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STABLE STATES AND STABLE PATHS OF PROPAGATION OF DAMAGE ZONES AND INTERACTIVE FRACTURES

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ABSTRACT

Localization of damage and propagation of damage zones or interacting cracks in structures is characterized by response path bifurcations and instabilities. Stability of inelastic structures is analyzed in general on the basis of the second law of thermodynamics and criteria for stable states and stable paths are given. Examples of application of these criteria in finite element analysis of damage and fracture are presented. For the sake of illustration, analogous path bifurcations and instabilities are also analyzed for an elasto-plastic column. The results show that checks for stable path need to be introduced into finite element programs for damage.

INTRODUCTION

In the analysis of static propagation of crack systems or damage zones in structures, one typically encounters equilibrium paths with multiple bifurcations and loss of uniqueness. These problems require stability analysis. However, a general stability theory for inelastic structures is presently unavailable. This is true not only for damage but also for plasticity, and for interactive crack systems. The present paper, which is based on report [1], attempts to fill this gap.

General Criterion of Stable Equilibrium State of Inelastic Structure

Consider an arbitrary structure whose state is characterized by discrete displacements q_k $(k = 1,2,\ldots,n)$. The total energy and Helmholtz free energy of the structure are defined as $U = W + Q$, $F = U - TS$ where $dW = \sum P_k dq_k$ = work of the associated loads $P_1,\ldots,P_n$ (applied forces), T = absolute temperature (assumed to be uniform), dQ = heat that flows into the structure from the outside, and S = total entropy of the structure, which is defined as $dS = (dQ/T) + (dS)_{in}$ where $(dS)_{in}$ = internal entropy of the structure. From these relations, one can verify that (see Appendix 5):

$$dF = \textstyle\sum_k P_k dq_k - SdT - T(dS)_{in}, \quad dU = \sum_k P_k dq_k + TdS - T(dS)_{in}. \tag{1}$$

For infinitesimal isothermal $(dT = 0)$ or isentropic $(dS = 0)$ equilibrium $((dS)_{in} = 0)$ deformation increments, which must be path-independent in the small, we may write, on the basis of the structural properties,

$$dF = \sum_k f_{Tk}\, dq_k \text{ (for } dT = 0), \qquad dU = \sum_k f_{Sk}\, dq_k \text{ (for } dS = 0). \tag{2}$$

Here f_{Tk}, f_{Sk} are the associated equilibrium forces (reactions) which depend on q_k according to incremental isothermal ($dT = 0$) or isentropic ($dS = 0$) material properties. These forces must be distinguished from the applied loads P_k, whose dependence on q_k is specified independently of the properties of the structure (P_k obey their own law such as gravity, fluid pressure, centrifugal force, electromagnetic force, aerodynamic force, etc.; but the dynamic instabilities which can occur for nonconservative loads are not studied here). Note that since Eq. 2 applies to equilibrium deformation increments, for which $(dS)_{in} = 0$, we have for isentropic conditions $dQ = TdS = 0$. This means that for Eq. 2 the isentropic conditions are equivalent to adiabatic conditions, and so f_{Sk} are to be determined using adiabatic incremental material properties. (Eq. 2 implies a certain hypothesis — see Appendix 5.)

Eqs. 1-2 are valid with or without nonlinear geometric effects due to large deformations. If these effects are neglected, one may further use the principle of virtual work to show that (see also Appendix 2 and Appendix 5):

$$\sum_k f_{Tk}\, dq_k = \int_V \underset{\sim}{\sigma}_T : d\underset{\sim}{\epsilon}\, dV, \qquad \sum_k f_{Sk}\, dq_k = \int_V \underset{\sim}{\sigma}_S : d\underset{\sim}{\epsilon}\, dV. \tag{3}$$

Here V = volume of the body; $\underset{\sim}{\epsilon}$ = strain tensor whose field is compatible with q_k; and $\underset{\sim}{\sigma}_T$, $\underset{\sim}{\sigma}_S$ = stress tensors whose fields are in equilibrium with f_{Tk} or f_{Sk}. The dependence of $\underset{\sim}{\sigma}_T$ and $\underset{\sim}{\sigma}_S$ on $\underset{\sim}{\epsilon}$ is determined by isothermal or isentropic incremental material properties. Note that the term $T(dS)_{in}$ does not belong in Eq. 3 because $\underset{\sim}{\sigma}_T$ and $\underset{\sim}{\sigma}_S$ are evaluated to include both reversible and irreversible responses (see Eqs. 7-8 where K_{jk} and $\underset{\sim}{C}_T$ include plastic stiffness).

It is useful to introduce the Helmholtz free energy $\mathcal{F}$ and the total energy $\mathcal{U}$ of the structure-load system; $\mathcal{F} = F - W$ and $\mathcal{U} = U - W$. For an elastic structure, $\mathcal{F}$ and $\mathcal{U}$ obviously reduce to the potential energy. But for an inelastic structure they do not represent potentials in the mathematical sense since they are irreversible, path-dependent variables. According to Eqs. 1-3 we can check that

$$d\mathcal{F} = -SdT - TdS_{in} = \sum_k f_{Tk} dq_k - \sum_k P_k dq_k \tag{4}$$

$$d\mathcal{U} = TdS - TdS_{in} = \sum_k f_{Sk} dq_k - \sum_k P_k dq_k \tag{5}$$

Let us now consider a change from the initial equilibrium state q_k^0 to a neighboring state $q_k^0 + \delta q_k$. We may suppose the inelastic response to be path-independent in the small, and introduce Taylor series expansion of functions $f_k(q_m)$. Assuming the applied loads P_k to be constant (dead loads), we have:

$$\Delta\mathcal{F} = \int_{q_k^0}^{q_k^0+\delta q_k} \sum_k [f_k^0 + \sum_j f_{k,j}^0 (q_j - q_j^0) + \tfrac{1}{2}\sum_j \sum_m f_{k,jm}^0 (q_j - q_j^0)(q_m - q_m^0) + \cdots] dq_k - \sum_k P_k \delta q_k \tag{6}$$

where f_k^0, $f_{k,j}^0$, $f_{k,jm}^0 + \cdots$ are the initial values of the equilibrium forces, and the subscripts separated by a comma denote partial derivatives with respect to q_j and q_m. Since the initial state is an equilibrium state, we have $f_k^0 = P_k$. This means that the first-order work term $\delta W = \sum_k P_k\, \delta q_k$ cancels out.

Consider that the change of the structure state is isothermal ($dT = 0$), and let us assume that $f_{k,j}^0$ are nonzero. Then, after integrating Eq. 6 up to second-order terms in δq_k and setting $\sum_j f_{k,j}^0 \delta q_j = \delta f_k$ = equilibrium force changes, we obtain

$$\Delta F = -T(\Delta S)_{in} = \sum_k \tfrac{1}{2}\delta f_k \delta q_k = \sum_j \sum_k \tfrac{1}{2} K_{jk}(\underset{\sim}{\nu})_T \, \delta q_j \delta q_k = \delta^2 W. \quad (7)$$

$K_{ij}(\underset{\sim}{\nu})_T$ is the tangential stiffness matrix which is associated with q_k and must be evaluated on the basis of isothermal material properties ($dT = 0$). This matrix depends in general on the direction vector $\underset{\sim}{\nu}$ of the displacements δq_k. (Introduction of tangential stiffness implies a certain hypothesis — see Appendix 5.)

If the nonlinear geometric effects are negligible [18,19], the principle of virtual work may be applied to show that (see also Appendix 2):

$$\Delta F = -T(\Delta S)_{in} = \int_V \tfrac{1}{2}\, \delta \underset{\sim}{\sigma}_T : \delta \underset{\sim}{\epsilon}\, dV = \int_V \tfrac{1}{2}\, \delta \underset{\sim}{\epsilon} : \underset{\sim}{C}_T : \delta \underset{\sim}{\epsilon}\, dV = \delta^2 W. \quad (8)$$

Here $\underset{\sim}{C}_T$ = tensor of isothermal tangential moduli of the material, which depends on whether the material loads or unloads. Note that Eq. 8 can also be obtained from $\int \underset{\sim}{\sigma} : d\underset{\sim}{\epsilon}$ by Taylor series expansion of the function $\underset{\sim}{\sigma}(\underset{\sim}{\epsilon})$.

By the same argument, one obtains similar results for isentropic deformations ($dS = 0$):

$$\Delta U = -T(\Delta S)_{in} = \tfrac{1}{2}\, \delta \underset{\sim}{f} \cdot \delta \underset{\sim}{q} = \sum_j \tfrac{1}{2}\delta f_j \delta q_j = \sum_j \sum_m \tfrac{1}{2} K_{jk}(\underset{\sim}{\nu})_S \, \delta q_j \delta q_k = \delta^2 W, \quad (9)$$

and if nonlinear geometric effects are negligible,

$$\Delta U = -T(\Delta S)_{in} = \int_V \tfrac{1}{2}\, \delta \underset{\sim}{\sigma}_S : \delta \underset{\sim}{\epsilon}\, dV = \int_V \tfrac{1}{2}\, \delta \underset{\sim}{\epsilon} : \underset{\sim}{C}_S : \delta \underset{\sim}{\epsilon}\, dV = \delta^2 W \quad (10)$$

in which $K_{jk}(\underset{\sim}{\nu})_S$ and $\underset{\sim}{C}_S$ must be evaluated on the basis of isentropic incremental material properties (which are equivalent to adiabatic incremental material properties).

Now consider stability. The second law of thermodynamics tells us that the changes for which $(\Delta S)_{in} > 0$ will occur, and those for which $(\Delta S)_{in} < 0$ cannot occur. Therefore, the structure is <u>stable</u> (i.e., it remains in its initial state) if $(\Delta S)_{in} < 0$ for all possible δq_k or all possible δf_k. The structure is <u>unstable</u> if $(\Delta S)_{in} > 0$ for some δq_k or some δf_k. This criterion, whose essence for fluids was stated already by Gibbs, is the fundamental criterion of stability of the state of any system, including structures, whether reversible (elastic) or irreversible (inelastic). According to Eqs. 7-10, this criterion may also be stated as follows:

The structure is <u>stable</u> if the second-order work given by Eqs. 7-8 (for $dT = 0$) or Eqs. 8-9 (for $dS = 0$) is positive for all possible δq_k (or all possible δf_k), in other words, if it is positive definite. The structure is <u>unstable</u> if this second-order work is negative for some δq_k (or some δf_k).

When the second-order work is zero for some δq_k (or some δf_k), we have the critical state, defined as the limit of stable states. Whether the critical state is stable or unstable can be decided only on the basis of the higher-order terms of the Taylor series expansion in Eq. 6, which yields higher-order terms to be added to Eqs. 7-10.

When the loads P_k are not constant but variable (e.g., as a function of q_k), higher-order terms involving P_k need to be added to Eq. 6, and consequently also to Eqs. 7-10. The stability criterion $(\Delta S)_{in} < 0$ remains valid whether loads P_k are conservative or nonconservative, but it of course can not detect dynamic instabilities such as flutter.

The special case of the foregoing criterion of stability of equilibrium states for boundary conditions of fixed displacements was stated, without thermodynamic derivation, by Hill [2].

