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STRAIN-SOFTENING CONTINUUM DAMAGE: LOCALIZATION AND SIZE EFFECT*

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ABSTRACT

The paper reviews recent work by the authors in the modeling of strain-softening arising from damage, such as cracking, in heterogeneous brittle materials. Attention is focused on the concept of localization limiters - mathematical methods which ensure that strain-softening zones cannot localize into a region of zero volume. Localization limiters make it possible to achieve proper convergence with mesh refinement, with a finite energy dissipation in the limit of vanishing element size. Certain recently obtained exact solutions for strain-softening wave propagation problems are reviewed and the convergence of a finite element model to these solutions is discussed. As an essential consequence of the localization limiting properties of heterogeneous materials, the size effect observed in the failure of geometrically similar specimens or structures of different sizes is analyzed, and certain new extensions of the previously identified approximate size effect law are presented. Consequences of the size effect for the validity of finite element programs for strain-softening or continuum damage mechanics are reviewed and the application of the size effect for identifying the localization limiting properties of the material, and especially its fracture energy, are analyzed. The conclusions are supported by test results. Finally, recent work on simulation of heterogeneous strain-softening materials by interface elements with a random arrangement is discussed, and its localization limiting properties are emphasized.

INTRODUCTION

The topic of strain-softening has recently attracted considerable attention because efforts to model the damage of materials and analyze the failure of structures are inevitably linked with stress-strain curves which have negative slope. Some mechanicians, such as Hegemier and Read [1984], have attributed negative slopes entirely to decreases in the area over which the stress acts, and have consequently argued that strain-softening does not exist. While a decrease in the effective area which results from damage may be a mechanism which is involved in strain-softening in brittle materials, this is not the only class of materials in which strain-softening is observed.

For example, in certain types of projectile penetration, the generation of heat associated with adiabatic plasticity results in a decrease in the yield stress and hence in a stress-strain curve which exhibits strain-softening. Similarly, in two-phase materials and van der Waals fluids, the pressure-volume curves exhibits a local strain softening domain. Furthermore, the argument that microstructural considerations may allow one to bypass the difficulties associated with strain-softening do not appear to be sound since regardless of whether the stress-strain behavior is based on microstructural considerations or on phenomenological grounds, it leads to the same type of mathematical behavior in the governing equations of the continua.

*This contribution represents a combination of two invited papers.

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1. The strain at the point in which strain-softening first is initiated becomes infinite, and the displacement at that point becomes discontinuous.

2. *Strain-softening in this type of material* is limited to a single point so that the dissipation associated with this damage mechanism vanishes.

3. The fact that strain-softening is limited to a point overcomes the objection of Hadamard about the ellipticity of the equations in the strain-softening domains inasmuch as this domain is always a set of measure zero.

In view of the character of the closed form solution, the mesh dependence which was reported by Sandler and Wright is not surprising. As the size of the element in which strain-softening is initiated is reduced, the strain in that element becomes larger and larger in its attempt to match the closed form solution, where the strain is infinite. The major drawback of the local strain-softening model is not in fact this mesh dependence, but the fact that the energy dissipation associated with the damage processes in the strain-softening model tends to zero as the mesh is refined. This characteristic was identified by Bazant, Belytschko and Chang [1985] who also proposed a nonlocal type of formulation to overcome this pathology; this will be discussed subsequently.

Further studies of the properties of finite element solutions with strain-softening by Belytschko et al. [1984, 1986] showed that the remedy of damping as proposed by Sandler and Wright is not sufficient to eliminate the pathological behavior in finite element solutions. This was exhibited in a spherically converging wave in a sphere. The character of a strain-softening solution in this geometry is especially complex when the load is a ramp function in time. In this case, a portion of the stress wave always passes a surface before strain-softening is initiated. This portion of the wave then grows because of the spherically convergent geometry and again triggers strain-softening. Thus, it is surmised that strain-softening in this case actually occurs on an infinite number of surfaces, although closed form solution could not be constructed for this case. Finite element solutions for these problems are shown in Fig. 4. As the meshes are refined, very large strains are generated at more and more points inside the domain; the location of these high strains in the finite element solution seems to be completely chaotic and there is no trend with mesh refinement. Furthermore, this pathological behavior is not eliminated by adding damping to the material.

Recently Belytschko et al. [1986] have presented closed form solutions for materials of the stress-strain laws shown in Fig. 5. The first is similar to the stress-strain law considered by Bazant and Belytschko [1985] except that the stress takes on a nonzero value σ_b as the strain becomes infinite. In the second material law, the negative portion of the stress-strain law is limited to the point between the strains ϵ_a and ϵ_b, and for the remainder of the stress-strain law the modulus is positive. A solution for one of these two materials and for the problem in Fig. 1 is sketched in Fig. 6. The following are the important characteristics of these solutions:

1. For the first stress-strain law, termed strain-softening-perfectly-plastic, a singularity in the strain and a discontinuity in the displacement is generated even if the stress asymptotically approaches a nonzero value. Hence the appearance of any strain-softening without a subsequent positive modulus will lead to a discontinuity in the displacements. This type of stress-strain law would approximately describe

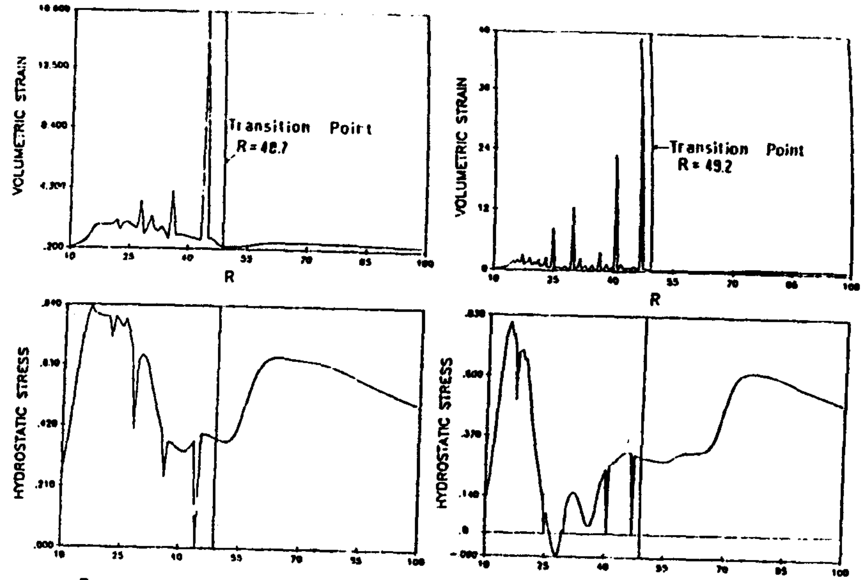

FIG. 4. Solutions for spherical converging wave with different finite element meshes.

