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Size Effect on Flexural Strength of Fiber-Composite Laminates

The size effect on the flexural strength (or modulus of rupture) of fiber-polymer laminate beams failing at fracture initiation is analyzed. A generalized energetic-statistical size effect law recently developed on the basis of a probabilistic nonlocal theory is introduced. This law represents asymptotic matching of three limits: (1) the power-law size effect of the classical Weibull theory, approached for infinite structure size; (2) the deterministic-energetic size effect law based on the deterministic nonlocal theory, approached for vanishing structure size; and (3) approach to the same law at any structure size when the Weibull modulus tends to infinity. The limited test data that exist are used to verify this formula and examine the closeness of fit. The results show that the new energetic-statistical size effect theory can match the existing flexural strength data better than the classical statistical Weibull theory, and that the optimum size effect fits with Weibull theory are incompatible with a realistic coefficient of variation of scatter in strength tests of various types of laminates. As for the energetic-statistical theory, its support remains entirely theoretical because the existing test data do not reveal any improvement of fit over its special case, the purely energetic theory—probably because the size range of the data is not broad enough or the scatter is too high, or both. [DOI: 10.1115/1.1631031]

Introduction

A quintessential property of the classical theories of solid mechanics, particularly plasticity and elasticity with a strength limit, is the absence of size effect. In other words, the nominal strength σ_N of a structure (which is a load parameter defined as P/bD where $b=$structure width) is independent of the characteristic size (dimension) D of the structure. During the 1980s, however, this property has been shown to be invalid when, instead of plastic yielding, the material exhibits softening damage such as distributed cracking [1–6]. In that case, a strong (non-statistical) size effect may be caused by stress redistribution causing energy release from an elastically unloaded zone of material at the flanks of a propagating band of softening damage or large cohesive fracture, taking place before the maximum load is reached. When this deterministic size effect occurs, it normally prevails over the size effect due to the randomness of strength, described by the classical Weibull theory.

However, the existing textbooks on composites and sandwich structures, as well as the current design practice, ignore the deterministic size effect due to the energy release. The recent researches at Northwestern University have shown that this can be dangerous when large structures are designed.

The recent studies have been focused on fiber-polymer laminates which are used for the skins of sandwich structures. The size effect in sandwich shells must be well understood if the Navy's current plans to build very large ship hulls, decks, bulkheads, masts and antenna covers entirely of composites should succeed. Experiments show that a laminate typically fails only after a large damage (or fracturing) zone has developed. This causes a very strong deterministic size effect. The first demonstration was given for the case of tensile (mode I) fracture of notched specimens [7]. Subsequently, it was shown that the same is true for the compression failure of fiber composites caused by the propagation of a kink band, in which the fibers undergo microbuckling [8]. For both cases, the classical theories, based on plasticity or strength criteria, exhibit no size effect. This is acceptable only for very small structures. For the design of sandwich structures contemplated for the construction of large ships, it is imperative that the quasibrittle cohesive fracture and scaling properties be taken into account. Only then it will be possible to extrapolate to such large sizes on the basis of the values of material strength deduced from laboratory tests of relatively small specimens.

The previously reported experimental investigations of size effect in unnotched laminates, however, were not evaluated taking into account the possibility of a deterministic size effect. Only the purely statistical classical Weibull theory was considered in evaluating the test results. To capture the effect of stress redistribution due to cracking and the associated energy release, this paper (which expands on a previous conference presentation [9]) uses a newly developed energetic-statistical size effect formula [10,11] to fit the existing test data on the flexural strength (or modulus of rupture). Since the size effect of the purely statistical theory is a special case of this formula, the optimum fitting of the test data would have to converge to that special case if this classical theory were applicable and provided a unique explanation. However, as will be seen, this does not occur. In fact, the formula parameters for the optimum fits are very different from the values that correspond to the special case of a purely statistical Weibull theory.

There are two basic simple types of the deterministic energetic size effect in quasibrittle materials, which obey different laws [1,2,4–6]: (1) The size effect in structures with notches, or with large stress-free (or fatigued) cracks that have formed before the maximum load [12]; and (2) the size effect due stress redistribution engendered by a boundary layer of cracking in structures that fail at the initiation of fracture from a smooth surface. Here we are concerned only with the latter, which is important, e.g., for safe design of very large and thick composite structures.

Important theoretical advances in simulating the effect of stress redistribution have been achieved by introducing various load-sharing concepts into the statistical weakest-link model [e.g. [13,14]] (Appendix I). These new advances, however, do not yet appear ready for practical applications to flexure of laminates.

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Available Experimental Evidence for Size Effect on Flexural Strength

The size effect on the flexural strength of fiber-polymer laminates, which is the focus of this paper, has been the subject of a host of studies during the last decade [15–19]. Wisnom [17] conducted four-point bending tests and pin-ended buckling tests on unidirectional XAS/913 carbon fiber-epoxy specimens with 25, 50, and 100 plies. Both types of tests showed a significant decrease in strength with increasing specimen size, scaled in three dimensions (three-dimensional). However, not all the specimens failed in tension; the small ones tended to do so, but the large ones tended to fail in compression. Among 6 100-ply specimens in the four-point bending tests, only one failed in tension, doing so in a brush mode. However among the 25-ply and 50-ply specimens, the tensile failures numbered 7 out of 10, and 9 out of 11, respectively. Fitting the data by the pure Weibull theory indicated Weibull modulus (shape parameter) $m=25.4$.

The results of Wisnom and Atkinson [18] obtained with four-point bending of three-dimensional-scaled unidirectional E glass/913 specimens of 16, 32 and 64 plies also showed a clear size effect. The optimum value of Weibull modulus was $m=24.2$.

Jackson [15] investigated the effects of specimen size on the flexural response and strength by performing tests on unidirectional, angle-ply, cross-ply and quasi-isotropic ply-level AS4/3502 carbon fiber-epoxy beams in a hinged axial loading fixture, with all the dimensions of the specimen and the rig properly scaled. Apparent size effects were found for all the stacking sequences. Weibull modulus values of 25.5, 21.4, 11.2, and 16.8 were reported for unidirectional, angle-ply, cross-ply and quasi-isotropic laminate beams, respectively.

