

Which Formulation Allows Using a Constant Shear Modulus for Small-Strain Buckling of Soft-Core Sandwich Structures?

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Although the stability theories energetically associated with different finite strain measures are mutually equivalent if the tangential moduli are properly transformed as a function of stress, only one theory can allow the use of a constant shear modulus G if the strains are small and the material deforms in the linear elastic range. Recently it was shown that, in the case of heterogeneous orthotropic structures very soft in shear, the choice of theory to use is related to the problem of proper homogenization and depends on the type of structure. An example is the difference between Engesser's and Haringx's formulas for critical load of columns with shear, which were shown to be energetically associated with Green's and Almansi's Lagrangian finite strain tensors. In a previous brief paper of the authors in a conference special issue, it was concluded on the basis of energy arguments that, for constant G , Engesser's formula is correct for sandwich columns and Haringx's formula for elastomeric bearings, but no supporting experimental results were presented. To present them, is the main purpose of this technical brief. [DOI: 10.1115/1.1979516]

Introduction

The question of a correct formula for critical load, P_{cr} , of elastic structures such as sandwich columns, composite columns, lattice columns, helical springs, and elastomeric bearings, in which shear deformations dominate, has been subject of polemics for decades; see [[1], and references therein]. The best known examples are the formulas of Engesser and Haringx. The polemics were settled in 1971 [2] by the demonstration that these two formulas are equivalent if the shear modulus G is properly transformed as a function of the axial stress; see also [3]. This showed that, if the material behavior is nonlinear and the stress-strain curve is defined in terms of the finite strain tensor as that used to interpret test results, it does not matter which formula is chosen because the difference is only in the form of stress dependence of tangent shear modulus G . This conclusion, however, leads to apparently paradoxical properties when the strains are so small that the material is in the linear range of response.

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In [3], it was shown that such a situation may arise for built-up (lattice) columns with very weak shear bracing, and that only the Engesser formula is correct for such columns. In [1], it was shown that such a situation also arises for sandwich columns very soft in shear.

For such columns, the equivalence of Engesser's and Haringx's formulas requires that the tangent shear modulus G cannot be constant in both of these two formulas even if the strains are so small that the core is in the linear range of response.

Because all of the axial load in a sandwich is carried by the skins and all of the shear force is resisted by the core, the stress dependence of G of the core means that the properties of the core depend on the axial normal stress in the skins (or facings). This may seem to be paradoxical. But the variational energy analysis in [1] showed that this is simply a consequence of the fact that the cross sections of the core may be assumed to remain plane.

In [4], variational energy analysis of the same kind as in [1] was used to prove that, if all the strains at critical load, P_{cr} , are small, and if a constant G (as measured in small-strain torsion) is to be used, then, for sandwich beams, only Engesser's formula is correct, and, for elastomeric bearings, only Haringx's formula is correct. However, experimental results supporting this conclusion have not been presented. To present them is the main objective of this Technical Note. A secondary objective is to briefly outline an extension of this analysis to arbitrary soft-in-shear structures under multiaxial stress, such as layered or highly orthotropic bodies. In full detail, this extension is found in [5].

Background on Variational Analysis of Homogeneous Orthotropic Columns With Shear

The variational energy analysis presented in [2] showed that the differences between various stability theories for buckling with shear arise from different choices of the finite strain measure. All the finite strain tensors used in stability theories so far belong to the class of Doyle-Ericksen finite strain tensors $\epsilon^{(m)} = (U^m - I)/m$ where m =real parameter, I =unit tensor, and U =right-stretch tensor. In particular, $m=2$ gives Green's Lagrangian strain tensor, and $m=-2$ Almansi's Lagrangian strain tensor. For calculating the critical load, only the second-order approximation of these tensors matters; it reads [2]:

$$\epsilon_{ij}^{(m)} = e_{ij} + \frac{1}{2}u_{k,i}u_{k,j} - \alpha e_{ki}e_{kj}, \quad e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad \alpha = 1 - \frac{1}{2}m \quad (1)$$

where e_{ij} =small-strain tensor (linearized), u_i =displacement components, and the subscripts refer to Cartesian coordinates x_i ($i = 1, 2, 3$). The finite strain tensors $\epsilon^{(n)} = (U^n - U^{-n})/2n$ proposed in [6] do not fit the class of Doyle-Ericksen tensors but the second-order approximation of $\epsilon^{(n)}$ for any n coincides with (1) for $m \rightarrow 0$ (Hencky strain tensor). The stability criteria obtained from any of these strain measures have been shown in [2] to be mutually equivalent if the tangential moduli $C_{ijkl}^{(m)}$ associated with different m -values satisfy the relation:

$$C_{ijkl}^{(m)} = C_{ijkl}^{(2)} + \frac{2-m}{4}(S_{ik}\delta_{jl} + S_{jk}\delta_{il} + S_{il}\delta_{jk} + S_{jl}\delta_{ik}) \quad (2)$$