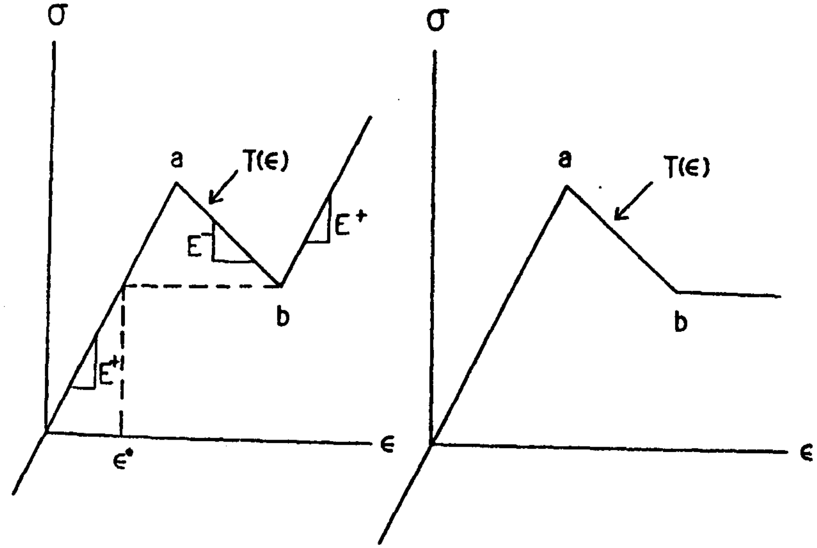

FIG. 5. Stress-strain laws 1 and 2 (strain-softening perfectly-plastic and strain-softening-rehardening).

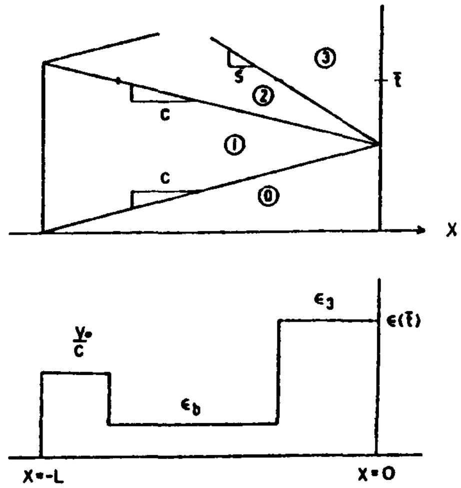

FIG. 6. Solution to problem in Fig. 2 with stress-strain law 2.

TABLE 1. Comparison of size effect law (with r=0.44, B=306, λ_0=0.608) and Hillerborg's finite element results with fictitious crack model.

$df_t'^2/EG_f$	Hillerborg	σ_N/f_t'
0.02	2.43	2.44
0.05	2.22	2.21
0.1	2.01	2.00
0.2	1.77	1.78
0.5	1.46	1.46
1	1.222	1.222
2	0.992	0.995
5	0.725	0.739

the behavior of a composite with an elastic-brittle matrix or concrete with a perfectly plastic reinforcement that begins to yield when the matrix begins to soften.

2. In the second stress-strain law, termed strain-softening with rehardening, the strain-softening point moves from the center in a manner similar to a shock except that the velocity of this discontinuity is slower than the elastic wave speed.

3. In the second stress-strain law, the solution is not unique; instead, a single parameter family of solutions which satisfy the equations of motion, the stress-strain law, and the strain-displacement equations conditions can be constructed.

A finite element solution for strain-softening with rehardening is given in Belytschko, et al. (1986) and it shows that the presence of strain-softening induces pathological behavior into the numerical solution. The oscillations are far greater than those associated with Gibb's phenomenon and would generally make an interpretation of the results extremely difficult.

NONLOCAL FORMULATIONS AND LOCALIZATION LIMITERS

It became quite clear early in the research in strain-softening models that it is necessary to limit the localization in order to reduce mesh sensitivity and to develop models which are more representative of reality. The term *localization limiter* was first coined by Belytschko [1986], but localization limiters based on the crack band model were reported much earlier [Bazant, Cedolin and Oh, 1976, 1979, 1980, 1983]. In these models, the damage was assumed to occur in a band of finite size; the size of this band was based on experimental evidence and it was assumed to be a material parameter, depending on the aggregate size. (Bazant's work was extended in the composite damage model of Pietruszczak and Mroz [1981] and Willam et al. [1984], and in the variable width crack-layer model of Chudnovsky et al. [1983]).

An alternative approach to limiting localization was taken by Bazant, Belytschko and Chang [1985] who essentially adopted a non-local formulation in which the key variables in the constitutive equations are the non-local strain

$$\bar{\epsilon}(x) = \int_{-\ell/2}^{\ell/2} w(\xi)\, \epsilon(x + \xi)\, d\xi \qquad (1)$$

and the non-local stress

$$S(x) = \int_{-\ell/2}^{\ell/2} w(\xi)\, \sigma(x + \xi)\, d\xi \qquad (2)$$

where $\sigma = \tau(\bar{\epsilon})$ is the constitutive law, $w(\xi)$ is the given weight function, x = coordinate, and S appears in the differential equation of motion, where $S_{,X} - \rho \ddot{u} = 0$ where u = displacement. One material parameter in this type of a formulation is the physical domain over which the local strain is averaged. Bazant [1984] also showed that this type of non-local continuum can be represented numerically by an overlapping (imbrication) of finite elements, which he termed an imbricate continuum. It was shown by Bazant et al. [1984], and Belytschko et al. [1986a] that this type of formulation spreads the strain-softening behavior over a finite domain and results in a finite dissipation of energy within that domain which is relatively independent of element size.

An alternative procedure for limiting localization is to introduce higher-order spatial derivatives into the governing equations. One approach, which was illustrated by Bazant [1984], is to use a strain displacement expression which in one dimension becomes as follows

$$\varepsilon = u_{,x} + \alpha u_{,xxx} \tag{3}$$

As shown by Bazant, this corresponds to expanding the strain in Eq. (1) in a Taylor series and truncating it after the second term. Alternatively, as shown by Belytschko and Hyun [1986], the localization limiting properties can be implemented by inserting additional gradient terms into the equilibrium equation. Thus, the one-dimensional equation of motion can be replaced by the following equation:

$$\sigma_{,x} + \sigma_{,xxx} = \rho \ddot{u} \tag{4}$$

An alternative to this approach has been taken by Schreyer and Chen [1984] who introduced the gradient of the yield function into the constitutive equation, and by Mang [1985] who considered the yield limit or strength to depend on the stress gradient. This again has the effect of limiting the localization so that the damage process takes place over a finite domain. Although closed-form solutions are not available for this case, it could be surmised that the dissipation is non-zero. Triantafyllidis and Aifantis [1984] have shown that introduction of the gradient of the deformation into a Blatz-Ko material results in finite localized deformation zones.