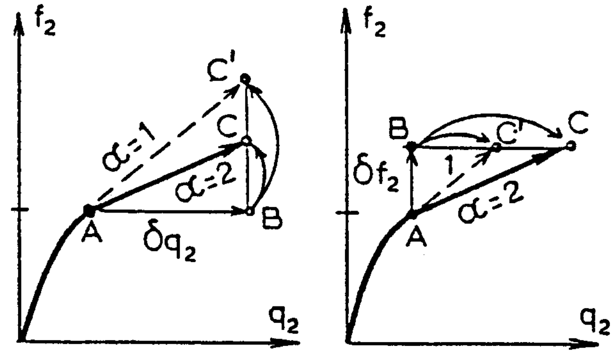

Figure 1. Bifurcation of Equilibrium Path under Displacement Control (left) or Load Control (right).

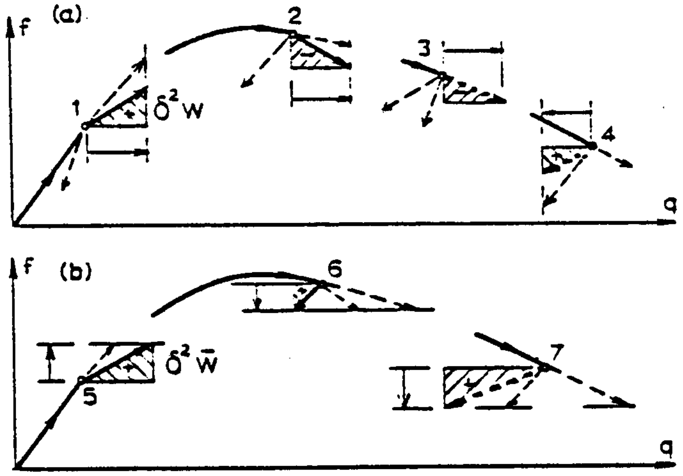

Figure 2. Stable and Unstable Paths for a Structure with Single Controlled Displacement or Single Load.

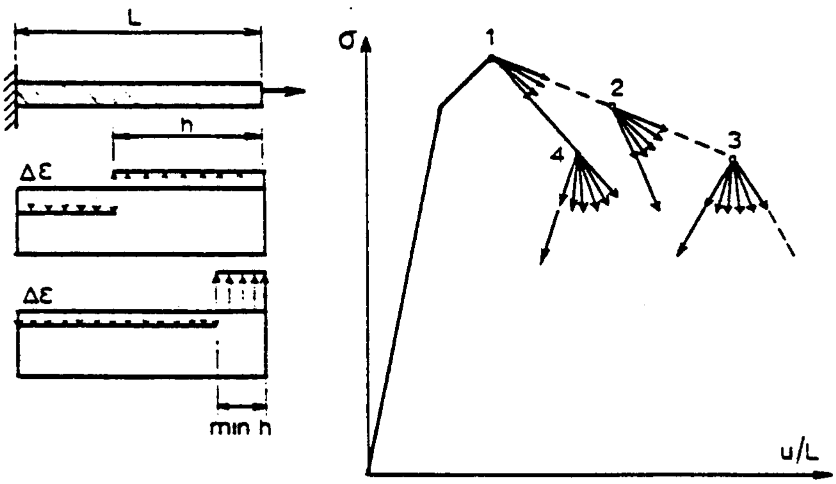

Figure 3. Bifurcations due to Strain Localization in a Strain-Softening Uniaxially Stressed Specimen.

General Criteria of Stable Equilibrium Path of Inelastic Structure

In contrast to elastic structures, the equilibrium states on all the equilibrium path branches emanating from a common bifurcation point (Fig. 1) can be stable. So, which branch will be followed by the structure? This question cannot be decided on the basis of stability analysis of equilibrium states. As proposed by Bažant [3], one must analyze stability of the equilibrium path as a whole.

An equilibrium path of a structure represents a series of infinitesimal deviations from equilibrium and restorations of equilibrium. Let us consider for an arbitrary (irreversible) inelastic structure a small loading step along the equilibrium path α ($\alpha = 1$ or 2) which starts at the bifurcation state A (at which the applied load is P_0, and $X = 0$) and ends at another state C on path α (Fig. 1). We decompose this step into two substeps, the first one (I) away from the initial equilibrium state A and ending at some intermediate nonequilibrium state B, and the second one (II) toward a new equilibrium state, ending on one of the equilibrium paths at state C; see Fig. 1. (This decomposition was introduced in Ref. 2, which presented a special case of the analysis that follows.)

The displacements or forces which are controlled are denoted as q_m, f_m. If q_m is controlled, we consider δq_m to be changed in the first substep (Fig. 1 left) while f_m are kept constant, which of course destroys equilibrium. Displacements δq_m are kept constant during the second substep in which δf_m are allowed to change so as to regain equilibrium. If f_m are controlled, we consider δf_m to be changed in the first substep (Fig. 1 right) while q_m are frozen (constant), which destroys equilibrium. Forces δf_m are kept constant during the second substep in which δq_m are allowed to change so as to restore equilibrium.

1a) Consider now displacement control (i.e., δq_m prescribed, Fig. 1 left) and isothermal conditions ($dT = 0$). Then δq_m are the same for all the paths α but the equilibrium force increments $\delta f_m^{(\alpha)}$ are different. According to Eq. 2, the increment of Helmholtz free energy of the structure over the entire step is, up to second-order terms, $\Delta F = \sum(f_m^0 + \frac{1}{2}\,\delta f_m^{(\alpha)})\delta q_m = \Delta F_I + \Delta F_{II}^{(\alpha)}$. Here $\Delta F_I = \sum f_m^0 \delta q_m$ = increment of F over the first substep, which is the same for both paths $\alpha = 1,2$, and $\Delta F_{II}^{(\alpha)} = \sum \frac{1}{2}\,\delta f_m^{(\alpha)}\,\delta q_m = \delta^2 W^{(\alpha)}$ = increment of F in the second substep in which q_m are constant, while the forces f_m change by $\delta f_m^{(\alpha)}$ to find new equilibrium on path α; $\delta^2 W^{(\alpha)}$ is the second-order work along path α. According to Eq. 1 with $dT = 0$, we have for the second substep (in which $\delta q = 0$) $\Delta F_{II}^{(\alpha)} = -T(\Delta S)_{in}$. The second law of thermodynamics indicates that the structure will approach the equilibrium state which maximizes $(\Delta S)_{in}$ [4], i.e., minimizes $\Delta F_{II}^{(\alpha)}$. Hence, the path α which actually occurs is that for which

$$-T\Delta S_{in} = \delta^2 W^{(\alpha)} = \sum_m \tfrac{1}{2}\,\delta f_m^{(\alpha)}\,\delta q_m = \sum_m \sum_j \tfrac{1}{2} K_{mj}^{(\alpha)}\,\delta q_m \delta q_j = \underset{(\alpha)}{\mathrm{Min}} \tag{11}$$

(if q_m is controlled). $K_{mj}^{(\alpha)}$ is the tangential stiffness matrix for path α, which must be based on isothermal incremental material properties.

1b) Next consider displacement control (δq_m prescribed, Fig. 1 left) and isentropic conditions ($dS = 0$). According to Eq. 2, the increment of the total energy of the structure over the entire step is, up to the second-order terms, $\Delta U = \sum(f_m^0 + \frac{1}{2}\delta f_m^{(\alpha)})\delta q_m = \Delta U_I + \Delta U_{II}^{(\alpha)}$. Here $\Delta U_I = \sum f_m^0 \delta q_m$, which is the same for both paths α, and $\Delta U_{II}^{(\alpha)} = \sum \frac{1}{2}\,\delta f_m^{(\alpha)}\,\delta q_m = \delta^2 W^{(\alpha)}$ = increment of U in the second substep in which q_m are constant while f_m change by $\delta f_m^{(\alpha)}$ to find new equilibrium on path α. According to Eq. 2, with $dS = 0$, we have for the second substep (in which $\delta q = 0$) $\Delta U_{II}^{(\alpha)} = -T(\Delta S)_{in}$. The second law of thermodynamics indicates that, on approach to equilibrium, $\Delta U_{II}^{(\alpha)}$ must be minimized [4]. Hence, the path which occurs is again determined by Eq. 11, in which however $K_{mj}^{(\alpha)}$ must be based on isentropic rather than isothermal material properties.

2a) Consider now load control (Fig. 1 right) and isothermal conditions. The proper thermodynamic function is now Gibbs' free energy, which is defined as $G = F - \sum P_k q_k$. From Eq. 1 it follows that

$$dG = -\sum_k q_k \, df_k - SdT - T(dS)_{in}. \tag{12}$$

Here $\sum q_k df_k$ represents the complementary work. At the same time, in terms of the corresponding equilibrium displacements q_{Tk}, we have according to Eq. 12 (for $dT = dS_{in} = 0$), $dG = -\sum q_{Tk} df_k$ (in similarity to Eq. 2); subscript T indicates quantities calculated on the basis of isothermal incremental material properties ($dT = 0$). According to the last relation we have, for both substeps combined, $\Delta G = -\sum(q_m + \frac{1}{2}\delta q_m^{(\alpha)})\delta f_m = \Delta G_I + \Delta G_{II}$. Here $\Delta G_I = -\sum q_m \delta f_m$ = increment of G over the first substep, which is the same for both paths $\alpha = 1, 2$, and $\Delta G_{II}^{(\alpha)} = -\sum \frac{1}{2}\delta f_m^{(\alpha)} \delta q_m = -\delta^2 \bar{W}^{(\alpha)}$ = increment of G over the second substep in which f_m are constant while q_m are allowed to change so as to restore equilibrium by reaching path α; $\delta^2 \bar{W}^{(\alpha)}$ is the second-order complementary work along path α. According to Eq. 12 with $dT = 0$, we have for the second substep (in which $\delta \underset{\sim}{f} = 0$) $\Delta G_{II}^{(\alpha)} = -T(\Delta S)_{in}$. Based on the second law of thermodynamics, the approach to the new equilibrium state must maximize $(\Delta S)_{in}$, i.e., minimize $\Delta G_{II}^{(\alpha)}$. Hence, the path α which actually occurs is that for which

$$T(\Delta S)_{in} = \delta^2 \bar{W}^{(\alpha)} = \sum_m \tfrac{1}{2}\delta f_m \delta q_m^{(\alpha)} = \sum_m \sum_j \tfrac{1}{2} D_{mj}^{(\alpha)} \delta f_m \delta f_j = \underset{(\alpha)}{\text{Max}} \tag{13}$$

(if f_m is controlled). $D_{mj}^{(\alpha)}$ is the tangential compliance matrix for path α, which must be based on isothermal material properties. Note that, in contrast to Eq. 11, the path label (α) now appears with δq_m rather than δf_m.

2b) Finally consider load control (Fig. 1 right) and isentropic conditions ($dS = 0$). The proper thermodynamic function is now the enthalpy, which is defined as $H = U - \sum P_k q_k$. From Eq. 1 it follows that

$$dH = -\sum_k q_k df_k + TdS - T(dS)_{in}. \tag{14}$$

In terms of equilibrium displacements q_{Sk} calculated from isentropic (adiabatic) incremental material properties we have (according to Eq. 14) $dH = -\sum q_{Sk} df_k$. Thus, for both substeps combined, $\Delta H = -\sum(q_n + \frac{1}{2}\delta q_n^{(\alpha)})\delta f_n = \Delta H_I + \Delta H_{II}$. Here $\Delta H_I = -\sum q_m \delta f_m$ = increment of enthalpy H over the first substep, which is the same for both paths $\alpha = 1, 2$; and $\Delta H_{II}^{(\alpha)} = -\sum \frac{1}{2}\delta f_m^{(\alpha)} \delta q_m = -\delta^2 \bar{W}^{(\alpha)}$ = increment of H over the second substep in which f_m are constant (Fig. 1 right) while q_m are allowed to change so as to restore equilibrium. According to Eq. 14 with $dS = 0$, we have for the second substep (in which $\delta \underset{\sim}{f} = \underset{\sim}{0}$) $\Delta H_{II}^{(\alpha)} = -T(\Delta S)_{in}$. In view of the second law of thermodynamics, the approach to new equilibrium must maximize $(\Delta S)_{in}$, i.e. minimize $\Delta H_{II}^{(\alpha)}$. Hence, the path which occurs is again that indicated by Eq. 13, in which however $D_{mj}^{(\alpha)}$ must be based on isentropic rather than isothermal material properties. For a generalization to mixed load and displacement controls, see Appendix.

