have been used since the early 1950's (e.g., Hill, Il'yushin, Rivlin and Pipkin; cf. Ref. 24).

Although the inelastic strain increments in endochronic theory may be derived by differentiation of a loading function F, the concept of yield surface is lacking, which suggests physical interpretations in terms of damage, microcracking, pore collapse and internal friction. This appears to make the endochronic formulations particularly suited for geological materials, including concrete. The essential features which had to be modeled for these materials [21] are the pressure sensitivity, inelastic dilatancy and strain-softening, along with the fact that the response curves are smooth through the point of peak stress (strength).

The most conspicuous feature of the ordinary endochronic formulation is that no distinction is made between loading and unloading. The "trick" that makes this possible is illustrated in Fig. A2.1 in which the stress increment $d\sigma$ corresponding to given $d\epsilon$ is shown as a sum of elastic increment $E d\epsilon$ and inelastic decrement $|d\sigma^{pl}| = -\sigma|dz|$. When the sign of $d\epsilon$ is reversed, $E d\epsilon$ changes sign but $d\sigma^{pl}$ does not, which automatically makes the unloading slope steeper than the loading slope, so the irreversibility, the salient feature of inelastic behavior, is built in a priori, without any inequalities.

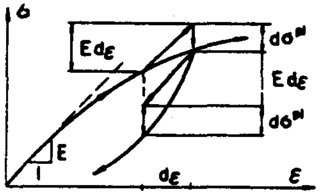


Fig. A2.1 Superposition of Elastic and Inelastic Stress Increments at Loading and Unloading in Ordinary Endochronic Theory.

The mere fact that endochronic theory works even without unloading criteria lends the theory great versatility in modeling the response to repeated and cyclic loads. However, some unsatisfactory predictions are obtained for cyclic loads of small amplitude superimposed on a large static load, and it appears that for these situations certain unloading-reloading criteria should be introduced [15]. The absence of such criteria in the ordinary endochronic models is not the basic feature of theory; rather, it is the incremental nonlinearity.

The intrinsic time is a convenient variable for describing the hardening or softening of the material. This may be accomplished by setting

$$d\zeta = \frac{d\eta}{f(\eta)}, \quad d\eta = F(\epsilon,\sigma)d\xi, \quad d\xi = \sqrt{\tfrac{1}{2}\, de_{ij}de_{ij}} \quad \text{.............. (A2.9)}$$

Function $f(\eta)$ is called hardening function and it must primarily describe the fact that the hysteresis loops are getting narrower as cyclic loading progresses. Since in cyclic loading ζ may be considered as a parameter for the number of cycles, η may be considered to be a function of ζ or of η, which is equivalent. This function must grow with η and the simplest possible choice is $f(\eta) = 1 + \beta_1\eta$, as first introduced by Valanis for metals [24]. A somewhat more complex form is needed to model the response for a great number of cycles, and a slight dependence on the stress and strain invariants must be introduced to model cyclic behavior in the strain softening range. Function $F(\epsilon,\sigma)$ must reflect the softening of concrete at

high strain, especially the strain-softening range. The softening is caused by microcracking which is independent of volumetric compression and depends mainly on deviator deformation. So F must increase with $J_2(\epsilon)$, and this increase must fade as confining hydrostatic pressure is introduced. Adding various secondary effects, a suitable form of function F was given in [21].

Another essential property of concrete is the inelastic dilatancy, λ. It is caused by microcracking, which results chiefly from deviator strains. So, it was postulated in [21] that

$$d\lambda = F_1(\lambda,\sigma,\epsilon)d\xi \quad \text{.. (A2.10)}$$

Function F_1 reflects the fact that $d\lambda$ diminishes with increasing confining pressure, increases with $J_2(\epsilon)$, especially in the strain-softening range, and decreases if large dilatancy has already been produced; see [21]. Elastic moduli should also be considered to depend on λ because they in general are affected by porosity changes [21]. To describe the inelastic behavior at high hydrostatic compression, which is caused by pore collapse and closing of voids, one must introduce additional volumetric inelastic strain $\sigma dz'/3K$ where dz' is related to $d\zeta'$ (Eq. (A2.8)) [25]. Furthermore, it is proper to distinguish another inelastic volume change, $d\lambda'$, which represents the compaction caused by additional pore collapse that is induced at the beginning of shear straining under high confining pressure [25].

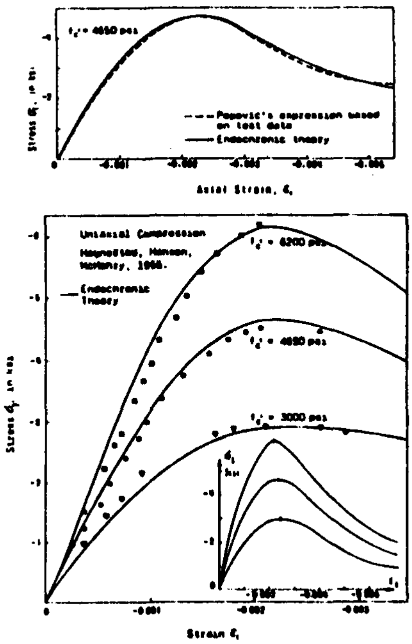
Combining all terms, and choosing the endochronic loading function F to be of Drucker-Prager type (with von Mises deviator part), the stress-strain relations of endochronic theory may be written as [25]:

$$de_{ij} = \frac{ds_{ij}}{2G} + \frac{s_{ij}}{2G}dz \quad \text{...................................... (A2.11)}$$

$$d\epsilon = \frac{d\sigma}{3K} + d\lambda + \frac{\sigma}{3K}dz' + d\lambda' \quad \text{............................. (A2.12)}$$

Available test data appear to be insufficient to decide whether a more complex function F brings about any improvement or not.

Using the endochronic formulation as just outlined, an unprecedented success has been achieved in modeling the test data reported in the literature [25,26]. The formulation has been shown [21,25] to model quite closely



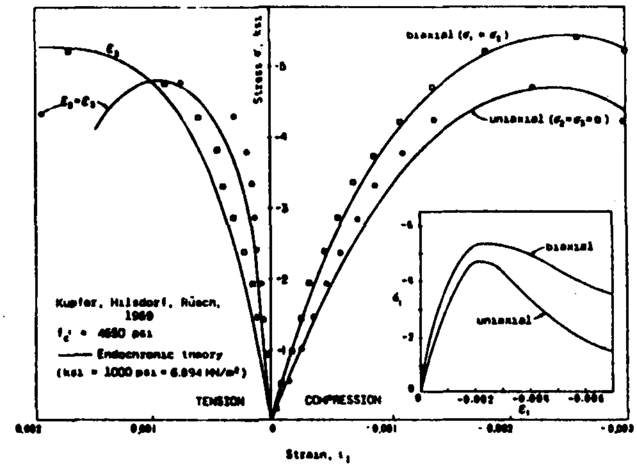


Fig. A2.2 Endochronic Theory Fits (after [25]) of Uniaxial Test Data of Popovics [58] (1973), Hognestad, Hanson and McHenry [55] (1955) and Biaxial Test Data of Kupfer, Hilsdorf and Rüsch [56] (1969).