Johnson et al. [16] studied the size effect in the four-point flexural response using ply-level and sublaminate-level angle-ply and quasi-isotropic scaled AS4/3502 graphite-epoxy laminate beams. They found that the flexural strength of ply-level scaled laminates decreased with the specimen size significantly, but the sublaminate-level scaled specimens did not show a pronounced size effect. The Weibull modulus values for the angle-ply and quasi-isotropic specimens were reported as 50.0 and 26.7, respectively.

Other studies considered the ratio of the nominal strength values measured in three-point and four-point loaded flexural tests, or in flexural tests and tensile coupon tests. Bullock [20] concluded in 1974 that, as far as the inevitable scatter permits it to be judged, this ratio compared favorably with the Weibull theory (the only theory for deviations from classical non-statistical mechanics of materials available at that time). However, comparisons to Weibull modulus values corresponding to the scatter, and to the size effect, which would represent much more severe tests of the theory, have not been made.

All the previous studies of flexural failure of laminates assumed a priori that the size effect is purely statistical, as described by Weibull's statistical theory of random local material strength [21,22]. The validity of this classical theory for fine-grained ceramics and fatigue-embrittled metals is beyond doubt, however, this is not the case for fiber-reinforced laminates, except for structure sizes far exceeding the practical range.

Energetic Size Effect on Flexural Strength of Laminates

It has been well established [10,11] that the Weibull theory is valid only (1) if the structure has a positive geometry (i.e., the energy release rate is increasing, rather than decreasing, with the crack extension), and (2) if the failure of one macroscopic small material element causes the whole structure to fail, as described by the weakest link model [23–25] and Weibull distribution [23]. The first condition excludes certain structural geometries, independently of the type or material. The second condition is generally not satisfied for heterogeneous quasibrittle materials, fiber-reinforced composites included, except in the infinite size limit [6]. For such materials, the energy release caused by the stress redistribution due to the growth of fracture is a potent source of size effect which overwhelms the statistical size effect on the mean strength.

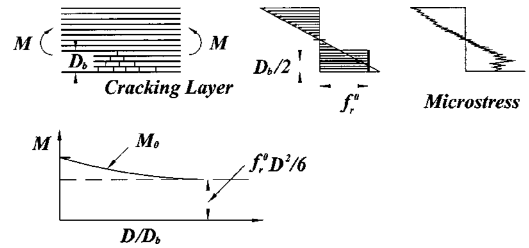

Fig. 1 Stress redistribution in flexure caused by a boundary layer of cracking

For the sake of simplicity, we consider the laminate cross section to be homogeneous, in which case the elastic bending stress diagram is linear (Fig. 1). As a crucial point of our analysis, we note that the peak bending moment M_0 is not reached when the elastically calculated bending stress at tensile face reaches the material strength f_t. Rather, before reaching the peak moment, a boundary layer of a certain finite thickness $2D_b$ that is a property of the fiber composite develops at the tensile face, causing stress redistribution and energy release.

The simplest way to take the boundary layer into consideration (a way that has been shown accurate up to the first two terms of the power series expansion in terms of $1/D$ [2,6,26]) is to consider that f_r^0 is approximately decided by the average elastically calculated stress within the boundary layer of thickness $2D_b$. Consequently, if the laminate is assumed to fail in tension rather than in compression, the bending stress formula gives $f_r^0=M_0(D-D_b)/2I$ where $D=$beam depth, $M_0=$bending moment, and $I=D^3/12$; f_r^0 represents the average tensile strength of the boundary layer, which is considered to be constant. The flexural strength (or modulus of rupture) of a laminate, f_r, chosen to represent the nominal strength σ_N, is defined as the elastically calculated maximum stress in the beam, $f_r=\sigma_N=M_0D/2I$. Therefore,

$$\sigma_N=f_r^0\left(1-\frac{D_b}{D}\right)^{-1} \tag{1}$$

This formula gives a negative σ_N for small enough D, and so it is usable only for large enough sizes $D\gg D_b$. This limitation is not surprising because our derivation has been correct only up to the first two terms of the asymptotic series expansion in terms of the powers of $1/D$ [27]. Therefore, any other formula that exhibits the same first two asymptotic terms in $1/D$ is equally justified as (1).

This observation suggests the use of asymptotic matching. This is a technique to obtain an approximate solution for hard problems in which two opposite asymptotic behaviors are much easier to solve than the intermediate behavior. This technique, borrowed from fluid mechanics [28–31], can be regarded as an interpolation between the opposite asymptotic solutions, corresponding here to $D\to 0$ and $D\to\infty$. In this spirit, formula 1 needs to be modified such that the first two terms of the large-size asymptotic expansion remain unaffected while a realistic small-size asymptotic behavior is matched.

Through a power series expansion in $1/D$, one may check that by making the replacement $(1-D_b/D)^{-1}\approx(1+rD_b/D)^{1/r}$ (r being any positive constant signifying a transition slowness parameter), the first two terms of the asymptotic expansion in terms of $1/D$ are not affected while, at the same time, σ_N, becomes positive, finite and monotonically decreasing through the entire range of D. With this replacement, (1) leads to the size effect formula [2,26]:

$$\sigma_N=f_r=f_r^0 q(D), \quad q(D)=\left(1+\frac{rD_b}{D}\right)^{1/r} \tag{2}$$

where $q(D)$ is a positive dimensionless decreasing function of size D having a finite limit for $D\to\infty$.

A more realistic starting hypothesis is to consider that, up to the maximum load, the cracking remains distributed (the discrete crack being formed only at, or after, the maximum load), and that the cracking is described by some effective stress-strain diagram characteristic of the fracture process zone (the size of which, related to D_b, is a material constant, the characteristic length). For the sake of simplicity, we consider a bilinear stress-strain diagram with postpeak strain-softening characterized by some tangent modulus E_t. The distributed cracking at maximum load is assumed to occupy a boundary layer at the tensile face of laminate, having a certain fixed thickness denoted by l_f. The corresponding stress distribution is sketched in Fig. 1. The result of such a calculation [6,27] is a formula that coincides with $\sigma_N=f_r^0(1-D_b/D)^{1/r}$ up to the second term of the asymptotic expansion of σ_N as a power series in $1/D$, provided that one sets $l_f=D_b/2$, differences being found only in the third and higher order terms.