Displacements $\delta q_k^{(\alpha)}$ are compatible with the field of strain increment tensor $\delta \underset{\sim}{\varepsilon}^{(\alpha)}$, and forces $\delta f_k^{(\alpha)}$ are in equilibrium with the field of stress increment tensor $\delta \underset{\sim}{\sigma}^{(\alpha)}$. Consequently, if nonlinear geometric effects are negligible, the principle of virtual work may be applied to Eqs. 11 and 13, which shows that (see also Appendix 2):

$$\delta^2 W^{(\alpha)} = \sum_m \tfrac{1}{2}\delta f_m^{(\alpha)} \delta q_m = \int_V \tfrac{1}{2}\delta \underset{\sim}{\sigma}^{(\alpha)} : \delta \underset{\sim}{\varepsilon}^{(\alpha)} dV = \underset{(\alpha)}{\text{Min}} \tag{15}$$

$$\delta^2\bar{W}^{(\alpha)} = \sum \tfrac{1}{2}\delta f_m \delta q_m^{(\alpha)} = \int_V \tfrac{1}{2}\delta\underset{\sim}{\sigma}^{(\alpha)}:\delta\underset{\sim}{\varepsilon}^{(\alpha)} dV = \underset{(\alpha)}{\text{Max}} \tag{16}$$

where V = volume of the structure. For computations, however, the discrete form is usually more efficient than the volume integral, especially since both $\underset{\sim}{\sigma}$ and $\underset{\sim}{\varepsilon}$ are different for various paths α. Moreover, the discrete form is more general since it applies also if the problem is geometrically nonlinear.

Structures with Single Load or Single Controlled Displacement

In this case the structural response is characterized by tangential stiffness $K = df/dq$. Expressing the second-order work and complementary work, we have for the stable path the conditions

$$\delta^2W^{(\alpha)} = \tfrac{1}{2}\,\delta f^{(\alpha)}\delta q = \frac{K^{(\alpha)}}{2}\,\delta q^2 = \underset{(\alpha)}{\text{Min}} \tag{17}$$

$$\delta^2\bar{W}^{(\alpha)} = \tfrac{1}{2}\,\delta f\delta q^{(\alpha)} = \frac{1}{2K^{(\alpha)}}\delta f^2 = \underset{(\alpha)}{\text{Max}} \tag{18}$$

From this the following theorem ensues:

If the critical state is stable, the stable path is that for which the tangential stiffness $K^{(\alpha)}$ is minimum, regardless of whether the displacement or the load is controlled. This theorem holds not only for $K^{(\alpha)} > 0$ (hardening) but also for $K^{(\alpha)} \to 0$ (perfect plasticity) and $K^{(\alpha)} < 0$ (softening). Note that for $K^{(\alpha)} < 0$ the load control is excluded because such an initial state is unstable.

Based on this theorem, examples of stable paths are marked by solid arrows in Fig. 2 for both displacement control (top) and load control (bottom).

For uniaxial test specimens, this theorem implies that strain must start to localize right after the peak stress point, even though nonlocalized (uniform) strain states in the softening range may be stable way beyond the peak as proven in Ref. 5.

Criterion of Path Bifurcation

Consider now an arbitrary structure with an n-dimensional column matrix of displacements $\underset{\sim}{q} = (q_1,\ldots,q_n)$. Let L be the loading-only sector in the n-dimensional spaces of δq_k, U the adjacent loading-unloading sectors, and $\underset{\sim}{K}^L$, $\underset{\sim}{K}^U$ the corresponding matrices $\underset{\sim}{K}$. If there are two paths under load control, then $\underset{\sim}{K}^L\delta \underset{\sim}{q}^{(1)} = \delta\underset{\sim}{f}$ and $\underset{\sim}{K}^U\,\delta \underset{\sim}{q}^{(2)} = \delta\underset{\sim}{f}$ where $\delta\underset{\sim}{f}$ is given. Assume the direction $\underset{\sim}{\nu}^{(1)}$ of $\delta \underset{\sim}{q}^{(1)}$ lies in sector L. Prior to the first bifurcation, the direction $\underset{\sim}{\nu}^{(2)}$ of $\delta \underset{\sim}{q}^{(2)}$ lies outside the corresponding sector U for all possible sectors U, i.e. no path 2 exists. After the first bifurcation, $\underset{\sim}{\nu}^{(2)}$ lies within the corresponding sector U at least for one U, and then path 2 exists.

Suppose now that the tangential material properties vary continuously along the loading path. Then the direction $\underset{\sim}{\nu}^{(2)}$ should also vary continuously. So, at first bifurcation, the direction $\underset{\sim}{\nu}^{(2)}$ must coincide with the boundary of sector L (as illustrated by Shanley's column). But then we must have not only $\underset{\sim}{K}^U\delta \underset{\sim}{q}^{(2)} = \delta\underset{\sim}{f}$ but also $\underset{\sim}{K}^L\delta \underset{\sim}{q}^{(2)} = \delta\underset{\sim}{f}$. Subtracting this from $\underset{\sim}{K}^L\delta \underset{\sim}{q}^{(1)} = \delta\underset{\sim}{f}$, we get

$$\underset{\sim}{K}^L(\delta\underset{\sim}{q}^{(2)} - \delta\underset{\sim}{q}^{(1)}) = \underset{\sim}{0}. \tag{19}$$

where $\delta\underset{\sim}{q}^{(2)} \neq \delta\underset{\sim}{q}^{(1)}$. Consequently, the first bifurcation is indicated by singularity of matrix $\underset{\sim}{K}^L$, i.e. by the fact that $\det \underset{\sim}{K}^L = 0$, or that the

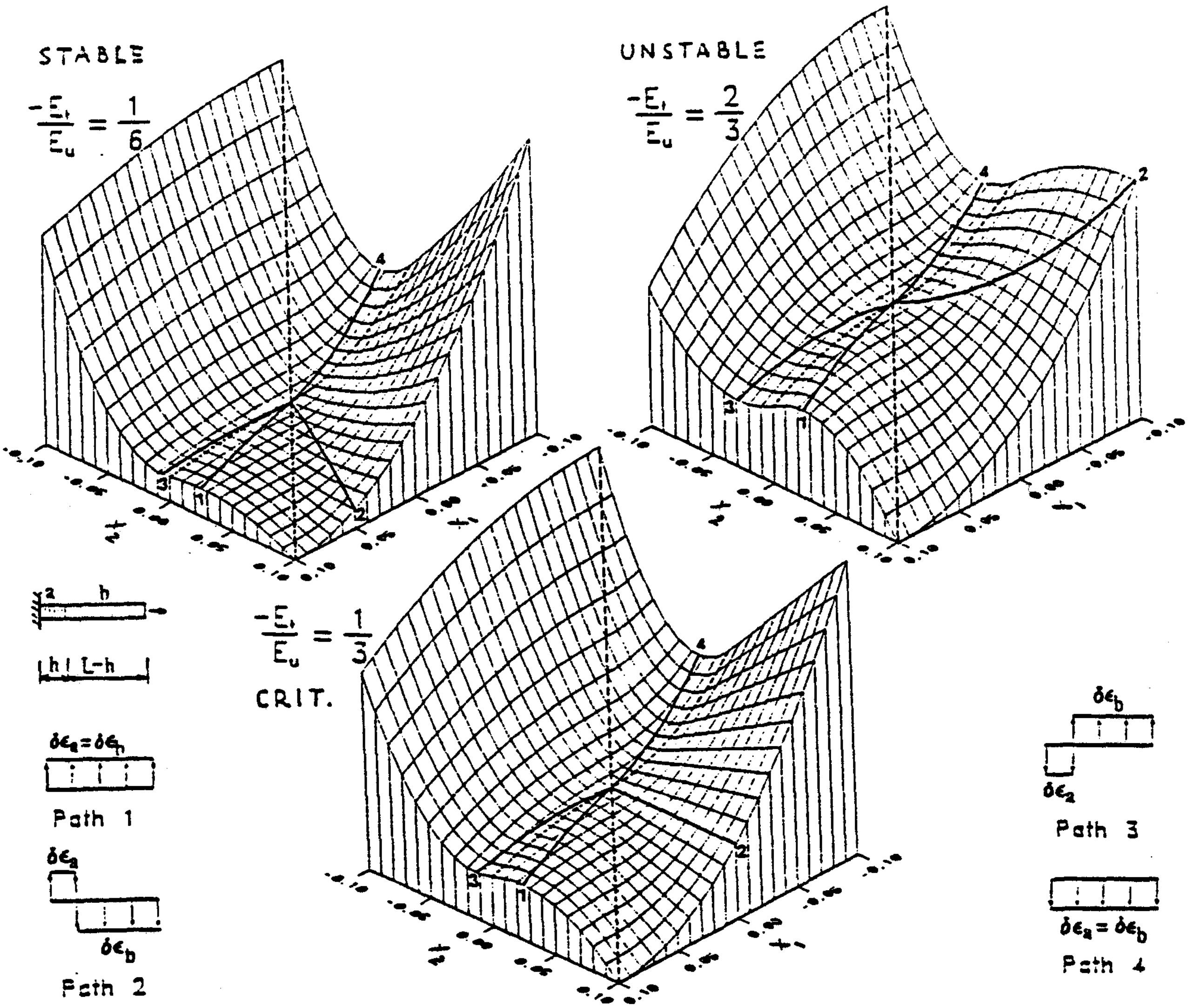

Figure 4. Surface of $-\Delta W = -\delta^2 W$ for Localization in a Strain-Softening Uniaxial Test Specimen.

smallest eigenvalue λ_1 of matrix $\underset{\sim}{K}^L$ vanishes. This is the well-known condition of Hill (and the solid corresponding to matrix $\underset{\sim}{K}^L$ for which multiple solutions exist is called the linear comparison solid [2]; see also Refs. 6-9.) If, however, the tangential material properties change along the loading path discontinuously then the first bifurcation occurs when λ_1 changes from positive to negative without $\underset{\sim}{K}^L$ ever becoming singular.

The eigenvector q^* of the singular matrix $\underset{\sim}{K}^L$ at the first bifurcation can lie either inside or outside sector L. 1) If q^* lies inside L, then there exists path 2 such that $\underset{\sim}{K}^L \delta q^{(2)} = \underset{\sim}{0}$, where $\delta q^{(2)} \sim q^*$. This means that there is neutral equilibrium, which represents the limit point instability (or snapthrough). 2) If q^* lies outside sector L, then $\delta q^{(2)}$ cannot coincide with q^* but must lie at the boundary of sector L; then $\underset{\sim}{K}^L \delta q^{(2)} = \delta \underset{\sim}{f}$ where $\delta \underset{\sim}{f}$ is nonzero. This means that the secondary path at the first bifurcation occurs at a changing load. An example is the Shanley-type bifurcation, known from the buckling theory of elastoplastic columns [7-10].

