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SCALING IN NONLINEAR FRACTURE MECHANICS

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Abstract. The paper[1] presents a review of recent results on the problem of size effect (or the scaling problem) in nonlinear fracture mechanics of quasibrittle materials and on the validity of recent claims that the observed size effect may be caused by the fractal nature of crack surfaces. The problem of scaling is approached through dimensional analysis and asymptotic matching. Large-size and small-size asymptotic expansions of the size effect on the nominal strength of structures are presented, considering not only specimens with large notches (or traction-free cracks) but also structures with no notches. Simple size effect formulas matching the required asymptotic properties are given. Regarding the fractal nature of crack surfaces, it is concluded that it cannot be the cause of the observed size effect.

1. Introduction

Scaling is a salient aspect of all physical theories. Nevertheless, little attention has been paid to the problem of scaling or size effect in solid mechanics. Up to the middle 1980's, observations of the size effect on the nominal strength of a structure have generally been explained by Weibull-type theory of random strength. However, recent in-depth analysis (Bažant and Xi, 1991) has shown that this Weibull-type theory does not capture the essential cause of size effect for quasibrittle materials such as rocks, toughened ceramics, concretes, mortars, brittle fiber composites, ice (especially sea ice), wood particle board and paper, in which the fracture process zone is not small compared to structural dimensions and large stable crack growth occurs prior to failure. The dominant source of size effect in these materials is not statistical but consists in the release of stored energy from the structure engendered by a large fracture.

By approximate analysis of energy release from the structure, a simple

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size effect law (Bažant 1983, 1984) has been derived for quasibrittle fracture. This law subsequently received extensive justifications, based on: (1) comparisons with tests of notched fracture specimens of concretes, mortars, rocks, ceramics, fiber composites (Bažant and Pfeiffer, 1987; Bažant and Kazemi, 1991, 1992; Bažant, Gettu and Kazemi, 1990; Gettu, Bažant and Karr, 1991, Bažant, Ožbolt and Eligehausen, 1994; Bažant, Daniel and Li, 1995) as well as unnotched reinforced concrete structures, (2) similitude in energy release and dimensional analysis, (3) comparison with discrete element (random particle) numerical modeling of fracture (e.g. Jirásek and Bažant, 1995), (4) derivation as a deterministic limit of a nonlocal generalization of Weibull statistical theory of strength (Bažant and Xi, 1991), and (5) comparison with finite element solutions based on nonlocal model of damage (Bažant, Ožbolt and Eligehausen, 1994). The simple size effect law has been shown useful for evaluation of material fracture characteristics from tests. Important contributions to the study of size effects in quasibrittle fracture have also been made by Carpinteri (1986), Planas and Elices (1988a,b, 1989, 1993), van Mier (1986), and others.

Recently, the fractal nature of crack surfaces in quasibrittle materials (Mandelbrot et al. 1984; Mecholsky and Mackin, 1988; Mosolov and Borodich, 1992; Borodich, 1992; Xie, 1993; etc.) has been studied intensively. It has been proposed that the crack surface fractality might be an alternative source of the observed size effect (Carpinteri 1994; Carpinteri et al. 1993, 1995; Lange et al., 1993, and Saouma et al., 1990, 1994).

This paper outlines a generalized asymptotic theory of scaling of quasibrittle fracture and also explores the possible role of the crack surface fractality in the size effect.

2. Large-Size Asymptotic Expansion of Size Effect

For the sake of brevity, the analysis will be made in general for fractal cracks and the nonfractal case will then simply be obtained as a limit case. Consider a crack representing a fractal curve (Fig. 1a) whose length is defined as $a_\delta = \delta_0(a/\delta_0)^{d_f}$ where d_f = fractal dimension of the crack curve (≥ 1) and δ_0 = lower limit of fractality implied by material microstructure, which may be regarded as the length of a ruler by which the crack length is measured (Mandelbrot et al., 1984). Unlike the case of classical, nonfractal fracture mechanics, the energy $\mathcal{W}_f$ dissipated per unit length of a fractal crack cannot be a material constant because the length of a fractal curve is infinite. Rather, it must be defined as

