Asymptotic Prediction of Energetic-Statistical Size Effect from Deterministic Finite-Element Solutions

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Abstract: An improved form of a recently derived energetic-statistical formula for size effect on the strength of quasibrittle structures failing at crack initiation is presented and exploited to perform stochastic structural analysis without the burden of stochastic nonlinear finite-element simulations. The characteristic length for the statistical term in this formula is deduced by considering the limiting case of the energetic part of size effect for a vanishing thickness of the boundary layer of cracking. A simple elastic analysis of stress field provides the large-size asymptotic deterministic strength, and also allows evaluating the Weibull probability integral which yields the mean strength according to the purely statistical Weibull theory. A deterministic plastic limit analysis of an elastic body with a through-crack imagined to be filled by a perfectly plastic "glue" is used to obtain the small-size asymptote of size effect. Deterministic nonlinear fracture simulations of several scaled structures with commercial code ATENA (based on the crack band model) suffice to calibrate the deterministic part of size effect. On this basis, one can calibrate the energetic-statistical size effect formula, giving the mean strength for any size of geometrically scaled structures. Stochastic two-dimensional nonlinear simulations of the failure of Malpasset Dam demonstrate good agreement with the calibrated formula and the need to consider in dam design both the deterministic and statistical aspects of size effect. The mean tolerable displacement of the abutment of this arch dam is shown to have been approximately one half of the value indicated by the classical deterministic local analysis based on material strength.

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Introduction

Although the importance of size effect for safe design of large concrete structures is now widely accepted, its consideration is still quite limited. This is especially true for the combined energetic-statistical size effect in structures failing at fracture initiation, for which an effective computational approach has been lacking.

Significant though the recent progress has been in stochastic nonlinear fracture modeling of concrete structures (Bažant and Xi 1991; Carmeliet 1994; Carmeliet and Hens 1994; de Borst and Carmeliet 1996; Gutiérrez and de Borst 1999, 2001; Bažant and Novák 2000a,b; Pukl et al. 2002, 2003; Bergmeister et al. 2004; Novák et al. 2003b, 2005; Vořechovský and Matesová 2006, Vořechovský 2006), no effective and simple method nevertheless exists for incorporating the combined energetic-statistical size effect in computer analysis of structures, avoiding the computational burden of direct Monte Carlo simulations. This work, whose major part was carried out at Northwestern University during 2003 (and received detailed coverage in Vořechovský's 2004 dissertation), has three main objectives: (1) derivation of an extended mean size effect formula capturing two independent size effect sources (energetic and statistical); (2) a method to predict the size effect with no stochastic simulations; and (3) full computational probabilistic verification of the proposed formula and asymptotic prediction. The known analytical scaling law (Bažant 2001, 2002, 2004a,b; RILEM 2004) of the energetic-statistical size effect will be extended and exploited to completely avoid stochastic finite-element analysis by matching of the asymptotic behaviors of structures much smaller and much larger in size. The deterministic finite-element analysis exhibiting both the energetic and statistical size effect will be simplified by employing previously proposed "random blocks" of finite elements whose mean strength is reduced according to the block size on the basis of the weakest-link model. The proposed method will be demonstrated by analyzing the failure of Malpasset Dam.

The analysis will deal exclusively with the Type 1 size effect (Bažant 2002, 2004a), which occurs for quasibrittle structures of initially positive geometry (Bažant and Planas 1998), failing (under load control) at crack initiation. For Type 2 size effect, which occurs in quasibrittle structures with large notches or deep stress-free (fatigued) cracks, there is no statistical size effect on the mean strength (Bažant and Xi 1991), i.e., the mean size effect is independent of material randomness. As usual, the size effect will be characterized in terms of the nominal strength $\sigma_N = cP/bD$ where P is the maximum load or load parameter,

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D is the characteristic structure size (or dimension), b is the structure width, and c is the arbitrary convenience parameter.

Nonlocal and Classical Weibull Statistical Theories

The probability P_f of failure of a quasi-brittle structure is realistically approximated by nonlocal generalization of Weibull statistical theory (Bažant and Xi 1991), in which

$$P_f = 1 - \exp\left(-\int_V \langle\bar{\sigma}(\mathbf{x})/s_0\rangle^m \mathrm{d}V(\mathbf{x})/l_s^n\right) \tag{1}$$

in which $\langle X\rangle = \max(X,0)$ (Macauley bracket); V=structure volume, area or length, depending on whether the fracture growth is scaled in one, two, or three dimensions, $n=1$, 2 or 3; s_0, m=positive constants called the scaling parameter and Weibull modulus; l_s=chosen reference size, such that l_s^n is the volume or area for which parameter s_0 has been measured (this volume cannot be taken smaller than the depth of the fully developed fracture process zone (FPZ) at fracture initiation); $\sigma(\mathbf{x})$=local stress at point of coordinate $\mathbf{x}$; and $\bar{\sigma}(\mathbf{x})$ is the nonlocal stress, defined (in one-dimensional simplification) as $\bar{\sigma}(\mathbf{x}) = E[\epsilon_e + \int_\Omega \alpha(\mathbf{x},\mathbf{s})\bar{\epsilon}''(\mathbf{x})\mathrm{d}V(\mathbf{s})]$ where ϵ_e=local elastic strain; ϵ''=local inelastic strain; E=Young's modulus, and the overbar denotes nonlocal averaging with a certain weight function $\alpha(\mathbf{x},\mathbf{s})$ over a nonlocal characteristic volume Ω centered at point $\mathbf{x}$. The power function $p_1 = \langle\bar{\sigma}(\mathbf{x})/s_0\rangle^m$ represents the failure probability of volume Ω. This is a power function of stress, with zero threshold, and (according to Bažant and Pang 2005a,b) this is an inevitable consequence of the fact that the failure of interatomic bonds is governed by Maxwell-Boltzmann distribution of thermal energies of atoms and the stress dependence of activation energy. The averaging in Eq. (1), of course, cannot introduce a length scale for a body under uniform stress, in which case other, purely statistical, length scales may arise due to autocorrelation. The averaging physically captures the fundamental property that the representative volume element (RVE) of material must act, due to its heterogeneity, as one unit dominated by parallel coupling (essentially equivalent to averaging, as in Daniels' fiber bundle model, but different from statistical correlation). This fact is usually the main reason for deviations from the classical (local) Weibull theory (although autocorrelation of local material strength may also engender deviations). The nonlocal averaging, however, does not give a realistic distribution tail for small structures (Bažant and Pang 2005b, 2006).