If matrix $\underset{\sim}{K}^L$ has a negative eigenvalue λ_1, we have $(\underset{\sim}{K}^L - \lambda_1 \underset{\sim}{I})q^* = \underset{\sim}{0}$ where $\underset{\sim}{I}$ = unit matrix. It follows that $q^{*T} \underset{\sim}{K}^L q = q^{*T}\lambda_1 \underset{\sim}{I} q = \lambda_1 q^{*T} < 0$. But this does not imply instability of state if the associated eigenvector q^* lies outside L, contrary to the comments in some recent papers. However, the existence of negative λ_1 means that a bifurcation point must have been passed and that the state might not lie on a stable path.

If only one displacement, say δq_n, is controlled and $\delta f_1 = \cdots = \delta f_{n-1} = 0$, one may take the foregoing case of load control for which $\underset{\sim}{K}^L$ is singular at the first bifurcation point, and then scale δf_n and δq_n by a common factor so as to make δq_n for both paths mutually equal. Since such a scaling does not change the eigenvalues of $\underset{\sim}{K}^L$, the condition det $\underset{\sim}{K} = 0$ also characterizes the first bifurcation point under displacement control (provided the tangential properties vary continuously).

Application to Softening Structures

The theorem on structures with single controlled displacements (Eq. 18) may be applied to a uniaxial (tensile or compressive) test specimen in the strain-softening range loaded under displacement control. The main path corresponds to uniform strain (path $\overline{123}$ in Fig. 3). Every point after the peak is a bifurcation point, with infinitely many paths emanating from each point, depending on the ratio h/L of the length h of the strain localization segment to the specimen length L (Fig. 3 left). As argued before [5,11], the requirement that failure cannot occur at vanishing energy dissipation makes it necessary that the size h of the localization zone has a certain minimum, h_{min}, which must be a material property. In Fig. 3 (right), the path corresponding to $h = h_{min}$ is always the leftmost descending path. Now, according to the theorem on structures with a single controlled displacement (Eq. 17), the stable path is always the path which corresponds to $h = h_{min}$ (i.e. path $\overline{14}$).

Hence strain localization must begin right after the peak stress state (point 1), even though the states on both the nonlocalized ($\overline{123}$) and localized ($\overline{14}$) response paths in Fig. 3 are under displacement control stable, as proven in Ref. 5. The states become unstable at displacement control only after the snapback point (point 3 or point 4) is passed. However, if the control is switched at the snapback point from displacement control to load control, the states on the branches with positive slope after the snapback point are stable. The path which occurs is determined by Eq. 18, and it is again the leftmost path.

The second-order work for the specimen in Fig. 3 is obtained as

$$\delta^2 W = \tfrac{1}{2} E_u \left[\frac{\xi}{h} \delta u + \frac{\eta}{L-h} (\delta u - \delta v)^2 \right] \tag{20}$$

in which E_u = unloading modulus (> 0), E_t = tangent modulus for further loading (< 0); $\xi = 1$ for $\delta v \le 0$, $\xi = -k$ for $\delta v > 0$, $\eta = 1$ for $\delta u \le \delta v$, $\eta = -k$ for $\delta u > \delta v$; $k = -E_t/E_u$; u, v = axial displacements at the end and at the interface of segments h and L - h. The surface in Eq. 20 is plotted in Fig. 4, in which $X_1 = \delta u/L$ and $X_2 = \delta v/L$.

Fig. 5 shows an example of a tensioned rectangular panel. Its material undergoes strain-softening described by the Tresca plasticity criterion whose yield limit decreases as a function of the effective plastic strain increment. This example was calculated at Northwestern University by F.-B. Lin [12] and cited in Ref. 3 and 11. A square finite element mesh is used, with the element size h equal to the characteristic length ℓ of the material. The initial yield limit in the two central elements on each side of

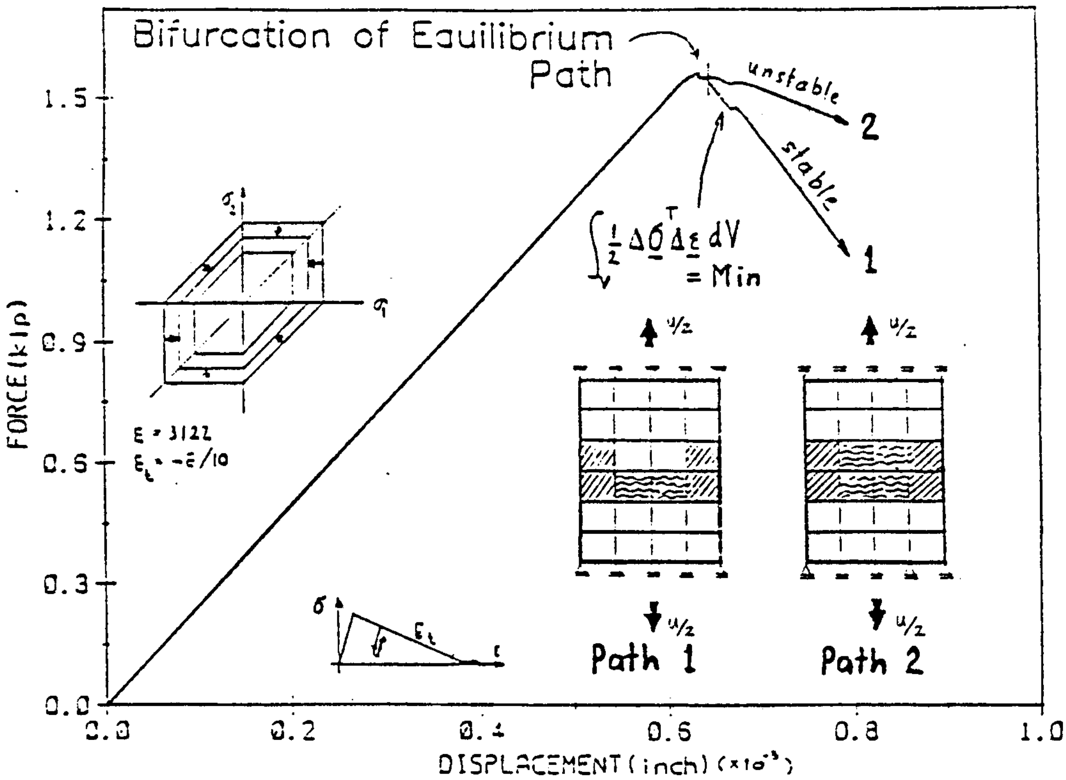

Figure 5. Finite Element Results of F.-B. Lin [27] on Crack Band Propagation in a Panel.

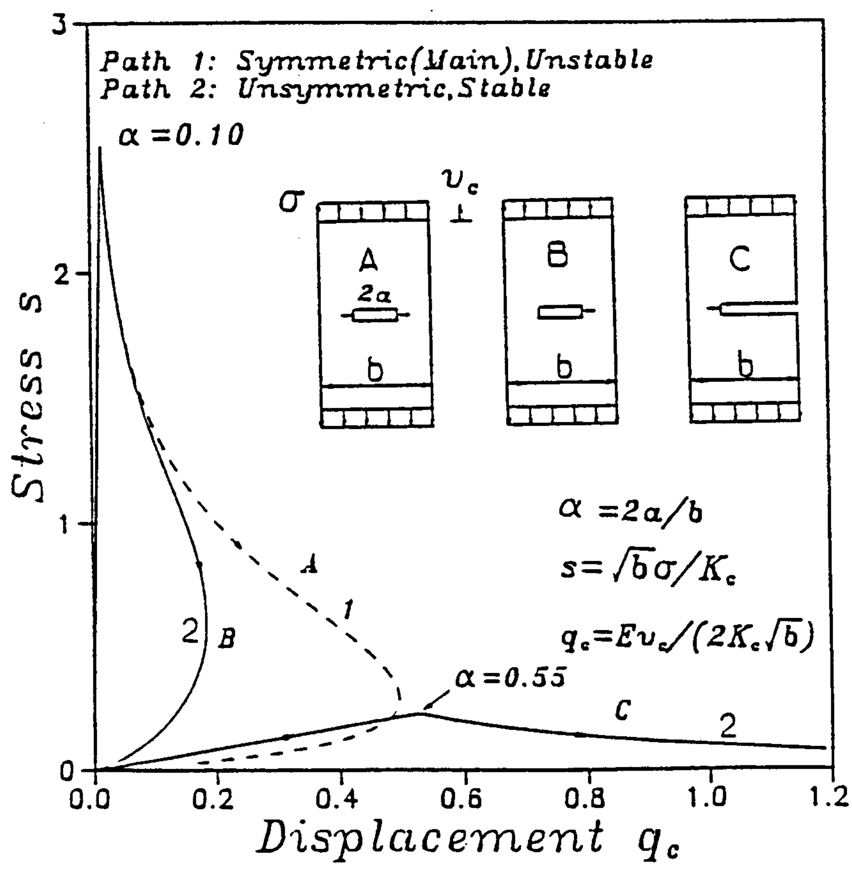

Figure 6. Stable (solid curve) and Unstable (dashed curve) Paths of Center-Cracked Panel [15].

the panel is assumed to be slightly less than in the other elements.

At each stage of softening, one can find two solution paths for the load increment: (1) the softening band grows symmetrically, being two elements wide, and (2) the softening extends asymmetrically in only one row of elements while the adjacent elements start unloading. The states obtained in each of these solutions are stable if the loading is displacement controlled, as has been checked from the condition $\delta^2 W^{(\alpha)}$ = Min (Eq. 8).

Yet only path (2), i.e. the localized path of a single-element width (asymmetric softening band) is stable. This follows from the fact that its response path descends steeper than path (2) for the symmetric band; see Fig. 5. However, unless the present conditions of path stability are included in the finite element program, the result obtained with the usual form of these programs is path (1); the iterations in each loading step converge well and the computer happily marches along the incorrect, unstable path, being unaware that a path toward a higher internal entropy of the structure exists nearby.

Exact analytical solutions are also possible for localization of softening into ellipsoidal regions and planar bands (Bažant, 1987, Bažant and Lin, 1987). It is found that stable uniform strain-softening states are possible, but Shanley-type localization at increasing displacement occurs right at the start of strain-softening.

These examples illustrate that checks for path bifurcation and path stability need to be introduced into finite element programs for damage. The same is true of interactive crack systems which we discuss next.

Interactive Crack Systems

Damage often consists of cracks which interact. The simplest, albeit usually not too realistic, approach is to treat crack propagation according to linear elastic fracture mechanics. If there are two crack tips which are simultaneously critical, two response paths normally exist: (1) both cracks extend, or (2) only one crack extends while for the other crack the stress intensity factor K decreases below the critical value K_c.

One such problem is the growth of parallel equidistant cooling (or shrinkage) cracks into a halfspace. At a certain crack length-to-spacing ratio a/s, an unstable state is reached, after which only every other crack grows (by a jump) while the intermediate cracks unload; see Ref. 13 and 14. A bifurcation state after which every other crack grows, in a stable-path mode and at increasing load (i.e. continued cooling), is reached even earlier.

Figs. 6-8 show some other recent results for some elementary cases with two interacting crack tips, obtained by Bažant, Tabbara and Kazemi [15]. Initially the system is symmetric and the stress intensity factors at the crack tips are equal to the fracture toughness K_c ($K_1 = K_2 = K_c$). The stress intensity factors used in Figure 6 were obtained from Murakami's handbook [16]. Squaring the stress intensity factor yields the energy release rate of each crack tip, integration along the crack length then yields the total energy release, and its differentiation with respect to the applied load then yields, according to Castigliano's theorem, the additional displacement due to cracks, which may then be used to evaluate $\delta^2 W^{(\alpha)}$. For Figs. 7-8 $\delta^2 W^{(\alpha)}$ was calculated using finite elements.