A more general and more fundamental approach, pursued in [10,11,27], is an asymptotic analysis based on expanding the energy release function $g(\alpha)$ of fracture mechanics (α=relative crack length). As far as the first two terms of the asymptotic expansion of the size effect are concerned, the result of such analysis happens to be again the same, except that a slight effect of specimen geometry is brought about through function $g(\alpha)$.

As a further improvement, (2) may be modified as follows:

$$\sigma_N=f_r^0 q(D), \quad q(D)=\left(1+\frac{rD_b}{D+rsD_b}\right)^{1/r} \tag{3}$$

where s is a non-negative constant [2,26]. This modification achieves that, unlike (2), the size effect formula gives a finite strength for $D\to\infty$ while remaining, for large sizes, asymptotically equivalent to the original formula $\sigma_N=f_r^0(1-D_b/D)^{-1}$ up to the second term of power series expansion in $1/D$. One can verify it by the following approximations, which are accurate up to the second term of the asymptotic power series in terms of ξ, with $\xi=D_b/D$;

$$\frac{f_r^0}{\sigma_N}=\left(\frac{1+rs\xi}{1+r(s+1)\xi}\right)^{1/r}\approx\frac{1+s\xi}{1+(s+1)\xi}\approx(1+s\xi)[1-(s+1)\xi]$$
$$\approx 1-\frac{D_b}{D} \tag{4}$$

Combined Energetic-Statistical Size Effect on Flexural Strength of Laminates

Because the local strength of material elements is random and the minimum of random strength encountered in the structure decreases with the structure size, the Weibull statistical size effect must also play some role. A general and fundamental approach to the combined energetic-statistical size effect is the nonlocal Weibull theory [2,10,11,32–34]—a theory in which the material failure probability at a given point of the body depends not on the stress at that point (as in the classical Weibull theory) but on the weighted average of strain within a certain characteristic volume of material surrounding the point. This theory has been used in numerical studies of flexural failure [10,11,34], which confirmed that the size effect for very thin plates is almost totally energetic (deterministic) and for very thick plates almost totally statistical, of the classical Weibull type. These two asymptotic conditions may be satisfied by the following simple size effect formula:

$$\sigma_N=f_r^0\left[\left(\frac{rD_b}{D+rsD_b}\right)^{rn_d/m}+\frac{rD_b}{D+rsD_b}\right]^{1/r} \tag{5}$$

where n_d=number of spatial dimensions in which the structure is scaled ($n_d=1$, 2 or 3), m=material constant=Weibull modulus. For lack of test data of broad enough size range, the value of s cannot be determined experimentally and, therefore, we assume $s=0$. Note that for $m\to\infty$ (and $s=0$) Eq. (5) becomes $\sigma_N=f_r^0(1+rD_b/D)^{1/r}$, which coincides with the foregoing energetic (deterministic) formula 2; For $D\gg D_b$, or $D_b=0$ (and $s=0$), Eq. (5) becomes

$$\sigma_N=f_r^0(D_b/D)^{n_d/m} \tag{6}$$

and thus one recovers the classical formula for Weibull size effect as the limit case (experience with concrete, however, showed that the statistical part of the size effect on flexural strength is not significant except for extremely thick structures such as arch dams).

A detailed derivation of the deterministic and energetic-statistical size effect formulas can be found in [2,10,11].

Reinterpretation of Previous Experimental Studies of Size Effect in Laminates

To check and calibrate the energetic-statistical theory of flexural strength of laminates, a systematic study of the numerous test data that exist in the literature [15–18] has been initiated. The specimen dimensions and stacking sequences are listed in Tables 1–3 (in which the scale factor represents the ratio of the characteristic dimensions of the scaled specimen and the full-size (largest) specimen). The corresponding mean values of flexural strength f_r for different test data are summarized in Table 4. There are two points that need to be explained.

First, in Wisnom's [17] four-point bending test, only the failure strains were reported (as listed in Table 4). Nevertheless, the present analysis can use the strains rather the stresses because the material behaves almost linearly up to the peak load, i.e., the nonlinear effects which cause deviation from the elastic stress-strain curve may be neglected. Consequently, the stress at failure can be inferred from the reported strain.

Second, in Jackson's experiments [35], the testing method was convenient but the maximum bending moment was not measured. Very slender laminate strips were loaded by axial force causing them to buckle, and the deflections were so large (on the average about 40 percent of the specimen length) that the laminate was almost under pure bending (the maximum compressive and tensile strains almost equal). The axial shortening and the axial force

Table 1 Jackson's (1992) AS4/3502 beam-column test

Scale factor	Beam dimension (mm)	Unidirectional	Angle-ply	Cross-ply	Quasi-isotropic
1/6	1×12.7×127	$[0]_{8T}$	$[45_2/-45_2]_S$	$[0_2/90_2]_S$	$[-45/0/45/90]_S$
1/4	1.5×19.5×190.5	$[0]_{12T}$	$[45_3/-45_3]_S$	$[0_2/90_3]_S$	-
1/3	2×25.4×254	$[0]_{16T}$	$[45_4/-45_4]_S$	$[0_4/90_4]_S$	$[-45_s/0_2/45_2/90_2]_S$
1/2	3×38.1×381	$[0]_{24T}$	$[45_6/-45_6]_S$	$[0_6/90_6]_S$	$[-45_3/0_3/45_3/90_3]_S$
2/3	4×50.8×508	$[0]_{32T}$	$[45_8/-45_8]_S$	$[0_8/90_8]_S$	$[-45_4/0_4/45_4/90_4]_S$
3/4	4.5×57.15×571.5	$[0]_{36T}$	$[45_9/-45_9]_S$	$[0_9/90_9]_S$	-
5/6	5×63.5×635	$[0]_{40T}$	$[45_{10}/-45_{10}]_S$	$[0_{10}/90_{10}]_S$	$[-45_5/0_5/45_5/90_5]_S$
6/6	6×76.2×762	$[0]_{48T}$	$[45_{12}/-45_{12}]_S$	$[0_{12}/90_{12}]_S$	$[-45_6/0_6/45_6/90_6]_S$