$$\mathcal{W}_f/b = G_{fl}\, a^{d_f} \tag{1}$$

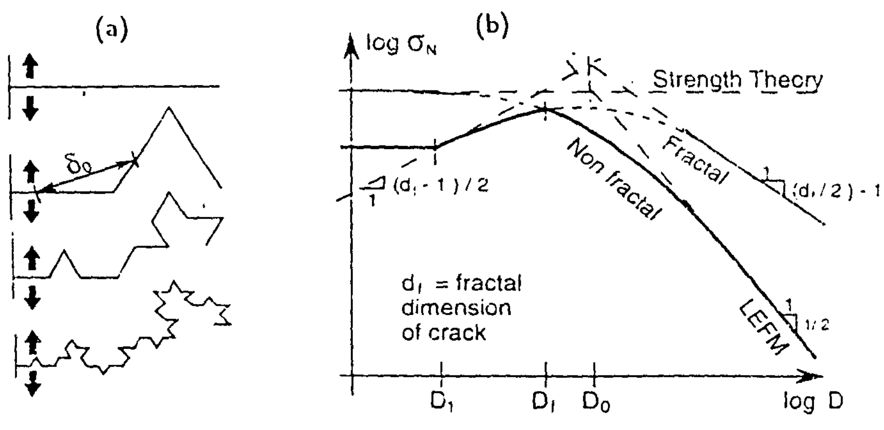

Figure 1: (a) Von Koch curves as examples of fractal crack at progressive refinement. (b) Size effect curves obtained for geometrically similar specimens with nonfractal and fractal cracks and finite size of fracture process zone (possible transition to horizontal line for nonfractal behavior is shown for $D < D_1$)

where b = thickness of the structure (considered to be two-dimensional), and G_{fl} = fractal fracture energy, of dimension Jm^{-d_f-1}. A nonfractal crack is the special case for $d_f = 1$, and then $G_{fl} = G_f$, representing the standard fracture energy, of dimension Jm^{-2}).

The rate of macroscopic energy dissipation $\mathcal{G}_{cr}$ with respect to the 'smooth' (projected, Euclidean) crack length a is:

$$\mathcal{G}_{cr} = \frac{1}{b}\frac{\partial \mathcal{W}_f}{\partial a} = G_{fl} d_f a^{d_f - 1} \tag{2}$$

(e.g., Borodich, 1992; Mosolov and Borodich, 1992). To characterize the size effect in geometrically similar structures of different sizes D, we introduce the usual nominal stress $\sigma_N = P/bD$ where D = characteristic size (dimension) of the structure, and P = applied load. If $P = P_{max}$ = maximum load, σ_N is called the nominal strength.

The problem will be analyzed under the following three hypotheses: (1) Within a certain range of sufficiently small scales, the failure is caused by propagation of a single fractal crack. (2) The fractal fracture energy, G_{fl} is a material constant correctly defining energy dissipation. (3) The material may (but need not) exhibit a material length, c_f.

The material length, c_f, may be regarded as the size (smooth, or projected) of the fractal fracture process zone in an infinitely large specimen (in which the structure geometry effects on the process zone disappear). The special case $c_f = 0$ represents the fractal generalization of linear elastic fracture mechanics (LEFM). Alternatively, if we imagine the fracture process zone to be described by smeared cracking or continuum damage mechanics, we may define $c_f = (G_{fl}/W_d)^{1/(2-d_f)}$ in which W_d = energy dissipated per unit volume of the continuum representing in a smeared way the fracture process zone (area under the complete stress-strain curve with strain softening). As still another alternative, in view of nonlinear fracture mechanics such as the cohesive crack model, we may define $c_f = (EG_{fl}/f_t^2)^{1/(2-d_f)}$ in which E = Young's modulus and f_t = material tensile strength.