(see also [3]) where $C_{ijkl}^{(2)}$ =components of tangential moduli tensor $C^{(m)}$ associated with Green's Lagrangian strain ($m=2$) and S_{ij} =components of current stress tensor S (Cauchy stress).

According to the standard simplifying hypothesis that the cross sections of the core remain plane, which makes the problem one-dimensional, the second-order accurate expression for the incremental potential energy of a column that is initially in equilibrium in an undeflected position is

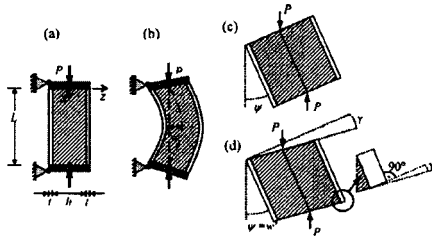


Fig. 1 Column in (a) initial state, (b) deflected state; contribution to the shear deformation of a sandwich beam element: (c) bending, and (d) shear

$$\delta^2 \mathcal{W} = \int_0^L \int_A \left[S^0(y, z) (\epsilon_{11}^{(m)} - e_{11}) + \frac{1}{2} E^{(m)}(y, z) e_{11}^2 + \frac{1}{2} G^{(m)}(y, z) \gamma^2 \right] dA dx \quad (3)$$

where A = cross-section area; $S^0(y, z) = -P/A$ = initial axial normal stress; P = axial compressive load; $E^{(m)}$, $G^{(m)}$ = tangent elastic moduli in the axial direction and shear, associated with finite strain tensor $\epsilon^{(m)}$; $\gamma = u_{1,3} + u_{3,1}$ = shear angle; $\psi(x)$ = rotation angle of cross section; $x_1 = x$ = axial coordinate, $x_3 = z$ = transverse coordinate in the plane of buckling, $u_1 = u(x)$ = axial displacement (in x -direction), and $u_3 = w(x)$ = transverse deflection (see Figs. 1(a) and 1(b)). Applying the Trefftz condition of critical state to Eq. (3), one can derive the critical load formulas associated with finite strain tensor of any parameter m [1]. In particular for $m=2$ and $m=-2$, one obtains the well-known formulas:

$$\text{for } m=2: \quad P_{cr} = \frac{P_E}{1 + (P_E/G^{(2)}A)} \quad (\text{Engesser}) \quad (4)$$

$$\text{for } m=-2: \quad P_{cr} = \frac{G^{(-2)}A}{2} \left[\sqrt{1 + \frac{4P_E}{G^{(-2)}A}} - 1 \right] \quad (\text{Haringx}) \quad (5)$$

where $P_E = \pi^2 EI/l^2$ = Euler's critical load of the column. It can be checked that (5) is obtained from (4) if the following substitution (a special case of (2) [2,3]) is made:

$$G^{(2)} = G^{(-2)} + P/A \quad (6)$$

Correct Critical Load Formula for Sandwich Columns and Experimental Evidence

If the tangent modulus G in the presence of normal stress happens to be constant with respect to Green's Lagrangian strain measure (corresponding to Engesser's theory, $m=2$), it cannot be constant with respect to the Almansi's Lagrangian strain measure (corresponding to Haringx's theory, $m=-2$), and vice versa. Thus the main question is how to use the constant shear modulus G that is measured by small-strain shear tests, for example by torsion of a hollow circular tube. Should this constant modulus be used in Engesser's formula, or in Haringx's formula, or in a formula for some other m -value? The correct answer for a sandwich is Engesser's formula. This answer was not reached in [1] because the physical interpretation of the variational analysis for a sandwich was incomplete. The correct interpretation, leading to the correct conclusion, was presented in [4] (and is also reviewed in [5]). It may be summarized as follows.

