Asymptotic Matching Analysis of Scaling of Structural Failure Due to Softening Hinges. II: Implications

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Abstract: The asymptotic matching analysis in the preceding Part I has established a new kind of size effect, caused by postpeak softening of inelastic hinges in beams and frames. The present Part II analyzes various implications, particularly the size effects on the maximum loads under load control or displacement control, the design loads, and the energy absorption of the structure.

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Introduction

The complexity of analysis of beams and frames with softening hinges was circumvented in the preceding Part I (Bažant 2003) by adopting the asymptotic matching philosophy. This permitted formulating approximate size effect laws for the peaks and the troughs of the load–deflection diagram. The present Part II will analyze the implications of these laws for the size dependencies of the maximum loads under load control or displacement control, of the design loads, and of the energy absorption of the structure. All the definitions and notations made in Part I (Bažant 2003) are retained.

Maximum Load or Displacement Attainable Under Load or Displacement Control

When the load–deflection diagram has multiple peaks P_i, it might seem that the design should be based on the highest one. But it is not so simple.

Consider first the case of *load control* (which may, for instance, be imagined as loading the structure by a bucket that is gradually being filled with water). Figs. 1(a–c) illustrate various possible responses for structures large enough so that the hinges soften one by one. The equilibrium (static) response paths are marked by single arrows, the dynamic path by double arrows. The vertical difference between the load values on the static and dynamic paths, divided by the associated mass of the structure, represents the acceleration of the load point. When peak P_i is reached, the structure undergoes dynamic snapthrough along the horizontal line P_iAB or 1357. Moving from peak P_i to point A [Fig. 1(a)], the structure accelerates and at point A its kinetic energy is given by the area W_i of the triangle $AP_iP_i'A$, representing an energy released by the structure

$$W_i=\frac{P_i}{2}\left(\frac{1}{K_{i+1}}-\frac{1}{K_i}\right)(P_i-P_i') \tag{1}$$

Continuing from point A to point B, the structure decelerates, and at B its kinetic energy is W_i-W_i', where W_i' is the area of the triangle $ABP_{i+1}A$, representing the energy stored in the structure upon moving from A to B;

$$W_i'=\frac{(P_{i+1}-P_i)^2}{2}\left[\frac{1}{K_{i+1}}-\frac{1}{P_{i+1}-P_{i+1}'}\left(\frac{P_{i+1}}{K_{i+1}}-\frac{P_{i+1}'}{K_{i+2}}\right)\right] \tag{2}$$

If $W_i-W_i'>0$, as shown in Fig. 1(a), and if the damping is not negligible, the kinetic energy has not been fully absorbed by the structure. So the structure continues to increase its load-point displacement w, i.e., the snapthrough continues, and the peak P_{i+1}' can never be reached. In that case, the maximum load attainable under load control is P_i. If finite damping is taken into account, then the condition of continuing snapthrough at load P_i may be written as

$$\chi W_i>W_i' \tag{3}$$

where χ=empirical damping factor, $\chi<1$.

If, on the other hand, $W_i-W_i'<0$ or, with damping, $\chi W_i-W_{i+1}'<0$, as shown in Fig. 1(b), then the kinetic energy gained before point A is insufficient for the structure to swing all the way to point B. The forward movement stops short of point B at some point B' for which the triangular area above the segment AB' is equal to W_i or, with damping, to χW_i, and then the structure begins to swing back. Without any damping it would permanently oscillate between P_i and B', but since there always is damping, the oscillation will come to a standstill at point A. Then the load can be stably increased to the next peak P_{i+1}. So in this case the snapthrough gets arrested and the maximum load attainable under load control is P_{i+1}.

The behavior can get more intricate if there are many subsequent peaks. In Fig. 1(c), for instance, the second peak 4 is, under load control, unattainable because triangle 3543 is smaller than triangle 1231. However, the combined area of triangles 3543 and 7987 is smaller than the combined area of triangles 1231 and 5675, and so the snapthrough cannot reach all the way to point 9; rather, the structure swings back, and after its oscillations get damped, the third peak P_{1+2} can be reached stably under load control.