Table 2 Wisnom's (1991,1997) four-point bending test

Scale factor	Unidirectional, E glass/913 Beam dimension (mm)	Unidirectional, E glass/913 Stacking sequence	Unidirectional, carbon XAS/913 Beam dimension (mm)	Unidirectional, carbon XAS/913 Stacking sequence
1/4	$2\times5\times45$	$[0]_{16}$	$3.175\times10\times102$	$[0]_{25}$
2/4	$4\times10\times90$	$[0]_{32}$	$6.350\times20\times204$	$[0]_{50}$
4/4	$8\times20\times180$	$[0]_{64}$	$12.70\times40\times408$	$[0]_{100}$

were measured but the maximum deflection was not. Fortunately this is not too serious a problem because, from the stress-deformation curve reported by Jackson [35], it may be inferred that the specimens behaved almost linearly up to failure. In that case, the deflection curve must have been the well-known "elastica" (Fig. 5), for which the exact ordinates are given by elliptic integrals according to a classical solution due to Kirchhoff [e.g., [36]]. Buckling in the form of elastica was previously introduced for laminate testing by Bažant and Skupin [37,38],[2] and a table of this curve computed by them can be used here (Table 5) and exploited to calculate the maximum bending moments in Jackson's bent strips from the reported maximum axial forces and relative end displacements (i.e., shortenings of the chord). The deformation is assumed to be nearly elastic up to the peak load. From the bending moment, the flexural strength σ_N is calculated for each specimen tested.

[2]In these works, the table of elastica was used for developing a very simple testing method of long-time (multi-year) stress relaxation in polymeric laminates in various aggressive environments. In this method, a laminate strip is strongly bent and its ends are supported on a fixed base that keeps the chord length of the arc constant. Despite stress relaxation, the shape of the deflection remains approximately constant, because of the linearity of viscoelastic behavior. At periodic intervals, the axial force is measured as the force needed to effect a very small increase of the shortening displacement between the ends. This method has been widely used in Czech Republic since the 1960s to measure the differences in environmental degradation between stressed and unstressed laminates of various types.

The efficient Levenberg-Marquardt nonlinear optimization algorithm has been applied to fit the test data by minimizing the sum of squared errors of the formula. First, the energetic (deterministic) formula (2) is fitted to test data, separately for each of the data sets listed in Table 4. This yields the optimum values of constants D_b and f_r^0 for each case. Then the relative strengths σ_N/f_r^0 (or f_r/f_r^0) are plotted versus the relative size D/D_b, where the optimum values of f_r^0 and D_b, different for each case, are used. It is seen that these data agree with the optimized energetic formula quite well; see Fig. 2(a), in which ω is an unbiased estimate of the coefficient of variation corresponding to the standard error of regression, i.e., $\omega=[\Sigma_i(y_i-\hat{y}_i)^2/(n-2]^{1/2}/\overline{y},$ where $\hat{y}_i$ are the ordinates of measured data points $(i=1,2,\ldots n)$, y_i are the corresponding formula predictions, and $\overline{y}=\Sigma_i\hat{y}_i/n$.

Subsequently the same fitting procedure is repeated with the more general energetic-statistical formula (5). The resulting optimum fit is shown in Fig. 2(b). As seen from the values of ω indicated in the figures, the more general formula yields no noticeable improvement of the fit this data set. In view of the previous experience with concrete, as well as the fact that Weibull theory must theoretically apply for $D\rightarrow\infty$, the lack of improvement is doubtless due to the fact that the size range is too limited in comparison to the width of the scatter band. A broadening of

Table 3 Johnson's (2000) AS4/3502 four-point bending test

Scale factor	Beam dimension (mm)	Angle-ply	Quasi-isotropic	Cross-ply
1/4	$2\times5\times45$	$[45/-45/45/-45/]_{2s}$	$[45/-45/0/90]_{2s}$	$[90/0/90/0]_{2s}$
2/4	$4\times10\times90$	$[45_2/-45_2/45_2/-45_2]_{2s}$	$[45_2/-45_2/0_2/90_2]_{2s}$	$[90/0/90/0]_{4s}$
4/4	$8\times20\times180$	$[45_4/-45_4/45_4/-45_4]_{2s}$	$[45_4/-45_4/0_4/90_4]_{2s}$	$[90/0/90/0]_{8s}$

Table 4 Means of flexural strength for various test data used in study

Size D, mm	Mean, MPa	Size D, mm	Mean, MPa	ε_f	Size D, mm	Mean, MPa	ε_f
Jackson, unidirectional		Jackson, angle-ply			Jackson, cross-ply		
1	1886	1	215		1	1858	
1.5	2044	1.5	198		1.5	1514	
2	1729	2	154		2	1523	
3	1751	3	99		3	1060	
4	1882	4	84		4	932	
4.5	1923	4.5	83		4.5	801	
5	1785	5	69		5	763	
6	1534	6	62		6	546	
Jackson, quasi-isotropic		Johnson, quasi-isotropic			Johnson, angle-ply		
1	777	2	566		2	1772	
1.5	-	4	547		4	1339	
2	634	8	342		8	902	
3	619						
4	511						
4.5	-						
5	471						
6	452						
Johnson, cross-ply		Wisnom, E glass/913			Wisnom, carbon XAS/913		
2	2814	2	198	4.395%	3.175	83	1.848%
4	2736	4	190	4.221%	6.350	77	1.703%
8	2674	8	167	3.711%	12.70	71	1.582%

Table 5 Geometrical properties of elastica (after Bažant and Skupin [29,30])

l_0/ρ	$\Delta(l_0/\rho)$	f/l_0	l/l_0	θ^0	$d(l/l_0)/d(l_0/\rho)$	l_0/ρ	l/l_0	f/l_0	θ^0	$d(l/l_0)/d(l_0/\rho)$
1.000	0.204	0.100	0.975	18.20	0.050	5.100	0.3772	0.480	87.80	0.157
1.207	0.207	0.120	0.963	20.94	0.061	5.355	0.3842	0.440	91.65	0.158
1.418	0.211	0.140	0.948	25.75	0.072	5.608	0.3899	0.400	95.37	0.158
1.634	0.216	0.160	0.932	29.63	0.082	5.860	0.3945	0.360	98.99	0.157
1.856	0.222	0.180	0.912	33.60	0.091	6.116	0.3981	0.320	102.57	0.155
2.086	0.230	0.200	0.891	37.69	0.100	6.382	0.4007	0.280	106.20	0.151
2.325	0.239	0.220	0.866	41.91	0.108	6.649	0.4023	0.240	109.75	0.148
2.575	0.250	0.240	0.838	46.30	0.115	6.921	0.4030	0.200	113.25	0.145
2.839	0.264	0.260	0.806	50.89	0.120	7.202	0.4027	0.160	116.75	
3.121	0.282	0.280	0.772	55.48	0.128	7.498	0.3988	0.120	120.32	
3.427	0.306	0.300	0.734	60.94	0.125	8.445	0.3913	0.000	130.72	
3.765	0.338	0.320	0.692	66.58						
4.148	0.383	0.340	0.648	72.85						
4.602	0.454	0.360	0.602	80.11						

the size range to much thicker laminates, or a reduction of scatter, or both, would be needed to see any significant improvement of data fit with the more general theory.