As the basic path ($\alpha = 1$) we consider the symmetric crack propagation

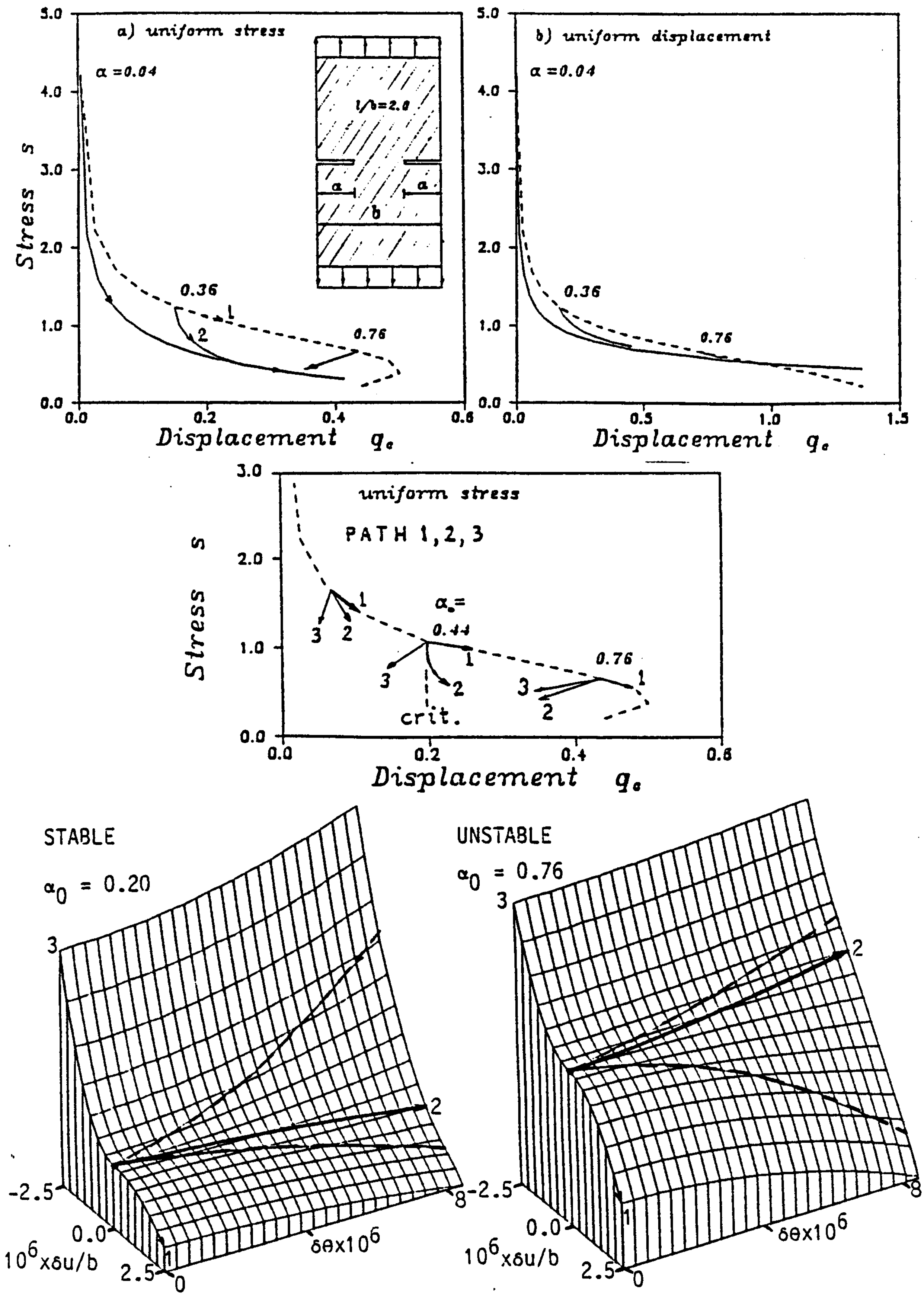

Figure 7. Stable and Unstable Paths of Edge-Cracked Panel and Surfaces of δ^2W (δu, $\delta\theta$ = axial displacement and rotation at specimen ends) [15].

with equal crack extensions ($\delta a_1 = \delta a_2 > 0$ and $\delta K_1 = \delta K_2 = 0$). The bifurcated path ($\alpha = 2$) corresponds to one crack growing while the other is stationary and its stress intensity factor drops below the critical value ($\delta a_1 > 0$, $\delta a_2 = 0$ and $\delta K_1 = 0$, $\delta K_2 < 0$). The third path ($\alpha = 3$) is the unloading path ($\delta a_1 = \delta a_2 = 0$ and $\delta K_1 = \delta K_2 < 0$). In general the states on the symmetric response path are unstable ($\delta^2 W < 0$ for some admissible deviations from the initial state. However, if the relative displacement near the crack tip is controlled, then the state could be stable.

Fig. 8a shows the surface of $\delta^2 W$ as a function of $\delta u/b$ and $\delta\theta$, where δu = average displacement variation due to the crack extension and $\delta\theta$ = end rotation variation due to the crack extension (Fig. 8a). Denoting δm and δf as the corresponding moment and force variations for $\delta\theta$ and δu, respectively, we therefore have $\delta^2 W = \frac{1}{2}\,\delta m \delta\theta + \frac{1}{2}\,\delta f \delta u$. The equilibrium paths for $\delta m = 0$ are shown in Figs. 8b,c and are labeled as 1, 2 and 3. Depending on the control variable, the stable path may be obtained according to Eqs. 17-18. It is found that while the symmetric crack states exemplified in Figs. 6-8 are stable, under displacement control conditions, the loading path which leads to them is unstable unless the cracks are long enough. The states on the unstable path cannot be obtained in a continuous loading process. They can only arise by other means. It seems that stability of the crack path generally requires localization of crack systems into a single crack with a single crack tip, a row of cracks on a line, or multiple rows of cracks; symmetry is not kept on a stable loading path. Similar results have been obtained for other crack systems, including cracks on parallel planes in strips or a space.

Analogous Problem: Buckling of Elasto-Plastic Columns

A classical problem exhibiting similar path bifurcations is the buckling of elasto-plastic columns. Although stability of state and paths of such columns has not been solved, the practical behavior of such columns is well understood on the basis of test results and imperfection analysis. This behavior agrees with the present theory and thus provides its corroboration, as shown in Ref. 1, in which Shanley's column [10] was studied in detail. This column (Fig. 9) is hinged and geometrically perfect. It consists of two rigid bars of lengths $\ell/2$, which are connected by a very short elasto-plastic link (point hinge) of length $h \ll \ell$ and width h, having an ideal I-beam cross section of area A. The lateral deflection and the axial displacement at the load point (positive if shortening) are denoted as q_1 and q_2, respectively. Initially the column is perfectly straight and is loaded by an axial centric load P (positive if compressive). The initial equilibrium under load $P = P_0$ at zero lateral deflection is disturbed by increasing the axial load to $P = P_0 + \delta f_2$ and applying a small lateral load δf_1. This causes in the flanges of the elasto-plastic link the strains $\delta\varepsilon_1 = -\delta\theta - \delta q_2/h$ and $\delta\varepsilon_2 = \delta\theta - \delta q_2/h$ (on the concave and convex sides, respectively; $\delta\theta = 2\delta q_1/\ell$ = rotation of the rigid bars, assumed to be small.

The incremental moduli for loading and unloading, both positive, are denoted as E_t and E_u. Always $E_t < E_u$. It is convenient to express the moduli at the concave and convex faces as $E_1 = \eta E_t$ and $E_2 = \eta\xi E_t$ and define the nondimensional displacements $X = \delta q_1/\ell$ and $Y = \delta q_2/2h$. For buckling to the right ($X > 0$), one has the following loading - unloading criteria (Fig. 9e):

$$\left.\begin{array}{lll} \text{1) for } Y > X \text{ (loading only):} & \xi = 1, & \eta = 1 \\ \text{2) for } -X \le Y \le X \text{ (loading - unloading):} & \xi = \xi_u, & \eta = 1 \\ \text{3) for } Y < -X \text{ (unloading only):} & \xi = 1, & \eta = \xi_u \end{array}\right\} \quad (21)$$

where $\xi_u = E_u/E_t$. The moment and axial conditions of equilibrium yield:

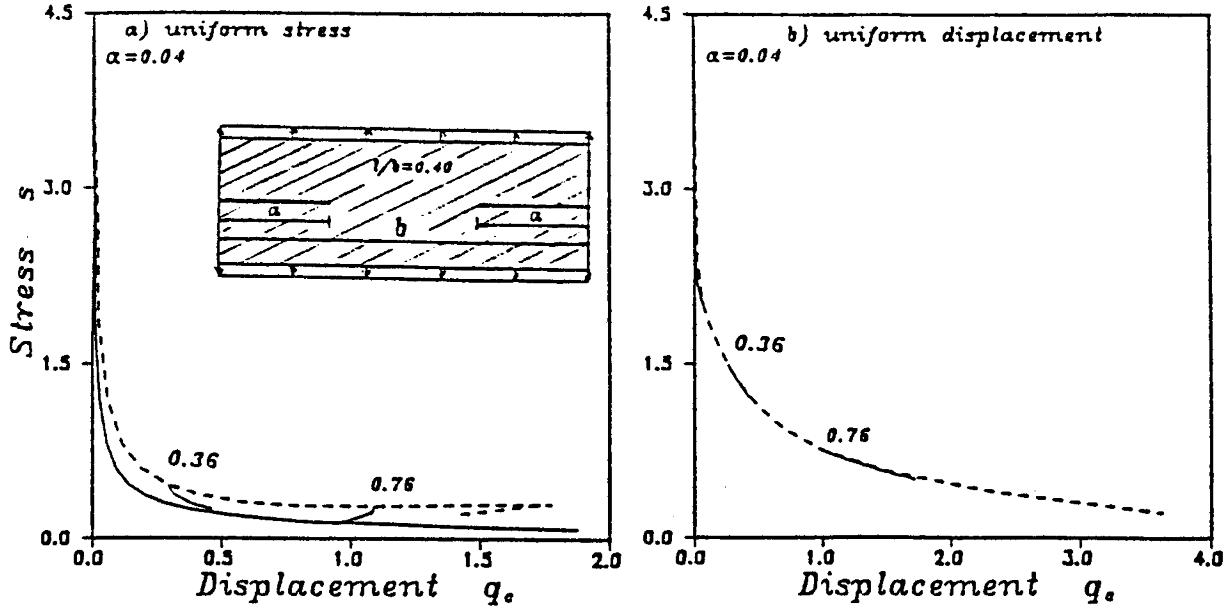

Figure 8. Stable and Unstable Paths of Short Edge-Cracked Panel [15] .