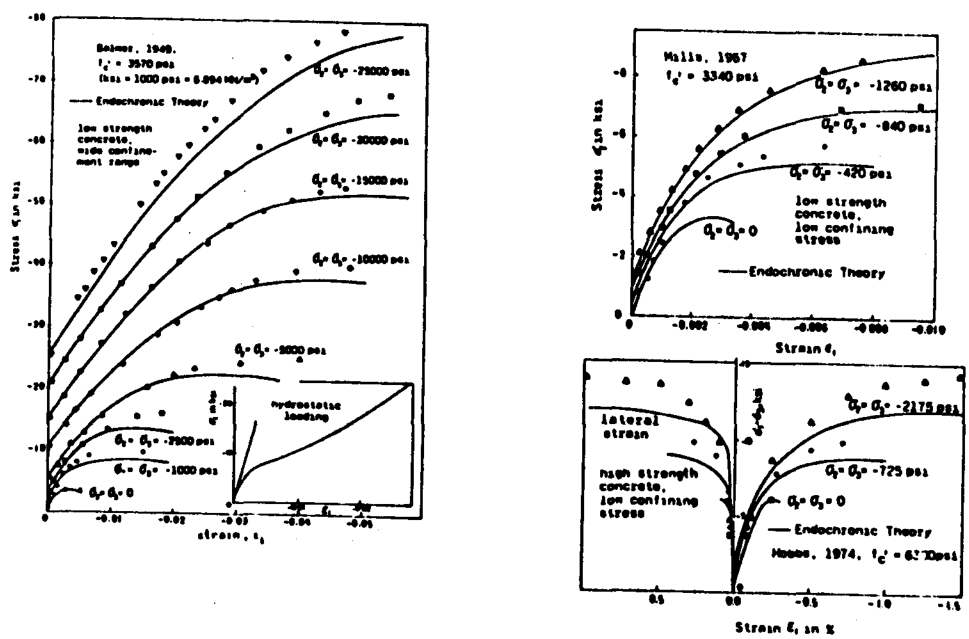


Fig. A2.3 Endochronic Theory Fits (after [25]) of Triaxial Test Data of Balmer[52] (1949), Mills [57] (1967) and Hobbs [54] (1974).

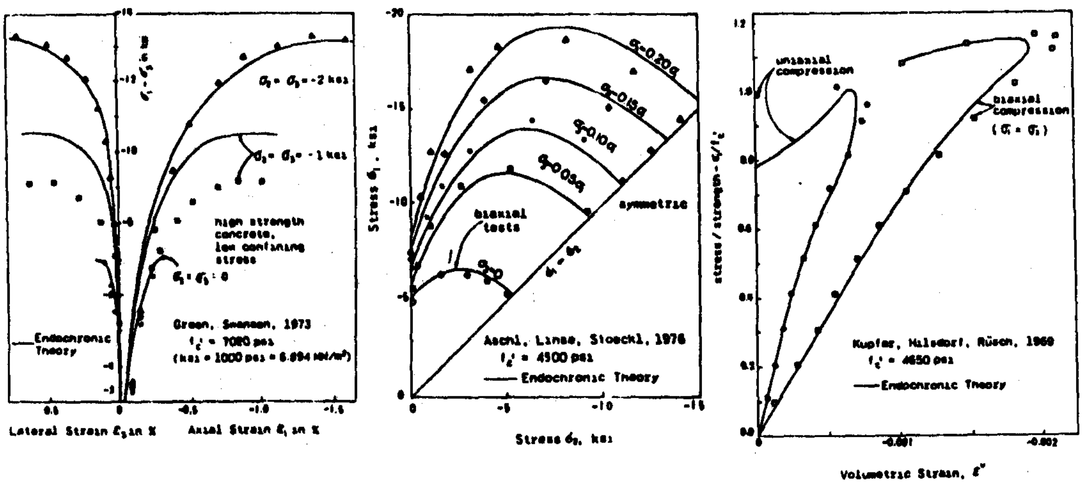


Fig. A2.4 Endochronic Theory Fits (after [25]) of Triaxial Test Data of Green and Swanson [53] (1973), Biaxial and Triaxial Failure Envelopes from Proportional Loading Tests of Aschl, Linse and Stöckl [51] (1976), and Volume Change Data of Kupfer, Hilsdorf and Rüsch [56] (1969).

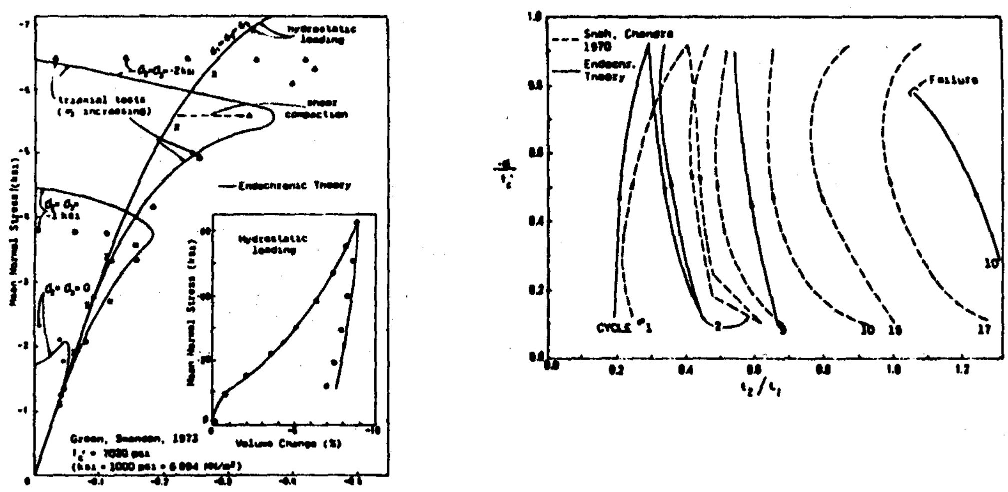


Fig. A2.5 Endochronic Theory Fits (after [25]) of Volume Change Data from Triaxial Tests of Green and Swanson [53] (1973) and of Lateral-to-Axial Strain Ratio from Cyclic Uniaxial Tests of Shah and Chandra [59] (1970).

the uniaxial stress-strain diagrams, including the strain-softening tail, biaxial and triaxial stress-strain diagrams, lateral strains and volume changes in all these tests, failure envelopes for biaxial and triaxial proportional loadings, shear-compression failure envelopes, uniaxial cyclic loading tests at small as well as high strain, including lateral strains and a high number of cycles. Examples of fits of some basic test data [51-59] are given in Figs. A2.2-5. Furthermore, with the inclusion of the term dt^2 in Eq. (A2.6), the rate sensitivity of uniaxial compression tests, the nonlinear creep isochrones, and the phenomenon of cyclic creep have been also modeled satisfactorily [21]. The material parameters have been expressed as a function of concrete strength, f_c'.

A2.2 Refinements and Comparisons of Endochronic and Incremental Plastic Formulations

As has been made clear in Sec. 2.3, the fracturing material theory yields inelastic strains which are particularly suitable to describe the strain-softening behavior. This is because these terms depend on strain ϵ_{ij}, which increases along the softening branch, rather than σ_{ij}, which decreases along the softening branch. Guided by this idea, it has been tried to introduce into the endochronic model additional inelastic strains

$$de_{ij}^{fr} = e_{ij}\frac{d\kappa}{\bar{\gamma}}, \quad d\epsilon^{fr} = \frac{2}{3}\alpha\, d\kappa \quad \text{(A2.13)}$$

with $d\kappa = F_0(\underline{\epsilon}, \underline{\sigma}, \xi)d\xi$ .... (A2.14)

which are analogous to those in Eqs. (A1.22) and (A1.23), but are related to path length ξ. This refinement allowed simplification of the functions that define the remaining inelastic terms [26], as well as modeling of the decrease of the unloading slope as strain grows into the falling branch.

Another logical refinement of endochronic theory consists in introducing a stress-induced anisotropy of coefficients $\tilde{p}_{ijkm}$ in Eq. (A2.6) defining the intrinsic time [15]. One possible form is

$$d\xi = \sqrt{\tfrac{1}{2}(de_{ij}de_{ij} + p_1 s_{ij} de_{ik} de_{ik})} \quad \text{(A2.15)}$$

However, the presently available test data are not sufficient for determining coefficient p_1.