It has been demonstrated by asymptotic analysis as well as numerical simulations that, for large enough structures ($D\to\infty$), the nonlocal Weibull theory reduces to the classical (local) Weibull statistical theory, for which $\bar{\sigma}(\mathbf{x})$ in Eq. (1) is replaced by local stress $\sigma(\mathbf{x})$ (see, e.g., Bažant and Planas 1998). For geometrically similar structures, it is convenient to write $\sigma(\mathbf{x}) = \tau\, s(\boldsymbol{\xi})$ where $s(\boldsymbol{\xi})$=dimensionless field of maximum principal stress. τ=nominal stress parameter; $\boldsymbol{\xi} = \mathbf{x}/D$=dimensionless coordinate vectors. Often only the positive (tensile) maximum principal stresses matter, and then

$$P_f = 1 - e^{-Cl_s^n(\tau/s_0)^m} \tag{2}$$

where $C = \int_V \langle s(\boldsymbol{\xi})\rangle^m \mathrm{d}V(\boldsymbol{\xi})$ (which is independent of D and nominal stress). In the purely statistical classical Weibull theory (Weibull 1939), the mean nominal strength

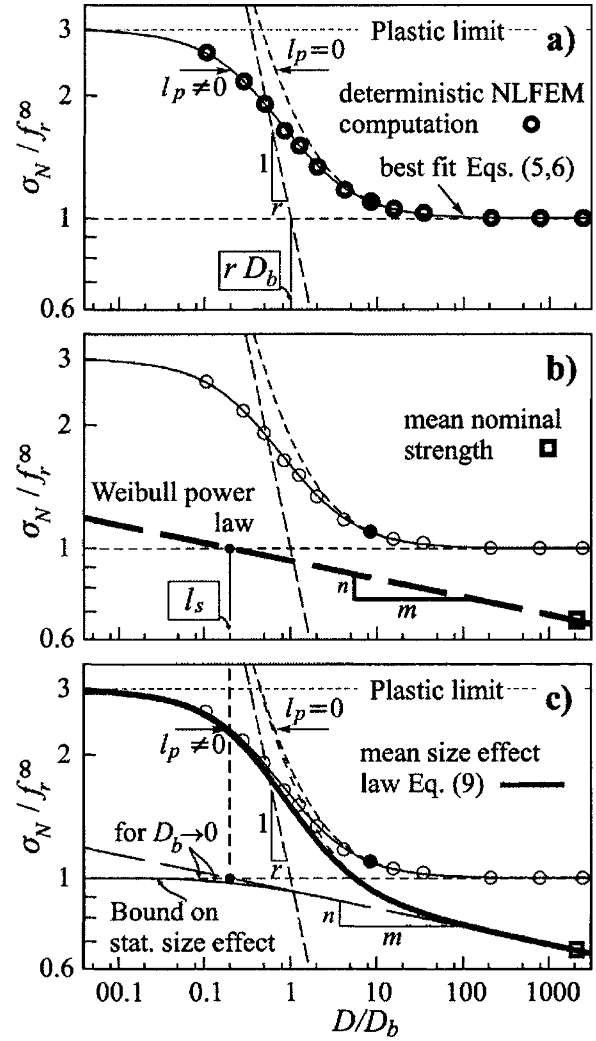

Fig. 1. Computational sequence: (a) Steps 1–4: fit of nonlocal computations by deterministic formula; (b) Step 5: determination of parameter l_s; and (c) final size effect formula (9) for predicting strength for the real size D_a of the dam. Limiting case of energetic and energetic-statistical size effect curves when $D_b \to 0$.

σ_N for any chosen reference size $D = D_{st}$ is $\sigma_{N_{st}} = \int_{-\infty}^{\infty} \tau \mathrm{d}P_f(\tau) = \cdots = C^{-1/m} l_s^{-n/m} s_0 \int_0^\infty t^{1/m} e^{-t}\mathrm{d}t$, i.e.,

$$\sigma_{st} = C^{-1/m} l_s^{-n/m} s_0 \Gamma(1 + m^{-1}) \tag{3}$$

The coefficient of variation ω of σ_N, calculated as $\omega^2 = \int_0^1 (\tau/\sigma_N - 1)^2 \mathrm{d}\tau$, is $\omega = [\Gamma(1+2m^{-1})\Gamma^{-2}(1+m^{-1}) - 1]^{1/2}$, which is independent of size D. If $\sigma_{N_{st}}$ denotes σ_N corresponding to $D = D_{st}$, the purely statistical size effect on the mean nominal strength may be written as

$$\sigma_N = \sigma_{N\mathrm{st}}(D_{st}/D)^{n/m} \tag{4}$$

Energetic Size Effect Formula and Its Statistical Generalization

A simple energetic (deterministic) size effect formula, sufficient for heterogeneous structures of any size, reads (Bažant 1995, 1997, 2002; Bažant and Chen 1997; Bažant and Novák 2000b; RILEM 2004):

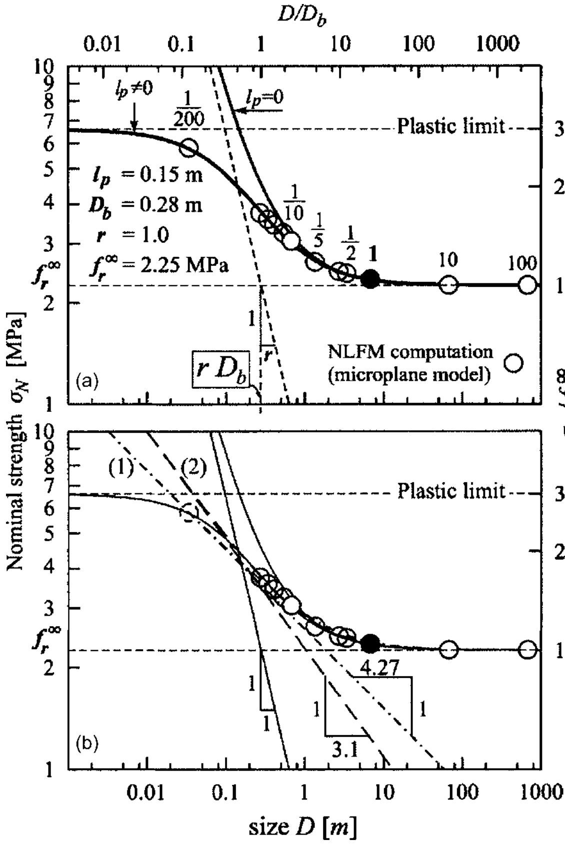

Fig. 2. Best fit of extended deterministic formula (6): (a) for a wide range of structural sizes (Malpasset Dam here); (b) examples of inadequate fitting of the simplified deterministic formula (5) to 1) the whole range of data; and 2) to the data without the smallest size

$$\sigma_N(D) = f_r^\infty(1 + rD_b/D)^{1/r} \tag{5}$$

[Fig. 1(a) or 2(a)] in which $l_p=0$]; and f_r^∞, D_b, r=positive empirical parameters. For three-point bending of a beam of span L, a convenient choice is $c=3L/2D$, in which case σ_N represents the maximum elastically calculated bending stress. The intersection of the small-size and large-size asymptotes of $\log \sigma_N$ versus $\log D$ [Fig. 1(a) or 2(a)] occurs at $D=rD_b$. Exponent r controls the initial slope of the size effect curve while having no effect on the first two terms of the asymptotic expansion of σ_N for $D\to\infty$ (Bažant and Planas 1998); D_b=deterministic characteristic length interpreted as the thickness of the boundary layer of cracking causing stress redistribution within the cross section; $D\approx 1.5$ to 2 grain sizes.