As we have seen, there exist numerous possibilities for the formulation of localization limiters, which apparently can all ensure that energy dissipation remains finite at arbitrary mesh refinement and that damage localization to a region of zero volume does not take place. However, our understanding of these procedures is only embryonic, both from the viewpoint of the computational implementation and the constitutive models. The computational models are quite complex. Many introduce difficulties such as extra boundary conditions; this is noted in Bazant [1984], Belytschko and Hyun [1986], and these extra boundary conditions, though not noted, also probably are necessary in the models of Schreyer and Chen [1984] and Triantafyllidis and Aifantis [1984]. The additional complexity of these models is also a cause for concern.

From the constitutive viewpoint, the localization limiting properties are undoubtedly related to size effects. Measurements of the effect of size on the failure loads of geometrically similar specimens offer a serious test for the correctness of failure theories, and at the same time the size effect represents practically the most important manifestation of the nonlocal character of strain-softening damage. Analysis of size effects through micromechanics provides another avenue for understanding and perhaps quantitative characterization.

CONTINUUM DAMAGE FORMULATIONS

A concept which offers a more general, and no doubt more fundamental, description of strain-softening damage is continuum damage mechanics. This field has flourished during the last decade (Janson and Hult, [1977]; Lemaitre and Chaboche, [1978]; Loland, [1980]; Mazars, [1981]; Krajcinovic and Fonseka [1981]; Krajcinovic, [1983]; Mazars, [1984]; Resende and Martin, [1984]; and others). These researches have led to various sophisticated formulations cast in a thermodynamic framework, which have been shown capable of describing test data from specimens assumed to be in homogeneous stress and strain states.

The continuum damage mechanics has been applied to two types of problems: (1) those for which the incremental stiffness matrix does not lose its positive definiteness, and (2) those where it does. The latter formulations, which represent in fact a form of strain-softening and inevitably occur when complete failure is to be analyzed, inevitably suffer from the strain- localization difficulties that we discuss here. These problems have so far been ignored in the published works on continuum damage mechanics, but they have to be addressed if these sophisticated theories are to be applied in the solution of continuum problems, whether it be by finite element codes or any other methods.

To illustrate the existence of strain localization and the associated spurious mesh sensitivity, continuum damage mechanics has been applied first in a layered finite element analysis of beams, unreinforced or reinforced, typical of concrete structures; Bazant, Pijaudier-Cabot, and Pan, [1986]. In particular, the following continuum mechanics formulation has been used as the starting point of the localization studies:

$$\underset{\sim}{\sigma} = (1 - D)\, \underset{\sim}{C} : \underset{\sim}{\varepsilon} \tag{5}$$

$$\text{For } F(y) = 0 \text{ and } \dot{F}(y) = 0: \quad \dot{D} = g(y,D)\ (>0)$$
$$\text{For } F(y) < 0, \text{ or for } F(y) = 0 \text{ and } \dot{F}(y) < 0: \quad \dot{D} = 0 \tag{6}$$

$$y = \frac{1}{2}\, \underset{\sim}{\varepsilon} : \underset{\sim}{C} : \underset{\sim}{\varepsilon} \tag{7}$$

in which $\underset{\sim}{\sigma}$ and $\underset{\sim}{\varepsilon}$ are the local stress and strain tensors, $\underset{\sim}{C}$ is the tensor of elastic constants (positive-definite), D = damage (assumed here as a scalar, for the sake of simplicity,$F(y)$ (< 0) is the loading function, g and F are empirical functions characterizing the material properties, $y = -\partial\rho\psi/\partial D$ in which ρ = mass density and ψ = elastic potential, $y\dot{D}$ = energy dissipation rate (> 0), $F(y) = y - \kappa(y)$ with $\kappa(y)$ = Max y up to the current time. As a special case, calculations were based on an integrable form of function g describing damage evolution: $D = 1 - a\exp(-by)^n$ in which n, a and b are positive material constants.

Some results obtained with a layered finite element code for the failure load of a beam whose material follows the foregoing continuum damage mechanics formulation are exemplified in Fig. 7. It is seen that an increase in the number of finite elements, N, has an enormous effect on the response, and this continuum damage mechanics solution converges for

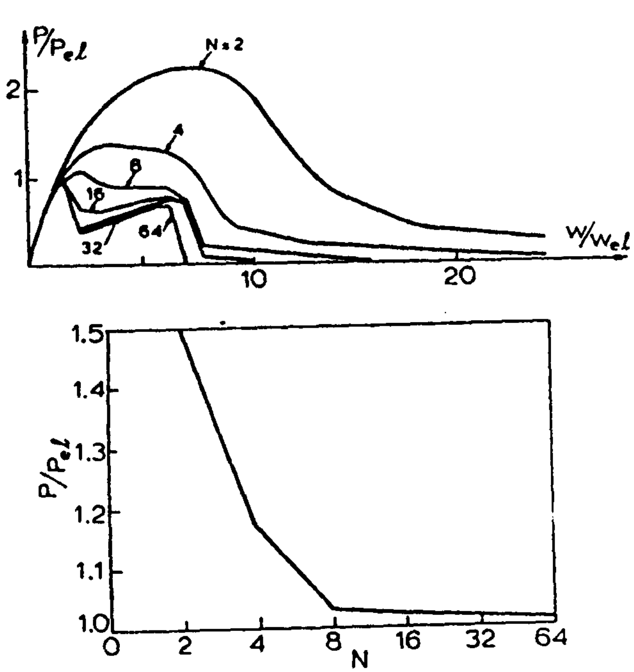

FIG. 7. Load-deflection diagrams of a beam in a material described by a continuous damage mechanics model, for various numbers N of finite elements over the beam (normalized with respect to the elastic state at which the tensile strength is first reached) - top, and dependence of maximum load on N - bottom (after Bažant and Pijaudier-Cabot and Pan [1986].

vanishing element size to a solution characterized by zero energy dissipation due to failure. This is the same as previously demonstrated for finite element solutions based on strain-softening formulations for cracking. Obviously, suitable localization limiters would have to be introduced into the existing continuum damage mechanics theories if these theories should be applied in practice. To this end, various types of nonlocal concepts have been considered by Pijaudier-Cabot and Bazant [1986]. Due to the greater sophistication of the continuum damage mechanics models, there are many possible ways in which the nonlocal concept can be introduced, and their comparisons are presently under investigation.

The continuum damage mechanics models are also being studied in dynamic wave propagation problems of the type discussed before; Pijaudier-Cabot and Bazant [1986]. It is already clear that the same problems as discussed before arise for these formulations. In particular, the damage zone localizes to a zero volume and the energy dissipation tends to zero at vanishing element size. On the other hand, for a nonlocal formulation, a finite damage zone, with a finite energy dissipation, is obtained regardless of mesh refinement.

The results in Fig. 7 are plotted in a nondimensional form which is also applicable when the element size is kept constant and the beam length is increased. In that viewpoint, Fig. 7 represents the size effect corresponding to a fixed size of the strain-localization region.