A further generalization, which did prove to be useful for widening the hysteresis loops at unload-reload cycles, is an introduction of kinematic hardening into the endochronic theory. Similarly as in plasticity, the endochronic loading function is assumed to be centered in the stress space at point α_{ij} which moves, i.e., $F = F(\sigma_{ij} - \alpha_{ij})$. This concept gives e.g., endochronic inelastic strain increments of the form

$$de''_{ij} = \frac{1}{2G}(s_{ij} - \alpha_{ij})d\zeta \quad \text{(A2.16)}$$

The rule for the variation of α_{ij} is the central question which is hard to decide on the basis of available test data. Nevertheless, one type of kinematic hardening proves to be appropriate for unloading and cyclic loading, on purely theoretical grounds. This will be discussed later.

The available test data have been fitted almost equally well by an endochronic formulation and by a plastic-fracturing formulation. What is then the difference between these two formulations? Since the mathematical forms are quite different, where is the difference manifested? These questions can be answered by postulating the concept of "inelastic stiffness locus" [15]. This locus is defined as the locus of the tips of all strain increments $d\epsilon_{ij}$ (vectors) which give the same magnitude of inelastic strain increment, $|d\underline{\epsilon}''|$. (The magnitude may be defined, e.g., as $|\underline{\epsilon}| = (\epsilon_{ij}\epsilon_{ij})^{\frac{1}{2}}$ = length of vector ϵ_{ij} in the six-dimensional strain space.) The distance of the inelastic stiffness locus from the initial stress point is proportional to inelastic stiffness modulus.

Using the tangent stiffness moduli, one can show that for the plastic as well as plastic-fracturing formulations, the inelastic stiffness locus in any two-dimensional section of the six-dimensional strain space is a straigh line (Fig. A2.6). If the normality rule is satisfied, this line is parallel to the tangent of the loading surface F in the initial state (or the fracturing surface). Otherwise it is inclined.

For the endochronic formulations (of ordinary type, with $d\xi = (\frac{1}{2} de_{ij}de_{ij})^{\frac{1}{2}}$), it can be shown that the inelastic stiffness locus is a circle about the initial state point in any two-dimensional cross section of the deviatoric strain space (while in the $\bar{\gamma} - \epsilon$ plane it is a set of two

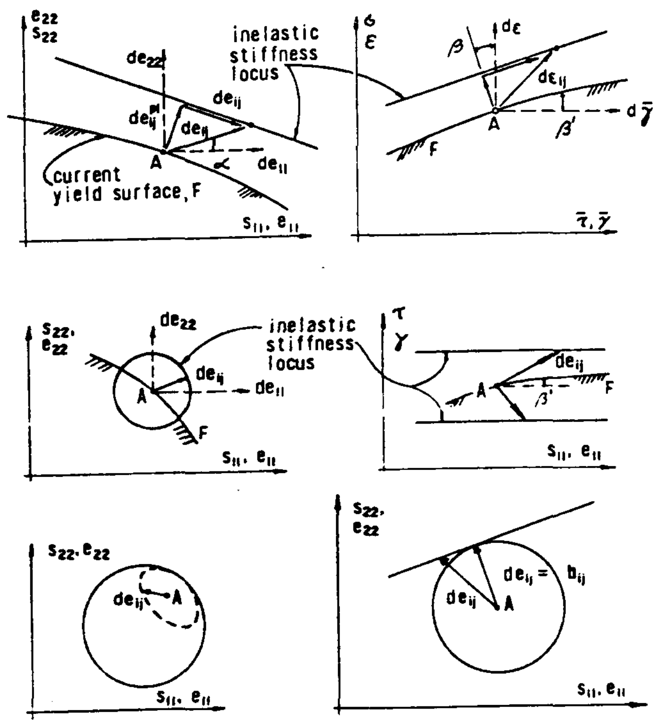


Fig. A2.6 **Inelastic Stiffness Locus for Plasticity (top), Ordinary Endochronic Theory (middle), Endochronic Theory with Stress-Induced Anisotropy (bottom left) and Tangential Linearization (bottom right).**

parallel lines); see Fig. A2.6.

If the material parameters are fitted to predominantly proportional loading data, the circular locus and the straight line locus will touch at a point which lies in the direction normal to the current loading surface (straight-ahead direction). The difference between the two inelastic stiffness loci becomes larger as the strain direction deviates from the straight-ahead direction, and it becomes largest when the strain direction is tangent to the current loading surface (straining-to-the-side). In that case, the inelastic strain $d\epsilon''_{ij}$ for the endochronic theory is as significant as in the straight-ahead direction, whereas in the plastic and fracturing theories $d\epsilon''_{ij}$ is zero, i.e., the response for straining-to-the-side is purely elastic.

The latter feature is definitely not correct. This has been realized by researchers in plasticity of metals, and to remedy the situation they have introduced the so-called vertex-hardening, in which one assumes the formation of a local vertex on the loading surface at the current state point. One particularly simple and versatile form of vertex hardening has been introduced by Rudnicki and Rice [27] and applied to rocks. By virtue of the vertex, one can achieve that the response for loading to-the-side becomes also inelastic, as test data as well as micromechanics models suggest.

For plasticity with vertex hardening, the inelastic stiffness locus assumes either a vertex shape or a curved shape. Thus, the trend toward vertex-hardening models brings plasticity closer to endochronic theory, in that both exhibit inelastic strain for the loading to-the-side. Nevertheless, there is still one noteworthy difference. In endochronic theory, the straining-to-the-side produces an inelastic strain increment whose vector has the straight-ahead (normal) direction, while in all vertex hardening models (e.g., [27]), this straining produces inelastic strain increments whose vector deviates from the straight-ahead direction.

The differences in the response to-the-side have little effect on the fits of most test data available. This is because well controlled tests performed thus far are either proportional or almost proportional loading tests. It might seem that the triaxial tests, in which all three principal stresses are first raised simultaneously and then they are varied individually, are an exception. Not so, however, because the initial hydrostatic loading produces no oriented damage in the microstructure. What one chiefly needs are tests where one raises first one or two stresses, so as to produce oriented damage, and then varies the other stresses, creating in the deviator strain space a path that has a sharp corner. Such tests are, of course, difficult to carry out in a well-controlled manner.

The inelastic strain for loading to-the-side has the greatest importance for predictions of material instability, such as strain localization, as has been shown by Rudnicki and Rice [27]. The larger the inelastic strain for loading to-the-side, the lower the critical stress for such instabilities. In this light, plasticity without vertex hardening is the least safe assumption that can be made if the test data on straining paths with a corner are lacking. Thus, the use of endochronic theories as well as vertex-hardening theories in finite element programs is in the interest of safety of instability predictions [15].

It is interesting to observe in this context that for loading to-the-side the deformation theory of plasticity (the variable secant moduli theory for concrete) gives a response that is softer than elastic. Hence, this theory is more correct than incremental plasticity in this respect. The micromechanics models for plastic behavior all indicate that there should be inelastic strain for loading-to-the-side [8,10].

The concept of inelastic stiffness locus illuminates the method of linearization of endochronic theory [15]. Linearization requires that the circular locus be replaced by a tangent straight line (Fig. A2.6). This can be done only if the direction b_{ij} of vector $d\epsilon_{ij}$ is approximately known, and the linearization will be acceptable only for straining directions that are close to $d\epsilon_{ij}$. It can be shown that the linearization of the endochronic formulation may be achieved by replacing $d\xi$ with the expression [15,22]:

$$d\xi = B_{ij} de_{ij} \quad \text{where } B_{ij} = \frac{b_{ij}}{\sqrt{2 b_{km} b_{km}}} \quad \ldots\ldots\ldots\ldots \quad (A2.17)$$

Alternatively, an equivalent linearization can be carried out directly in the stiffness coefficients of finite elements [30]. The linearization leads to a more effective numerical step-by-step algorithm for structural analysis.