Quasibrittle fracture can be deterministically well simulated by the cohesive crack model. This is a continuum model which, by contrast to Eq. (5), can be extrapolated below the size of FPZ and all the way to $D=0$. The following generalization of Eq. (5), conforming to the cohesive crack model over the entire size range $D\in(0,\infty)$, is advantageous because the zero size limit of the cohesive crack model (or nonlocal damage model) can be easily calculated and used for calibration (Bažant and Planas 1998; Bažant 2002)

$$\sigma_N(D) = f_r^\infty\left(1 + \frac{rD_b}{D + l_p}\right)^{1/r} \tag{6}$$

where, aside from D_b, l_p=second deterministic characteristic length controlling the center of transition to a horizontal asymptote of the $\ln \sigma_N$ versus $\ln D$ curve [Fig. 1(a) or 2(a)]. Formula (6) gives the transition from perfectly plastic behavior for $D/l_p\to 0$ (corresponding to an elastic body in which the crack is filled with a perfectly plastic glue, Bažant 2002), through quasi-brittle behavior, to perfectly brittle behavior for $D/D_b\to\infty$. The limiting strength for $D/l_p\to 0$ (and $l_p>0$) (horizontal asymptote in Fig. 1 or 2) is $f_r^\infty(1+rD_b/l_p)^{1/r}$.

In the special case of bending without normal force, the ratio η_p of the bending moments M for $D\to 0$ and $D\to\infty$ is $\eta_p=M_{\mathrm{pl}}/M_{\mathrm{el}}=(1+rD_b/l_p)^{1/r}$. Knowing η_p, one gets from Eq. (5)

$$l_p = rD_b/(\eta_p^r - 1) \tag{7}$$

E.g., for a rectangular cross section, $\eta_p=3$. In this limit case ($D\to 0$), the entire cross section is under uniform tensile stress $\sigma=f_r^\infty$, balanced by a compressive concentrated force at the compression face. This force is, of course, fictitious (if a finite compression strength f_c' were introduced, a finite zone of finite compressive stress would appear but then the size effect curve for $D\to 0$ would cease to be a smooth extension of the size effect curve for normal sizes, and thus become ineffective for calibration.

For most purposes, $l_p=0$ can be assumed. Then the small-size asymptotic size effect is the same as for Eq. (5), i.e., $[\sigma_N(D)]_{l_p\to 0}=f_r^\infty(rD_b/D)^{1/r}$; see Fig. 2(a) in which $l_p=0$).

Energetic-Statistical Size Effect Formula

The large-size asymptote of the deterministic Eqs. (5) and (6) is horizontal, i.e., $\sigma_N=f_r^\infty$; see Figs. 1 and 2. Except for the Bažant and Novák (2000b, 2001) formula, the large-size asymptote of all the classical formulas for the modulus of rupture is horizontal (Bažant and Planas 1998), but according to Bažant and Novák's (2000a,b, 2001) analysis based on the nonlocal Weibull theory (Bažant and Xi 1991) it must be inclined, with the slope of $-n/m$; i.e., $\sigma_N\propto D^{-n/m}$. The reason that this property is not readily detected in modulus of rupture tests is their limited size range. Nevertheless, a modified formula with an inclined Weibull-type asymptote has been shown to give a closer fit of the existing test data and be necessary for a close fit of new data of very broad size range (Bažant and Novák 2000b, 2001). Hence, a statistical generalization of Eq. (5) is needed. Adapting slightly the derivation of Bažant and Novák (2000b), one may argue as follows.

Since the Weibull size effect dominates for $D\to\infty$, for which the stress redistribution in the boundary layer is negligible and the structure is far larger than the FPZ (or RVE), we need to adjust the horizontal asymptote of Eq. (5).

Since the deterministic part of size effect vanishes for $D\to\infty$, it consists of the difference Δ of formula (5) from the horizonal asymptote, for which

$$\Delta^r = (f_r/f_r^0)^r - rD_b/(D + l_p) \tag{8}$$

where $\Delta=1$. If material strength randomness is taken into consideration, difference Δ should conform to the size effect of Weibull statistical theory, $\sigma_N\propto D^{-n/m}$ ($n=2$ for two-dimensional fracture). Therefore, instead of $\Delta=1$, we set $\Delta=(D/l_s)^{-n/m}$. But this applies only for $D\gg$FPZ size, for which the FPZ occurs randomly in any of the elemental volumes of material. Otherwise, there is no chance for the failure to begin at different random locations, and so the statistical size effect must disappear for small D. This feature [which is similar to Eq. (33) of Bažant 2004a and of an identical T&AM Report No. 05-03/C728s at Northwestern University 2003; and to $f(l)$ in Eq. (6.10) of Vořechovský 2004] may be captured by replacing $\Delta=(D/l_s)^{-n/m}$ with $\Delta=[(D+\kappa l_s)/l_s]^{-n/m}$

where, as shown later, $\kappa=1$. Substitution into (8) then furnishes the final formula for energetic-probabilistic size effect

$$\sigma_N(D)=f_r^{\infty}\left[\left(\frac{l_s}{D+l_s}\right)^{rn/m}+\frac{rD_b}{D+l_p}\right]^{1/r} \tag{9}$$

For the special case $l_s=D_b$, this formula was derived in Eq. (17) in Bažant (2004a) by dimensional analysis with asymptotic matching, and in Eq. (33) of Bažant (2004b) by nonlocal Weibull theory. Using the foregoing procedure, Bažant and Novák (2000b) derived the formula

$$\sigma_N(D)=f_r^{\infty}[(D_b/D)^{rn/m}+(rD_b/D)]^{1/r} \tag{10}$$

which represents a still more special case of Eq. (9) and suffices for all purposes except theoretical extrapolations to cross section sizes smaller that the aggregate size.

Because normally $rn/m \ll 1$, formula (9) satisfies three asymptotic requirements: (1) for small but not too small sizes, it asymptotically approaches the power law $[\sigma_N(D)] \to f_r^{\infty} r^{1/r}(D_b/D)^{1/r} \propto D^{-1/r}$ for $D/l_s \to 0$ and $D/l_p \to \infty$ while the simplest deterministic energetic formula (5) approximately holds true for $l_p \ll D \ll l_s$. (2) For large sizes, $D \gg l_s$ (and for $l_s \geq l_p$), Eq. (9) asymptotically approaches the Weibull size effect

$$[\sigma_N(D)]_{D/l_s\to\infty(l_s\geq l_p)}=f_r^{\infty}(l_s/D)^{n/m}\propto D^{-n/m} \tag{11}$$

(3) For $m\to\infty$, the limit of Eq. (9) is the deterministic energetic formula (5). Eq. (9) is in fact the simplest formula with these three asymptotic properties. Moreover, it also agrees with the second-order terms of the asymptotic expansions of cohesive crack model and nonlocal Weibull model, which require that: (1) $\sigma_N=c_1-c_2D+O(D^2)$ for $D\to 0$; (2) $\sigma_N=b_1D^{-n/m}+b_2D^{-1}+O(D^{-2})$ for $D\to\infty$ (where b_1,b_2,c_1,c_2 are constants); and (3) the power law $\sigma_N \propto D^{-1/r}$ ought to be approached as an intermediate asymptote for $l_s/l_p\to 0$ (Bažant 2002, 2004a).