We have two basic variables, a and c_f, both of the dimension of length. Let us now introduce two dimensionless variables: $\alpha = a/D$ and $\theta = c_f/D$. In view of Buckingham's theorem of dimensional analysis, the complementary energy Π^* of the structure with a fractal crack may be expressed in the form:

$$\Pi^* = \frac{\sigma_N^2}{E} bD^2 f(\alpha, \theta) \tag{3}$$

in which f is a dimensionless continuous function characterizing structure geometry.

The energy balance during crack propagation (first law of thermodynamics) must be satisfied by nonfractal as well as fractal cracks. The energy release from the structure as a whole is a global characteristic of the state of the structure and must be calculated on the basis of the smooth (projected, Euclidean) crack length a rather than the fractal curve length a_δ, i.e.

$$\frac{\partial \Pi^*}{\partial a} = \frac{\partial W_f}{\partial a}. \tag{4}$$

Substituting (3) and differentiating, we obtain an equation (see Bažant, 1995a,b) containing the derivative $g(\alpha, \theta) = \partial f(\alpha, \theta)\partial\alpha$, which represents the dimensionless energy release rate. The derivative of (3) must be calculated at constant load (or constant σ_N) because, as known from fracture mechanics, the energy release rate of a crack is the derivative of the complementary energy at constant load, i.e. $\partial\sigma_N/\partial a = 0$. In this manner (Bažant, 1995a,b) one obtains the equation $\sigma_N = \sqrt{E\mathcal{G}_{cr}/Dg(\alpha_0, \theta)}$ where α_0 = relative crack length α at maximum load. Because $g(\alpha_0, \theta)$ ought to be a smooth function, we may expand it in a Taylor series about the point $(\alpha, \theta) \equiv (\alpha_0, 0)$. This leads to the result (Bažant, 1995a,b,c):

$$\sigma_N = \sqrt{\frac{E\mathcal{G}_{cr}}{D}} \left[g(\alpha_0, 0) + g_1(\alpha_0, 0)\frac{c_f}{D} + \frac{1}{2!} g_2(\alpha_0, 0) \left(\frac{c_f}{D}\right)^2 + ... \right]^{-1/2} \tag{5}$$

in which $g_1(\alpha_0, 0) = \partial g(\alpha_0, \theta)/\partial\theta$, $g_2(\alpha_0, 0) = \partial^2 g(\alpha_0, \theta)/\partial\theta^2, ...$, all evaluated at $\theta = 0$. This equation represents the large-size asymptotic series expansion of the size effect. To obtain a simplified approximation, one may truncate the asymptotic series after the linear term, i.e.

$$\sigma_N = Bf_t' D^{(d_f - 1)/2} \left(1 + \frac{D}{D_0}\right)^{-1/2} \tag{6}$$

in which D_0 and B are certain constants depending on both material and structure properties, expressed in terms of function $g(\alpha_0, 0)$ and its derivative. For the nonfractal case, $d_f \to 1$, this reduces to the size effect law deduced by Bažant (1983, 1984, 1993), which reads $\sigma_N = Bf_t'/\sqrt{1+\beta}$, $\beta = D/D_0$, in which β is called the brittleness number (Bažant and Pfeiffer, 1987).

If only geometrically similar fracture test specimens are considered, α_0 is constant (independent of D), and so is D_0. For brittle failures of geometrically similar quasibrittle structures without notches, it is often observed that the crack lengths at maximum load are approximately geometrically similar. For concrete structures, the geometric similarity of cracks at maximum load has been experimentally demonstrated for diagonal shear of beams, punching of slabs, torsion, anchor pullout or bar pullout, and bar splice failure, and is also supported by finite element solutions (e.g. ACI, 1992; Bažant et al. 1994) and discrete element (random particle) simulations (Jirásek and Bažant, 1995), albeit for only a limited size range of D. Thus, k, c_0, D_0, σ_N^0 and Bf_t' are all constant. In these typical cases, (6) describes the dependence of σ_N on size D only, that is, the size effect. Fig. 1b shows the size effect plot of $\log \sigma_N$ versus $\log D$ at constant α_0. Two size effect curves are seen: (1) the fractal curve and (2) the nonfractal curve (for the latter, the possibility of termination of fractality at the left end is considered in the plot).