If stable equilibrium at peak 4 in Fig. 1(c) were reached under

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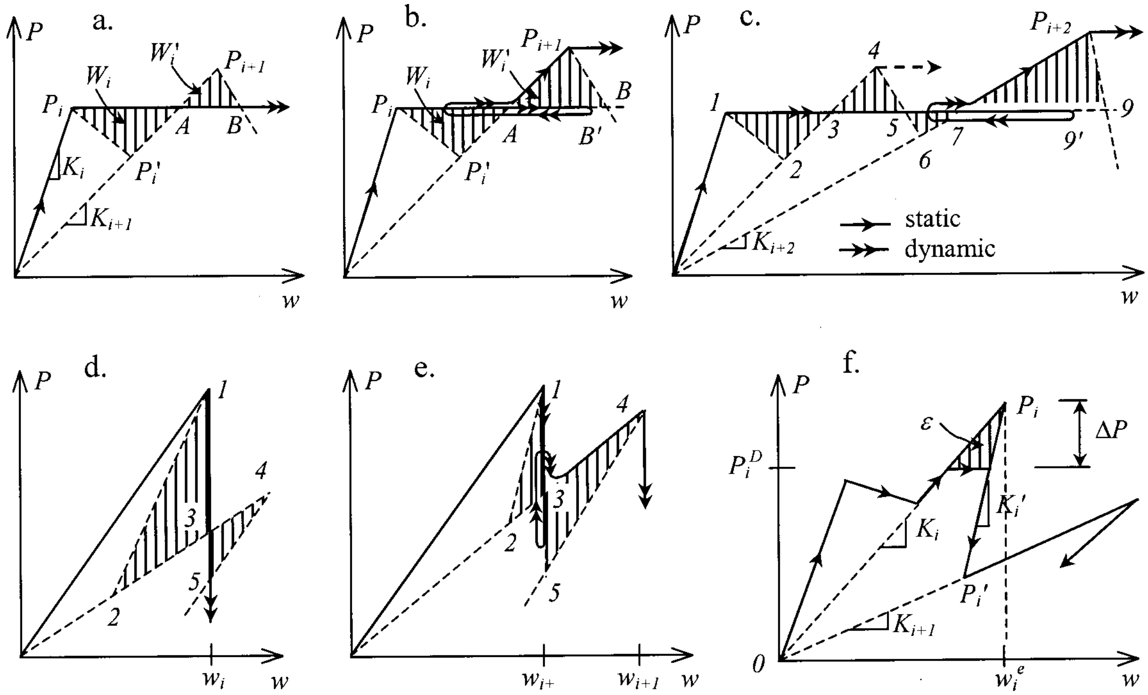

Fig. 1. (a)–(c) Snapthrough under load control and load capacity; (d)–(e) snapdown under controled displacement and maximum attainable deflection; (f) design load P_i^D corresponding to snaptrough caused by dynamic disturbance whose kinetic energy is $\mathcal{E}$

load control, and then a switch to load control were made, the structure would exhibit an unbounded snapthrough and the third peak P_{i+2} could never be reached. Thus, a displacement control during the initial loading can reduce the load attainable under subsequent load control. A switch from displacement control to load control may be destabilizing.

When there are snapbacks, similar phenomena occur under displacement control. In Fig. 1(d), after reaching snapback point 1 (local maximum of w), the load, at constant load-point displacement, drops dynamically (due to motion of the rest of the structure, except the load point); this is called the snapdown (Bažant and Cedolin 1991, Chap. 4). Passing from 1 to 3, the structure accelerates, and the complementary energy (enthalpy) given by area 1231 (minus whatever energy has been lost by damping), gets converted into kinetic energy of the structure. Continuing below point 3, the structure decelerates and at point 5 the complementary energy given by area 3543 (minus whatever is lost by damping) has been transferred from kinetic energy of the structure into its strain energy. For small enough damping, the snapdown in Fig. 1(d) does not become arrested because the triangle 3543 is smaller than the triangle 1231. So the displacement cannot exceed its value at point 1, and the second snapback point 4 can never be reached.

On the other hand, when area 3543 is larger than area 1234 (reduced by whatever energy is lost by damping), then point 5 cannot be reached [Fig. 1(e)]. The structure (except the load point) swings back; the load value (actually a reaction at the load point) increases at constant load-point displacement (an upward movement toward point 1) and oscillates until the load value stabilizes at point 3. Then, under displacement control, the displacement can be increased stably to the next snapback point 4.

Design Load and Its Size Dependence

If the structure is large enough for the load-deflection diagram to involve softening segments, designing for the highest peak, $\max P_i$, becomes questionable because, under load control, the structure is unstable during each softening segment. Such a design becomes even more questionable when the structure is large enough for snapbacks to occur, in which case it is unstable even under load-point displacement control (Bažant and Cedolin 1991, Chap. 7).