Finally, the same data are optimally fitted using the purely statistical Weibull size effect formula, $\sigma_N = f_r = f_r^0 (D_b/D)^{n_d/m}$; see Fig. 2(*c*,*d*). It is found that the complete data set cannot be fitted well using the same value of Weibull modulus m (shape parameter) for all the tests. For some tests, the optimum fit is obtained with Weibull modulus, $m=5$ (column c), and for others with 30 (column d). It is seen that the match of the purely statistical Weibull theory with the complete data set is quite poor if the m value is kept the same for all the test. Each of the two m-values is suitable just for some part of the data and not for the rest.

Figure 3 presents the optimum fits of various theories to nine individual size effect data for various types of laminates, obtained in different laboratories. The data are the same as those shown (and listed in the same sequence) in Fig. 2; column a pertains to the purely energetic theory, column b to the energetic-statistical theory, and columns c and d to the purely statistical Weibull theory. The value $m=5$ was found to give the optimum fit of some tests (cases c2, c3 and c5 in Fig. 3) while $m=30$ of some other tests (cases c1, d8, d9), and therefore both values are used in columns c and d to fit all the data.

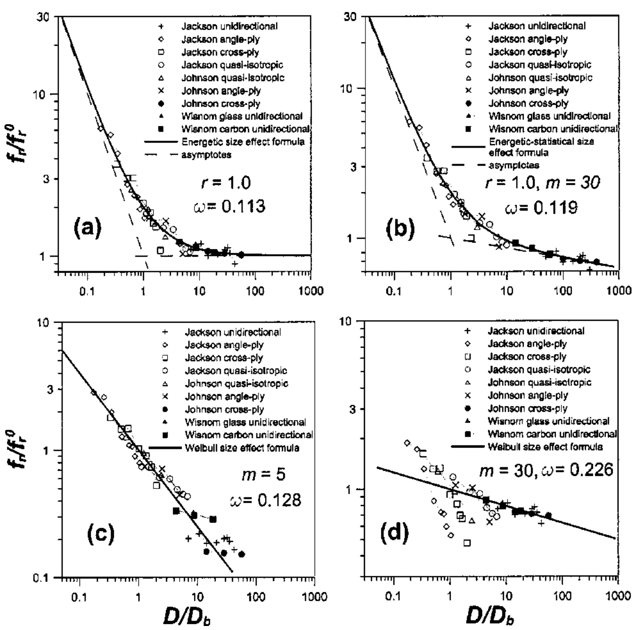

Fig. 2 Optimum fits of existing test data on modulus of rupture versus relative size, in dimensionless coordinates, by (*a*) deterministic energetic formula; (*b*) energetic-statistical formula; (*c*) Weibull size effect formula with $m=5$; and (*d*) Weibull size effect formula with $m=30$

Comparing columns and b in Fig. 3, note that, similar to Fig. 2, the fits in these two columns are about equally good. This confirms again that the existing test data do not suffice to document any advantage of the energetic-statistical theory over its special case, the purely energetic theory. The likely reason is that the size range of the existing data is not broad enough in relation to the scatter band width, causing that the different curvatures of the energetic and energetic-statistical formulas in the logarithmic plot (Fig. 2 top) cannot be distinguished due to data scatter within the limited size range. The energetic-statistical theory so far rests entirely on theoretical arguments.

Because laminates with various ply arrangements are macroscopically different materials, there is, in principle, no reason why m, as an effective property of the laminate, could not be different for each. Even though the cross-ply and angle ply laminates may be identical, they are loaded in different directions, and this could, in theory, also cause differences in m because m can depend on the loading direction in anisotropic materials. However, the differences in the optimum m-values seen in Fig. 2 are irrational, for two reasons:

1. First, the strength of an angle-ply laminate depends strongly on the polymeric matrix (and its bond to fibers), while the strength of an axially loaded unidirectional laminate depends mainly on the fibers. Since the failure of fibers is more brittle than the failure of matrix, one would expect for a unidirectional laminate that $m=30$ should give a better fit than $m=5$ but the opposite is seen in Fig. 3, cases c1, d1, c8, d8, c9 and d9. Likewise, for an angle-ply laminate, one would expect a higher optimum m value than for a cross-ply laminate, but the opposite is seen comparing cases c6, d6, c7, and c8 in Fig. 3.
2. Second, unlike the energetic and energetic-statistical theories, the purely statistical Weibull theory is characterized by a unique relationship between the size effect exponent and the coefficient of variation ω_W of the random scatter of the strength values for identical specimens (Appendix 2). This relationship is given by the well-known formula

$$\omega_W = \sqrt{\frac{\Gamma(1+2/m)}{\Gamma^2(1+1/m)} - 1} \tag{7}$$

[e.g., 6]. In principle, laminates of different layups could, of course, exhibit differences in scatter, corresponding to different ω and m. However, the values for the optimum m obtained by data fitting with formula (7) differ far too much:

$$\omega_W = 22.8 \text{ percent for } m=5 \tag{8}$$

$$\omega_W = 4.18 \text{ percent for } m=30 \tag{9}$$

Although the standard deviation of the strength tests has not

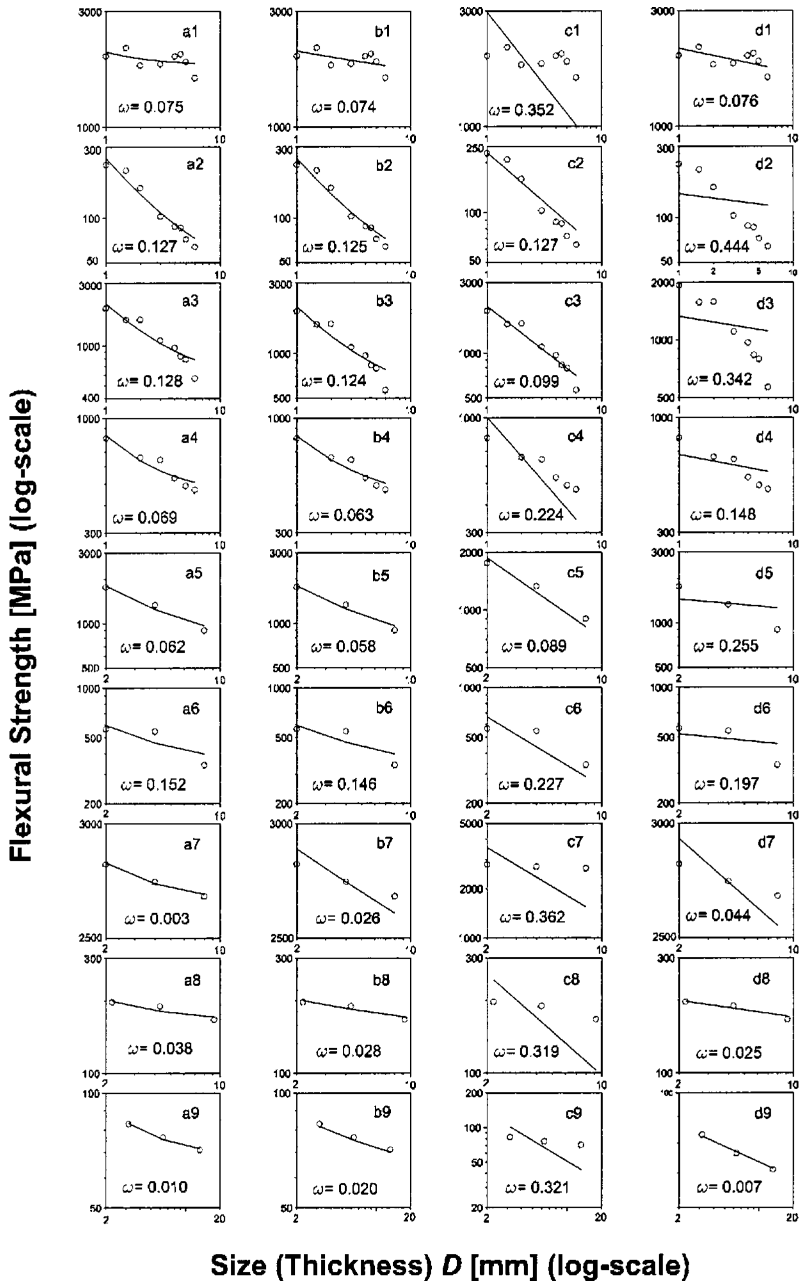

Fig. 3 Optimum fits of individual data sets by different formulas. (*a*) deterministic energetic size effect formula; (*b*) energetic-statistical size effect formula; (*c*) Weibull theory for $m=5$; and (*d*) Weibull theory for $m=30$. Numbers from 1 to 9 correspond to the data sets showed in Fig. 2.

been reported for the test data considered, such huge differences in scatter go against all experience and are impossible. Were the statistical Weibull theory applicable, the coefficient of variation of scatter for the angle ply laminate would have to be about 23 percent and for the unidirectional laminate about 4 percent, but this is not the case. The angle-ply laminate, due its greater ductility (lower brittleness), would be expected to give a somewhat smaller scatter (larger m) than the unidirectional, cross-ply or quasi-isotropic laminate, but the opposite is systematically noted in Fig. 3 by comparing case c2 with d1 and c6 with d5 and d7.

Therefore, the classical hypothesis that the size effect is purely statistical is untenable.

Similar observations can be made by returning to the overall fits in Fig. 2(c,d). It is seen that the small-size and large size data are well fitted by Weibull statistical theory with $m=5$ and $m=30$, respectively. This implies that the coefficient of variation would be 22.8 percent for small sizes and 4.16 percent for large sizes. Such a huge difference is not verified by experiments. Anyway, the purely statistical theory could be valid only if the coefficient of variation of strength were independent of the size.

Noting that Weibull theory is well established for flexure of fine-grained ceramics, which are very homogeneous, one might suspect whether a different conclusion might result if one would specifically take into account the fact that, according to the lamination theory, the stresses change discontinuously between the lamina and have different nonlinear distributions for different lay-ups. However, exponent $-n/m$ of Weibull size effect is inde-

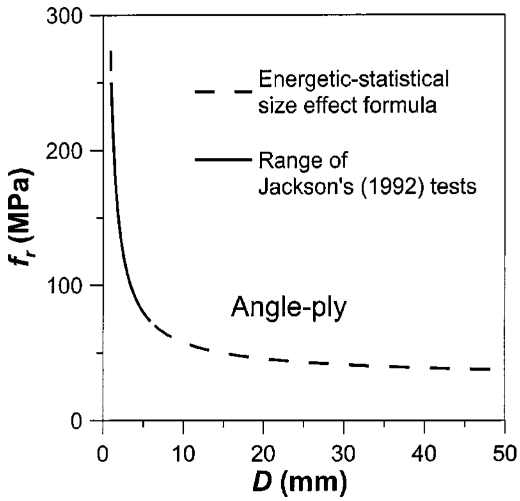

Fig. 4 Energetic-statistical formula of Jackson's angle-ply data (actual scale)

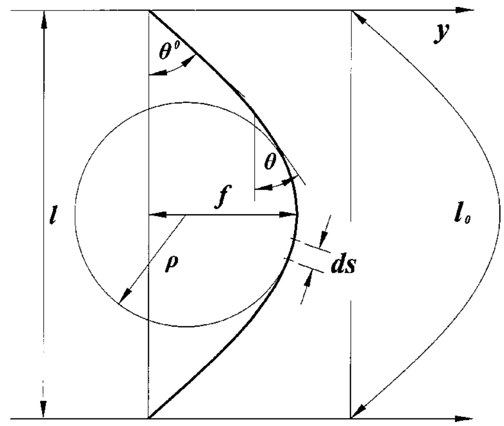

Fig. 5 The curve of elastica used in calculations for Table 5

pendent of the stress distribution across the laminate, as long as the stress distributions across the laminates of different thicknesses are similar (which is assured because the lay-ups in Jackson's tests were similar); see Appendix 2 where a proof of this point is given.