The special case in Eq. (10) does not satisfy the small size asymptotic limit of the cohesive crack model, and thus cannot be calibrated by this model. Yet it is good enough for describing all the available test data, and has therefore been adopted in a proposal for an improved testing standard for the modulus of rupture (Bažant and Novák 2001).

The physical cause of the energetic part of size effect [given by the second term in Eq. (9)], is the stress redistribution and energy release caused by a sizable boundary layer of cracking (or FPZ), and its characteristic length D_b is set principally by the material inhomogeneity size, i.e., the aggregate size in concrete. On the other hand, the physical cause of the statistical part of size effect [given by the first term in Eq. (9)] is the material strength randomness.

Compared to the pure Weibull theory, parameter l_s has here a different physical meaning than in the asymptotic limit of Eq. (11). In that classical theory, there is no material characteristic length (because the size effect law is a power law). Rather, l_s in Weibull theory is simply a chosen unit of measurement to which the spatial density of failure probability is referred (e.g., Bažant and Planas 1998). It represents the size of test specimens on which Weibull modulus m has been calibrated from statistical scatter. Obviously, l_s can be arbitrarily changed as long as the $f_r^{\infty} l_s^{n/m}$ remains the same. However, in Eq. (9), l_s is no longer arbitrary and acquires the physical meaning of a statistical characteristic length. This length must be roughly proportional to the width of the damage localization band, which is in turn proportional to the size of material inhomogeneities or the FPZ.

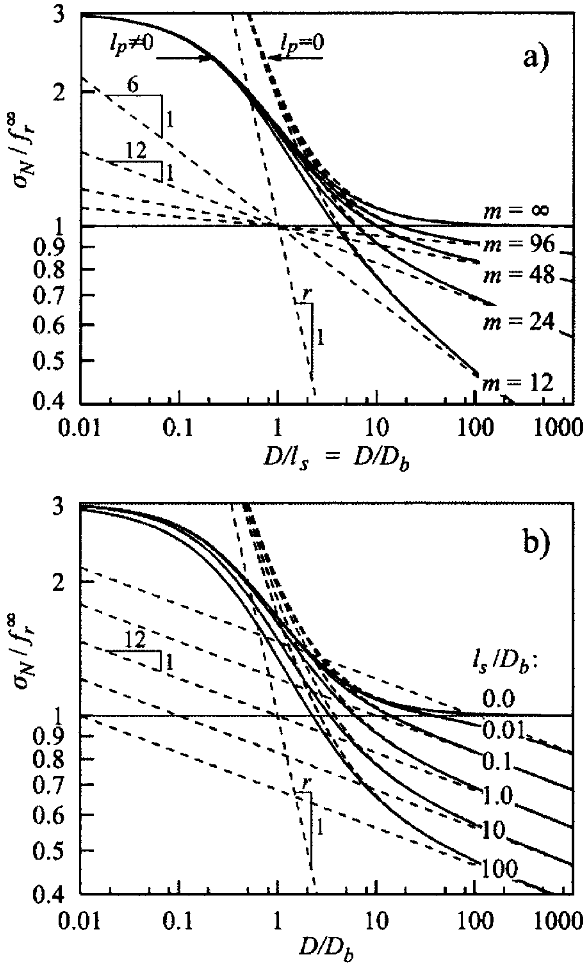

Fig. 3. Dependence of the size effect law on the variations in (a) Weibull modulus m; (b) scaling lengths l_s (scaled for bending in 2 dimensions, $n=2$)

Because the statistical and energetic physical causes of size effect are different and independent, l_s cannot be affected by changes D_b. This property, along with the fact that σ_N must be bounded when $D\to 0$, can be exploited to deduce the value of l_s. To this end, consider that D_b is reduced to 0. Then the energetic term becomes a constant, approaching the horizontal line $\sigma_N=f_r^{\infty}$ [see Fig. 1(c)] which must represent an upper bound on σ_N. The same bound must also limit the statistical part of size effect because it is based on the spatial density of failure probability, which is a continuum concept and thus cannot operate for sizes $D\ll l_s$. The equality of these two bounds is crucial. Another reason for the upper bound of statistical part of size effect is based on distribution of extremes (minima) of random fields representing local material strength (Vořechovský 2004; Vořechovský and Chudoba 2006).

Consequently, the statistical term must describe (for $D\to 0$) a smooth gradual transition between the Weibull asymptote given by Eq. (4) and the horizontal asymptote $\sigma_N=f_r^{\infty}$ (Fig. 3). The coordinate $D=l_s$ of the intersection of these two asymptotes, representing the center of the transition in the logarithmic scale, is obtained, according to Eq. (11), from the relation $\sigma_{N_{\mathrm{st}}}=f_r^{\infty}(l_s/D_{\mathrm{st}})^{n/m}$; hence

$$l_s=D_{\mathrm{st}}(\sigma_{N\mathrm{st}}/f_r^{\infty})^{m/n} \tag{12}$$

Note that $\kappa=1$ is necessary to ensure that the center of transition [Fig. 1(c)] between the small-size and large-size asymptotes for the case of no deterministic size effect ($D_b \to 0$) would lie at $D = l_s$ (and thus to ensure that the statistical size effect be bounded by f_r^∞ when $D \to 0$). This is what has already been assumed in Eq. (9). For the special case of $l_s = D_b$, a more fundamental derivation of Eq. (9) was given in Bažant (2004a), where it was also shown that, up to the second term of power series expansion in $1/D$, the probability-density function (PDF) remains Weibull.

How to Predict the Energetic-Probabilistic Size Effect without Any Stochastic Simulations

Bažant and Novák's (2000b) formula (10) for size effect prediction agrees quite well with computer simulations by nonlocal Weibull theory and can be used as a starting point. The stochastic finite-elements simulations, which are usually complex and tedious, can be avoided by calibrating the parameters of this formula, or better the improved formula (9). To do this, the real structure should be scaled down and up, using the following procedure.