With the substitution of Eq. (A2.17), the formulation becomes identical to a plastic-fracturing formulation, and it is also possible to express the

tensor of tangential moduli C_{ijkm}. Observe, however, that this term is different for each straining direction b_{ij}. This is rather unpleasant when the loading path exhibits a sharp corner (as is usually the case for material instability) because no good estimate of b_{ij} can be made and many possible directions b_{ij} must be tried. Solving the eigenvalue problem of instability for one chosen direction b_{ij} gives no information on the situation for other possible b_{ij}, a fact which must be kept in mind in interpreting finite element claculations for materials whose inelastic stiffness locus is not straight.

The circular shape of the inelastic stiffness locus in endochronic theory can be nothing more than an approximation. In general one must expect the distances to the inelastic stiffness locus in the backward direction and the tangential direction to be different from that in the straight-ahead direction. An endochronic theory which allows one to control these two properties may be obtained by replacing $d\xi$ with Eq. (A2.15) or with

$$d\xi = [\tfrac{1}{2}(d\epsilon_{ij} + p_0 s_{ij} d\xi')(de_{ij} + p_0 s_{ij} d\xi')]^{\frac{1}{2}}$$

$$\text{where } d\xi' = \sqrt{\tfrac{1}{2} de_{ij} de_{ij}} \quad \ldots\ldots\ldots\ldots \quad (A2.18)$$

respectively. These two expressions are examples of possible stress induced anisotropy of tensor p_{ijkm} defining the intrinsic time. The corresponding inelastic stiffness locus is an eccentric circle or an ellipse [15] (Fig. A2.6). The available test data, however, seem to be insufficient for identifying constants p_1 and p_0 in Eqs. (A2.16) and (A2.17).

The concept of intrinsic time has recently been generalized in a different manner by Valanis [28]. Considering the uniaxial case, he proposed that $d\zeta = |d\epsilon - k\, d\epsilon_{el}|$ where $d\epsilon_{el} = d\sigma/E_0$, and showed that for $k = 1$ an endochronic material with this intrinsic time definition becomes equivalent to an elastic-perfectly plastic material. In that case the theory also satisfies the normality rule and Drucker's postulate. However, the case $k = 1$ is not of practical interest, while for $k < 1$ Drucker's postulate is not satisfied (and in the case of cyclic loading discussed in the next section, the reloading slope is still predicted smaller than the unloading slope).

A2.3 Cyclic Loading by Endochronic Theory

The properties of endochronic formulations in cyclic loading have recently been criticized by Sandler [29], who claimed that numerical instabilities would result from inevitable numerical errors in dynamic finite element analysis. Yet, the endochronic formulations outlined have already been tried in dynamic finite element analysis, and the instabilities expected by Sandler did not materialize.

The question is as follows. Consider the case of a small cyclic stress s superimposed on static stress σ_0, i.e., $\sigma = \sigma_0 + s \sin \omega t$. It can be found that according to the endochronic theory the response $\epsilon(t)$ may deviate within sufficient time from the static value ϵ_0 by any amount no matter how small is amplitude s. This means that the endochronic theory does not guarantee stability in the Liapunov sense, which is the essence of Sandler's criticism. However, this fact per se cannot be objectionable because in classical viscoplasticity, a physically well founded theory, the situation is precisely the same. Moreover, the endochronic formulation does satisfy a weaker condition of stability, which may be stated as

$$\lim_{s \to 0} |\epsilon(t) - \epsilon_0| = 0 \quad \text{for any given } t \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (A2.19)$$

This condition represents essentially a continuity condition, a requirement that the problem be "properly posed."

The lack of stability in the Liapunov sense is connected to the fact that endochronic theories may violate Drucker's postulate. In particular, for a small unload-reload cycle of stress the second-order work figuring in the postulate can be negative according to the ordinary endochronic formulations. This is because in endochronic theory the slope at the start of reloading can be less steep than the previous unloading slope, which is normally not observed in tests. Yet, as far as the mean slope of the unloading and reloading branches is concerned, this behavior is correct because otherwise the phenomenon of cyclic strain accumulation (cyclic creep) could not be modeled. Nevertheless, Drucker's postulate should not be violated unless there is some clear physical reason for doing so, such as the frictional effects, and for uniaxial cycling no such reason is seen.

A way to modify the endochronic theory so as to satisfy the Drucker's

postulate in cyclic loading of arbitrarily small amplitude, yet allow representing the cyclic creep, has been found [15]. One must use kinematic hardening such that the center α_{ij} of the loading surface jumps to the extreme stress point whenever loading reverts to unloading or unloading reverts to reloading (jump-kinematic hardening). For unloading, $d\zeta$ must be replaced by $c_u d\zeta$, and for reloading, by $c_r d\zeta$. It can be shown that Drucker's postulate is satisfied for the unload-reload cycle if $c_u < c_r \leq 2c_u$, giving response with $\Delta W > 0$ as shown in Fig. A2.7 [15].

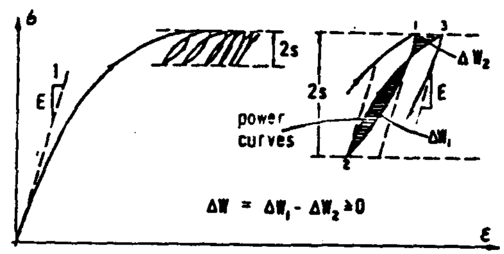


Fig. A2.7 Hysteresis Loop of Small Amplitude, Satisfying Drucker's Postulate and Giving Cyclic Strain Accumulation (after [15]).

As for the unloading and reloading criteria, the sign of the increment of total work, $W = \sigma_{ij} d\epsilon_{ij}$, is a suitable indicator [15]. A refined endochronic theory which can always satisfy Drucker's postulate has been formulated. It allowed an excellent representation of numerous test data for concrete (as well as sand).

CHAPTER A3

CRACKING, STABILITY AND STRUCTURAL ANALYSIS

A3.1 Tensile Cracking and Net-Reinforced Concrete

Tensile cracking is perhaps best treated separately from other inelastic behavior. One way to approach the problem is to superimpose the strain due to tensile cracks upon the strain of the concrete between the cracks, which can be calculated according to the endochronic theory (or plasticity) [30]. This approach, albeit plausible, is not entirely satisfactory as it does not answer the problem of gradual and partial tensile cracking and of partial shear transfer due to aggregate interlock.

Representation of gradual tensile cracking is especially important for dynamic finite element analysis of concrete under blast loads. If one assumes in such an analysis the tensile cracks to form suddenly, a wave of steep front is emitted at each crack formation, which may cause the adjacent finite elements to crack and emit further waves causing the structure to disintegrate in a chain reaction. Such behavior is spurious and does not occur in reality because cracks form gradually and concrete exhibits tensile softening. These properties must be reflected in a proper mathematical model. A relatively simple treatment of tensile softening in a large explicit finite element analysis of a concrete reactor vessel was introduced in Ref. [31].

A more sophisticated model may be achieved by introducing tensor a_{ij} which defines the uncracked area fraction for various directions. The stiffness matrices for concrete fully cracked in one, two or three directions may then be linearly combined according to tensor a_{ij} [49]. Another way to treat the problem is to introduce Dougill-type fracturing material theory.

Endochronic theory combined with tensile cracking has been used quite successfully to predict the response of beams to seismic-type cyclic loading [32]. An example of the agreement with test data is given in Fig. A3.1. The theory allowed modeling realistically the effect of stirrups on ductility.

The problem of tensile cracking is of particular importance to concrete that is reinforced by a steel bar net which stabilizes the cracks.