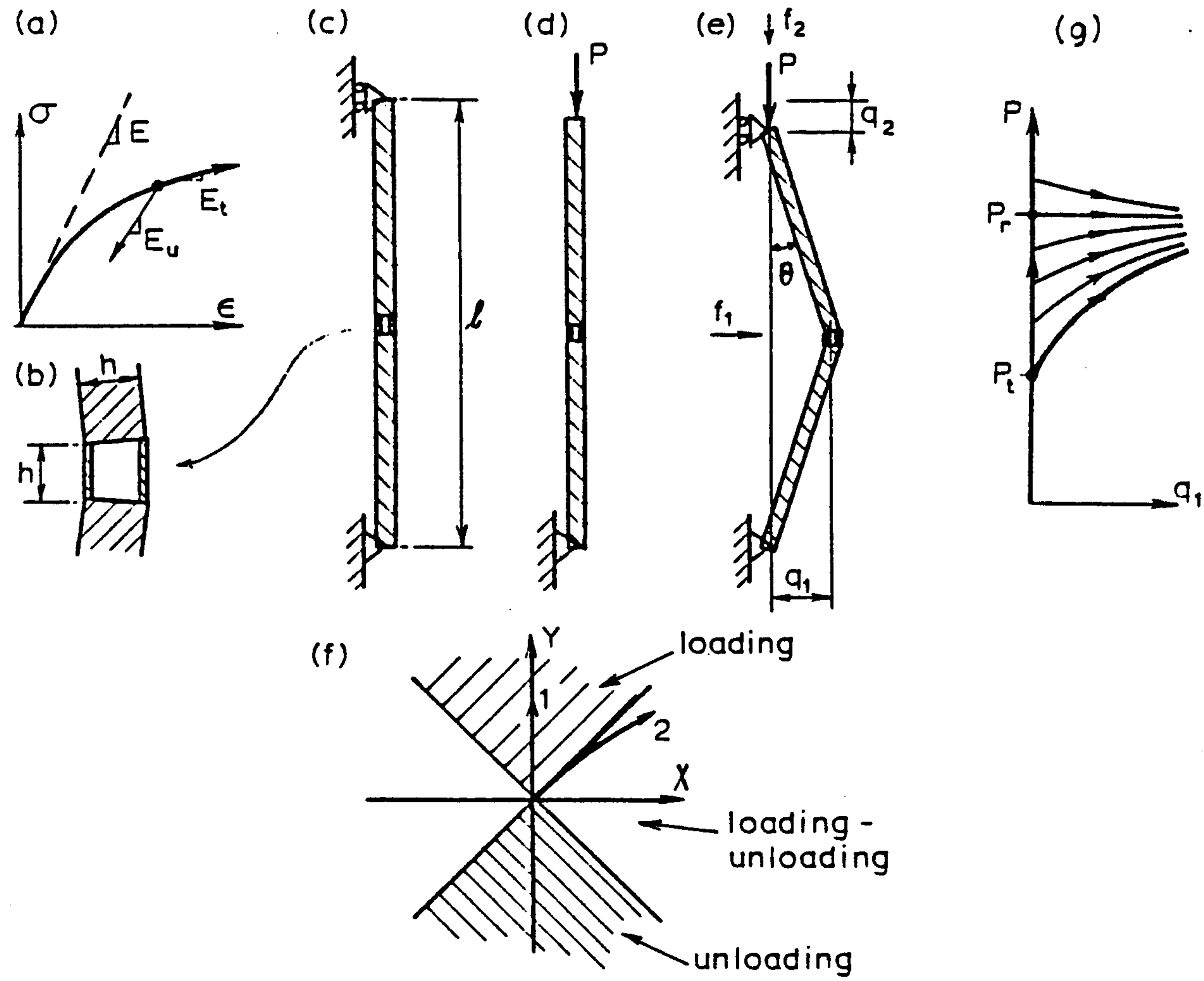

Figure 9. Shanley's Elastoplastic Column.

$$\begin{Bmatrix} \ell\delta f_1 \\ 2h\delta f_2 \end{Bmatrix} = 2\eta P_t \ell \begin{bmatrix} 1 + \xi - \frac{2P_0}{\eta P_t}, & 1 - \xi \\ 1 - \xi & 1 + \xi \end{bmatrix} \begin{Bmatrix} X \\ Y \end{Bmatrix} \tag{22}$$

in which $P_t = E_t A\, h/\ell$ = Shanley's tangent modulus load [10]. If $\delta f_2 = 0$ (P = const.), then the only nonzero solution of Eq. 22 is $P_0 = 2\xi_u P_t/(\xi_u + 1) = P_r$, which represents the reduced modulus load of Engesser and von Kármán, at which there is neutral equilibrium (Fig. 9f). Path 1 is characterized by zero deflection and $\xi = \eta = 1$; $X^{(1)} = 0$, $\delta f_2^{(1)} = 2P_t Y^{(1)}\ell/h$. For $P_0 \to P_t$, however, $X^{(1)}$ is arbitrary since the matrix of Eq. 22 becomes singular (but $X^{(1)} < Y^{(1)}$ because $\xi = 1$). For path 2 characterized by $\xi > 1$, one gets

$$X^{(2)} = \frac{\xi_u - 1}{\xi_u + 1 - 2P_0/P_t} Y^{(2)}, \qquad \delta f_2^{(2)} = \frac{P_t \ell}{h} \frac{4\xi_u P_t - 2(\xi_u + 1)P_0}{(\xi_u + 1)P_t - 2P_0} Y^{(2)}. \tag{23}$$

Path 2 is possible only if $\xi > 1$, and $\xi > 1$ is possible only if $P_0 \geq P_t$. Both paths exist for each point $P_0 \geq P_t$ (solid curves in Fig. 9f), and the main path for $P_0 \geq P_t$ represents a continuous sequence of bifurcation points. At the first bifurcation ($P_0 = P_t$) we have $Y^{(2)}/X^{(2)} = 1$, i.e., the secondary path starts along the boundary of the loading-only sector in the (X,Y) plane. The second-order work we obtain as $\delta^2 W = \frac{1}{2}(\delta f_1 \delta q_1 + \delta f_2 \delta q_2)$, which furnishes (for buckling to either left or right):

$$\delta^2 W(X,Y) = \frac{P_t \ell \eta}{\xi + 1} \left\{ [(\xi + 1)Y - (\xi - 1)|X|]^2 + 4\xi\left(1 - \frac{(\xi + 1)P_0}{2\xi P_t \eta}\right) X^2 \right\}. \tag{24}$$

Under a displacement-controlled mode of loading, we have $\delta q_2 = 0$, and then $\delta^2 W = \frac{1}{2}\delta f_1 \delta q_1$, i.e. (for $Y \geq 0$):

$$[\delta^2 W]_{Y=\text{const}} = \delta^2 W(X) = \eta P_t \ell \left(\xi + 1 - \frac{2P_0}{\eta P_t}\right) X^2. \tag{25}$$

On the other hand, under a load-controlled mode of loading ($\delta f_2 = 0$):

$$[\delta^2 W]_{P=\text{const}} = \delta^2 W(X) = \eta P_t \ell \left(\frac{4\xi}{\xi + 1} - \frac{2P_0}{\eta P_t}\right) X^2. \tag{26}$$

For the difference of the works $\delta^2 W$ done along the equilibrium paths 1 and 2 we obtain, when Y is the same,

$$\delta^2 W^{(1)} - \delta^2 W^{(2)} = \frac{(P_0 - P_t)(\xi_u - 1)P_t \ell}{\left(1 + \frac{(\xi_u - 1)^2}{4\xi_u}\right)P_r - P_0} Y^2 \tag{27}$$

and when δf_2 is the same,

$$\delta^2 W^{(1)} - \delta^2 W^{(2)} = -\frac{(P_0 - P_t)(\xi_u - 1)h^2}{2(P_r - P_0)(\xi_u + 1)P_t \ell} \delta f_2^2. \tag{28}$$

The condition of stability is that the expressions in Eq. 24 (for load control) or Eq. 25 (for displacement control) must be positive for all possible X and Y (i.e., positive definite). From this it follows that under load control the column is stable if $P_0 < P_{cr}^L$ and unstable if $P_0 > P_{cr}^L$, and under displacement control the column is stable if $P_0 < P_{cr}^D$ and unstable if $P_0 > P_{cr}^D$, in which

$$P_{cr}^L = P_r = \frac{2\xi_u}{\xi_u + 1} P_t, \qquad P_{cr}^D = \frac{\xi_u + 1}{2} P_t = \left(1 + \frac{(\xi_u - 1)^2}{4\xi_u}\right) P_r > P_r. \tag{29}$$

P_t has in the past been considered to be the critical load, but this is not true.

The main aspects of the present stability problem can be illustrated by the surfaces of $T(\Delta S)_{in} = -\delta^2 W(X,Y) = -\Delta W$ in Fig. 10 (for $E_u = 3E_t$). These surfaces consist of quadratic portions separated by the lines $X = \pm Y$ at which ξ or η changes discontinuously. These are lines of curvature discontinuity. In contrast to elastic potential energy, the present surfaces are not smooth. They nevertheless have continuous slopes. The limit of stable states ($P_0 = 1.5\ P_t = P_{cr}$) is manifested on these surfaces by the existence of a horizontal path emanating from the origin 0 (lines B or B'). Instability is characterized by the existence of a path for which $-\delta^2 W$ rises while moving away from the origin. Absence of any such path ensures stability.

As for path stability (Eqs. 11, 13), we now find from Eqs. 27-28 that under displacement control (same Y) we always have $\Delta W^{(2)} << \Delta W^{(1)}$ if $P_0 > P_t$, and under load control (same δf_2), we always have $\Delta W^{(2)} > \Delta W^{(1)}$ if $P_0 > P_t$. This means that, for $P_0 > P_t$, path 2 must occur and is, therefore, stable, while path 1 cannot occur and is, therefore, unstable. So the column must deflect for $P_0 > P_t$. What is then the meaning of the stable states of a perfect column for $P_0 > P_t$? They can be reached if temporary restraints are placed at the sides of the column to prevent it from buckling. If $P_0 < P^D_{cr}$ at axial displacement control, or if $P_0 < P^L_{cr}$ at axial load control, this column will not deflect when the lateral restraint is removed. So the column is stable. Deflection occurs only if the load is increased further. If E_t decreases discontinuously, an undeflected equilibrium state $P_0 > P_t$ can be reached even without lateral restraints. This occurs, e.g., if the stress-strain diagram is bilinear, or if a change of E_t is caused by temperature change, change of moisture content, hysteretic loop of unloading, irradiation damage, etc.

The equilibrium paths leading away from the origin are marked in Fig. 10 as 1, 2 and 2'. In the plots of $-\delta^2 W$, the structure follows the path that descends less steeply. The limit of stability of the main path is characterized by the fact that points 1 and 2 (of equal Y) are at equal altitude ($P_0/P_t = 1$). Instability of the main path is characterized by the fact that point 2 in the plots of $-\delta^2 W$ lies at a higher altitude than point 1.

Eqs. 21-29 are also valid when the column is softening, i.e., $E_t < 0$, in which case $P_t < 0$. One finds that such a column is always unstable because both Eqs. 25 and 26 are violated for at least one direction $\underset{\sim}{\nu}$ of vector $\delta \underset{\sim}{q}$.

Concluding Remarks

The static structural stability studies in the literature, even those conducted in the most general sense of catastrophy theory [17], have so far been confined to elastic structures which possess a smooth potential surface. The surface of $\delta^2 W$ which determines stability of inelastic structures is unsmooth, exhibiting lines of curvature discontinuity. This causes two striking properties: 1) The first bifurcation point on the equilibrium path of an inelastic structure does not have to represent the limit of stability, i.e. the states on all the branches emanating from the bifurcation point can be stable. 2) Yet the stable states of one branch beyond the first bifurcation point cannot be reached by a continuous loading process. Hence, for inelastic structures it is essential to distinguish between: (1) a stable equilibrium state, and (2) a stable equilibrium path. Stability of the state is decided on the basis of deviations away from equilibrium, while stability of the path is decided on the basis of approaches toward equilibrium. For elastic (reversible) structures, both concepts are equivalent, but not for irreversible systems. For such systems, examination of stable states is insufficient.