- **Step 1**. Using, e.g., a standard elastic finite-element program, first compute the elastic stress field for a chosen structure size D_{st}, which may, but need not, be the actual structure size $D = D_a$. The computed stress field is then used for four purposes: (1) To obtain the large-size asymptotic deterministic nominal strength $\sigma_N = f_r^\infty$, by setting the maximum elastically calculated stress in the structure equal to the material strength, f_t' (the classical design procedure stopped here); (2) to evaluate from this field the local Weibull integral in Eq. (2), which yields constant C; (3) to determine the mean nominal strength σ_{st}, using the nonlocal Weibull probability integral (3); and (4) to calculate l_s from Eq. (12).
- **Step 2**. Next calculate σ_N for $D \to 0$, and η_p, using simple plastic limit analysis.
- **Step 3**. For the actual size D_a of the real structure, prepare then a deterministic finite-element model for nonlinear fracture analysis (paying attention to proper meshing and objectivity with respect to mesh size and orientation). If the crack path is known and is made to coincide with a mesh line, the cohesive crack model or crack band model is satisfactory (otherwise one must use mesh angle correction, or better a nonlocal damage model with a greatly refined mesh in the cracking zone). The computations, based on the mean material properties, yield the deterministic load-deflection curve. Its peak is σ_N, which gives point (D_a, σ_N) on the size effect plot (solid circle in Fig. 1).
- **Step 4**. Then, preserving geometry, scale the computational model down or up, or both, to obtain σ_{N_i} for a set of fictitious geometrically similar structures of sizes D_i $(i = 1, \ldots, N)$ [shown as circle points in Fig. 1(a)]. Since the deterministic part of the size effect formula in Eq. (9) has four parameters f_r^∞, D_b, r, and η_p, and since two parameters f_r^∞ and η_p, are already known, one needs, in theory, only two σ_{Ni} values to identify all the four parameters, one for the actual size and one for a scaled size. However, such two σ_{Ni} values would have to lie roughly at the thirds of the transitional range spanning the interval $\sigma_N \in (f_r^\infty, \eta_p f_r^\infty)$ in the $\log \sigma_N$ scale [Fig. 1(a)]. It is unlikely to have the luck of selecting D_i giving such values of σ_{Ni}. So, the deterministic nonlinear analysis program normally needs to be run for a greater number of sizes D_i (by experience, typically about six), until the σ_{Ni} values cover uniformly, in the scale of $\log D$ [Fig. 1(a)], the lower two thirds of the interval $(f_r^\infty, \eta_p f_r^\infty)$ (in the upper third, the fictitious scaled-down structure is usually smaller than the size of the material inhomogeneities, and thus of no practical relevance).
- **Step 5**. Next fit optimally the set of N pairs $(D_i, \sigma_{N_i}$, $i = 1, \ldots, N)$ [points in Fig. 1(a)] by the deterministic-energetic formula (6). Since this formula cannot be converted to linear regression (except for $r = 1$), nonlinear regression, e.g., the Levenberg-Marquardt optimization algorithm, is required. Since η_p is known from Step 2, one may substitute Eq. (7) for l_p into Eq. (6), which yields

$$\sigma_N = f_r^\infty \left(1 + \frac{(\eta_p^r - 1) r D_b}{(\eta_p^r - 1) D + r D_b}\right)^{1/r} \quad (i = 1, \ldots, N) \qquad (13)$$

  Since f_r^∞ is also known from Step 1, there are only two unknown parameters D_b and r to determine by the optimization algorithm. If there are only two data points (D_i, σ_{Ni}) (which must, of course, be properly located), the fit is exact. To obtain it easily, first eliminate f_r^∞ and then solve the resulting single nonlinear equation for r by Newton iterations. However, more than two points are appropriate [Fig. 1(a)], to allow minimizing the approximation errors by regression.
- **Step 6**. In Eq. (9), n/m is either known from the strength scatter or assumed from experience. So, there is only one unknown parameter l_s. It may simply be taken from the previously justified Eq. (12). Thus, any need for stochastic numerical simulation is circumvented [Fig. 1(b)] .
- **Step 7**. Once the energetic-statistical formula (9) has been calibrated, one can use it to evaluate σ_N for any D, particularly the real size. This prediction will generally be below the deterministic prediction [Fig. 1(c)]. The larger the structure, the greater the difference.
- **Step 8**. Once the mean combined size effect is predicted, the distribution of nominal strength for each size have to be determined. If Weibull distribution of σ_N is justified, the coefficient of variation ω_N of σ_N and failure probability P_f as a function of load can be determined from standard formulas in which the scaling parameter $s_0 = \sigma_N / \Gamma(1 + m^{-1})$ where σ_N is the mean nominal strength obtained by the foregoing procedure.

It must be warned, though, that the Weibull distribution is realistic only for large enough structures, such that, approximately, $N_{eq} \geqslant 10^4$ where N_{eq}=equivalent number of representative volume elements (RVE) of material in the structure (adjusted according to the stress field) =number of RVEs in a uniformly stressed specimen giving the same probability distribution. Here RVE must be defined not by homogenization (implying-low moment statistics) but by extreme value statistics; thus RVE is the smallest volume of material that causes the whole structure to fail; see Bažant and Pang (2005a,b, 2006), who further show that for smaller N_{eq}, the failure probability is given by

$$P_f(\sigma_N) = 1 - [1 - P_R(\sigma_N)]^{N_{eq}} \qquad (14)$$

where $P_R(\sigma_N)$=cumulative probability distribution of strength of one RVE, provided that the free constant c (in defining $\sigma_N = cP/bD$) is selected so that σ_N coincide with the maximum elastic principal stress in the structure; $P_R(\sigma_N)$ must be Gaussian except for a far out left tail, which must be of Weibull (or power-law) type up to the probability of roughly 0.001.

Alternatively, instead of Step 1(a), one may run the deterministic nonlocal nonlinear code to get the σ_N values for two very large structures of different sizes. If they converge, they are also equal to f_r^∞. If they do not, it could mean that the computational model implemented is not mesh objective. But it could also mean that the structure geometry is not positive (Bažant and

Planas 1998). For such geometry, a macrocrack of finite relative depth develops before the maximum load, causing a size effect of Type 2 (Bažant 2002).

The value of m and the meaning of n call for comments. The traditional thinking was that, for concrete, $m \approx 12$ (Zech and Wittmann 1977). However, Bažant and Novák (2000b) showed that if the energetic and statistical size effects are properly separated $m \approx 24$. As for n, note that, for mechanical reasons, a crack front must propagate simultaneously over the entire width of a beam (except for microscopic unevenness of the front). Despite the randomness of local material strength, one particularly weak point along the fracture front across the thickness cannot propagate forward (on the macroscale) while another point at which the strength is higher is stalled. So, the bending fracture propagation is two-dimensional although the structure is three-dimensional. Therefore, in bending fracture $n \approx 2$ even if the beam width is scaled in the third dimension (for this reason, the classical interpretation of Weibull size effect as a "volume" effect is misleading; often it actually is an "area" effect).

By the same argument, however, variation of beam width is likely to exert a different kind of size effect. The material elements on a fracture front developing across the beam are forced to deform simultaneously even though they have different random strengths. Such behavior is characterized by Daniels' fiber bundle model. This implies that an increase of beam width b, should have no effect on the mean σ_N but should cause a reduction of coefficient of variation of elemental strength in two-dimensional modeling of the beam. According to Daniels' model, this reduction would be proportional to $1/\sqrt{b}$ if the crack front remained perfectly straight (which is only approximately true). Furthermore, because of parallel coupling along the front (Bažant and Pang 2005a,b), the Weibull tail of the distribution of an area element in two-dimensional modeling should shrink and the Weibull tail should expand. Such behavior is doubtless the reason why no statistical size effect on mean σ_N has been reported for bending fracture of plates or shells in which the crack front is very long.