The ambiguity of data interpretation due to scatter and limited size range may be clarified by Bažant and Novák's study of similar but much more extensive data for concrete [10,11]. It documents how the fitting of individual data can yield wide-ranging results. Instead of separate fitting of each data set for one concrete, one must simultaneously optimize the fit of the combination of all the available data in one plot while the value of Weibull modulus m is forced to correspond to a common asymptotic value for very large sizes ($D/D_b \rightarrow \infty$). In the energetic-statistical formula, the quasibrittle behavior implies a decrease of the modulus of rupture with the size, while the pure statistical Weibull size effect line of slope $-n_d/m$ is approached asymptotically (as in Fig. 2(*b*)). Such apparent Weibull modulus m represents the large-size asymptotic Weibull modulus. It was shown that its value may be considered as common for all the data sets for concrete. Then, in the overall plot of the data, the individual data sets are positioned in the relative size range of D/D_b corresponding to their "individual" apparent Weibull modulus—those with a small apparent m-value in the individual Weibull fit will be close to the small-size asymptote, while those with a large apparent m will be close to the large-size asymptote in the overall combined size effect plot such as Fig. 2(*b*).

Because the available data are mainly limited to the range of small sizes, the automatic iterative fitting procedure that was used for concrete [11] to identify parameter r could not be applied here. To avoid the arbitrary guessing of r and of the asymptotic Weibull modulus m, we choose a set of m values, namely 5, 15, 25, 30, which include the (so far) widely accepted value of 25 for carbon fiber composites and 15 for glass fiber composites, and a set of r values, namely 0.6, 0.8, 1.0, 1.2. After comparing the coefficient of variation ω of regression errors for different combinations of r and m, the values

$$r = 1, \quad m = 30 \tag{10}$$

appear to be optimum. Because of limited data, no definite conclusion can be made about the coefficient of variation.

Is is not clear whether the failure of Jackson's specimens was initiated by tension or compression. However, the Weibull as well as the energetic theories are valid for either case. The only condition it that small and large specimens should fail in the same manner.

For the sake of illustration, Fig. 4 shows the energetic-statistical size effect for angle-ply specimens in the actual linear scales. The size range of Jackson's (1992) data is indicated in the figure.

It may be concluded that the size effect in the tests studied here must be primarily deterministic energetic, caused by stress redistribution within the cross section of the laminate, with the corresponding energy release. The influence of strength randomness in this size range of data is small, although it is likely to become strong for much larger sizes.

Previous Viewpoints

1. Johnson et al. [16] pointed out that the failure of a flexed laminate may start in the second (or next) ply from the surface if it is sufficiently weaker that the first ply. This may for example be the case for a ply with fibers in a transverse direction because the transverse tensile strength of fiber-polymer lamina is much lower than the longitudinal strength. Such behavior, however, is not in conflict with the nonlocal theory because this theory in intended to provide only a homogenized macroscopic description of a microscopically irregular failure process in a heterogeneous medium.

2. Wisnom [39] listed several possible factors that may influence the size effect in unnotched fiber composites: (1) material defects; (2) free-edge effects; (3) stress gradient; (4) specimen manufacture and preparation; and (5) testing procedure. The last two effects cannot be covered by any material model and require a separate consideration. The material defects are what gives rise to randomness of strength in the homogenizing macroscopic continuum and leads to the statistical part of size effect, which is included in the present formulation. The stress gradient effect is part of the present theory and represents a simplified way to look at the energetic size effect—simplified, because what matters is the energy release rate which depends on the entire stress field in the structure (and thus also on the free-edge effects), and not just the local gradient. Referring to the classical tests of Daniel and Weil [40] and his own [41], Wisnom argued that the stress gradient effect (in our approach, the energetic effect) need not be the primary cause of size effect. This is possible but must be qualified by structure size. Indeed, according to the present theory, the size effect in a large enough structure is governed primarily by statistics. Wisnom's arguments are based on a set of mere 7 data points, which is far smaller set than the presently analyzed set of 45 data points. As for [40], it should be noted that these classical flexural tests were made on ceramics (aluminum oxide and beryllium oxide) rather than fiber-polymer composites, and that the size range of these tests was very limited (3.175 mm–12.7 mm).

3. Some researchers, doubting the applicability of Weibull theory to scaled fiber-polymer laminates [16,39], argued that the failure modes for different sizes are not the same. This is of course possible, but in a certain sense it is reflected in the preceding proposed energetic-statistical formula. For the small size range,

this formula reduces to the purely deterministic formula, which implies the size effect to be governed by a stable spread of cracking. For the large size range, the size effect is due to the random occurrence of defects at or near the tensile face, which produce a different mode of failure—a sudden brittle collapse.

4. Johnson et al. [16] noted that the stacking sequence may influence the size effect. This is of course valid, but all that it means is that the value of D_b may depend on the stacking sequence and that the distribution of local material strength across the laminate may have to be considered as nonuniform (see Appendix 2). No data that would suffice to assess this effect seem to exist.

5. Jackson [15] compared the normalized failure loads to a scale factor considering either the pure statistical Weibull theory or the fracture based strength-size relation of Atkins and Caddell [42]. The former has already been commented on. The latter did not fit the test data well, which is not surprising because the strength-size relation was based on linear elastic fracture mechanics, the applicability of which to Jackson's experiments is questionable.

Conclusions

1. In textbooks as well as practice, the failure of fiber-polymer laminates has so far been treated according to the strength theory or plastic limit analysis, which exhibits no size effect, and all the size effects have been considered as purely statistical. The present analysis of existing experimental data indicates that this approach needs to be fundamentally revised.

2. The size effect on the flexural strength of laminates appears to be primarily energetic (deterministic) rather than statistical, except possibly for very large thicknesses for which the statistical size effect might also be significant. This further implies that fracture mechanics, rather than some strength criterion (or material failure criterion expressed in terms of stresses and strains), needs to be used for evaluating the strength of laminates. The fracture mechanics approach must take into account the quasibrittle (or cohesive) nature of fracture.

3. Fitting of the existing size effect test data by the statistical theory implies excessive and unrealistic differences in the coefficient of variation of strength tests. This experimentally demonstrates the inapplicability of that classical theory to laminates.