The curve of fractal scaling obtained in Fig. 1b disagrees with the bulk of experimental evidence (for concrete, see e.g. the review in Bažant et al. 1994); for carbon fiber epoxy composites used in aerospace industry, see (Bažant, Daniel and Li, 1995). It follows that crack fractality cannot be the cause of the observed size effect.

What aspect of the fracture process causes the crack fractality to have no significant effect on scaling of failure? The fracture front in quasibrittle materials does not consist of a single crack, but a wide band of microcracks, which all must form and dissipate energy before the fracture can propagate. Only very few of the microcracks and slip planes eventually coalesce into a single continuous crack, which forms the final crack surface with fractal characteristics. Thus, even though the final crack surface may be to a large extent fractal, the fractality cannot be relevant for the fracture process zone advance. Most of the energy is dissipated in the fracture process zone by microcracks (as well as plastic-frictional slips) that do not become part of the final crack surface and thus can have nothing to do with the fractality of the final crack surface.

3. Generalizations and Ramifications of Asymptotic Analysis

Material length c_f can be defined as the LEFM-effective length of the fracture process zone, measured in the direction of propagation in a specimen of infinite size. In that case, $\theta = c_f/D = (a - a_0)/D = \alpha - \alpha_0$, and so $g(\alpha, \theta)$ reduces to the LEFM function of one variable, $g(\alpha)$. Thus Eq. (6) yields (Bažant, 1995a,b,c):

$$D_0 = c_f \frac{g'(\alpha_0)}{g(\alpha_0)}, \qquad Bf'_t = \sqrt{\frac{EG_f}{c_f g'(\alpha_0)}}, \qquad \sigma_N^0 = \sqrt{\frac{EG_{fl} d_f \alpha_0^{d_f - 1}}{c_f g'(\alpha_0)}} \tag{7}$$

and so Eq. (6) takes the form:

$$\sigma_N = \sqrt{\frac{EG_{fl} d_f \alpha_0^{d_f - 1}}{g'(\alpha_0) c_f + g(\alpha_0) D}} \tag{8}$$

The advantage of this equation is that its parameters are directly the material fracture parameters. For $d_f = 1$, Eq. (8) reduces to the form of size effect law derived in a different manner by Bažant and Kazemi (1990, 1991) (also Eq. 12.2.11 in Bažant and Cedolin, 1991). Fitting this equation to size effect data, which can be done easily by rearranging the equation to a linear regression plot, one can determine G_f or G_{fl} and c_f. This serves as the basis of the size effect method for measuring the material fracture parameters, which has been adopted by RILEM as an international standard for concrete.

More generally, one may introduce general dimensionless variables $\xi = \theta^r = (c_f/D)^r, h(\alpha_0, \xi) = [g(\alpha_0, \theta]^r$, with any $r > 0$. Then, expanding the function $h(\alpha_0, \xi)$ in a Taylor series with respect to ξ, one obtains by a similar procedure as before a more general large-size asymptotic series

expansion (whose nonfractal special case was derived by Bažant in 1985 (see ACI, 1992):

$$\sigma_N = \sigma_P \left[\beta^r + 1 + \kappa_1 \beta^{-r} + \kappa_2 \beta^{-2r} + \kappa_3 \beta^{-3r} + \ldots\right]^{-1/2r} \tag{9}$$

in which $\beta = D/D_0$ and $\kappa_1, \kappa_2, \ldots$ are certain constants. However, based on experiments as well some limit properties, it seems that $r = 1$ is the appropriate value for most cases.