A completely safe design is that for the lowest trough preceding the overall load maximum, $\min P_i'$. But then again the safety margin would often be unnecessarily high by far. A realistic design load P lies somewhere between $\max P_i$ and $\min P_i'$, for a smaller structure closer to $\max P_i$ and for a larger one closer to $\min P_i'$.

To decide the proper design load rationally, one should take into account the geometric imperfections and possible dynamic disturbances. In structures with plastic hinges, a dynamic disturbance in which the peak load is reached only temporarily leads merely to a permanent deflection but in the case of postpeak softening or even snapback it may trigger stability loss and sudden explosive failure.

The problem is similar to that of an axially compressed thin cylindrical shell (or a thin spherical dome), for which the critical load of a theoretically perfect shell (unattainable in practice) is followed by a snapback to a very low residual load. It is now well understood (e.g., Bažant and Cedolin 1991) that, due to inevitable imperfections and dynamic disturbances, one must design the shell for that residual load (typically between 1/8 and 1/3 of the theoretical critical load). An effective semiempirical method has been developed to deal with this problem for shells.

Taking dynamic disturbances into account usually guards against geometric imperfections as well. So let us now focus solely on the latter, characterized by a given kinetic energy $\mathcal{E}$ that might be accidentally imparted to the structure, e.g., by impact or vibrations of some carried object.

In the light of size effect, the question arises whether the same energy $\mathcal{E}$ should be considered for all the sizes D. Obviously it should not. For instance, one would not design for a bulldozer passing over a pedestrian bridge. So it makes sense to assume that the larger the structure, the larger is the $\mathcal{E}$ value that should be considered in design.

With this intuition, it will be assumed, somewhat arbitrarily, that structures should be designed for an imparted energy proportional to the strain energy stored in the structure at load peak P_i, which is $P_i^2/2K_i$; therefore

$$\mathcal{E}=\eta P_i^2/2K_i \tag{4}$$

where η=constant (size-independent) disturbance factor (such as 20%) to be decided on the basis of design experience (or, better, probabilistic reliability analysis).

If the structure is large enough for P_i to be a peak, a safe design load, P_i^D, is that for which the horizontal "snapthrough" line shown in Fig. 1(e), lying at distance ΔP below the peak, cuts off a triangle of area $\mathcal{E}$. This area may be calculated as follows:

$$\mathcal{E}=\frac{\Delta P^2}{2\bar{K}_i}, \quad \frac{1}{\bar{K}_i}=\frac{1}{K_i}-\frac{1}{K_i'}, \qquad \frac{1}{K_i'}=\frac{1}{P_i-P_i'}\left(\frac{P_i}{K_i}-\frac{P_i'}{K_{i+1}}\right) \tag{5}$$

where K_i' is the slope from load peak P_i to the next load trough P_i'. Substituting this into Eq. (4), we obtain the design nominal strength

$$\sigma_N^D=\max_i \sigma_{Ni}(1-\sqrt{\eta \bar{K}_i/K_i}), \quad \sigma_{Ni}=P_i/bD \tag{6}$$

Here P_i, K_i', and $\bar{K}_i$=functions of size D and an additional size effect is caused by the fact that, for different sizes D, peaks of different number i yield the maximum. Plotting the size effect curves of $\log \sigma_N^D$ versus $\log D$, one observes, as expected, that the larger the η, the stronger the size effect. Due to differences in the sharpness of the subsequent spikes, the overall maximum σ_{Ni} can be associated with different i for different D, even if D is large.

Dependence of Energy Absorption on Structure Size

The size effects of softening hinges are important not only for the load capacity of redundant brittle beams and frames (as well as plates), which is the focus of this study, but also for the energy absorption capability of structures, which is the most important characteristic for the resistance to earthquake, blast, shock, and impact). The energy dissipated by the failure of a structure with N softening hinges is

$$\mathcal{W}=\sum_{i=1}^{N} G_f b_i D_i=D^2\sum_{i=1}^{N} G_f \frac{b_i}{D}\frac{D_i}{D}\propto D^2 \tag{7}$$

where b_iD_i=cross section area of hinge i; and the ratios b_i/D and D_i/D=constant for geometrically similar structures.