An interesting question is whether a finite damage zone can be replaced by a surface or a line, or a point on the beam, characterized by a stress-displacement relation, or in a beam by a moment-rotation relation considered as a material property. This type of approach, motivated by ductile fracture models of Dugdale-Barrenblat type, such as Hillerborg's fictitious crack model of concrete, have been widely used to obtain mesh insensitive solutions. However, it has not been recognized that this approach is objective with respect to the mesh choice only for single fracture (single localized damage zone). In certain problems, many interacting fractures (interacting damage zones) can form, and in those problems Hillerborg's type approach is not objective with regard to the mesh choice. Essentially, what is lacking from these formulations is some characteristic measure of the spacing of the major fractures (or of the localized damage zones), which indicates that some form of nonlocal concept must be introduced into these theories if they should yield results which are properly convergent with mesh refinement. Otherwise these models cannot describe cracking or damage zones which are not fully localized and may be much larger than the minimum possible size of the damage zone. A numerical demonstration of the nonobjectivity of the stress-displacement fracture models can be found in Crisfield [1984] and Bažant [1985a,b].

A counterpart to the replacement of damage zone with a line described by a stress-displacement relation is for a beam a replacement of a moment-curvature relation with strain-softening by a moment-rotation relation. They can be calibrated so that for uniform curvature they give the same results. However, such equivalence can hold only within a limited range. This is illustrated by the diagram in Fig. 8 from Bazant, Pijaudier-Cabot and Pan [1986], which shows the ductility of the beam (relative deformation at failure) as a function of the beam length (slenderness) at the same cross-section, with the constant C of the elastic rotation restraint at the beam ends as a parameter. It is seen from this figure that while the predictions of the moment-rotation formulation for a softening hinge and a moment-curvature formulation for a strain-softening segment coincide for very slender beams, they differ substantially for short beams. This illustrates that even for a single isolated damage zone, the use of stress displacement characteristics for cracking or damage is only a partial answer to the problems that exist.

SIZE EFFECT

The problem of strain localization, and in particular the strain-localization limiters, is closely intertwined with the problem of the size effect. The size effect may be isolated from other infuences by considering geometrically similar structures of different sizes. For failures due to distributed damage (cracking), the simplest description of the size effect is provided by the size effect law (Bazant, [1984]):

$$\sigma_N = Bf_t'\left(1 + \frac{d}{\lambda_0 d_a}\right)^{-1/2} \qquad (8)$$

in which σ_N is the nominal stress at failure (the failure load divided by the characteristic dimension and structure thickness), d is the characteristic dimension of the structure, d_a is the maximum size of material inhomogeneities, (e.g., the aggregate size in concrete); f_t' = tensile strength (from direct tensile tests), and B, λ_0 = empirical constants. The size effect law has been shown to follow by dimensional analysis and similitude arguments from the following two simplifying hypotheses:

1. The energy release W due to failure is a function of the length a of the localized band of strain-softening damage.

2. At the same time, W is a function of the volume of the zone of strain-softening damage.

The second hypothesis alone leads to plastic limit analysis, which exhibits no size effect, and the first hypothesis alone leads to classical linear elastic fracture mechanics. Eq. (7) represents the simplest formula for the transition between failures dominated by the strength limit, and failures dominated by strain localization and characterized by fracture energy. It has been shown that Eq. (7) correctly describes the results of fracture tests of concrete within the scatter range of measurements, Bazant and Pfeiffer, [1986], and that it also can be applied to the failure of various concrete structures which fail in a brittle manner, such as the diagonal shear failure of prestressed and unprestressed reinforced concrete beams without stirrups, the torsional failures of concrete beams, the beam and ring failures of unreinforced pipes, the punching shear failure of reinforced concrete slabs, etc. In making these applications, two further hypotheses are implied in the use of the size effect law in Eq. (7):

1. The shapes of the final fractures at failure in specimens of different sizes are geometrically similar; and

2. The failure does not occur at crack initiation.

The latter hypothesis is practically always satisfied since it is prohibited by codes to design concrete structures which fail at the first crack initiation. The approximate applicability of the first hypothesis appears to be verified by the existing test data.

The scatter of existing measurements is not sufficiently small to make it possible to detect significant deviations from the size effect law in Eq. (1). However, finite element computations can be carried out to make

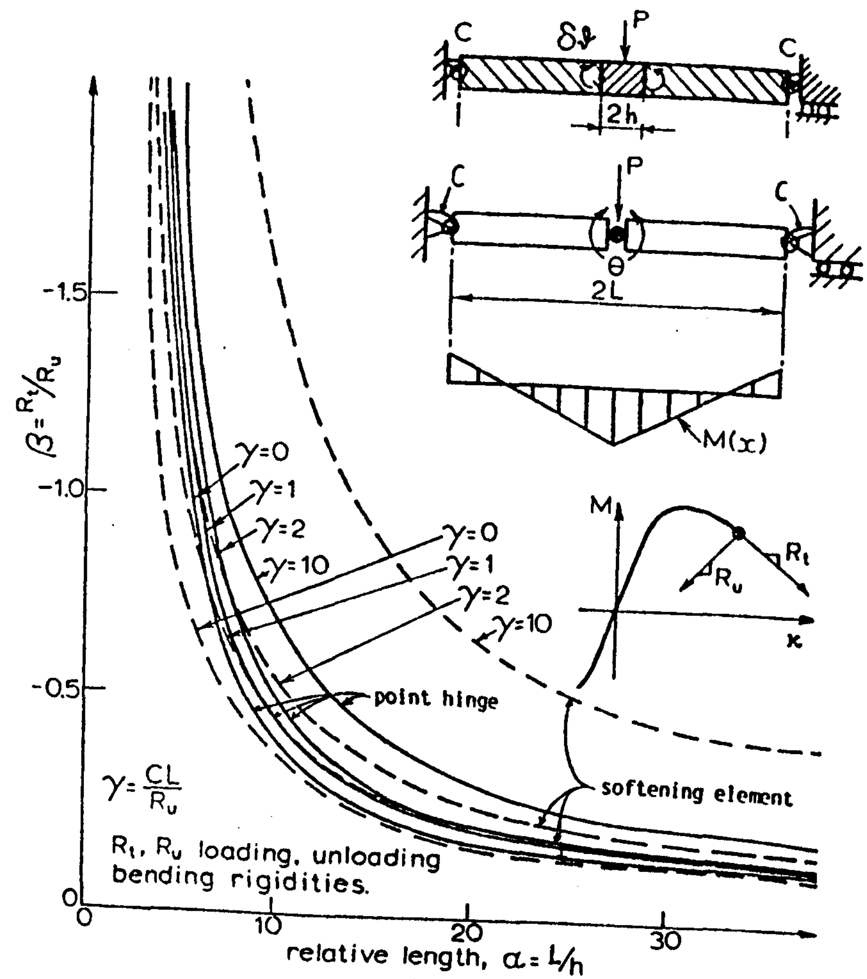

FIG. 8. Softening slope R_t at which the beam becomes unstable, for different beam lengths and different element stiffnesses.