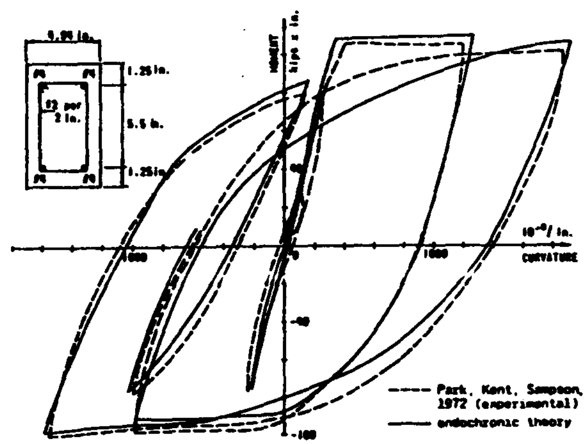


Fig. A3.1 Fits of Park, Kent and Sampson's Test Data [60] (1972) on Cyclic Response of a Beam by Endochronic Theory Enhanced by Cracking Rules (after Bažant and Bhat [13]).

This problem has in the past been studied in terms of limit design (e.g., Ref. 33), in which the possible friction on the cracks has been neglected. Recently, limit design in which friction on the crack is not neglected has been explored [34] and, curiously, it appeared that a heavier rather than lighter reinforcement is required when friction on the cracks is accounted for. Thus, a neglect of friction is not here on the side of safety. This is because the tensile forces in yielding reinforcement must not only balance the applied load but must also provide the compressive force across the crack that is needed to develop friction. This force is always induced by dilation on the cracks upon slipping of rough, interlocked crack surfaces.

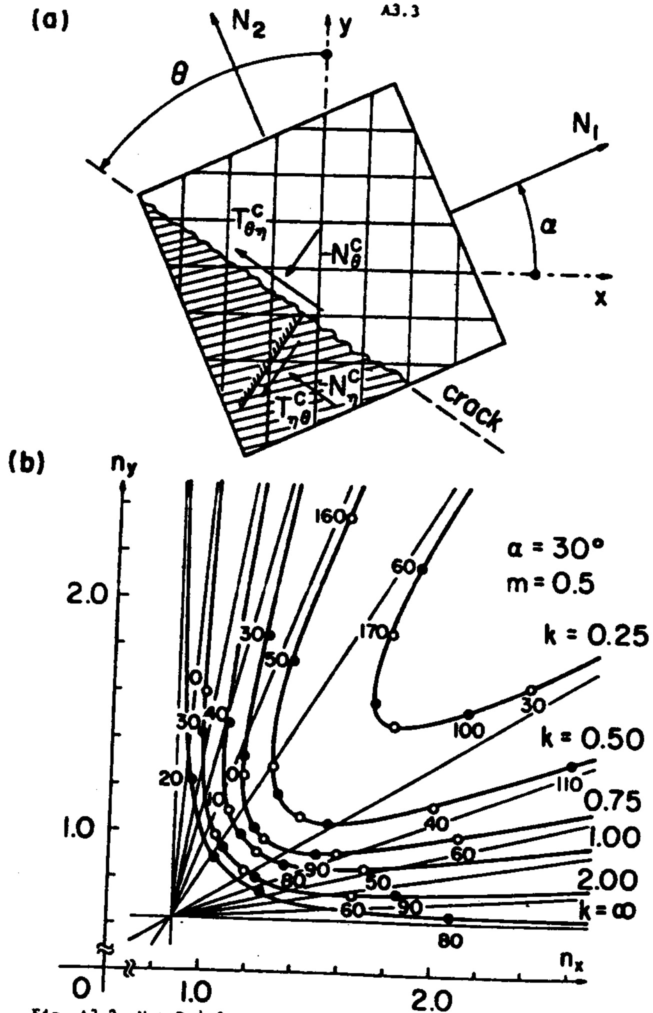


Fig. A3.2 Net Reinforced Slab with Friction on the Cracks (a) and Corresponding Limit Design Envelopes (b) for Various Friction Coefficients k. (Typical value is k = 1.4; existing theory implies k = ∞.) (After Bažant and Tsubaki [34]).

Design diagrams for this type of optimum limit design have been developed [34] (Fig. A2.8), and the resulting envelope of safe designs is given by

$$[(n_x - n_x^0) - \beta_1(n_y - n_y^0)][(n_y - n_y^0) - \beta_1(n_x - n_x^0)] = [\beta_2(1-m)\sin 2\alpha]^2 \qquad (A3.1)$$

in which $\beta_1 = (1-\sin\beta)/(1+\sin\beta)$, $\beta_2 = 1/(1+\sin\beta)$, β = angle of friction on the crack (probably $\beta \approx 1.4$ to 1.7), $n_x = N_x^s/N_1$, $n_y = N_y^s/N_1$, $m = N_2/N_1$; N_x^s, N_y^s = yield forces in the steel reinforcement in orthogonal directions x and y; N_1, N_2 = principal internal forces due to applied load, α = angle of N_1 with respect to axis x. Geometrically, Eq. (A3.1) represents a hyperbola in the (n_x, n_y) design plane, and it is noteworthy that its axes are inclined whereas for zero friction considered thus far its axes are orthogonal.

A3.2 Strain-Localization Instability Due to Strain-Softening

It is well known that, except at very rapid dynamic loading, a continuum can exhibit no strain softening because it is unstable. Yet, strain-softening can be observed experimentally even under static loading, and all our previously discussed models exhibit strain-softening as an essential property. The reason is that, although we have no other choice but to use continuum concepts of stress and strain, the material is actually nonhomogeneous, preventing localization of strain into regions that are too small. If one assumes that the width ℓ_a of a strain-localization band cannot be narrower than the size of aggregate, d, material instability is obtained at a certain non-zero, finite value of the downward slope E_t of the strain-softening diagram [35], depending on the size of the specimen relative to aggregate size, the stiffness of the testing machine, and possible parallel elastic restraints. This may be most simply demonstrated in a uniaxial model which gives a rather simple condition for stability [35,37]:

$$-\frac{E_t}{E_u} > \frac{1}{\frac{\ell}{\ell_a} - 1 + \frac{AE_u}{\ell_a(C_s + C_p)}} \quad .... \quad (A3.2)$$

where E_u is the unloading modulus ($E_u > 0$), C_s = stiffness (spring constant) of the testing machine, C_p = stiffness of eventual elastic rods parallel to the specimen, A = cross section area of specimen, ℓ = its length, ℓ_a = nd, n ≈ 2. From this formula we can calculate the critical value of tangent modulus E_t, from which the failure point on the stress-strain diagram can then be located. For a typical stress-strain diagram the results are shown in Fig. A3.3 where ductility is understood as the ratio of the strain at instability to the strain at peak stress.

A similar simple formula can be derived for curvature localization in concrete frames which exhibit softening [35]. Various aspects of such frames can also be described assuming softening plastic hinges, and adopting this approach penetrating analysis of stability has been carried out by Maier et al. (e.g., Ref. 36).

A deterministic approach to localization instability can only depict the dependence of ductility on size and stiffness of supports (stored energy). However, in a further investigation [37] in which a model consisting of a system of parallel strain-softening elements whose properties are randomly distributed over the system was considered, it was found that the localization instability due to strain-softening can also describe the effect of size and support stiffness upon the strength (peak stress) of a concrete specimen. This is because in the parallel system the localization instabilities occur gradually and sequentially as the strain is increased. This statistical effect on strength is fundamentally different from that modeled by the weakest link in a chain, as used so far.

A study of the effect of parallel restraint provided by steel reinforcement upon the strain-localization instability will be crucial for the theory of deformation and cracking in net-reinforced concrete.