Numerical Example Partially Reinterpreting Malpasset Dam Catastrophe

To demonstrate applicability, the collapse of the Malpasset Dam in the French Maritime Alps has been analyzed. This arch dam of record-breaking slenderness and size was built in 1954 and failed at its first complete filling in 1959 (see Fig. 4(b) for illustration). The flood killed 412 people and wiped out the town of Fréjus (Bartle 1985; Levy and Salvadori 1992; Pattison 1998). The failure is believed to have started by vertical flexural cracks engendered by lateral displacement of abutment (due to slip of thin clay-filled seam in schist). This conclusion is not disputed, but appears incomplete. At the time of design, the energetic size effect was unknown and the Weibull statistical size effect was not yet established for concrete. To what extent might have the size effect contributed, weakening the resistance of dam to displacement?

From simplified analysis of a single cross section, Bažant and Novák (2000b) already inferred that the energetic-statistical size effect must have been a significant aggravating factor. To assess it more precisely, the dam is now analyzed using the proposed procedure.

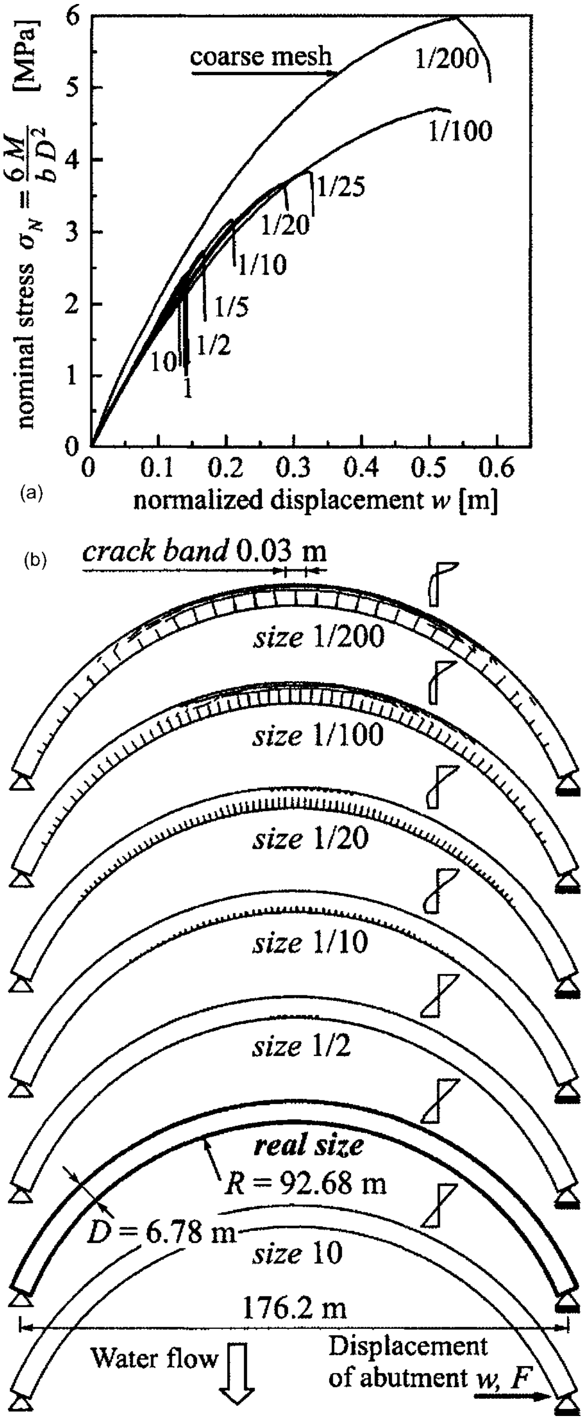

Fig. 4. Results of deterministic computations based on nonlinear fracture mechanics, for different scaled sizes of the dam; (a) nominal strength versus normalized displacement; (b) crack patterns at failure

Deterministic Nonlinear Fracture Analysis

Commercial finite-element code ATENA (Červenka and Pukl 2006) is used. It approximates cohesive fracture by smeared cracking, and employs the crack band model (Bažant and Oh 1983) to capture the size effect and ensure objectivity with respect to mesh refinements (this capability was demonstrated by Pukl et al. 1992; Červenka and Pukl 1994; and by Novák et al. 2001 for commercial code SBETA, a simpler predecessor similar to

ATENA). The dam is simulated only by a horizontal arch, discretized by two-dimensional finite elements in plane strain; the arch angle $2\alpha=133°$, inner radius $R=92.68$ m, and thickness at the base $D=6.78$ m. The damage constitutive law is the three-dimensional microplane model M4 (Caner and Bažant 2000), which has been incorporated into ATENA (Model M5 would have been more realistic, but it is not yet available in ATENA). The crack band model is combined with Model M4 using the so-called equivalent localization element (Červenka et al. 2005). The arch is supported by a sliding hinge at one abutment and fixed hinge at the other, and is loaded by prescribed displacement increments, assumed to be in the chord direction. The loading by water is disregarded, which corresponds to reality near the dam top and is on the safe side at lower elevations. The compressive strength and Young's modulus measured during construction were $f_c'=32.5$ MPa and $E=31.3$ GPa. From this, Model M4 generates (Caner and Bažant 2000) the default values of its four free parameters: $k_1=0.000119$, $k_2=500$, $k_3=15$, $k_4=150$. To integrate over spherical angles, 21 microplanes are used. The minimum crack band width is assumed to be 30 mm, which implies the initial fracture energy $G_f=55\ \mathrm{J/m^2}$. From computations, the corresponding $D_b\approx 0.28$ m. The crack band model allows changing the element size (with post-peak softening adjusted to ensure the same G_f). Nevertheless, to improve accuracy, the element sizes in the fracturing zone were kept the same for all sizes, except (inevitably) the two smallest.

In Step 4, the real dam size is scaled down by ratios 1/2, 1/5, 1/10, and 1/200, and up by ratios 10, 100, and 1,000 (size range 1:200,000). The resulting diagrams of the reaction versus abutment displacement are shown in Fig. 4(a), in which $\sigma_N=6M_{\max}/D^2$, with $M_{\max}$=maximum (midspan) bending moment in the arch at maximum load (per unit height of dam). For the real dam size ($D=6.78$ m), computations furnish $\sigma_N=2.35$ MPa. For sizes 10, 200, and $1{,}000\times$ larger, the computed σ_N is the same, 2.25 MPa. This agrees with the hand-calculated $f_r^\infty=2.25$ MPa. For the scaled-down sizes D, computations give increasing σ_N, and for the minimum ($D=33.9$ mm), the increase of σ_N is 2.9-fold. (Why is this less than the theoretical ratio $\eta_p=3$ for $D\to 0$?—because η_p implies a concentrated force at the face, which cannot be captured by elements of finite size, with a finite compressive strength.) Using $\eta_p=3$ gives $l_p=0.14$ m. Fig. 4 shows the computed midspan cross-wall distributions of normal stresses σ_x at peak load. For small sizes, compressive fracturing also occurs during pre-peak response.