comparisons with the size effect law. One of such calculations has been recently carried out by Hillerborg using his fictitious crack model for a three-point bent fracture test. Considering specimen size range of the ratio 1:250 (which is much larger than it can ever be carried out in practice, due to cost limitations), he detected appreciable deviations from Eq. (1); see the results in Table 1. It has been shown, however, (Bazant, [1985]) that a refinement of the size effect law is possible such that it closely agrees with Hillerborg's finite element results; see the last column in Table 1 which is so close to Hillerborg's results that a graphical distinction is hardly possible. These results are based on the following generalized size effect law [Bazant, 1985]:

$$\sigma_N = Bf_t^*\left[1 + \left(\frac{d}{\lambda_0 d_a}\right)^r\right]^{-1/2r} \qquad (9)$$

which itself is a special case of the general asymptotic series expansion

$$\sigma_N = Bf_t^*(c_0\xi^{-1} + 1 + c_1\xi + c_2\xi^2 + c_3\xi^3 + \ldots)^{-1/2r}, \quad \xi = \left(\frac{d_a}{d}\right)^r \qquad (10)$$

in which $f_t^* = f_t'$ if the aggregate size d_a is the same for all specimens, and B, λ_0, r are empirical parameters, and so are the coefficients c_0, c_1, c_2,...

It has now been established be Bazant, Kim and Pfeifer [1986], that there is a one-two-one relationship between the size effect law and the shape of the softening portion of the stress displacement diagram used in Hillerborg's type models. When one of these relations is known, the other one can be determined. The same is true for the crack band model, in which the front of strain-softening damage is assumed to have a certain constant width which is a material property; to each shape of the strain-softening stress-strain diagram there corresponds a certain size effect law and vice versa. No doubt this is also true of the damage laws. Furthermore, if the front of the band of strain-softening damage is variable, this has a direct effect on the size effect law, and from size effect observation it is possible to make inferences on the size of the strain-softening zone.

In the light of these results, the size effect law emerges as the key indicator of the nonlocal properties of damage, both from the viewpoint of practical application to structures as well as from the viewpoint of identifying material properties from test data. (A similar one-to-one relationship was previously established between the size effect law and the R-curves from blunt fracture tests.)

In view of the foregoing observations, the size effect observed on geometrically similar specimens appears as the best means for identifying the material properties that restrict the localization of strain-softening damage. The most important among these properties is the fracture energy. The fracture energy of materials such as concrete has proven difficult to determine as well as define. Various testing methods currently in use yield results which may differ by several hundred percent, and aside from that, none of the existing definitions of the fracture energy appears to yield unique results.

Based on the size effect, it now appears that a unique definition of

fracture energy can be provided as follows:

The fracture energy G_f of a heterogeneous brittle material is the specific energy required for fracture propagation in a geometrically similar specimen of infinite size.

It has been shown by Bazant, Kim and Pfeiffer [1986], that this definition leads to the formula

$$G_f = \frac{g_F}{B^2\lambda_0 d_a E} \qquad (11)$$

in which B, λ_0, d_a are the parameters of the size effect law, E is the elastic modulus of the material, and g_F is the nondimensional energy release rate for a sharp fracture, calculated according to linear elastic fracture mechanics. It can be theoretically shown that the fracture energy for specimen size extrapolated to infinity must be the same for all specimen shapes. Indeed, for an infinitely large specimen, the relative size of the strain-softening damage zone is infinitely small, and the zone is surrounded by the asymptotic elastic field known from linear elastic fracture mechanics, which is the same regardless of the structure geometry. Therefore, at extrapolation to infinity, the detailed picture of the fracture process zone (zone of strain-softening damage) must be the same for all structure geometries.

This theoretical conclusion has been verified experimentally by Bazant and Pfeiffer [1986]. Specimens made of the same concrete were cast in different sizes, and different types of notches with different loading were used. The test series included triple point bent specimens, centrically tensioned edge-notched specimens, and eccentrically compressed specimens. These shapes include just about the extreme of the range of conditions to which the ligament cross section may be exposed; bending moment over the ligament, tensile force over the ligament, and a combination of bending moment and force.

The results of these tests conducted at Northwestern University are exhibited in Fig. 9 in terms of the plots of $f_t'^{-2}$ versus d/d_a. According to the size effect law, these plots should ideally be straight lines, which makes it possible to use linear regression for the determination of the parameters of the size effect law. The slope of the regression line is proportional to the inverse of the fracture energy, in view of Eq. (11). Based on the slopes of the regression lines of the test results in Fig. 9, it is found that the fracture energy for the three types of specimens are about the same, and do not deviate from each other more than is inevitable for a heterogeneous material such as concrete (the deviations from the mean are within ± 3%). Noting that at the same time, by definition, this method of determining fracture energy is independent of specimen size, it appears that the determination of fracture energy on the basis of the size effect indeed yields unique results, which cannot be said of other existing methods.

It remains to be seen whether refinements of the size effect law can yield unambiguous information on the further parameters which govern strain localization. Important work in this direction has been recently carried out by Planas and Elices at Technical University in Madrid (private communication, 1986). They considered various strain-softening formulations and calculated the corresponding size effect curves for certain types of specimens. By matching such curves to test results it should be possible, in principle, to gain further information on the material parameters that