The question of instability due to strain-softening is an important outstanding question in the finite element analysis of concrete structures. Special stability checks need to be developed for these programs. The structural tangent stiffness matrix, as usually set up, is incapable of revealing all instabilities, especially those which involve "straining to-the-side" discussed before, because the entire structural stiffness matrix is different if a different strain increment direction in the strain space is considered for one element. This means that one would have to set up a

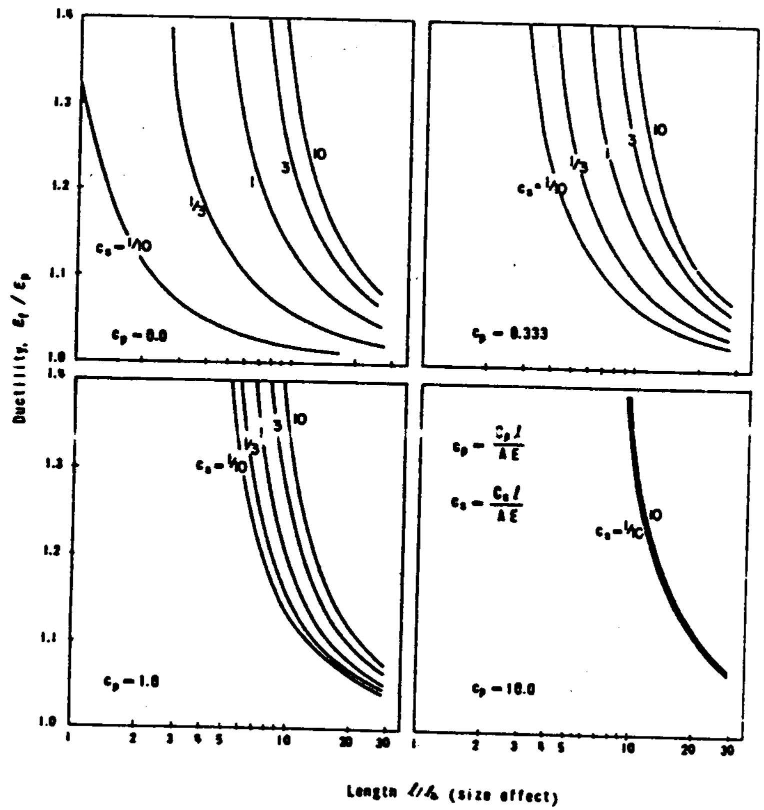


Fig. A3.3 Ductility of Concrete as Function of Specimen Size, Machine Stiffness $C_s = c_s A E_u/\ell$ and Coupled Parallel Bar Stiffness $C_p = c_p A E_u/\ell$ (after Bažant and Panula [37]).

different structural stiffness matrix for each possible strain increment for each finite element, obviously a prohibitive task. For this reason, simple physical concepts, such as the propagation of softening strain-localization bands, must be developed.

Tensile cracking is a special, limiting case of strain-softening. The currently used method of smeared cracks within the finite element suffers from a lack of convergence when the size of the finite element is reduced to zero. Consider the elements at the blunt tip of a smeared crack band. The stress σ_1 in the element ahead of the blunt crack band tip is usually taken as the condition of cracking, yet this stress could have any value (even for a fixed load) depending on the size of the finite elements, and theoretically tends to infinity as this size shrinks to zero. Thus, the analyst could make the crack band propagate at an arbitrarily small load, just by making the finite elements small enough; this is a fundamentally objectionable property unless the finite element size is already the minimum size that is admissible in relation to the aggregate size and the steel mesh size. One way to remove this violation of the principle of objectivity is to use fracture mechanics criteria for the crack band propagation.

A3.3 Step-by-Step Numerical Algorithms

Much discussion could be devoted to this topic, but we must content ourselves with noting that an effective algorithm for static analysis with iterations in each step has been given in Ref. 21 and verified by finite element analyses in Ref. 22. Some improvements in the algorithm were made by Sørensen, Arnesen and Bergan [38]. Large finite element programs have by now been developed in several laboratories; see, e.g., Ref. [30] where a careful study of the effectiveness of various algorithms was made and further improvements for structural analysis based on endochronic theory were proposed.

CHAPTER A4

CREEP, SHRINKAGE AND EFFECTS OF MOISTURE AND TEMPERATURE

A4.1 Thermodynamics of Viscoelastic Solidifying Material

Creep of concrete is a fascinating subject of tremendous breadth, in which continuum mechanics, cement physics, physical chemistry, thermodynamics, probabilistic theory and numerical structural analysis intersect. Since an extensive survey of this subject was published not too long ago (1975 [39]), the review of the creep theory will be focused only on very recent developments.

A rigorous treatment of creep requires formulating the relevant continuum thermodynamics. The most important work in this respect has been that of Argyris, Pister, Szimmat and Willam [40], who used the internal variable formalism and expressed the complementary free energy as a function of not only the stresses and internal variables but also the moisture content and temperature. Further extensions of this excellent work must be carried out to account in a proper manner for the effect of aging due to solidification or hydration of cement paste. Just like in thermodynamics of chemical reactions, a proper approach requires that we identify and introduce time-invariant material components of fixed properties (e.g., constant elastic modulus E^0, constant viscosity η^0, and treat the variation of the overall material properties (e.g., modulus $E(t)$ of concrete) as a consequence of volume growth or bonding (polymerization of cement gel); i.e., $E(t) = E^0 v(t)$ where $v(t)$ = volume of cement gel participating in resisting the load. The complementary free energy Π is then obtained as a time-history integral involving volume $v(t)$, and it can be expressed as a function of current state only if v = constant. In consequence, strains cannot be derived as $\epsilon = \partial\Pi/\partial\sigma$ but one must use

$$\epsilon = \partial\dot{\Pi}/\partial\dot{\sigma} \quad \text{...} \quad (A4.1)$$

As an example, we may give the complementary free energy for linear aging creep governed by the Maxwell chain model:

$$\dot{\Pi} = \varepsilon\dot{\sigma} = \sum_{\mu}\left(\int_0^t \frac{d\sigma_\mu(\tau)}{E^0 v_\mu(\tau)}\right)\dot{\sigma}_\mu + \sum_{\mu} \varepsilon_\mu \dot{\sigma}_\mu \quad \text{(A4.2)}$$

in which ε_μ = internal variables = strain in the dashpot of μ-th Maxwell unit, σ_μ = associated stress, $v_\mu(t)$ = associated volume growth. The differential equations of the Maxwell chain model are obtained as $\varepsilon = \partial\dot{\Pi}/\partial\dot{\sigma}$ with $\sigma = \sum_\mu \sigma_\mu$ and

$$\dot{\sigma}_\mu = \partial\dot{\Pi}/\partial\varepsilon_\mu \quad \text{(A4.3)}$$

while the dissipation inequality is satisfied by setting

$$\partial\Pi/\partial\varepsilon_\mu = \eta_\mu^0 v_\mu(t)\dot{\varepsilon}_\mu \quad \text{(A4.4)}$$

in which η_μ^0 = intrinsic time-constant viscosity of the μ-th Maxwell unit.

An important fact is that for an aging material, the elastic stress-strain relation must be written as $d\sigma = E^0 v(t) d\varepsilon(t) = E(t) d\varepsilon(t)$. The relation $\sigma = E^0 v(t)\varepsilon = E(t)\varepsilon$ is inadmissible because it would violate the dissipation inequality for the strain energy change caused by volume growth. However, for a desolidifying material (dehydrating concrete above 100°C), the relation $\sigma = E^0 v(t)\varepsilon$ is the correct one.

To show that this is indeed so, we may express the stored instantaneously recoverable (reversible) energy U of the material governed by $\sigma = E^0 v(t)\varepsilon$; we have $U = \sigma^2/2E^0 v(t)$ and so

$$\dot{U} = \sigma \frac{d\sigma}{E^0 v(t)} + \dot{D}_{ch}, \quad \dot{D}_{ch} = -\frac{\sigma^2}{2E^0}\frac{\dot{v}(t)}{v(t)^2} \quad \text{(A4.5)}$$

where $\dot{D}_{ch}$ = chemical dissipation of stored elastic energy due to removal (dissolution) of stressed matter. The second law of thermodynamics requires not only that the frictional (viscous) dissipation $\dot{D}$ be non-negative, but also that $\dot{D}_{ch} \geq 0$. Hence the relation $\sigma = E(t)\varepsilon$ is possible only if $\dot{v} \leq 0$ or $\dot{E} \leq 0$, from which it follows further that $d\sigma = E(t)d\varepsilon(t)$ is the only possibility if $\dot{E} \geq 0$. The last conclusion also follows from a microstructural consideration, based on the fact that newly solidified matter when it is added within the microstructure to the loaded part must be initially in an unstressed state. In many works using rheological models for aging linear creep the foregoing inequalities have been violated.