For comparison, consider a plastic structure in which the moment-rotation diagrams have a horizontal plateau long enough for all the hinges to reach their plastic moment capacity simultaneously. In that case, the energy dissipated by the structure is

$$\mathcal{W}_{\mathrm{pl}}=\sum_{i=1}^{N} \sigma_p \epsilon_{\mathrm{pb}} b_i h_i l_i=D^3\sum_{i=1}^{N} \sigma_p \epsilon_{\mathrm{pb}}\frac{b_i}{D}\frac{h_i}{D}\frac{l_i}{D}\propto D^3 \tag{8}$$

where ϵ_{pb}=ductility limit of the plastic material (the strain at which the plastic material breaks); and l_i=effective length of the hinge region, which is known to be approximately similar to beam depth in the case of plastic behavior (i.e., $l_i/D\approx$const).

So the size effect on the energy absorption capability of a structure with softening hinges is

$$\mathcal{W}\propto \mathcal{W}_{\mathrm{pl}}/D \tag{9}$$

Summary and Conclusions

1. The main idea of this two-part study is to apply the technique of asymptotic matching to structures in which the inelastic hinges lack a yield plateau but exhibit progressive postpeak softening of the bending moment at an increasing rotation. The softening is typically caused by propagation of a tensile cohesive crack from the tensile face or by compressive fracture (or crushing) of the material at the compression face. The crack or compression crushing band is characterized by an approximately constant value of the energy dissipated per unit area, representing the tensile or compressive fracture energy of the material.
2. The postpeak softening of inelastic hinges implies the existence of a size effect, both on the dimensionless peak bending moment $\bar{M}_0$ in the hinge and on the dimensionless postpeak bending stiffness $\bar{R}_t$ of the hinge.
3. The size effect on $\bar{M}_0$ is similar to that amply verified by tests of the modulus of rupture and is caused by the development of a fracture process zone of a certain fixed depth D_b attached to the beam face.
4. The size effect of $\bar{R}_t$ is due to propagation of a crack, or to localization of compression crushing into a band of a fixed width. It causes the downward postpeak slope $\bar{R}_t$ of the moment-rotation diagram to increase in magnitude as D^3. By contrast, if a softening stress-strain relation with no localization were assumed, $\bar{R}_t$ would increase as D^2.
5. In a large enough statically indeterminate beam or frame, the softening hinges form one by one, i.e., only one hinge is softening at a time. In that case the load–deflection response is simple to solve. It represents a series of spikes whose width decreases to zero as $D\to\infty$. (Whether or not this kind of behavior occurs for real structures is irrelevant since the main purpose of solving this simple asymptotic case is to obtain a support for an asymptotic matching formula applicable through the entire range.)
6. The nominal strength for the peaks of the spikes on the load-deflection diagram in a large enough structure exhibits the same size effect as the modulus of rupture, i.e., the size effect disappears for large enough sizes. The ratio of the load drop from each peak to the next trough increases as $1/D$. This is a very strong size effect, such that a size increase rapidly leads to snapback instabilities and extreme sensitivity to geometric imperfections or dynamic disturbances, similar to those seen in shells.
7. For large enough structures, it may be dangerous to base the design on the maximum peak. A safe design load capacity should be based on the analysis of geometric imperfections or dynamic disturbances and lies somewhere between the maximum peak and the subsequent trough—the larger the structure, the deeper below the peak.
8. When D decreases, the troughs and multiple peaks on the load–deflection curve disappear, i.e., there is only one peak and no trough). Thus, a small enough structure is insensi-

tive to geometric imperfections and dynamic disturbances, and a design based on the maximum load is safe.

9. Eventually, for small enough D, two or more hinges soften simultaneously. If that is the case, an exact yet simple analytical formula for the size effect is impossible. The idea of the present approach is to skip analyzing this complex case. Instead, an approximate and simple size effect formula is obtained by asymptotic matching (''interpolation'') between the nominal strength formula for $D \to 0$ and the nominal strength formula for $D \to \infty$. The former is easily obtained in the classical way—by plastic limit analysis.
10. The differences of the present scaling formulas from those proposed by Bažant (1984) are caused by two significant differences in the failure behavior and assumptions: (1) The failure is not due to propagation of one dominant fracture or damage band but requires several fractures to take place, in several hinge regions of beams; and (2) geometric similarity of the crack lengths in the softening hinges in similar structures of different sizes cannot reasonably be assumed.
11. The ratio of the energy absorption capability of a large structure with softening hinges to that of a structure with plastic hinges decreases inversely to the structure size D.

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