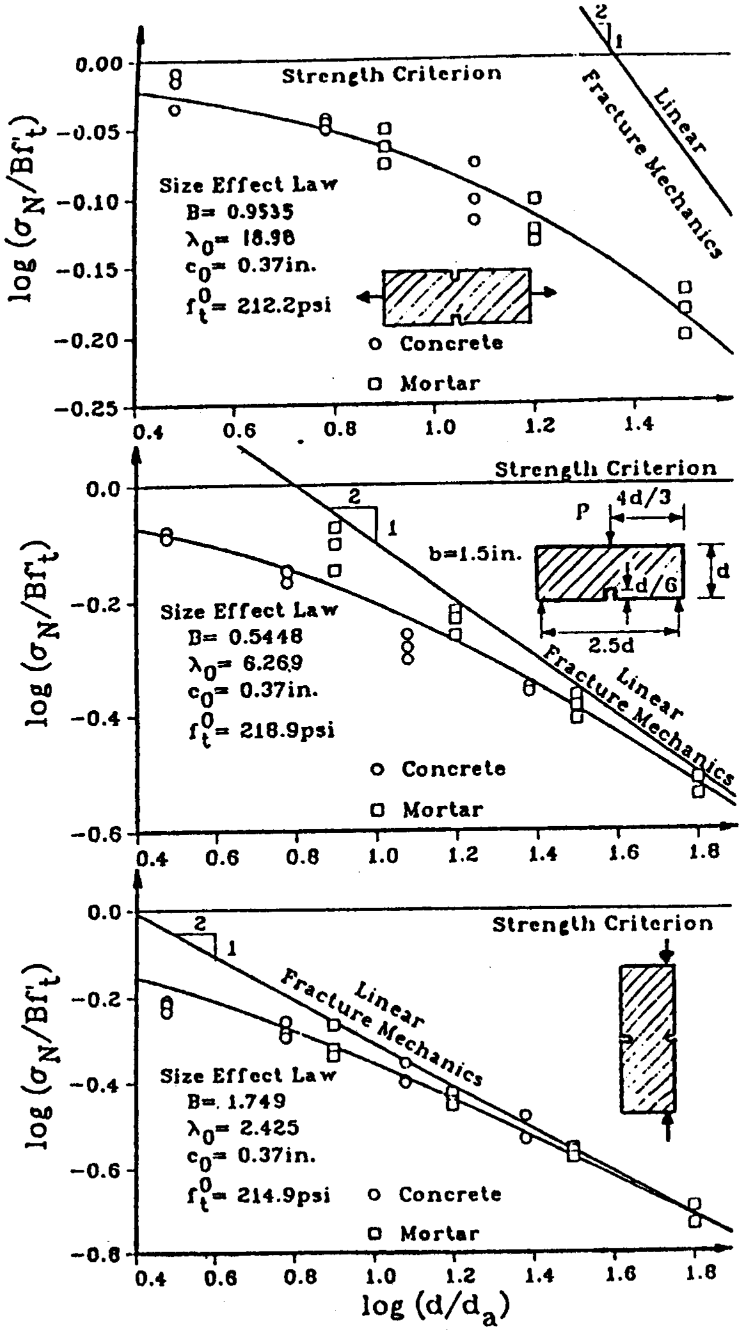

FIG. 9. Nominal stresses at failure vs. specimen size as measured by Bazant and Pfeiffer (1986), and fits by size effect law (Eq. (8) (3-point bent specimen - top, edge-notched tension specimen - middle, and notched accentric compression specimen - bottom).

govern strain-softening.

The availability of an unambiguous definition of the fracture energy of concrete makes it easier to study the effect on various influencing factors. An important factor is temperature. Various types of geometrically similar specimens of different sizes have been tested at Northwesten University at various temperatures, in a controlled temperature chamber Bazant and Prat [1986]. The specimens have been predried in order to avoid complicating the results with the effects of variable moisture content. As is seen from Fig. 9, the results were quite systematic and yielded for the dependence of fracture energy G_f on temperature T a consistent trend. It appears that this trend follows a rule which can be derived on the basis of the activation energy theory (rate process theory of Eyring, Glasstone and Leidler). This result is also quite interesting with regard to the effect of deformation rate. Since it is known that both the deformation rate (or crack growth rate) and the temperature effect are governed by the same physical mechanism, namely the breakages of activation energy barriers, one can make inferences from the temperature effect on the effect of the deformation rate. This connection deserves further careful investigation.

The size effect is of interest not only with regard to fracture testing and design of structures. The size effect is equally valuable for checking the soundness of finite element models. At present, models of cracking which are formulated strictly on the basis of stress-strain relations and pay no attention to strain-localization instabilities and energy aspects of failure are still in prevalent use. All these finite element codes predict the nominal stress at failure for structures of different sizes which are geometrically similar and are calculated with geometrically similar meshes to be the same. Experimental evidence clearly indicates that for brittle failures such predictions are incorrect. This may be one reason that despite two decades of effort, the existing finite element codes still cannot reliably predict brittle failures of concrete structures, except perhaps when the parameters of the model are calibrated for one structure size and predictions are made for roughly the same size. A check for the size effect, and its comparisons with the theoretically derived size effect law or experimental evidence, if available, should be an integral part of evaluation of the applicability of every finte element code to failures due to strain-softening damage.

INTERFACE ELEMENT SIMULATION WITH RANDOM MICROSTRUCTURE

The nonlocal properties of strain-softening materials have their origin in the heterogeneity of the microstructure, for which the maximum size of the inhomogeneities, such as the aggregate size, is the principal determining characteristic. Conversely, a nonlocal strain-softening continuum may be numerically modeled by an element system which simulates in some suitably simplified manner the microstructure. This approach has recently been explored by Zubelewicz and Bazant [1986] in their interface element model.

In this model, which is patterned after Cundall's distinct element method for frictional interaction of the grains of sand or gravel, one assumes that all inelastic deformation is concentrated into thin interface layers between rigid particles representing the major aggregate pieces, which are normally much stiffer than the mortar filling the space in-between. It has been found that a behavior which closely resembles that of concrete can be obtained even with rather simplified assumptions governing the inelastic deformation in the interface layers. The deformation has been assumed to be concentrated into isolated points, the centers of the interface layers, and has been described in terms of a stress-displacement diagram which exhibits a sudden stress drop after reaching the strength limit. A computer program has been written to generate a random system of rigid particles with their interface layers, and then solve the field problem in an incremental manner considering in each loading step the force and moment equilibrium conditions of each rigid particle loaded by the interparticle forces. Fig. 10 shows an example of this type of simulation for a rectangular specimen with a notch. An important aspect of this approach is that the microstructure is generated randomly.

The load-displacement diagram obtained under the assumption of a uniform displacement on the top and bottom sides of the specimen is shown in Fig. 11 (after Zubelewicz and Bazant, [1986), and Fig. 12 shows the development of cracking as well as the directions and magnitudes of the interparticle forces. The interparticle forces exceeding 80% of the strength limit are shown as the solid lines, those exceeding 60% by dashed lines, and those exceeding 40% by dotted lines. Despite the considerable simplifications in this model, the calculated tensile load-displacement diagram (Fig. 10), with a rapid initial drop after the peak load followed by a long tail of strain-softening response, is quite realistic. Noteworthy is the discontinuous propagation of cracks, as well as the fact that the cracking is neither localized in a single line nor distributed over the entire element, but propagates as a band of a width of approximately three particle sizes. The localization limiting properties of this model are obviously inherent to the random dicrete microstructure.

The interface element approach appears to have considerable potential for the simulation of cracking of smaller structures but would probably be too involved, with too many unknowns, for complete analysis of large structures.

CONCLUSION

To sum up, the modeling of failure which involves strain-softening damage presents a number of new challenges. Contrary to previously accepted opinions, strain-softening solutions which are in some cases unique can be found for various wave propagation problems as well as static problems, and numerical finite element solutions converge to these solutions on mesh refinement. The problem with these solutions is that they indicate a complete localization of strain-softening damage into a zone of zero volume, and consequently predict the failure to occur with a zero energy dissipation, which is physically unrealistic.

To obtain a realistic model, one must implement in some form certain new mathematical devices which can be broadly termed the localization limiters. A nonlocal continuum concept as well as some associated higher-order gradient formulations, can be used for this purpose.