A4.2 Constitutive Equations

Rate-type models are required not only for formulating the thermodynamics of creep but also, and mainly, for finite element creep analysis of structures. This is to avoid the prohibitive storage and computing time demands posed by the direct use of the creep law in the form of a hereditary integral. These integral-type laws can, fortunately, be approximated by the rate-type creep law as closely as desired, and effective computer algorithms for this purpose have been developed [41,39]. One such algorithm for aging creep of concrete has been incorporated in large computer code SAP [50].

In classical viscoelasticity, the Maxwell and Kelvin chain models are known to be equivalent. This is true for aging materials only if negative spring moduli and negative viscosities are allowed, which is, however, thermodynamically inadmissible. If these are required to be positive, it can be shown that the Maxwell chain is more general than the Kelvin chain, which is a rather surprising result. Namely, the latter is capable of representing only convergent creep curves for different loading ages, while the former can represent both divergent and convergent creep curves, a property which is typical of creep test data for concrete. It can be also shown that divergent creep curves do not violate any thermodynamic restrictions. Therefore, only Maxwell chain should be used in the finite element analysis or else an arbitrary unjustified limitation is implied. (As a consequence of this fact, the Maxwell chain can give, upon application of the superposition principle, non-monotonic creep recovery curves without violating any thermodynamic restrictions, while the Kelvin chain can yield only monotonic recovery curves.)

The afore-mentioned findings present a further argument against the use of the so-called "improved Dischinger method" in practical code recommendations. Adopting the formulation from the German DIN Code, this formulation has just been incorporated into the C.E.B. Model Code (1978). It was argued in its favor that it allows simpler structural analysis (essentially according to Dischinger's classical method), and that the separation of delayed elastic and flow terms in creep is appealing to some engineers. However, this separation is not justified by long-time creep data [42], and the agreement with test data is far inferior to other available practical models, such as the Branson's product type model (adopted by ACI

Committee 209) and the refined product-type model based on the double power law. Moreover, a new method for creep calculations in design offices has been developed (Trost method or the age-adjusted effective modulus method), allowing an even simpler calculation than the Dischinger-type formulations, and for any form of the creep function [42]. The improved Dischinger-method corresponds to a Burgers rheological model, which is a special case of Kelvin chain. According to the preceding discussion, this model is inherently incapable of representing divergent creep curves, and this is part of the reason why the improved Dischinger formulation is incapable of a satisfactory description of long-time creep data. Another reason is that this formulation implies the shape of the creep curve as well as the aging effect to be defined in terms of one and the same function, which does not correspond to the reality.

The double power law, along with the practical model for shrinkage and drying creep [43], have been recently developed into a full practical prediction model. By computer analysis of practically all test data available in the literature, formulas to predict the material parameters from concrete strength and composition have been developed [44].

Micromechanics models of the solidification process can be successfully utilized for deriving a proper form of creep function of concrete [45]. One such model [45], in which layers of unstressed material are being deposited upon loaded creeping solid, is shown in Fig. A4.1.

Rate-type models for creep are also needed to formulate the effect of variable humidity on creep (in addition to the effect on shrinkage). A model based on the thermodynamics of multiphase equilibrium between adsorbed water in the micropores of cement gel, capillary water and vapor, and on the hypothesis of coupled diffusion of water and solids within the micropores, has been developed and shown to give reasonable agreement with test data [39].

Based on an idealized concept of creep mechanism, as well as the additivity properties of linear creep, a stochastic process model for concrete creep has been deduced recently [46]. The basic creep is treated as an additive non-stationary process with independent increments and leads to a non-stationary local gamma process [46]. This model can be utilized for extrapolating short-time creep test data and for determining confidence limits of a given probability cut-off. An example of thirty extrapolation

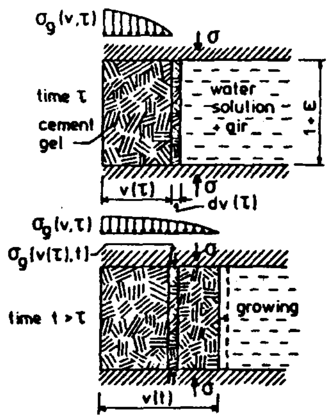


Fig. A4.1 Model for Creep of Solidifying Material (after [45]).

trials simulated on computer is shown in Fig. A4.2, in comparison with observed creep. The simulations are built up forward as well as backward from the last measured value.

A general constitutive equation for nonlinear creep of concrete, which is important at high stress levels as well as certain strain histories at low stress, must encompass both the linear creep formulation and the formulation for time-independent nonlinear triaxial behavior. For this reason, it has been attempted to extend the endochronic theory to nonlinear creep. This required using a different intrinsic time for each unit of a Maxwell chain [47], but otherwise very few additional parameters; see Fig. A4.3. The endochronic theory automatically yields not only creep increase at high stress but also the stiffening due to long-time compression of small value.

A simpler model which is applicable for nonlinear uniaxial creep can be obtained by generalizing the integral-type creep law based on the principle of superposition. The previous works (especially in Russian literature) attempted to generalize the integral expression for total strain $\epsilon(t)$. However, any physically conceivable mechanism of nonlinear creep increase must be expected to be weak in memory, and so a simple addition of a visco-

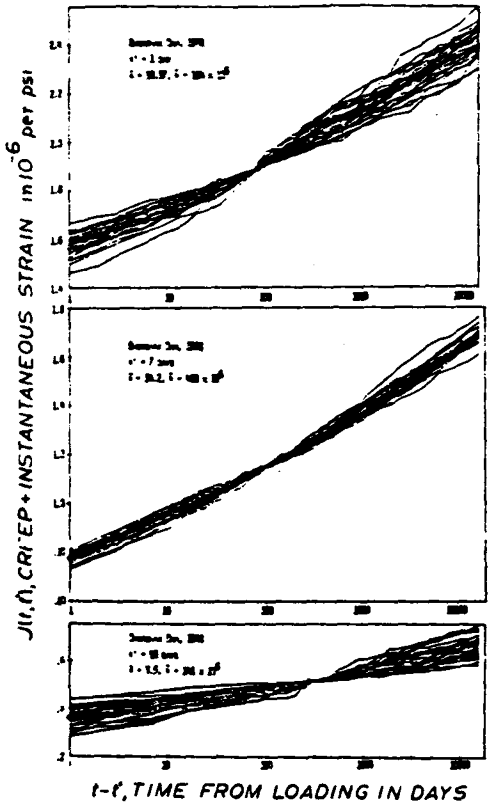


Fig. A4.2 Simulation of Stochastic Process for Concrete Creep (after Çinlar et al. [46]), Compared with Pirtz' Test Data [62] (1969) for Dworshak Dam.