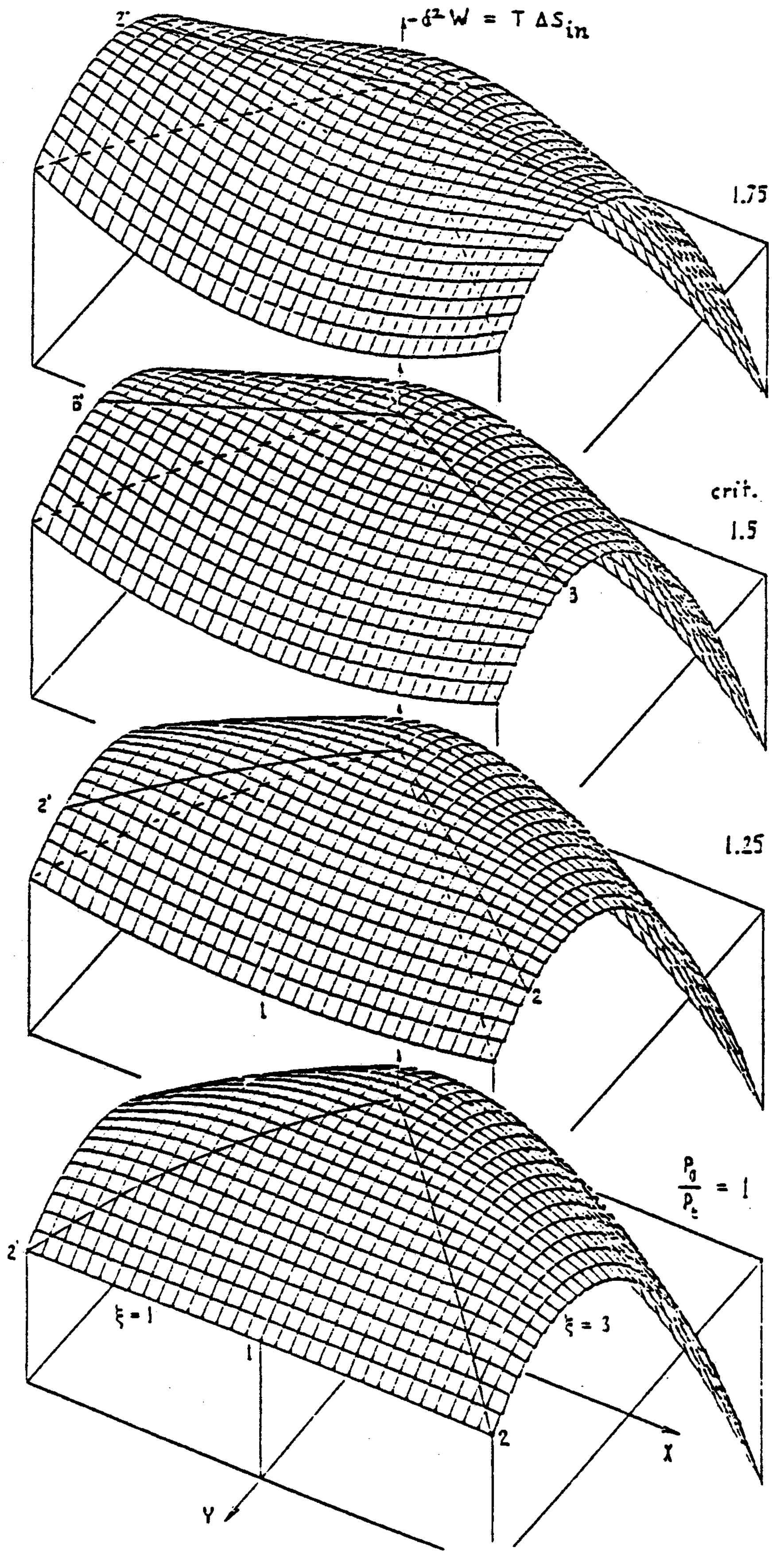

Figure 10. Surface of $- \Delta W = - \delta^2 W$ for Shanley's Elastoplastic Column.

The basic idea advanced here (and in Ref. 1) is that the stable path is that which 1) consists of stable states, and 2) maximizes $(\Delta S)_{in}$ compared to all other paths.

Propagation of damage or interacting cracks often leads to bifurcating equilibrium paths. Thermodynamic stability analysis can identify which path will actually occur. Checks for path stability need to be included in finite element programs for damage.

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Appendix 1. - Mixed Load and Displacement Controls

When load and displacement controls are mixed, a more general condition for stable path is required. Consider that displacements q_i $(i = 1,...,m)$ and loads f_j $(j = m+1,...,n)$ are controlled. The force and displacement responses, which are different for various paths α, are $f_i^{(\alpha)}$ and $q_j^{(\alpha)}$. To treat this case, it is necessary to introduce a semi-complementary thermodynamic function which involves the complementary work for $f_i^{(\alpha)}$ but the actual work $q_j^{(\alpha)}$. This function is obtained by Legendre transformation of f_j; $Z = F - \sum_j f_j q_j^{(\alpha)}$. If only the displacements are controlled, $Z = F =$ Helmholtz free energy, and if only the loads are controlled, $Z = G =$ Gibbs free energy. Using the expression $dS = -SdT + \sum_i f_i^{(\alpha)} dq_i + \sum_j f_j dq_j^{(\alpha)}$ which is valid for equilibrium changes, we obtain

$$\Delta Z = -\int S^{(\alpha)} dT + \int \sum_i f_i^{(\alpha)} dq_i - \sum_j q_j^{(\alpha)} df_j \tag{30}$$

where $f_i^{(\alpha)}$ and $q_j^{(\alpha)}$ vary to maintain equilibrium. Expanding $f_i^{(\alpha)}$ and $q_j^{(\alpha)}$ into Taylor series, integrating, and neglecting the terms of higher than second order, we get

$$\Delta Z = -\int S^{(\alpha)} dT + \Delta Z_I + \Delta Z_{II} \tag{31}$$

in which

$$\Delta Z_I = \sum_i \delta f_i^0 \delta q_i - \sum_j \delta q_j^0 \delta f_j \tag{32}$$

$$\Delta Z_{II} = \sum \tfrac{1}{2}\delta f_i^{(\alpha)}\delta q_i - \sum_j \tfrac{1}{2}\delta q_j^{(\alpha)}\delta f_j . \tag{33}$$

On the other hand, if the same change of controlled variables δq_i and δf_j is carried out at constant $f_j = f_j^0$ (dead loads) and constant displacements $q_i = q_i^0$, we have a nonequilibrium change for which ΔS_{in} must be nonzero. Hence,

$$\Delta Z = -\int S^{(\alpha)} dT + \sum_i f_i^0 \delta q_i - \sum_j q_j^0 \delta f_j - T(\Delta S)_{in} . \tag{34}$$

f_i^0 and q_j^0 are the same for all paths α because we consider increments from a common bifurcation point. Subtracting Eq. 34 from Eq. 31 (with 32 and 33), we finally obtain $T(\Delta S)_{in} = -\Delta Z_{II}$.

Note that this result is valid not only for constant temperature (dT=0) but also for variable temperature, provided that its variation is controlled, i.e. ΔT is the same for all the paths. If the temperature is not controlled, then ΔT can be different for various paths, in which case one cannot write $T\Delta S_{in} = -\Delta Z_{II}$.

The linear terms in Eqs. 31-33 and Eq. 34 are identical. Therefore, the internal entropy change during the first substep, in which the controlled variables q_i and f_j are changed while their responses f_i and q_j are kept constant (frozen), is the same for each path α. So the differences in internal entropy among paths $\alpha = 1,2,\ldots$ arise only during the second substep, in which the controlled variables q_i and f_j are kept constant while $f_i^{(\alpha)}$ and $q_j^{(\alpha)}$ change to find an equilibrium state on one of the paths α. Therefore, the equation $T\Delta S_{in} = -\Delta Z_{II}$, which corresponds to the second substep, represents the internal entropy change on approach to an equilibrium state. According to the second law of thermodynamics as stated by Gibbs, this entropy change must be maximized. This leads to the following theorem:

The stable path is that for which

$$T(\Delta S)_{in} = -\Delta Z_{II} = \sum_j \tfrac{1}{2}\delta q_j^{(\alpha)}\delta f_j - \sum_i \tfrac{1}{2}\delta f_i^{(\alpha)}\delta q_i = \underset{(\alpha)}{\text{Max}} \tag{35}$$

i.e. the difference between the second-order complementary work of controlled loads and the second-order actual work on controlled displacements done along the path is maximum.

A similar argument shows that this is also true when the total entropy S is constant (isentropic conditions) or even when it varies in a controlled manner, remaining the same for all paths α. To derive this result one needs to introduce, instead of Z, the semi-complementary thermodynamic function $\bar{Z} = U - \sum_j f_j q_j^{(\alpha)}$ which differs from Z only by the thermal term.

Note that the previously stated path stability conditions when only displacements are controlled or only loads are controlled result as special cases of Eq. 35.

The first bifurcation corresponds to the case when $\Delta Z_{II} = 0$. It is interesting that, in contrast, for cases where only the loads or only the displacements are controlled, it is not necessary that both second-order work terms in Eq. 35 vanish at the first bifurcation, even if the structure properties vary continuously.

Appendix 2. - Second-Order Work of Stresses in Geometrically Nonlinear Problems

In this case, instead of Eqs. 8 and 9, we may use the principle of conservation of energy to conclude that

$$\Delta F = -T\Delta S_{in} = \int_{V_0} \tfrac{1}{2}\,\delta\underset{\sim}{\sigma}_T : \delta\underset{\sim}{\varepsilon}\, dV_0 + \delta^2 W_\sigma = \int_{V_0} \tfrac{1}{2}\,\delta\underset{\sim}{\varepsilon} : \underset{\sim}{C}_T : \delta\underset{\sim}{\varepsilon}\, dV_0 + \delta^2 W_\sigma \qquad (36)$$

$$\Delta U = -T\Delta S_{in} = \int_{V_0} \tfrac{1}{2}\,\delta\underset{\sim}{\sigma}_S : \delta\underset{\sim}{\varepsilon}\, dV_0 + \delta^2 W_\sigma = \int_{V_0} \tfrac{1}{2}\,\delta\underset{\sim}{\varepsilon} : \underset{\sim}{C}_S : \delta\underset{\sim}{\varepsilon}\, dV_0 + \delta^2 W_\sigma . \qquad (37)$$

Here V_0 = initial volume of the structure before incremental displacements δu_i; $\delta\underset{\sim}{\sigma}_T$ or $\delta\underset{\sim}{\sigma}_S$ must be interpreted as the objective stress increment associated by work with $\delta\underset{\sim}{\varepsilon}$ (unless only the rotations are large, the strains remaining small); $\delta^2 W_\sigma = \sum_k \sum_m \tfrac{1}{2}\delta q_k G_{km} \delta q_m$, G_{km} = geometric stiffness matrix, which depends (linearly) on the initial stress tensor $\underset{\sim}{\sigma}^0$ of components σ^0_{ij} (or on P_0) but is independent of $\underset{\sim}{C}_T$. In absence of geometric nonlinearity ($G_{ij} = 0$), Eq. 36 reduces to Eq. 8 and Eq. 37 to Eq. 9. The geometric stiffness matrix can be deduced from the relation[1]

$$\delta^2 W = \int_{V_0} \underset{\sim}{\sigma}^0 : (\delta\underset{\sim}{\varepsilon} - \delta\underset{\sim}{e})\, dV_0 \qquad (38)$$

in which $\delta\underset{\sim}{e}$ = incremental small strain tensor, $\delta e_{ij} = \tfrac{1}{2}(\delta u_{i,j} + \delta u_{j,i})$, $\delta\underset{\sim}{\varepsilon}$ = finite strain tensor corresponding to $\delta\underset{\sim}{u}$, which needs to be accurate only up to second-order terms in $\delta u_{i,j}$. Choosing the Lagrangian strain tensor, we have $\delta\varepsilon_{ij} - \delta e_{ij} = \tfrac{1}{2}\delta u_{k,i}\delta u_{k,j}$ and $\delta\underset{\sim}{\sigma}_T$ or $\delta\underset{\sim}{\sigma}_S$ must be interpreted as the Truesdell's stress rate times the time increment δt. Choosing Biot's strain, we have $\delta\varepsilon_{ij} - \delta e_{ij} = \tfrac{1}{2}(u_{k,i}\, u_{k,j} - e_{ki}\, e_{kj})$ (also denote ω_{ij} = material rotation), and $\delta\underset{\sim}{\sigma}_T$ or $\delta\underset{\sim}{\sigma}_S$ must then be Biot's incremental stress[1]. The values of the associated moduli $\underset{\sim}{C}_T$ or $\underset{\sim}{C}_S$ depend on the choice of $\delta\varepsilon_{ij}$ (unless only $\delta\omega_{ij}$ are large, the strains remaining small, as was the case for our column example).