The most important practical manifestation of the localization limiting properties is the size effect in failure, observed on geometrically similar specimens of different sizes. Measurement of the size effect provides an effective means for determining the localization limiting properties of the material. At the same time the size effect is an important check on the develoment of finite element programs for strain-softening damage.

The formulations based on continuum damage mechanics, in their present form, lack the localization limiters and exhibit physically incorrect convergence on mesh refinement, with a zero energy dissipation for a vanishing

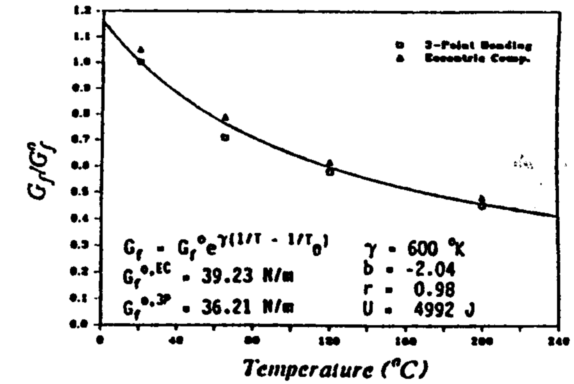

FIG. 10. Dependence of fracture energy on absolute temperature T, determined on the basis of the size effect (after Bažant and Prat [1986].

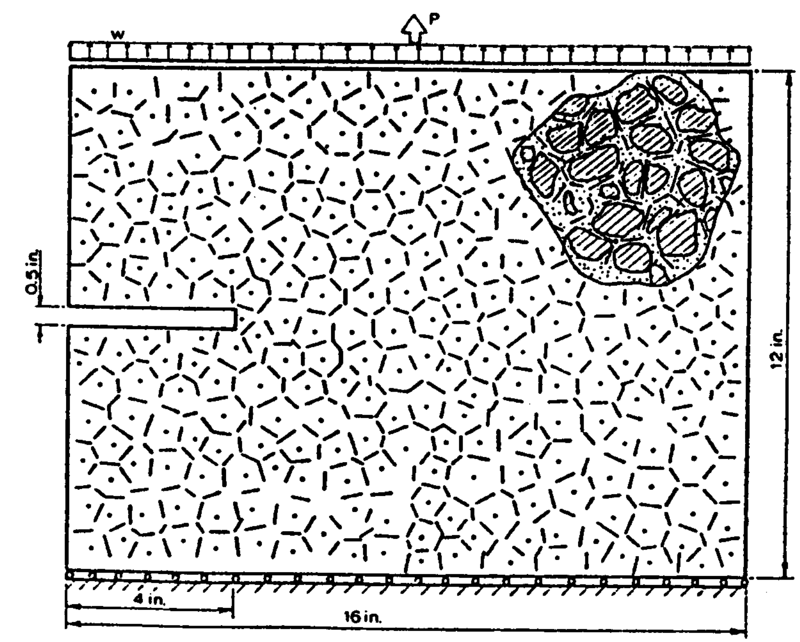

FIG. 11. Simulation of random microstructure of a concrete specimen with an interface element model.

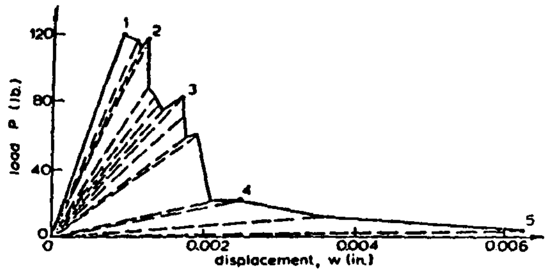

FIG. 12. Load-displacement diagram of the specimen shown in Fig. 11, calculated by the interface element model (Zubelewicz and Bažant, 1986).

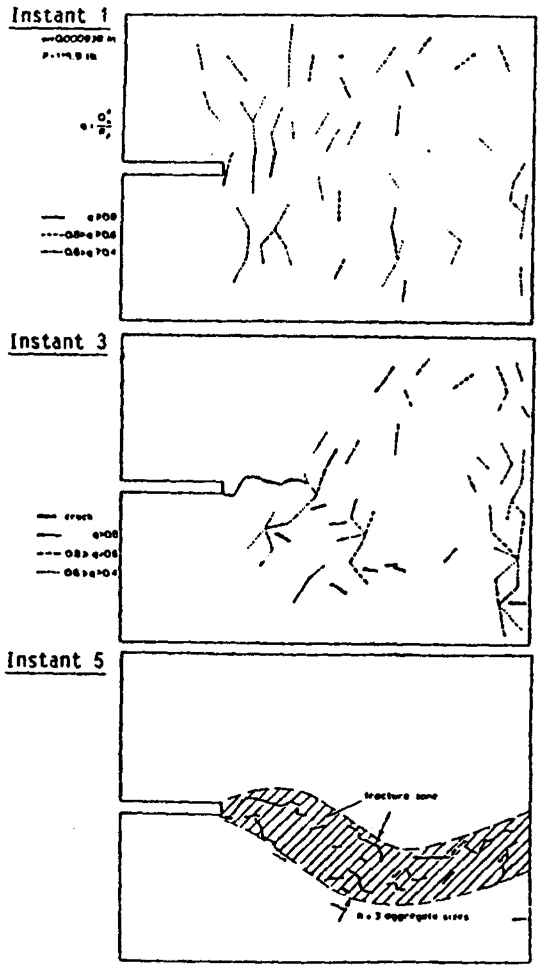

FIG. 13. Development of cracks and interparticle forces, shown for instants 1, 3, and 5 labeled in Fig. 12 (after Zubelewicz and Bažant [1986]).

element size.

The localization limiting properties have their source in the inhomogeneity of the microstructure, and by simulating these inhomogeneities with some simplified model, such as the interface element model, it is possible to obtain solutions in which the localization of strain-softening damage is limited and the energy dissipation at failure is finite.