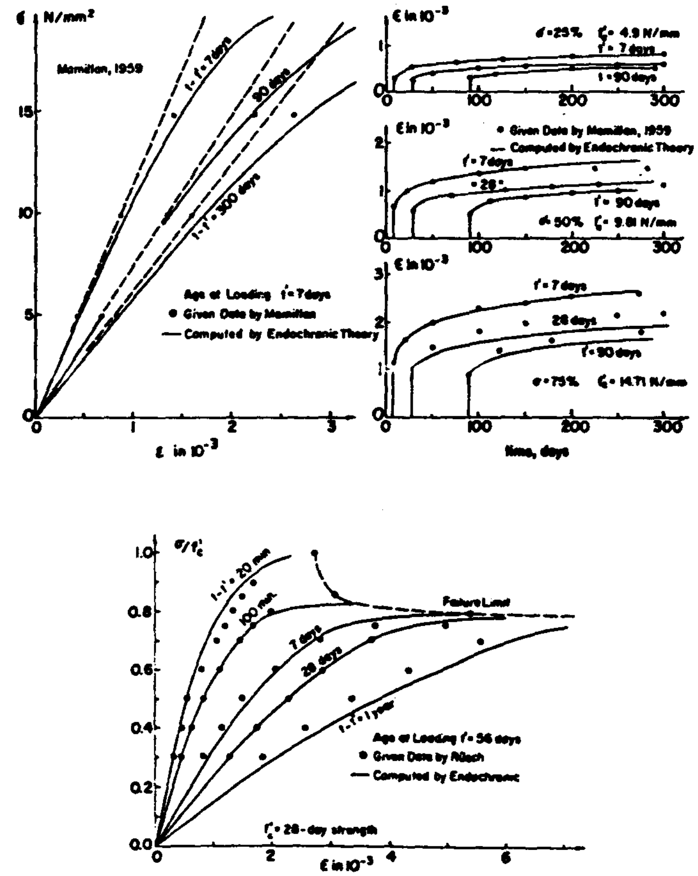


Fig. A4.3 Endochronic Theory Fits of Nonlinear Creep Data by Mamilian [61] (1959) (after [47])and by Rüsch [63] .

plastic flow term seems to be quite appropriate. Thus, a model based on the equation

$$\dot{\epsilon}(t) = \frac{\dot{\sigma}(t)}{E(t)} + \int_0^t \frac{\partial J(t,t')}{\partial t} \frac{d\sigma(t')}{1+a(t')} + f(\sigma-\alpha)\dot{\phi}(t) \quad \ldots\ldots\ldots\ldots \quad (A4.5)$$

has been tried [19] and appeared to give surprisingly good agreement with test data. Here $J(t,t')$ = linear creep function = linear strain at time t caused by stress $\sigma = 1$ applied at t'. The increase of creep at high stress is modeled by the last term, in which α = state variable that corresponds to kinematic hardening in plasticity, and $\dot{\phi}(t)$ decaying function giving the age effect. The integral term exhibits a nonlinearity that is opposite to that at high stress, namely, a decrease rather than increase of creep, due to gradual adaptation into long-time compression of small magnitude. The growth of state variables a and α is defined in terms of simple nonlinear first-order differential relations [19]. Acceleration of aging caused by sustained low compression is also accounted for in evaluating $J(t,t')$ for Eq. (A4.5). The consequent interplay of creep softening due to the last term in Eq. (A4.5) (high-stress nonlinearity) and of creep stiffening due to the memory term, is responsible for the complicated response observed in tests. Without both of these nonlinearities it is possible to fit only selected test data.

A4.3 Moisture Transfer and Thermal Effects

Calculation of moisture content and temperature is indispensable for a rational approach to creep and shrinkage. *Diffusion theory is applicable* for this purpose, but the problem is nonlinear because it was found that permeability as well as diffusivity decrease about twenty times as pore relative humidity drops from 100% to 60%; cf. [39].

At high temperatures (above 100°C), prediction of pore water pressure is of acute concern (e.g., for explosive spalling and nuclear accident analysis). Rather interesting phenomena are found to occur in this range [48]. The dependence of permeability on pore pressure seems to vanish and permeability exhibits an upward jump about 200-times as 100°C is passed. This

was chiefly deduced from the drying tests at Northwestern University [48]. Physical explanation of this phenomenon can be given by postulating certain changes in the necks in moisture migration passages. The relation of pore pressure and moisture content may be based on thermodynamic properties of capillary water, provided that one postulates: (a) an increase of pore space due to dehydration of cement paste, and (b) an increase of pore space available to free (capillary) water caused by pressure increase. The distributed mass source due to liberation of chemically held water hydrate water must be considered in the mass conservation equation. Deducing the permeability diagram and sorption isotherms shown in Fig. A4.3, satisfactory agreement with available measurements on heated concrete has been achieved [48].

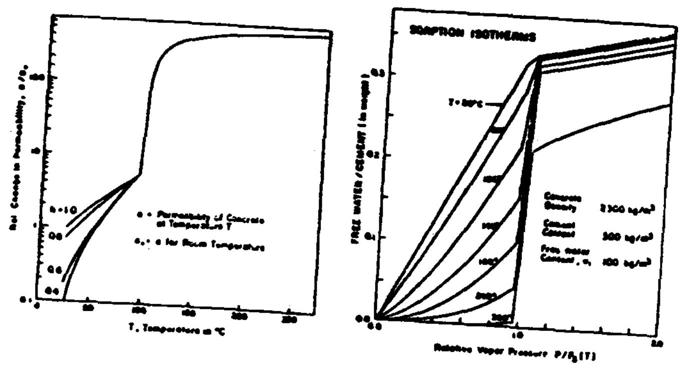


Fig. A4.4 Moisture Permeability and Sorption Isotherms Characteristic of Concrete at High Temperature (after [48]).

CONCLUDING REMARKS

As is clear from the preceding survey, a realistic and accurate description of the inelastic properties of concrete necessitated theories that cut across many disciplines. Thus, most significant future advances must be expected from an interdisciplinary approach.

The inevitable theoretical nature of various questions often tempts one to engage in research for the sake of theory. To mitigate this widespread malaise, it is, therefore, of paramount importance to keep in mind that all theoretical developments must be checked against experimental data, and not just a few conveniently selected favorable data, but all relevant data available in the literature. With the advent of computer optimization techniques for data fitting, this task has, fortunately, been greatly relieved of the tediousness it once involved. The pool of test data available in the literature is vast and must be tapped, even though many tests have not been as well documented and as closely controlled as one might have desired.

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PART II

FINITE ELEMENT ANALYSIS OF REINFORCED CONCRETE STRUCTURES

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SEMINAR PROGRAM

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* SEMINARIO 20-23 GIUGNO 1978 *

Martedì 20 Giugno 1978:

9,00 - 9,30	Prolusione del Direttore del Corso (Prof. A. Dei Poli)
9,30 - 11,00	Incremental Plasticity (Prof. Z.P. Bazant)
11,30 - 13,00	Plastic Fracturing Material Theory (Prof. Z.P. Bazant)
14,30 - 16,00	Endochronic Theory (Prof. Z.P. Bazant)
16,30 - 18,00	Comparison of Endochronic and Incremental Plastic Formulations (Prof. Z.P. Bazant)

Mercoledì 21 Giugno 1978:

9,00 - 10,30	Tensile Cracking and Net-Reinforced Concrete (Prof. Z.P. Bazant)
11,00 - 12,30	Creep Mechanism, Thermodynamics and Constitutive Equations for Creep (Prof. Z.P. Bazant)
14,30 - 16,00	Rate-Type Creep Formulations (Prof. Z.P. Bazant)
16,30 - 18,00	Practical Creep Analysis (Prof. Z.P. Bazant)

Giovedì 22 Giugno 1978:

9,00 - 10,30	Finite Element Modeling of Reinforced Concrete Structures (Prof. A.C. Scordelis)
11,00 - 12,30	The Solution of the Nonlinear Equations (Prof. W.C. Schnobrich)
14,30 - 16,00	Finite Element Study of Reinforced Concrete Beams with Diagonal Tension Cracks (Prof. A.C. Scordelis)
16,30 - 18,00	Slabs Problems (Prof. W.C. Schnobrich)

Venerdì 23 Giugno 1978:

9,00 - 10,30	Panel and Wall Problems (Prof. W.C. Schnobrich)
11,00 - 12,30	Analysis of Shells (Prof. W.C. Schnobrich)
14,30 - 16,00	Nonlinear Analysis of Reinforced and Prestressed Concrete Beams and Frames (Prof. A.C. Scordelis)
16,30 - 18,00	Analysis of Curved Segmentally Erected Prestressed Concrete Bridges (Prof. A.C. Scordelis)