The large-size asymptotic expansion (9) diverges for $D \to 0$. For small sizes, one needs a small-size asymptotic expansion. The previous energy release rate equation $(\sigma_N^2/E)Dg(\alpha, \vartheta) = \mathcal{G}_{cr}$ is not meaningful for the small size limit because the zone of distributed cracking is relatively large. In that case, the material failure must be characterized by W_f rather than G_f. In that case, the energy balance equation (first law) for $\partial\sigma_N/\partial a = 0$ (second law) must be written in the form $\sigma_N^2[\psi(\alpha, \eta)]^r/E = W_f$ where $\psi(\alpha, \eta)$ = dimensionless function of dimensionless variables $\alpha = a/D$ and $\eta = (D/c_f)^r = \vartheta^{-r}$ (variable ϑ is now unsuitable because $\vartheta \to \infty$ for $D \to 0$), and exponent $r > 0$ is introduced for the sake of generality, as before. Because, for very small D, there is a diffuse failure zone, a must now be interpreted as the characteristic size of the failure zone, e.g., the length of cracking band. The same procedure as before now leads to the result (Bažant, 1995a,c):

$$\sigma_N = \sigma_P \left[1 + \beta^r + b_2 \beta^{2r} + b_3 \beta^{3r} + \ldots\right]^{-1/2r} \tag{10}$$

in which $\beta = D/D_0$ and $\sigma_P, D_0, b_2, b_3, \ldots$ are certain constants depending on both material and structure properties and can be expressed in terms of function $\psi(\alpha_0, 0)$ and its derivatives. Eq. (10) represents the small-size asymptotic series expansion.

An important common characteristic of the large-size and small-size asymptotic series expansions in Eqs. (9) and (10) is that they have the first two terms in common. Therefore, if either series is truncated after the second term, it reduces to the same generalized size effect law derived by Bažant in 1985 (see ACI, 1992):

$$\sigma_N = \sigma_P (1 + \beta^r)^{-1/2r} \tag{11}$$

Because this law is anchored to the asymptotic cases on both sides and shares with both expansions the first two terms, it may be regarded as a matched asymptotic (e.g. Bender and Orszag, 1978), that is, an intermediate approximation of uniform applicability for any size. The value $r = 1$ appears, for several reasons, most appropriate.

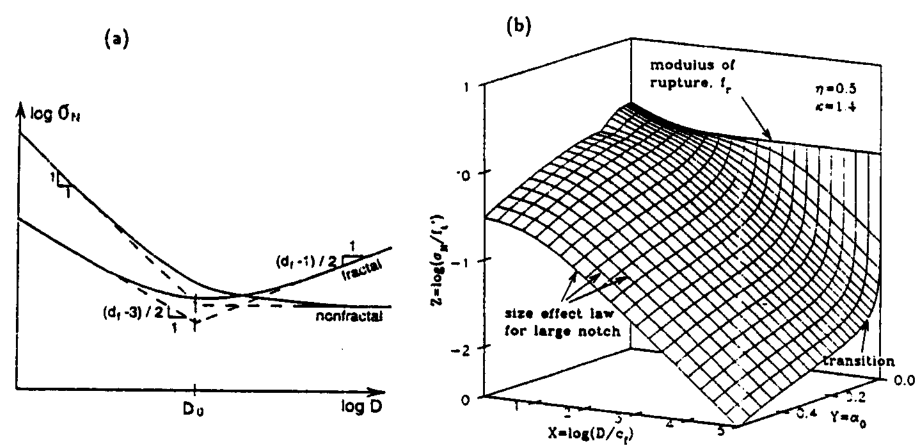

Figure 2: (a) Size effect curves obtained for unnotched specimens, nonfractal and fractal. (b) The surface of universal size effect law for notched as well as unnotched fracture specimens

A different approach is needed for unnotched quasibrittle structures that reach the maximum load when the crack initiates from a smooth surface, as exemplified by the standardized bending test of modulus of rupture f_r of a plain concrete beam. Applying the size effect law in Eq. (6 for the case $\alpha_0 \to 0$ is impossible because $g(\alpha_0, 0)$ vanishes as $\alpha_0 \to 0$. To deal with this case, one must truncate the large-size asymptotic series expansion only after the third term. Then, considering that $r = 1$ and $g(\alpha_0, 0) = 0$, restricting attention to the nonfractal case only, and using a similar procedure as that which led to Eq. (8), one obtains after some further asymptotic approximations (Bažant, 1995a,c) the following size effect law (Fig. 2a) for failures at crack initiation from a smooth surface:

$$\sigma_N = Bf_r^{\infty}\left(1 + \frac{D_b}{D}\right) = f_r^{\infty}\left[1 - 0.0634 g''(0)\frac{\bar{c}_f}{D}\right] \tag{12}$$