Size Effect Obtained from Formulas Calibrated by Deterministic Computations

For the deterministic two-parameter formula (13), the finite-element results are best fitted when $D_b=280$ mm and $r=1$; for the extended formula (6) when $l_p=D_b/2=0.14$ m [Eq. (7)]; see Fig. 2. If the computed results are closely fitted by Eq. (9) with $l_p=0$ and if η_p ignored, the fitting gives a significantly higher r than the complete formula fitted to the results for all the sizes [see Fig. 2(c) where $r=3.1$]. Moreover, if the small-size computer results are left out, the portion of size effect curve to the left of inflexion point does not get sampled. Such a fit leads to $r=4.27$ [this approach, with $l_p=0$, was taken in the Bažant and Novák (2000a) study of normal-size flexural strength tests, giving $r=1.14$]. However, even if small-size data are unavailable, attaining good agreement with the cohesive crack model necessitates calculating and using ratio η_p, with the full deterministic-energetic formula (6).

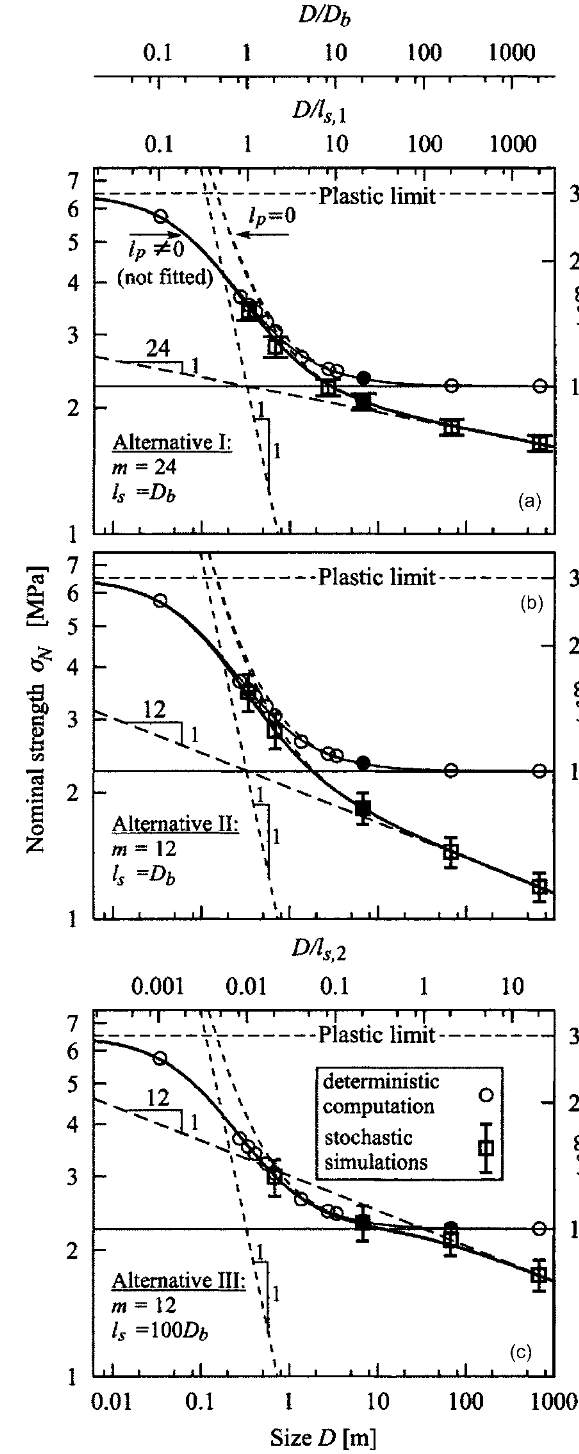

Fig. 5. Comparisons of three two-dimensional stochastic simulations (mean nominal strength ± standard deviation) and of deterministic results with predictions (not fits) by proposed size effect formula [Eq. (9), no fit]. (a) $m=48$, $l_s=D_b$ $(n/m=1/24)$, (b) $m=24$, $l_s=D_b$ $(n/m=1/12)$, (c) $m=24$, $l_s=D_b$ $(n/m=1/12)$; (b) is the most realistic estimate.

This point is documented in Fig. 2(c), showing how the optimum r depends on the size range of σ_N data. Fig. 2(c) shows examples of two erroneous data fits with the deterministic formula. For the combined energetic-statistical size effect Eq. (9), Fig. 5 shows the σ_N values for scaled dams and three different (m, l_s) pairs (corresponding to different l_s/D_b).

The tolerable foundation displacement of a dam can never be set as zero. The value considered for Malpasset Dam design is unknown. Nevertheless, the foregoing analysis with the energetic-statistical size effect formula (with $m=24$ and $l_s=D_b=0.28$ m) reveals that the mean tolerable displacement of Malpasset Dam abutment would today be about 50% of the value considered safe according to the design method that was standard in the early 1950s when this ill-fated dam was designed. If the size dependence of the understrength safety factor in quasibrittle structures (Bažant and Pang 2005b, 2006) were taken into account, the tolerable displacement would safely be reduced to much less than 50%, albeit to not less than 25% (of course, this leaves unaffected many other aspects of dam safety; Hartford and Baecher 2004).

Verification by Stochastic Finite-Element Simulations

It was conducted using ATENA with embedded probabilistic software FREET (Novák et al. 2003a, 2006), the effectiveness of which has been documented by reliability assessments of bridge structures (Pukl et al. 2002; 2003; Bergmeister et al. 2004), and by size effect studies of concrete specimens (Vořechovský and Matesová 2006). These tools provide information not only on the mean size effect but also the cumulative probability distributions. The differences of the mean σ_N from Eq. (9) are small and can be attributed to inevitable errors and the limited size of samples of stochastic variables. Note that the tail of the statistical distribution cannot be expected to be captured accurately, because Latin hypercube sampling, which cannot adequately sample the extreme values (Bažant et al. 2007), has been used.

The mean dam strength from the combined statistical-energetic size effect is found to represent the following percentages of the strength from the deterministic energetic size effect alone; i.e., ratios (Eq. (9)) f_r^{∞})=54, 60, 71, 77, 81, and 85% for $m=10$, 12, 18, 24, 30, and 40, respectively, provided that the all the strength distributions are Weibull. A realistic value for concrete is $m=24$, for which the statistical size effect reduces the strength of the dam to 77.8% of deterministic size effect. The energetic size effect from the size of typical modulus of rupture tests to size the dam size represents, according to Fig. 5, a strength reduction to about 64%, and the combined energetic and statistical size effects cause strength reduction to about $0.778\times0.64=50\%$ of the strength of laboratory specimens (bent beam of reasonable depth $D=0.27$ m). This is a significant effect indeed.