We consider a sandwich beam element to undergo a rigid-body rotation through a small angle w' followed by homogeneous pure shear deformation γ , as shown in Figs. 1(c) and 1(d). Using the notation shown in Figs. 1(a) and 1(b), we can describe the displacement and deformation fields as follows:

$$u_1 = u_{1,1} = u_{1,3} = e_{11} = 0, \quad u_{3,1} = \gamma, \quad e_{13} = e_{31} = \gamma/2 \quad (7)$$

According to the assumption of negligible skin thickness, the shear deformation within each skin vanishes (as shown in the zoomed region in Fig. 1(d)), although the skins as a pair, of course, do exhibit shear deformation (Fig. 1(d)). First we calculate the work as a special case of Eq. (3) for a sandwich column. After substitution of the homogeneous strain field (7) into (3), the flexural terms vanish and one obtains (for a cross section of width b in the y direction):

$$\delta^2 \mathcal{W} = \left(bhG + \frac{2-m}{4} P \right) \frac{\gamma^2}{2} - \frac{Pw'^2}{2} \quad (8)$$

Superscript m is here omitted because the core is in small strain, in which case the shear modulus G is independent of the specific choice of strain measure.

Second, we figure out the work on the sandwich beam element by a direct elementary reasoning. During the rotation of the beam element (Fig. 1(c)), the second-order approximation to the vertical shortening is $w'^2/2$, and the work of the axial force due to this shortening is $-Pw'^2/2$ (note that the work of loads must be taken as negative because the potential energy of internal stresses is taken as positive). During the subsequent shear deformation of the core (Fig. 1(d)), the work of shear stresses obviously is $bhG\gamma^2/2$. So the total work is

$$\delta^2 \mathcal{W} = bhG \frac{\gamma^2}{2} - \frac{Pw'^2}{2} \quad (9)$$

Expressions (8) and (9) must coincide. Their comparison immediately reveals that this happens if and only if $m=2$. So, this is the value of m that allows the use of a constant shear modulus in the small-strain analysis of a sandwich. For this, and only this, value of m , the variational result in (8) coincides with the second-order work obtained by the elementary reasoning based on (9). Therefore, it must be concluded that whenever the strains are small and the shear modulus is assumed to be constant, sandwich structures must be analyzed on the basis of Engesser-type theory.

This result does not come as a surprise. Already in 1966, Plantema [7] justified Engesser's theory on the basis of several experimental data. The Engesser-type theory has also been adopted by Zenkert [8].

The most extensive support for Engesser's theory is provided by the recent tests of Fleck and Sridhar [9] shown in Figs. 2(a)–2(c). These tests also validate the correctness of the energetic argument in [5] summarized here. In these tests, the sandwich columns were fixed at both ends. The skins were made of the same material and had the same width. The cores had the same in-plane thickness and were made of Divinycell PVC foam. Three different densities of the foam, with designations H30, H100, and H200, and several column lengths L , giving different slenderness $l/(h+2t)$, were used. The diagrams in Fig. 2 are plotted in logarithmic scales and dimensionless coordinates, as P_{cr}/P_E versus $l/(h+2t)$. The agreement with Engesser's formula is good (except for one anomalous point), while the formula of Haringx is seen to deviate significantly from the test results if a constant G is used.

Correct Critical Load Formula for Homogenized Elastomeric Bearings and General Layered or Orthotropic Structures

For elastomeric bearings and helical springs, discussed in detail in [5], a large number of studies and all experiments clearly favor Haringx-type theory. The mathematical argument is analogous to that pursued for sandwich columns. If one calculates the work as a special case of Eq. (3) for a pure shear deformation and compares it to the expression obtained by direct elementary reasoning (based on the sketch in Fig. 3), one concludes that the only m value for which the two expressions are the same is $m=-2$, which corresponds to Haringx's formula [5].

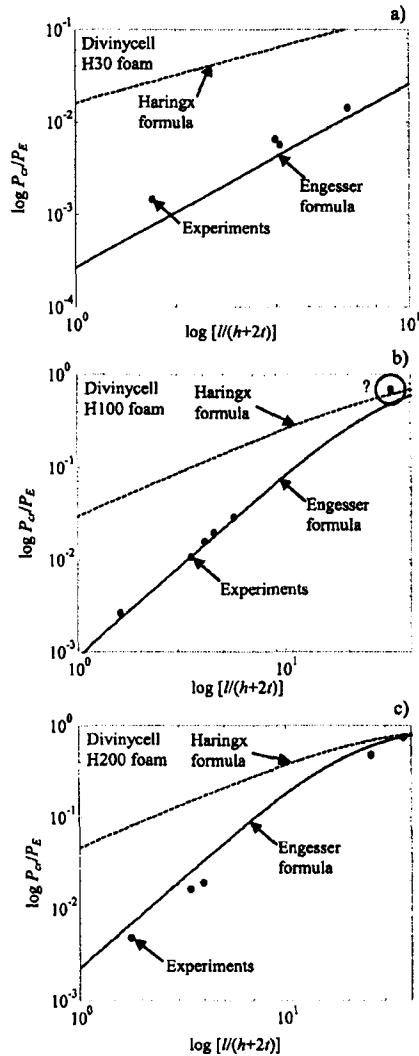


Fig. 2 Comparison of buckling formulas and experimental results for various density of Divinycell foam