In our example of elasto-plastic column, Eq. 36 or 37 may be written as $\delta^2 W = \delta^2 W_0 + \delta^2 W_\sigma$, in which $\delta^2 W = \tfrac{1}{2}(\delta\sigma_1\delta\varepsilon_1 + \delta\sigma_2\delta\varepsilon_2)h\,A/2$ = second-order work of stress increments, and $\delta^2 W_\sigma = -P_0\Delta\ell$, with $\Delta\ell = \ell(1 - \cos\theta) \simeq \ell(\delta\theta)^2/2 = 2\ell X^2$ = second-order axial displacement at the load point. Note that $\delta^2 W_\sigma = \tfrac{1}{2}(G_{11}X^2 + 2G_{12}XY + G_{22}Y^2)$ in which $G_{11} = -4P_0\ell$, $G_{12} = G_{21} = G_{22} = 0$. It may be checked that this indeed yields the same expression as Eq. 24.

Appendix 3. - Path Dependence and Symmetry of Tangential Stiffness Matrix

It has often been stated that symmetry of the stiffness matrix $\underset{\sim}{K}$ guarantees path independence, but this is not true because $\underset{\sim}{K}$ depends on the direction $\underset{\sim}{\nu}$ of vector $\delta\underset{\sim}{q}$. It follows from Eq. 2 and 7 that

$$f_{Tk} = \partial F/\partial q_k, \qquad f_{Sk} = \partial U/\partial q_k \qquad (39)$$

$$K_{jk}(\underset{\sim}{\nu})_T = \partial^2 F/\partial q_j \partial q_k, \quad K_{jk}(\underset{\sim}{\nu})_S = \partial^2 U/\partial q_j \partial q_k \qquad (40)$$

when the differentiations are carried out along path $\delta\underset{\sim}{q}$. Then K_{jk} is symmetric. However, if one considers various non-radial paths from point 0 to some adjacent point A such that $\overline{OA} = \|\delta q\|$ is infinitely small, then F or U is path-independent only for those paths for which the path direction of all the points of the path belongs to the same sector of $\underset{\sim}{\nu}$-directions in the $\delta\underset{\sim}{q}$-space, for which matrix $\underset{\sim}{K}$ is the same. For other paths there is path-dependence. For incrementally nonlinear theory, such as the endo-

[1]Z.P. Bažant (1971), "A correlation study of formulations of incremental deformations and stability of continuous bodies," ASME J. of Applied Mechanics 38 (Dec.), 919-928.

chronic theory, these sectors become infinitely narrow and infinitely numerous, and each different path yields in general a different $\delta^2 W$.

Eq. 22 for elasto-plastic column confirms that the stiffness matrix is symmetric. Yet the behavior is not path-independent. It may be verified that a short increment from point 0 ($P = P_t$, $q_1 = 0$) to an adjacent point A ($P = P_A$, $X = 0$) along the bifurcating path ($\alpha = 2$) gives a lower value of F or U (or of F or U) than a path for which P is first raised from P_0 to P_A at $X = 0$ and subsequently X is increased from 0 to X_A while $P = P_A$ = constant. (These two paths also give at point A different values of Y.) It is because of this path dependence that the state $P = P_A$ at $X = 0$ can be stable. The fact that the value of F at this state is in fact higher than the value of F at point A reached along path $\alpha = 2$ is immaterial.

Any point (X,Y) can be reached by moving first in the direction ϕ and then in direction ψ ($\psi \neq \phi$). For each pair (ϕ,ψ), one generally obtains at (X,Y) a different value of $\delta^2 W(X,Y)$. Thus, there exist infinitely many different surfaces of $\delta^2 W(X,Y)$ (or of $(\Delta S)_{in}$ if nonradial paths are considered.

In view of path dependence, the derivation of Eqs. 11 and 13 requires mentioning that $K_{ij}^{(\alpha)}$ must be evaluated in both substeps on the basis of the direction $\underset{\sim}{v}^{(\alpha)}$ of the equilibrium branch, and not on the basis of the directions of the two substeps introduced to derive these equations. This is justified by the fact that $\delta \underset{\sim}{q}$ is the only actual direction of loading, while the directions of the two substeps are fictitious, not actually experienced by the structure.

Appendix 4. - Numerical Aspects and Symmetry-Breaking Imperfections

Aside from providing a thermodynamic foundation to Hill's theory of bifurcation and generalizing the bifurcation theory from elasto-plastic to arbitrary irreversible structures, the usefulness of the present theory is that it makes it possible to decide which post-bifurcation branch will actually be followed. This decision can be made on the basis of $(\Delta S)_{in}$ calculated from the tangential stiffness matrices $\underset{\sim}{K}$ for the initial deformations along the post-bifurcation branches, although if $\underset{\sim}{K}$ is singular at the bifurcation state (first bifurcation), one must use matrix $\underset{\sim}{K}$ for a state shortly after the first bifurcation. This procedure obviously requires that the tangential stiffness matrix be actually computed in the numerical solution algorithm.

In some step-by-step methods for nonlinear finite element analysis, the tangential stiffness matrix is not computed - e.g., when iterations on the basis of the elastic stiffness or secant stiffness are used. For such algorithms, Hill's method of linear comparison solid cannot be implemented; however, the present method can, because ΔS_{in} (or $\delta^2 W$, $\delta^2 \overline{W}$) can be calculated directly from $\delta \underset{\sim}{q}^{(\alpha)}$ and $\delta \underset{\sim}{f}^{(\alpha)}$ without evaluating $\underset{\sim}{K}$. A complicating factor in this approach is that all the branching paths α must first be detected before ΔS_{in} can be calculated from $\delta \underset{\sim}{q}^{(\alpha)}$ and $\delta \underset{\sim}{f}^{(\alpha)}$.

In some finite element studies at Northwestern University, the calculation of the lowest eigenvalue of $\underset{\sim}{K}$ was avoided and the secondary path was obtained by introducing a slight symmetry-breaking imperfection in the nodal coordinate or a small disturbing load (see the paper by P. Droz and Z.P. Bažant on shear bands in this volume). Subsequently, $(\Delta S)_{in}$ could be compared.

Bifurcations generally represent some type of a breakdown of symmetry. Symmetry is eliminated by imperfections, and so an imperfect system does not exhibit bifurcations. Could this approach render the present theory unnecessary.

Not at all. Knowledge of the behavior of the perfect (or symmetric) system gives information on the structure of the solutions for an entire class of imperfect systems in its vicinity (similarly as Koiter's analysis of perfect structure at the limit of stability gives all the important information on imperfection sensitivity). Using the imperfection route, one would have to check, in principle, the behavior for all the possible types of imperfections, which are infinitely numerous. Computations for small imperfections are difficult as they require very small loading steps near the bifurcation state of perfect structure. By considering not only equilibrium states but also all the adjacent nonequilibrium states, thermodynamic analysis is in fact equivalent to considering all the imperfections.

Appendix 5
Thermodynamic Concepts, Equation of State and Tangential Stiffness

Eqs. 7 and 2 tacitly imply the hypothesis that, for the purpose of incremental analysis of a small loading step, the actual inelastic structure can be replaced by a tangentially equivalent elastic (i.e. reversible) structure whose elastic stiffnesses are equal to the tangential stiffnesses $K_{ij}(\underset{\sim}{\nu})_T$ or $K_{ij}(\underset{\sim}{\nu})$. We assume this elastic structure to behave in a small loading step the same way as the actual structure, and so the thermodynamic analysis need deal only with this elastic structure. It is important to realize this, since thermodynamic analysis must be based on an equation of state of the material or the structure, which must be reversible. Inelastic constitutive laws and equations describing inelastic structural properties are not equations of state (as pointed out to the writer by Professor Joseph Kestin of Brown University).

However, an elastic stress-strain relation or an elastic relation between structural displacements and forces is an equation of state. Therefore, the relations $df_{T_i} = K_{ij_T}(\underset{\sim}{\nu})dq_j$ or $df_{S_i} = K_{ij_S}(\underset{\sim}{\nu})dq_j$ do represent an equation of state for the tangentially equivalent elastic structure.

The tangential stiffness matrix K_{ij} has different values depending on the direction $\underset{\sim}{\nu}$. Adoption of one of them as the stiffness matrix of a tangentially equivalent elastic structure is of course valid as long as the solution dq_k of the loading step agrees with the assumed direction $\underset{\sim}{\nu}$ on which the tangential stiffness was based. If dq_k is found to have a different direction, another tangentially equivalent elastic structure, characterized by other K_{ij} values, must be tried in an iterative search, until agreement is reached.

The reason that $(dS)_{in}$ must not appear in Eq. 2 (a crucial fact) is that the response of the tangentially equivalent elastic structure that remains in mechanical equilibrium is an equilibrium thermodynamic process, i.e. reversible. This approach makes it possible to avoid dealing with internal variables (such as damage variables, plastic strains, etc.) which describe the dissipative (irreversible) process determining the tangential stiffness. Introduction of internal variables would greatly complicate analysis, and is not necessary for stability analysis. Moreover, the problem would be shifted to the realm of irreversible thermodynamics. The present problem, by contrast, belongs to classical thermodynamics because only states that are infinitely close to the equilibrium states of the tangentially equivalent elastic structure are considered. In this regard note that the present use of $(dS)_{in}$ does not represent an application of the principle of maximum entropy production, which is often applied in irreversible thermodynamics but is known not to be valid in general.

The term "equilibrium" in this paper means "mechanical equilibrium" (i.e. equilibrium of forces) and does not imply thermal equilibrium except where the term "thermodynamic equilibrium" is explicitly stated. The term "internal entropy" increment, $(dS)_{in}$, is a short form of saying the "internally produced entropy increment" (see Guggenheim, 1959). There is, of course, only one entropy S, and it is just for convenience that we decompose its increment as $dS = (dS)_{in} + (dS)_{ex}$ where $(dS)_{ex} = dQ/T$ = externally produced entropy = entropy increment due to heat influx dQ. It is of course impossible to integrate $(dS)_{in}$ to obtain S_{in}.