The stochastic simulations used the Latin hypercube sampling of all stochastic variables. In this efficient technique, the probability range (0, 1) of the mean strength of each of N random blocks, is divided into N layers of equal thickness, each of which is sampled once and only once in all of N_s stochastic simulations (hence, $N_s=N$). From all these values, one can estimate the mean and the coefficient of variation of σ_N for each size, and one can also plot the cumulative distribution for each size (see the scatter bands in Fig. 5).

A question might be raised regarding autocorrelation of the strength field, which surely exists and is here not considered. However, the Weibull theory needs no autocorrelation to be physically meaningful, and since the random blocks are in most cases larger than a conceivable autocorrelation length, this question is not relevant anyway. Nonlocality may invite another question. But the crack band model is a simpler equivalent of the nonlocal approach.

Size-Dependent "Random Block Method" for Efficient Stochastic Simulations

The foregoing stochastic simulations were facilitated by employing Bažant and Novák's (2003) method (elaborated on by Novák et al. 2003b) which uses a "stochastic mesh" consisting of so-called "random blocks" (originally called "macroelements") that generally differ from the finite elements and have a strength scaled depending on the block size. This method allows increasing the element size in proportion to structure size D, and so the number of elements in dam simulations can be kept constant (eight elements per wall thickness for any D). The properties of each random block (associated with one stochastic variable) are uniform but independent of the properties of other random blocks. If Model M4 is used, the mean (or deterministic) strength properties are characterized by parameter k_1.

For small structures, the random blocks may be considered identical to the finite elements, having roughly the size of l_s. But for large structures, this would lead to many thousands of stochastic variables, creating enormous computational burden. Although the random blocks must be small enough so that the stress would not vary greatly over each block, they can be scaled up with D because the stress field is scaled up, too. Each random block, of size l, is imagined to consist of n_e elemental material volumes (or RVEs), whose strengths are independent stochastic variables. The weakest-link model is assumed to be followed not only by the structure as a system of random blocks but also by each block as a system of RVEs—i.e., the failure of one RVE causes the whole random block, and thus the whole structure, to fail. So, the strength of each random block must follow Eq. (14). To prove it rigorously, recall that the structure is statistically equivalent to a chain of many RVEs. For the reference structure of size $D=D_0$, the stress in the ith RVE is $\gamma_i\sigma_N$, and the number of RVEs is N_0. When the structure is scaled up to size D, the ith RVE is scaled up to the ith random block, each consisting of $n_i=\rho^{n_d}$ RVEs having common stress $\gamma_i\sigma_N$; $\rho=D/D_0$=scaling ratio, assumed to be an integer, and the number of random blocks remains N_0 regardless of D. Since the structure (of positive geometry) survives if each random block survives, and each random block survives if each of its RVEs survives

$$1-P_f(\sigma_N)=\prod_{j=1}^{N}[1-P_R(\gamma_{i(j)}\sigma_N)]=\prod_{i=1}^{N_0}(1-P_i) \qquad (15)$$

$$1-P_i=[1-P_R(\gamma_i\sigma_N)]^{\rho^{n_d}}$$

where $\gamma_{i(j)}\sigma_N$ is the stress in the ith RVE within random block j, and $N=N_0\rho^{n_d}$. The last term indicates that scaling of the mean strength of each random block is sufficient and that it asymptotically approaches the Weibull scaling if the number ρ^{n_d} of RVEs in the random block is large ($\geqslant 1000$, Bažant and Pang 2006a,b).

In previous simulations (Bažant and Novák 2003; Novák et al. 2003b), the strength distribution of one RVE in Eq. (14) was assumed to be completely Weibull. This implied a Weibull size effect in the scaling of mean strength $\bar{\sigma}_b(l)$ of each random block from one RVE to the block size, i.e., $\bar{\sigma}_b(l)=f_t'(l/l_s)^{(n/m)}$, and also a constant value of the coefficient of variation. Such scaling, which is equivalent to using the stability postulate of random value statistics (Fréchet 1927; Fisher and Tippett 1928; Gumbel 1958; Ang and Tang 1984; Bouchaud and Potters 2000) is accurate for the random block strength only if the material inhomogeneities are negligible (or if $N_{\text{eq}}>10^4$, Bažant and Pang 2005a, b) and if spatial correlation can be disregarded (Vořechovský and

Novák 2004). This applies to fine-grained brittle ceramics except on the micrometer scale. For small concrete specimens, the existing experimental histograms are too limited to distinguish between the Gaussian and Weibull distributions because they differ significantly only in the far-out tail. Studies subsequent to the present computations, based on stress dependence of atomic activation energy (Bažant and Pang 2005a, b, 2006), showed that the distribution for one RVE must be Gaussian except for a far-out Weibull (or power-law) tail grafted on a Gaussian core at the failure probability of roughly 0.001 (if a significant part or the whole of strength distribution of an alleged RVE were Weibull, this alleged RVE could not really be a RVE since it would behave as a chain in which failure must localize into one link—the true RVE). The mean size effect based on Eq. (14) [Fig. (9a) and Eqs. (59)–(64) in Bažant and Pang 2005a, Fig. 3(b) in Bažant and Pang 2006] deviates to a large degree from the mean Weibull size effect only for $N_{eq} < 200$ while the real-size dam size with its stress field corresponds to $N_{eq} \approx 2000$ if the RVE size is taken as 5 cm.

Conclusions

1. The recently derived energetic-statistical formula for size effect on the strength of quasibrittle structures failing at crack initiation is extended and used to avoid stochastic nonlinear simulations of structural response.
2. The characteristic length for the statistical part of size effect can be deduced by considering the limiting case of energetic part of size effect for a vanishing thickness of the boundary layer of cracking. The elastically calculated stress field is used to obtain the large-size deterministic strength and to evaluate the Weibull integral for the failure probability for one chosen structure size, which gives one point on the curve of the statistical part of size effect. Deterministic plastic limit analysis of an elastic body with a through-crack imagined to be filled by a perfectly plastic "glue" yields the small size asymptote of size effect. Deterministic nonlinear simulations of several scaled structures, based for example on the crack band model, then suffice for calibrating the deterministic part of the size effect.
3. The present approach allows easy calibration of the mean energetic-statistical size effect.
4. Stochastic two-dimensional nonlinear simulation of the failure of Malpasset Dam demonstrates good agreement with the proposed procedure, and documents the necessity of considering the size effect in the design of arch dams and other structures failing at crack initiation. Although the abutment displacement considered tolerable in design is not known, one may conclude that, upon taking into account the mean combined energetic-statistical size effect, the mean tolerable displacement of the abutment of this ill-fated dam must have been only about 50% of the displacement considered tolerable at the time of design, when no type of size effect in concrete was known.

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