(the first part of this equation was derived by Bažant and Li (1995) in a different manner). Here f_r^{∞} is the modulus of rupture for an infinitely large beam (but not so large that Weibull statistical size effect would become significant), and B is a dimensionless parameter. This equation can be arranged as a linear regression plot of σ_N versus $1/D$, which is again helpful for easy identification of the constants from tests.

Asymptotic matching of the three asymptotic expansions, namely: (1) the large-size expansion for large α_0, (2) the large-size expansion for vanishing α_0, and (3) the small-size expansion for large α_0, leads (Bažant, 1995a,c) to the following approximated universal size effect law (Fig. 2b) valid for failures at both large cracks and crack initiation from a smooth surface:

$$\sigma_N = \sigma_0\left(1 + \frac{D}{D_0}\right)^{-1/2}\left\{1 + \frac{1}{s}\left[\left(\eta + \frac{D}{D_b}\right)\left(1 + \frac{D}{D_0}\right)\right]^{-1}\right\}^{s} \tag{13}$$

in which σ_0, D_0, D_b and $\bar{c}_f$ are constants expressed in terms of $g(\alpha_0)$ and its first and second derivatives and of EG_f, and η and κ are additional empirical constants.

4. Summary and Conclusion

In quasibrittle structures, the size effect can be generally characterized on the basis of asymptotic series expansions and asymptotic matching. Whereas for normal sizes the scaling problem is extremely difficult, it becomes much simpler both for very large sizes (LEFM) and for very small sizes (plasticity). Asymptotic matching is an effective way to obtain a simplified description of the size effect in the normal, intermediate range of sizes. The size effect at crack initiation from a smooth surface can also be described the basis of the asymptotic energy release analysis, and a universal size effect law comprising both types of size effect can be formulated The fractal morphology of crack surfaces in quasibrittle materials does not appear to play a significant role in fracture propagation and the size effect.

Appendix. Is Weibull-Type Size Effect Important for Quasibrittle Failure?

It is proper to explain at least briefly why strength randomness is not considered in the present analysis of size effect. The main reason is the redistribution of stresses caused by stable fracture growth prior to maximum load and localization of damage into a fracture process zone. If the Weibull probability integral is applied to the redistributed stress field, which has high stress peaks near the crack tip, the dominant contribution to the integral comes from the fracture process zone. The important point is that the size of this zone is nearly independent of structure size D. The contribution from the rest of the structure is nearly vanishing, which corresponds to the fact that the fracture cannot occur outside the process zone. Because, in specimens of different sizes, this zone has about the same size, the Weibull-type size effect must, therefore, disappear. In other words, the fracture is probabilistic, but only the random properties of the material in a zone of the same size decide the failure, even though the structures have different sizes.

A generalized version of Weibull-type theory, in which the material failure probability depends not on the local stress but on the average strain of a characteristic volume of the material, has been shown to yield the approximate size effect formula (Bažant and Xi, 1991):

$$\sigma_N = \frac{B f_t'}{\sqrt{\beta^{2n/m} + \beta}} \qquad (14)$$

in which m = Weibull modulus (exponent of Weibull distribution of random strength), which is typically about 12 for concrete, and $n = 1$, 2 or 3 for one-, two- and three-dimensional similarity. Typically, for $n = 2$ or 3, $2n/m \ll 1$, for concrete. Then, for $m \to \infty$, which is the deterministic limit, this formula approaches the size effect law in (6). Also, for $D \to 0$, this formula asymptotically approaches the classical Weibull size effect law, and for large sizes and any m, this formula asymptotically approaches Eq. (6). It has been shown that the difference between these two formulas for concrete structures is significant only for extremely small sizes, which are below the applicability of continuum modeling.

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