4. The improvement over the statistical theory that is achieved in the fitting of the existing experimental data supports the applicability of both the energetic theory and the energetic-statistical theory, Eq. (5). However, the available data do not suffice to demonstrate that the energetic-statistical theory is better than the energetic theory (which is a special case). Experimental data of a broader size range or lower scatter, or both, would be needed for that purpose. Superiority of the energetic-statistical theory so far rests only on theoretical arguments.

Appendix 1

Relation to Weakest-Link Statistical Models with Load-Sharing. To capture the interplay of stress-redistribution and statistical size effects, another, more classical, avenue of approach, aiming from an opposite side of the problem toward the same objective, has been pursued in theoretical research. It consists of a generalization of the extreme value statistics of the weakest link model by introduction of various phenomenological hypotheses about load-sharing in some critical cluster, the simplest prototype of which is Daniels' fiber bundle model [43].

The theory as well as massive Monte Carlo simulations show that the composite strength distribution deviates from Weibull's by a concave curvature in Weibull probability paper and that the apparent (effective) Weibull modulus increases with the critical cluster size, which are features resembling those of the nonlocal Weibull theory. Although this avenue of approach, pursued by S. Leigh Phoenix and co-workers [e.g., [13,14]] and others [44–46], might seem to retain a purely statistical description of the size effect, the load-sharing hypotheses of one kind or another in effect produce stress redistribution associated with energy release.

Rigorous treatment of statistics in these works has proven to be mathematically very challenging and has led to high mathematical sophistication. Although valuable mathematical results have been achieved for the statistical distribution of strength in tensioned parallel structural systems such as ropes, cables, yarns or fiber strands with statistical variation of strength, this avenue of approach has not yet made it possible to deal specifically with the load-sharing properties governed by cohesive fracture mechanics.

In particular, it is not yet clear how various load-sharing concepts could be generalized to capture the multi-dimensionality of stress redistribution caused by fracture and its fracture process zone, and how they could capture the disparity between the energy release and energy dissipation rates, which is the physical source of energetic size effect—particularly the fact that the energy release rate grows with increasing structure size roughly quadratically while the energy dissipation rate grows roughly linearly.

For practical applications, it thus seems more profitable to approach the problem of stress redistribution from the opposite side, as a probabilistic generalization of the energetic size effect theory [47].

The size effect has also recently been analyzed on the basis of a nonlocal continuum model enhanced by weakest-link statistics [47].

Appendix 2

Independence of Size Effect Exponent of Stress Distribution. The size effect exponent $-n_d/m$ is not affected by the fact that the lamination theory predicts a discontinuous nonlinear stress distribution for various lay-ups. To prove it, consider the well-known derivation of Weibull size effect in geometrically similar structures of various sizes D in which the stress distribution $S(\xi)$ is independent of D ($\xi = x/D$ = relative coordinates of material points, x actual coordinates). The structure is considered as an assembly of smallest elementary volumes V_0 for which the concept of stress makes sense. Denote P_k = failure probability of the k-th elementary volume ($k=1,2,\ldots,N$) and P_f = failure probability of the structure. If the failure of one small elementary volume V_0 is assumed to cause the whole structure to fail, then the probability of survival of the structure is the joint probability of survival of all its small elementary volumes, i.e.,

$$1-P_f=(1-P_1)(1-P_2)\cdots(1-P_N) \tag{11}$$

$$\text{or}\quad \ln(1-P_f)=\sum_{k=1}^{N}\ln(1-P_k)\approx-\sum_{k=1}^{N}P_k \tag{12}$$

where we took into account the fact normally $P_k \ll 1$. The basic idea of Weibull [21,22] was that the tail of the cumulative distribution of strength must be a power law, i.e., $P_k=[\sigma(x_k)/\sigma_0]^m$ where σ_0 and m are material constants called the scale parameter and Weibull modulus (shape parameter), and $\sigma(x)$ is the positive part of the maximum principal stress at point x. Substituting this into (12) and making a transition from a discrete sum to an integral over structure volume V, one gets the well-known Weibull probability integral:

$$-\ln(1-P_f)=\int_V[\sigma(x_k)/\sigma_0]^m \mathrm{d}V(x)/V_0 \tag{13}$$

Geometrically similar structures of different sizes D are identical in dimensionless coordinates $\boldsymbol{\xi}=\boldsymbol{x}/D$, and because their stress fields must be similar, one may set $\sigma(\boldsymbol{x})=\sigma_N S(\boldsymbol{\xi})$ where σ_N = nominal stress and $S(\boldsymbol{\xi})$ = dimensionless stress distribution, which is independent of D. Substituting this and $\mathrm{d}V(\boldsymbol{x}) = D^n \mathrm{d}V(\boldsymbol{\xi})$ into (13) (where n_d = number of spatial dimensions in which the structure is scaled, $n=1$, 2, or 3), we get, after rearrangements,

$$-\ln(1-P_f)=(\sigma_N/S_0)^m D^n \quad \text{where}$$

$$S_0^{-m}=\sigma_0^{-m}\int_V S^m(\boldsymbol{\xi})\mathrm{d}V(\boldsymbol{\xi})/V_0 \tag{14}$$

$$\text{or} \quad P_f(\sigma_N)=1-\mathrm{e}^{-(\sigma_N/S_0)^m D^n} \tag{15}$$

which is the Weibull cumulative distribution of nominal strength. From (14),

$$\sigma_N=C_0 D^{-n/m}, \quad C_0=S_0[-\ln(1-P_f)]^{1/m} \tag{16}$$

This equation, in which C_0 is independent of D, gives the scaling of nominal strength for a fixed failure probability (e.g., the median σ_N for $P_f=0.5$). Using (15), one gets the scaling of the mean nominal strength

$$\overline{\sigma_N}=\int_0^1 \sigma_N \mathrm{d}P_f=\int_0^\infty \sigma_N \frac{\mathrm{d}P_f}{\mathrm{d}\sigma_N}\mathrm{d}\sigma_N=S_0\Gamma\left(1+\frac{1}{m}\right)D^{-n/m} \tag{17}$$

The standard deviation δ_N is similarly obtained as $\delta_N^2=\int_0^1\sigma_N^2\mathrm{d}P_f-\overline{\sigma}_N^2$. The coefficient of variation of strength is then $\omega=\delta_N/\overline{\sigma}_N$, which gives formula (7).

The point to be noted is that the power law exponent in (17) is independent of the stress distribution across the laminate as long as this distribution is the same for different sizes. The same is true for ω.

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