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REFERENCES

1. Z.P. Bažant. Instability, Ductility and Size Effect in Strain-Softening Concrete, J. Engrg. Mech. ASCE 102, 331-344; discussions, 103, 357-358 (1976); based on Northwestern University Report, 1974.
2. Z.P. Bažant, and L. Cedolin. Blunt Crack Band Propagation in Finite Element Analysis, J. Engrg. Mech. ASCE 105(2), 297-315 (1979).
3. Z.P. Bažant and B.H. Oh. Rock Fracture Via Stress-Strain Relations, Center for Concret and Geomaterials, Report No. 82-11/665r. Northwestern University, Evanston, IL (1982); also, J. Engrg. Mech. ASCE 110, 1015-1035 (1984).
4. Z.P. Bažant and L. Cedolin. Finite Element Modeling of Crack Band Propagation, J. Struct. Engrg. ASCE 109(2), 69-92 (1983).
5. Z.P. Bažant, T.B. Belytschko and T-P. Chang. Continuum Theory for Strain-Softening, J. Engrg. Mech. ASCE 110, 1666-1692 (1984).
6. Z.P. Bažant. Imbricate Continuum and its Variational Derivation, J. Engrg. Mech. ASCE 110(12), 1693-1712 (1984)
7. Z.P. Bažant and L. Cedolin. Fracture Mechanics of Reinforced Concrete, J. Engrg. Mech. ASCE, 106(6), 1287-1306 (1980); discussion, 108 464-471 (1982).
8. Z.P. Bažant. Size Effect in Blunt Fracture: Concrete, Rock, Metal, J. Engrg. Mech, ASCE 110, 518-535 (1984).
9. Z.P. Bažant. Fracture Mechanics and Strain-Softening of Concrete, in Proceedings, U.S.-Japan Seminar on Finite Element Analysis of Reiforced Concrete Structure, Tokyo, May (1985) ASME, New York, (1986).
10. Z.P. Bažant. Distributed Cracking and Nonlocal Continuum, in Preprints, Europe-US Symposium on Finite Element Methods for Nonlinear Problems, P. Bergan, Ed. Norwegian Inst. of Tech. Trondheim, II.2-1-II.2.25 (1985).
11. Z.P. Bažant. Comment on Hillerborg's Comparison of Size Effect Law with Fictitious Crack Model, in Dei Poli Anniverisary Volume, L. Cedolin, Ed. Poltecnico di Milano (1985).
12. Z.P. Bažant and T.B. Belytschko. Wave Propagation in a Strain-Softening Bar: Exact Solution, J. Engrg. Mech. ASCE 111(3).
13. Z.P. Bažant, J-K. Kim and P. Pfeiffer. Nonlinear Fracture Properties from Size Effect Tests, J. Struct. Engrg., ASCE 112, 289-307 (1986).
14. Z.P. Bažant, P. Pfeiffer. Determination of Fracture Energy from Size Effect Law, Report, Center for Concrete and Geomaterials, Northwestern University (1986).
15. Z.P. Bažant, G. Pijaudier-Cabot and J. Pan. Strain-Softening Beams and Frames, Report, Center for Concrete and Geomaterials, Northwestern University (1986).
16. Z.P. Bažant and P.C. Prat. Dependence of Concrete Fracture Energy on Temperature - in preparation (1986).

17. T. Belytschko. Localization Limiters, Presented at U.S. National Congress, Austin, Texas, June (1986).
18. T. Belytschko and Y-W. Hyun. Numerical Implementation of Localization Limiters, in preparation (1986).
19. T. Belytschko, X.J. Wang, Z.P. Bazant and Y-W. Hyun. Transient Solutions for One-Dimensional Problems with Strain-Softening, Journal of Applied Mechanics, to be published (1986).
20. T. Belytschko T, Z.P. Bazant, Y-W. Hyun, and T-P. Chang. Strain-Softening Materials and Finite Element Solutions, Computers and Structures," 23(2), 163-180 (1986).
21. A. Chudnovsky, A. Moet, R.J. Bankert and M. Takemori. Effect of Damage Dissemination on Crack Propagation in Polypropylene, J. Appl. Phys., 54(10), 5562-5567 (1983).
22. M.A. Crisfield. Difficulties with Current Numerical Models for Reinforced Concrete and Some Tentative Solutions," Proceedings, Intl. Conf. on Computer Aided Analysis and Design of Concrete Structures, Split, Yugoslavia, September, F. Damjanic, E. Hinton, et al. Eds. Pineridge Press, Swansea, UK, 331-358 (1984).
23. J. Janson and J. Hult, Fracture Mechanics and Damage Mechanics, a Combined Approach, J. Mec. Appl., 1(1), 69-84 (1977).
24. D. Krajcinovic and G.U. Fonšeka. (1981) "The Continuous Damage Theory of Brittle Materials, Part I: General Theory," ASME J. Appl. Mech., 48, 809-815 (1981).
25. D. Krajcinovic. Constitutive Equations for Damaging Materials, J. Appl. Mech., 50, 355-360 (1983).
26. J. Lemaitre and J-L. Chaboche. Aspect phenomologique de al rupture par endommangement, J. Mec.Appl., 2, 317-365 (1978).
27. K.E. Loland. Continuous Damage Model for Load-Response Estimation of Concrete, Cement Concr. Res. 10, 395-402 (1980).
28. M. Lorrain and K.E. Loland. Damage Theory Applied to Concrete" in Fracture Mechanics of Concrete, F.J. Wittmann, Ed. Elsevier, Amsterdam, Chapt. 44, 341-369 (1983).
29. J. Mazars. Mechanical Damage and Fracture of Concrete Structures, in Advances in Fracture Research, Proc. 5th Int. Conf. Fracture, Cannes, D. Francois, Ed., 4, 1499-1506 (1981).
30. J. Mazars. Description of the Multiaxial Behavior of Concrete with an Elastic Damaging Model, in Proceedings, RILEM Symposium on Concrete Under Multiaxial Conditions, INSA-OPS Toulouse, 190-203 (1984).
31. S. Pietruszczak and Z. Mroz. Finite Element Analysis of Deformation of Strain-Softening Materials, Int. J. Numer. Meth. Egrg. 17, 327-334 (1981).
32. G. Pijaudier-Cabot and Z.P. Bažant. Nonlocal Damage Theory - in preparation (1986).
32. L. Resende and J.B.A. Martin. A Progressive Damage 'Continuum' Model for Granular Materials, Comput. Methods Appl. Mech. Egrg., 42, 1-18 (1984).
33. I.S. Sandler. Strain-Softening for Static and Dynamic Problems, in Proceedings of Symposium on Constitutive Equations: Micro, Macro, and Computational Aspects, ASME Winter Annual Meeting, New Orleans, December, K. Willam, Ed. ASME New York, 217-231 (1984).
34. H.L. Schreyer and Z. Chen. The Effect of Localization on the Softening Behavior of Structural Members, in Proceedings of Symposium on Constitutive Equations: Micro, Macro, and Computations Aspects, ASME Winter Annual Meeting, New Orleans, December, K. Willam, Ed. ASME, New York, 193-203 (1984).
35. N. Triantafyllidis and E.C. Aifantis. Mechanics of Microstructures, MM Report No. 6, A Gradient Approach to Localization of Deformation I. Hyperelastic Materials, Dept. Mech. Engrg. - Engrg. Mech., Michigan Technological University, Houghton, Michigan (1984).
36. K.J. Willam, N. Bicanic and S. Sture. Constitutive and Computational Aspects of Strain-Softening and Localization in Solids, in Proceedings of Symposium on Constitutive Equations: Micro, Macro and Computational Aspects, ASME Winter Annual Meeting, New Orleans, December, K.J. Willam, Ed. ASME, New York (1984).
37. A. Zubelewicz and Z.P. Bazant. Interface Element Modeling of Fracture in Aggregate Components, Report No. 85-12/428, Center for Concrete and Geomaterials, Northwestern University (1985).