The same type of argument may be applied to general homogenized soft-in-shear structures, e.g., layered bodies and unidirectional fiber reinforced composites. The Engesser-type or Haringx-type theory is found to apply if the compressive force is applied in the direction parallel or normal to the layers or fibers, respectively.

For the general case of a soft-in-shear orthotropic structure under biaxial stress, it is found [5] that a theory corresponding to a general m -value must be used. It has been shown [5] that m is given by the following equation:

$$m = \frac{2(S_3^0/S_1^0) - 2}{(S_3^0/S_1^0) + 1} \quad (10)$$

where S_1^0 and S_3^0 are the initial normal stresses in the directions of orthotropy.

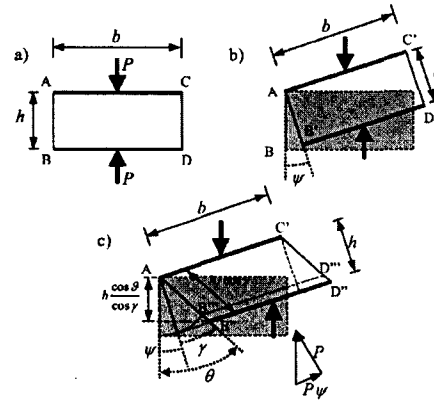


Fig. 3 Contribution to the shear deformation of an elastomeric bearing element: (a) initial state, (b) bending, and (c) shear

To permit applying finite element discretization to homogenized soft-in-shear structures of this kind, it would be useful to generalize the current commercial finite element programs for an arbitrary value of $m \neq 2$ [4,5]. At present, all of them use an updated Lagrangian algorithm energetically associated with Green's Lagrangian finite strain tensor, i.e., with $m=2$, and thus, in general, cannot give correct results for homogenized soft-in-shear structures [5].

Conclusion

The recent conclusion that for soft-in-shear sandwich columns in small-strain and in the linear elastic range the critical load is predicted correctly by Engesser's (but not Haringx's) formula with a constant shear modulus of core is shown to be in good agreement with recent extensive test results.

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References

- [1] Bažant, Z. P., 2003, "Shear Buckling of Sandwich, Fiber-Composite and Lattice Columns, Bearings and Helical Springs: Paradox Resolved," *ASME J. Appl. Mech.*, **70**, pp. 75–83.
- [2] Bažant, Z. P., 1971, "A Correlation Study of Incremental Deformations and Stability of Continuous Bodies," *ASME J. Appl. Mech.*, **38**, pp. 919–928.
- [3] Bažant, Z. P., and Cedolin, L., 1991, *Stability of Structures: Elastic, Inelastic, Fracture and Damage Theories*, Oxford University Press, New York; and republication with updates, Dover, New York, 2003.
- [4] Bažant, Z. P., and Beghini, A., 2004, "Sandwich Buckling Formulas and Applicability of Standard Computational Algorithm for Finite Strain," *Composites, Part B*, **35**, pp. 573–581.
- [5] Bažant, Z. P., and Beghini, A., 2004, "Stability and Finite Strain of Homogenized Structures Soft in Shear: Sandwich or Fiber Composites, and Layered Bodies," *Theoretical and Applied Mechanics Report No. 04-03/C441s*, Northwestern University, Evanston, IL; also *Int. J. Solids Struct.*, (in press).
- [6] Bažant, Z. P., 1998, "Easy-to-Compute Tensors with Symmetric Inverse Approximating Hencky Finite Strain and its Rate," *ASME J. Eng. Mater. Technol.*, **120**, pp. 131–136.
- [7] Plantema, F. J., 1966, *Sandwich Construction: The Bending and Buckling of Sandwich Beams, Plates and Shells*, Wiley, New York, p. 9.
- [8] Zenkert, D., 1997, *The Handbook of Sandwich Construction*, EMAS, UK.
- [9] Fleck, N. A., and Sridhar, I., 2002, "End Compression of Sandwich Column," *Composites, Part A*, **33**, pp. 353–359.